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Power-divergence copulas: A new family of Archimedean copulas

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Abstract: This paper demonstrates that, under a particular convention, the convex functions that characterise the phi divergences also generate Archimedean copulas in at least two dimensions. As a special case, we develop the family of Archimedean copulas associated with the important family of power divergences, which we call the *power-divergence copulas*. The properties of the family are extensively studied, including the subfamilies that are absolutely continuous or have a singular component, the ordering of the family, limiting cases (i.e., the Fréchet-Hoeffding lower bound and upper bound), the Kendall's tau and tail-dependence coefficients. We obtain partial results that allow a sub-family to extend to three dimensions. Two particular cases are also shown to extend to infinitely many dimensions. We then apply our new copulas to the analysis of the dependence between the DJIA and S&P500 stock-market indices from October 2022 to September 2026 to showcase their potential usefulness.

Key words and phrases: alpha divergence; bivariate; f divergence; multivariate; phi divergence.

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1. Introduction

Copulas are flexible mathematical tools for modelling the dependence between two or more random variables. By Sklar's theorem (Sklar, 1959), the joint distribution of two or more continuous random variables can be uniquely decomposed into a set of marginal cumulative distribution functions (CDFs) that characterise the marginal behaviour, and a copula function that characterises the dependence between them. A wide range of copulas exist to capture many different kinds of dependence structures. See Größer and Okhrin (2022) for a recent overview.

Archimedean copulas (e.g., Nelsen, 2006, Ch. 4) are a broad subclass of copulas characterised by a convex and strictly decreasing function called an Archimedean-copula generator. The applications of Archimedean copulas include actuarial science (Kuluratne et al., 2021), finance (Fenech et al., 2015), and many others. They have found varied and successful applications because they are easy to construct; their properties are well understood; and they can model asymmetric dependence and dependence between extreme events that are frequently observed in real-world data. This is in contrast to assuming, for example, that two or more variables are jointly Gaussian or follow a Gaussian copula model, which offers no possibility of capturing the aforementioned dependence structures. This can lead to negative real-

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world outcomes, such as systematic underestimation of loan-default risks in the financial sector (Fenech et al., 2015).

Generators of Archimedean copulas may be derived from other mathematical objects. For example, Spreeuw (2014) showed that Archimedean copulas can be derived from econometric utility functions. The phi (ϕ) divergences (Csiszar, 1963; Morimoto, 1963; Ali and Silvey, 1966) are statistical divergences between non-negative functions (e.g., probability densities) with myriad uses in information theory (Amari, 2016) and statistics (Cressie and Pardo, 2002). Recent work has established a relationship between copulas and the ϕ divergences (Geenens and Lafaye de Micheaux, 2022). While this paper also concerns (Archimedean) copulas and ϕ divergences, we consider a different link between them. Specifically, we demonstrate that, under a certain convention, the same convex functions that generate ϕ divergences are also valid Archimedean-copula generators. This observation allows us to develop new bivariate/multivariate distributions with practical utility.

An important and well known family of ϕ divergences is the power divergences (Cressie and Read, 1984; Read and Cressie, 1988). They are also called the alpha (α) divergences in information theory (Amari, 2016). This family, indexed by a single real-valued parameter, $\lambda \in (-\infty, \infty)$, connects several divergences, including the Kullback-Leibler divergence, the Pearson

χ^2 divergence, and the Hellinger distance. Using the convex generator of the power divergences (Cressie and Pardo, 2002) as an Archimedean-copula generator results in the family of *power-divergence* (PD) copulas. In this paper, we characterise the properties of the PD copulas. We show they have subfamilies that are absolutely continuous or have a singular part; that they are negatively ordered with respect to their parameter; that they are capable of modelling both negative and positive dependence; that they have the Fréchet-Hoeffding lower bound and upper bound as limiting cases; that they exhibit a constant moderate upper tail-dependence coefficient, while the lower tail-dependence coefficient varies with λ; and that certain values of the parameter λ also allow the PD copulas to be valid Archimedean copulas in more than two dimensions. We also show how to calculate the PD copulas and simulate from them. To illustrate practical applications, we use PD copulas to model the dependence between two stock-market indices.

The rest of the paper is structured as follows. Section 2 establishes the background for our theoretical developments and introduces the phi-divergence copulas. Section 3 develops the power-divergence (PD) copulas as a special case. Section 4 considers computational aspects. Section 5 uses the PD copulas to analyse a real dataset. Section 6 discusses and concludes. R code for this paper can be found at `https://github.com/apear9/PD_`

copula_paper.

2. Background

2.1 Copulas

A copula in d dimensions is a d-variate joint cumulative distribution function (CDF) for the random vector $\mathbf{U} \equiv (U_1, ..., U_d)^\top$, where U_j is uniformly distributed on $[0, 1]$ for $j = 1, ..., d$. We write $C(u_1, ..., u_d) \equiv \Pr(U_1 \leq u_1, ..., U_d \leq u_d)$, $u_1, ..., u_d \in [0, 1]$. If the copula is absolutely continuous, the *copula density* is defined as $c(u_1, ..., u_d) \equiv \partial^d C(u_1, ..., u_d)/\partial u_1 \cdots \partial u_d$. If the copula is a member of a parametric family indexed by parameter θ, we may write $C_\theta(u_1, ..., u_d)$ (copula) and $c_\theta(u_1, ..., u_d)$ (copula density, if it exists) to emphasise this dependence on θ. In this paper, our attention is primarily focused on cases where $d = 2$, but we will discuss general d-variate copulas for $d \geq 3$ where appropriate.

Copulas can be used to model the joint distribution of the continuous random variables $X_1, ..., X_d$. The joint CDF of $\mathbf{X} \equiv (X_1, ..., X_d)^\top$ is $F(x_1, ..., x_d) \equiv \Pr(X_1 \leq x_1, ..., X_d \leq x_d)$, where $x_1, ..., x_d$ take values in the support of $X_1, ..., X_d$. Now, let $F_j(x_j) \equiv \Pr(X_j \leq x_j)$ denote the CDF of X_j, and set $U_j \equiv F_j(X_j)$ since $F_j(X_j)$ follows a uniform distribution on $[0, 1]$ for $j = 1, ..., d$. Sklar's theorem (Sklar, 1959) states that we can uniquely

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represent the joint CDF of $\mathbf{X}$ as $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$.

At this juncture, it is convenient to define three special bivariate copulas. The first is the *product copula*, $C_{\text{PROD}}(u_1, u_2) \equiv u_1 \times u_2$; this copula represents independence of the marginal variables. The other two are the Fréchet-Hoeffding lower bound (FHLB) and upper bound (FHUB), which, respectively, represent perfect negative and perfect positive dependence between the margins. For $u_1, u_2 \in [0, 1]$, the FHLB is $C_{\text{FHLB}}(u_1, u_2) \equiv \max\{u_1+u_2-1, 0\}$, and the FHUB is $C_{\text{FHUB}}(u_1, u_2) \equiv \min\{u_1, u_2\}$, $u_1, u_2 \in [0, 1]$. All bivariate copulas, $C(u_1, u_2)$, satisfy $C_{\text{FHLB}}(u_1, u_2) \leq C(u_1, u_2) \leq C_{\text{FHUB}}(u_1, u_2)$ for all $u_1, u_2 \in [0, 1]$. It is necessary to specify 'bivariate copula' here because the FHLB cannot be extended to $d \geq 3$ dimensions and remain a valid copula (e.g., McNeil and Nešlehová, 2009, Ex. 2.1).

2.2 Bivariate Archimedean copulas

Archimedean copulas are a class of copulas constructed from an *Archimedean-copula generator* ψ (e.g., Nelsen, 2006, Sec. 4.1). Note that, in what follows, for function g, $\lim_{s\downarrow s_0} g(s)$ and $\lim_{s\uparrow s_0} g(s)$ denote one-sided limits where s_0 is approached from above and below, respectively.

Definition 1. Let $\psi : [0, 1] \to [0, \infty]$ be a convex and strictly decreasing function with $\psi(1) = 0$, where $\psi(0) \equiv \lim_{s\downarrow 0} \psi(s)$, and $\lim_{s\downarrow 0} \psi(s) = \infty$ if

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$\lim_{s\downarrow 0}\psi(s)=\infty$. Let $\psi^{[-1]}:[0,\infty]\to[0,1]$ be the pseudoinverse,

$$\psi^{[-1]}(t)\equiv\begin{cases}\psi^{-1}(t) & 0\le t\le\psi(0),\\ 0 & \psi(0)\le t\le\infty,\end{cases}$$

where $\psi^{-1}(t)$ is the inverse of $\psi(t)$ for $t\in[0,\psi(0)]$. If $\psi(0)=\infty$, the generator ψ has a *strict inverse*, and $\psi^{[-1]}(t)=\psi^{-1}(t)$ for all $t\in[0,\infty]$.

Then we construct an Archimedean copula as follows.

Definition 2. Assume the function ψ satisfies Definition 1 and has pseudoinverse $\psi^{[-1]}$. Then the two-dimensional Archimedean copula generated by ψ is $C(u_1,u_2;\psi)\equiv\psi^{[-1]}(\psi(u_1)+\psi(u_2))$. If ψ has a strict inverse, then the associated Archimedean copula, $C(u_1,u_2;\psi)\equiv\psi^{-1}(\psi(u_1)+\psi(u_2))$, is called a *strict Archimedean copula*.

If the Archimedean-copula generator is indexed by a parameter, say θ, then this generates a parametric family of Archimedean copulas.

Definition 3. Let $\theta\in\Theta$ be a parameter, where Θ is a parameter space. Then, if ψ_θ satisfies the conditions of an Archimedean-copula generator in Definition 2 for all $\theta\in\Theta$, then $\{C(u_1,u_2;\psi_\theta):\theta\in\Theta\}$ is a parametric family of Archimedean copulas.

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2.3 Multivariate Archimedean copulas

If a function ψ satisfies the requirements of Definition 1, we can construct the bivariate Archimedean copula, $C(u_1, u_2; \psi) \equiv \psi^{[-1]}(\psi(u_1)+\psi(u_2))$. The natural extension to $d \geq 3$ dimensions is $C(u_1, ..., u_d; \psi) \equiv \psi^{[-1]}(\psi(u_1) + \cdots + \psi(u_d))$. However, even if $C(u_1, u_2; \psi)$ is a valid bivariate copula, ψ might not produce a valid copula for $d \geq 3$. A generator ψ produces a valid d-dimensional Archimedean copula for all $d \geq 3$ if and only if the pseudoinverse $\psi^{[-1]}$ is *completely monotone* on $[0, \infty)$ (Kimberling, 1974).

Let I be an interval of $\mathbb{R}$, and let I^{O} denote the interior of I. A function $h(x),\ x \in I$, is called *completely monotone* on I if, for all $x \in I^{\mathrm{O}}$, derivatives of all orders exist, and the k-th derivative with respect to x, written as $h^{(k)}(x)$, satisfies $(-1)^k \times h^{(k)}(x) \geq 0$ for $k = 0, 1, 2, ...$ (Nelsen, 2006, Def. 4.6.1). The function $h(x)$ is called *absolutely monotone* on I if $h^{(k)}(x) \geq 0$ for all $x \in I^{\mathrm{O}}$ and $k = 0, 1, 2, ...$ The function $h(x)$ is completely monotone on I if and only if $h(-x)$ is absolutely monotone on I (Widder, 1946, Def 2c). If the function $h_a : I_a \to (-\infty, \infty)$ is absolutely monotone on interval I_a and the function $h_b : I_b \to I_a$ is completely monotone on the interval I_b, then $h_a(h_b(x))$ is completely monotone on I_b (Widder, 1946, Thm 2b).

As final notes, the requirement that $\psi^{[-1]}$ be completely monotone also implies that ψ must have a strict inverse, and $\psi^{[-1]}(t) = \psi^{-1}(t)$ for all

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$t \in [0, \infty)$ (Nelsen, 2006, pp. 151-152). Further, complete monotonicity of $\psi^{-1}(t)$ over $t \in [0, \infty)$ is sufficient and necessary to define an Archimedean copula in all dimensions $d \geq 3$. However, McNeil and Nešlehová (2009) show that, for a given $d \geq 3$, the sufficient and necessary condition to define an Archimedean copula with generator ψ in d dimensions is that $\psi^{[-1]}$ (the pseudoinverse, which need not be strict) is *d-monotone* on $[0, \infty)$. This means that $\psi^{[-1]}(t)$ has $(d-2)$ derivatives over $t \in (0, \infty)$ that satisfy $(-1)^k(\psi^{[-1]})^{(k)}(t) \geq 0$ for $k = 0, 1, ..., d-2$, and $(-1)^{d-2}(\psi^{[-1]})^{(d-2)}(t)$ is convex and non-increasing (McNeil and Nešlehová, 2009).

2.4 The family of ϕ divergences

A ϕ divergence (Csiszar, 1963; Morimoto, 1963; Ali and Silvey, 1966) is a directed, statistical divergence of one non-negative function (e.g., a potentially unnormalised probability density) from another. Let f_1 and f_2 be non-negative functions (e.g., unnormalised probability densities). Below, we define the ϕ divergence *of* f_1 *from* f_2 (e.g., Cressie and Pardo, 2002), which is written as $D_\phi(f_1 \| f_2)$.

Definition 4. Let $\phi : (0, \infty) \to [0, \infty)$ be a convex function satisfying (a) $\phi(1) = 0$, (b) $\phi'(1) = 0$, (c) $\phi''(1) > 0$. Then, the ϕ divergence *of* f_1 *from* f_2 is, $D_\phi(f_1 \| f_2) = \int f_2(s) \times \phi\left(\frac{f_1(s)}{f_2(s)}\right) \mathrm{d}s$, where, by convention, $0 \times \phi(0/0) = 0$,

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and $0 \times \phi(v/0) = v \times \{\lim_{x\to\infty} \phi(x)/x\}$.

For our purposes, we shall take an extended-value definition of the ϕ functions in Definition 4. That is, we define $\phi(0) = \lim_{s\downarrow 0} \phi(s)$ and $\phi(0) = \infty$ when $\lim_{s\downarrow 0} \phi(s) = \infty$, and we write $\phi : [0, \infty) \to [0, \infty]$.

There are other formulations of ϕ divergences. An alternative definition and its relationship to Definition 4, are explained in S1 of the Supplementary Material ('the Supplement'), where we also show that it does not allow a linkage with Archimedean copulas in general.

2.5 Archimedean copulas via ϕ divergence generators

For our first result, we show that any function ϕ satisfying the conditions in Definition 4 also satisfies Definition 1 and therefore defines a valid Archimedean-copula generator.

Proposition 1. *Let ϕ be a function that satisfies (a)-(c) in Definition 4, defining $\phi(0) \equiv \lim_{s\downarrow 0} \phi(s)$ and $\phi(0) = \infty$ when $\lim_{s\downarrow 0} \phi(s) = \infty$. Then $\phi : [0, 1] \to [0, \infty]$ is a valid Archimedean-copula generator.*

The proof and all other proofs are deferred to the Supplement. Proposition 1 then leads to the construction of bivariate ϕ-divergence copulas.

Definition 5. Let ϕ be a function as in Proposition 1. Let $\phi^{[-1]}$ be

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its pseudoinverse. Then, the associated ϕ-divergence copula is defined as $C(u_1, u_2; \phi) \equiv \phi^{[-1]}(\phi(u_1) + \phi(u_2))$.

The restriction to bivariate Archimedean copulas in Proposition 1 is necessary. We can find examples of ϕ functions that generate Archimedean copulas in $d = 2$ dimensions but not in $d \geq 3$ dimensions. A necessary (but not sufficient) condition for both complete monotonicity and d-monotonicity is that the pseudoinverse $\psi_\theta^{[-1]}$ be continuously differentiable over the interval $(0, \infty)$ (Nelsen, 2006; McNeil and Nešlehová, 2009). For a bivariate Archimedean copula, $C(u_1, u_2; \psi)$, the generator ψ need not be differentiable everywhere on $(0, 1)$; it only needs to be convex on $[0, 1]$ (Nelsen, 2006). However, when ψ is not continuously differentiable on $(0, 1)$, the pseudoinverse will also fail to be continuously differentiable over $(0, \infty)$. This implies such Archimedean-copula generators will not produce valid copulas in any given dimension $d \geq 3$ since d-monotonicity (McNeil and Nešlehová, 2009) requires the existence of $(d - 2)$ derivatives of $\psi^{[-1]}(t)$ over all $t \in (0, \infty)$.

According to Definition 4, ϕ need only be twice differentiable at $x = 1$ but not all other $x \in (0, 1)$ (c.f., the 'standard f divergences' of Amari, 2016, pp. 54-56). A function may satisfy the conditions in Definition 4 but lack a derivative at some point in $(0, 1)$; see Supplement S2. As a separate

issue, if $\phi(0)$ is finite, ϕ will not have a strict inverse. Then $\phi^{[-1]}$ cannot be completely monotone on $[0, \infty)$, and the associated ϕ-divergence copula will not exist for all $d \geq 3$, though it might for a particular value of $d \geq 3$.

3. Power-divergence copulas

3.1 Power divergences

Power divergences (Cressie and Read, 1984; Read and Cressie, 1988), or α divergences (e.g., Amari, 2016), are an important member of the class of ϕ divergences. Let $\lambda \in (-\infty, \infty)$ and, for $x \in (0, \infty)$, define the ϕ function,

$$\phi_\lambda(x) \equiv \begin{cases} \frac{1}{\lambda(\lambda+1)}\left(x^{\lambda+1} - x + \lambda(1-x)\right) & \lambda \neq -1, 0 \\ 1 - x + x\log(x) & \lambda = 0 \\ x - 1 - \log(x) & \lambda = -1, \end{cases} \tag{3.1}$$

which satisfies (a)-(c) in Definition 4. Then, for non-negative functions f_1 and f_2, the power divergence of f_1 from f_2 is given by $D_{\phi_\lambda}(f_1||f_2) \equiv \int f_2(s) \times \phi_\lambda(f_1(s)/f_2(s))\, \mathrm{d}s$.

Some elementary properties of (3.1) are needed for later. First, (3.1) is strictly convex and continuously differentiable over $(0, \infty)$. Second, the

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first derivative with respect to x is,

$$\phi'_\lambda(x) = \begin{cases} \lambda^{-1}(x^\lambda - 1) & \lambda \neq -1, 0, \\ \log(x) & \lambda = 0, \\ 1 - x^{-1} & \lambda = -1, \end{cases} \tag{3.2}$$

for all $x \in (0, \infty)$. Finally, at $x = 0$, we have,

$$\phi_\lambda(0) = \begin{cases} 1/(\lambda + 1) & \lambda > -1 \\ \infty & \lambda \leq -1. \end{cases} \tag{3.3}$$

3.2 Power-divergence (PD) copulas

The family of power-divergence (PD) copulas can be defined by using (3.1) with Proposition 1 and Definition 2; that is, $C_\lambda(u_1, u_2) \equiv C(u_1, u_2; \phi_\lambda)$, $u_1, u_2 \in [0, 1]$ for all $\lambda \in (-\infty, \infty)$. The set $\{C_\lambda(u_1, u_2) : \lambda \in (-\infty, \infty)\}$ forms the family of PD copulas. The bivariate PD copulas for $\lambda \in (-\infty, \infty)$ are,

$$\begin{aligned} C_\lambda(u_1, u_2) &\equiv \phi_\lambda^{[-1]}(\phi_\lambda(u_1) + \phi_\lambda(u_2)) \\ &= \begin{cases} \phi_\lambda^{-1}(\phi_\lambda(u_1) + \phi_\lambda(u_2)) & 0 \leq \phi_\lambda(u_1) + \phi_\lambda(u_2) \leq \phi_\lambda(0), \\ 0 & \phi_\lambda(0) \leq \phi_\lambda(u_1) + \phi_\lambda(u_2) \leq \infty. \end{cases} \end{aligned} \tag{3.4}$$

However, some technical challenges must be overcome to make (3.4) practically useful. The main problem is that $\phi_\lambda^{[-1]}$ cannot be written in closed

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form for general $\lambda \in (-\infty, \infty)$. Letting $t \in [0, \phi_\lambda(0)]$ for $\lambda \in (-\infty, \infty)$ and writing $t = \phi_\lambda(x)$, the inverse, $x = \phi_\lambda^{-1}(t)$, is found by the following analysis, which has three cases (i.e., $\lambda \neq -1, 0$, $\lambda = 0$, and $\lambda = -1$).

We begin with the two special cases, where $\lambda = -1$ and $\lambda = 0$. These have closed-form solutions, though they involve the Lambert W function (e.g., see Corless et al., 1996). When $\lambda = -1$, the (strict) inverse $\phi_{-1}^{-1}(t),\ t \in [0, \infty]$, is,

$$\phi_{-1}^{-1}(t) = -\mathfrak{W}_0(-\exp\{-(t+1)\}),\ t \in [0, \infty], \tag{3.5}$$

where $\mathfrak{W}_0$ denotes the principal branch of the Lambert W function. Then, the PD copula for $\lambda = -1$ can be written as,

$$C_{-1}(u_1, u_2) = -\mathfrak{W}_0(-u_1 u_2 \exp\{1 - (u_1 + u_2)\}),\ u_1, u_2 \in [0, 1]. \tag{3.6}$$

When $\lambda = 0$, $\phi_0(x)$ does not have a strict inverse since $\phi_0(0) = 1$ is finite. The pseudoinverse $\phi_0^{[-1]}(t),\ t \in [0, \infty]$, is given by,

$$\phi_0^{[-1]}(t) = \begin{cases} \exp\{\mathfrak{W}_{-1}((t-1)/\exp\{1\}) + 1\} & 0 \leq t \leq 1, \\ 0 & 1 \leq t \leq \infty, \end{cases} \tag{3.7}$$

where $\mathfrak{W}_{-1}$ is the lower branch of the Lambert W function. The PD copula

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for $\lambda = 0$ is,

$$C_0(u_1, u_2) = \begin{cases} \exp\{\mathfrak{W}_{-1}((t_{1,2} - 1)/\exp\{1\}) + 1\} & 0 \leq t_{1,2} \leq 1, \\ 0 & 1 \leq t_{1,2} \leq \infty, \end{cases} \tag{3.8}$$

where $t_{1,2} = 2 - u_1(1 - \log(u_1)) - u_2(1 - \log(u_2))$ for $u_1, u_2 \in [0, 1]$.

For general $\lambda \neq -1, 0$, the inverse of ϕ_λ is non-strict for $\lambda > -1$ and strict for $\lambda < -1$. In the non-strict cases (i.e., $\lambda > -1$ and $\lambda \neq 0$), we find $\phi_\lambda^{-1}(t)$ as the value of $x \in [0, 1]$ that solves,

$$0 = x^{\lambda+1} - (\lambda + 1)x + \lambda - \lambda(\lambda + 1)t, \tag{3.9}$$

for $t \in [0, \phi_\lambda(0)]$. In the strict cases (i.e., $\lambda < -1$), we find $\phi_\lambda^{-1}(t)$ as the value of $x \in (0, 1]$ that solves (3.9) for $t \in [0, \infty)$, and we have $\lim_{t\to\infty} \phi_\lambda^{-1}(t) = 0$ by construction. Proposition 2 guarantees (3.9) always has a unique solution in $[0, 1]$ for $t \in [0, \phi_\lambda(0)]$ when $\lambda > -1$ or for $t \in [0, \infty)$ when $\lambda < -1$.

Proposition 2. *For $\lambda > -1$ and $\lambda \neq 0$, there exists a unique solution of (3.9) in the interval $x \in [0, 1]$ for $t \in [0, \phi_\lambda(0)]$. For $\lambda < -1$, there exists a unique solution of (3.9) in the interval $x \in (0, 1]$ for $t \in [0, \infty)$.*

In some cases, (3.9) can be solved using methods devised for root-finding in polynomials. Consider the following examples. When λ is a

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positive integer, the right-hand side (RHS) of (3.9) becomes a $(\lambda+1)$-th degree polynomial in x, in which case the inverse is one of the roots of this polynomial. For $z \in \{2,3,...\}$, choosing $\lambda = (1-z)/z$ allows (3.9) to be rewritten as $0 = x^{1/z} - x/z + (1-z)(1-t/z)/z$ with $t \in [0, \phi_{(1-z)/z}(0)]$, where the RHS is a z-th degree polynomial in the variable $w \equiv x^{1/z}$. Then the solutions of (3.9), and hence also $\phi^{-1}_{(1-z)/z}$, are related to the roots of the polynomial in w. Similarly, with $z \in \{1,2,3,...\}$, choosing $\lambda = -(z+1)$ allows (3.9) to be rewritten as the $(z+1)$-th degree polynomial, $0 = zx^{z+1} - (z+1)[zt+1]x^z + 1$, where $t \in [0,\infty)$ since, in this case, $\lambda = -(z+1) < -1$. It follows that $\phi^{-1}_{-(z+1)}$ can be derived as one of the roots of this polynomial.

In a small number of cases where the RHS of (3.9) is a polynomial, there is a closed-form expression for ϕ^{-1}_λ (see Section 4.1 and Supplement S3). However, even for the polynomial cases, closed-form solutions remain elusive in general due to the Abel-Ruffini Theorem (Ruffini, 1813; Abel, 1881), which states that polynomials of degree five and higher have no general solutions in terms of radicals (i.e., n-th roots for positive integer n). For general $\lambda \neq -1, 0$, the inverse $\phi^{-1}_\lambda(t)$ (for $t \in [0, \phi_\lambda(0)]$), pseudoinverse $\phi^{[-1]}_\lambda(t)$ (for $t \in [0,\infty]$), and copula in (3.4), lack a closed form.

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3.3 Zero set and zero curve

For bivariate copula $C(u_1, u_2)$, $u_1, u_2 \in [0,1]$, the *zero set* is $\mathcal{Z}(C) \equiv \{(u_1, u_2) \in [0,1]^2 : C(u_1, u_2) = 0\}$. The zero set always contains the two lines, $\{(u_1, 0), (0, u_2) : u_1, u_2 \in [0,1]\}$, but some copulas have more extensive zero sets with positive area. The *zero curve* traces the boundary of the zero set and, for an Archimedean copula, $C(u_1, u_2; \psi)$, it is the curve defined by $\psi(u_1) + \psi(u_2) = \psi(0)$ for $(u_1, u_2) \in [0,1]^2$ (e.g., Nelsen, 2006). The result below establishes that the zero curve and zero set of a PD copula vary with the parameter λ.

Proposition 3. *When* $\lambda \leq -1$, $\mathcal{Z}(C_\lambda) = \{(u_1, 0), (0, u_2) : u_1, u_2 \in [0,1]\}$. *When* $\lambda > -1$, *the zero set has positive area.*

3.4 Absolutely continuous and singular components

Here, we analyse when the PD copulas are absolutely continuous and when they contain a singular component. We begin with the Kendall function (Genest and Rivest, 1993). For any pair of continuous random variables X_1 and X_2, the random vector $\mathbf{X} \equiv (X_1, X_2)^\top$ has a Kendall function that only depends on the copula of $\mathbf{X}$. If $\mathbf{X}$ has the Archimedean copula, $C(u_1, u_2; \psi_\theta)$, then the Kendall function is $K_\theta(s) = s - \psi_\theta(s)/\psi'_\theta(s)$ for $s \in (0,1)$, where $K_\theta(0) \equiv \lim_{s \downarrow 0} K_\theta(s)$ and $K_\theta(1) \equiv \lim_{s \uparrow 1} K_\theta(s) = 1$

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(Nelsen, 2006, p. 163). The term $\psi'_\theta(s)$ can be interpreted as the right-sided derivative of $\psi_\theta(s)$ at $s \in (0,1)$ if $\psi_\theta(s)$ lacks a derivative at some points $s \in (0,1)$. The Kendall function gives the measure of the set $\{(u_1,u_2) \in [0,1]^2 : C(u_1,u_2;\psi_\theta) \le s\}$, for $s \in [0,1]$.

For the PD copula and $s \in (0,1)$, we have

$$K_\lambda(s) = \begin{cases} \frac{\lambda}{\lambda+1}\left(\frac{s^{\lambda+1}-1}{s^\lambda-1}\right) & \lambda \neq -1, 0, \\ \frac{s-1}{\log(s)} & \lambda = 0, \\ \frac{s\log(s)}{s-1} & \lambda = -1. \end{cases} \tag{3.10}$$

From Genest and MacKay (1986), the Archimedean copula $C(u_1,u_2;\psi_\theta)$ is absolutely continuous if $\lim_{s\downarrow 0}\ \psi_\theta(s)/\psi'_\theta(s) = 0$. Otherwise, it has a singular component supported on the zero curve, with probability mass $K_\theta(0) = -\lim_{s\downarrow 0}\psi_\theta(s)/\psi'_\theta(s)$. This gives the following result for PD copulas.

Theorem 1. *The subfamily of power-divergence copulas,* $\{C_\lambda(u_1,u_2) : \lambda \le 0\}$*, is absolutely continuous. The subfamily of power-divergence copulas with* $\lambda > 0$ *have a singular part supported on the zero curve with probability mass* $\lambda/(\lambda+1) \in (0,1)$.

Absolutely continuous Archimedean copulas have copula densities, which are written as $c(u_1,u_2;\psi_\theta) \equiv \partial^2 C(u_1,u_2;\psi_\theta)/\partial u_1 \partial u_2$; see Genest and MacKay (1986) for a standard form. For $\lambda \le 0$ and $\{(u_1,u_2) \in [0,1]^2 : 0 \le$

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$\phi_\lambda(u_1) + \phi_\lambda(u_2) < \phi_\lambda(0)\}$, define,

$$\frac{\partial^2 C_\lambda(u_1,u_2)}{\partial u_1 \partial u_2} = \begin{cases} -\frac{\lambda C_\lambda(u_1,u_2)^{\lambda-1}(u_1^\lambda-1)(u_2^\lambda-1)}{(C_\lambda(u_1,u_2)^\lambda-1)^3} & \lambda \neq -1, 0, \\ -\frac{\log(u_1)\log(u_2)}{C_0(u_1,u_2)\log(C_0(u_1,u_2))^3} & \lambda = 0, \\ -\frac{(1-u_1^{-1})(1-u_2^{-1})}{C_{-1}(u_1,u_2)^2(1-C_{-1}(u_1,u_2)^{-1})} & \lambda = -1. \end{cases} \tag{3.11}$$

For the absolutely continuous subfamily of the PD copulas ($\lambda \leq 0$), we can define $c_\lambda(u_1,u_2) = \partial^2 C_\lambda(u_1,u_2)/\partial u_1 \partial u_2$ in (3.11) and interpret it as a density almost everywhere. On the other hand, if the copula has a singular component (i.e., $\lambda > 0$), then $\partial^2 C_\lambda(u_1,u_2)/\partial u_1 \partial u_2$ cannot be called a density as it does not integrate to unity over its domain of definition.

3.5 Ordering

Assume that θ is univariate. The copula family $\{C_\theta(u_1,u_2) : \theta \in \Theta\}$ may be 'ordered' with respect to parameter θ. The parametric family $\{C_\theta(u_1,u_2) : \theta \in \Theta\}$ is called *negatively ordered* if, for $\theta_1 \leq \theta_2$, $C_{\theta_1}(u_1,u_2) \geq C_{\theta_2}(u_1,u_2)$ for all $u_1,u_2 \in [0,1]$. It is called *positively ordered* if, for $\theta_1 \leq \theta_2$, $C_{\theta_1}(u_1,u_2) \leq C_{\theta_2}(u_1,u_2)$ for all $u_1,u_2 \in [0,1]$. Using a convenient test of ordering for families of Archimedean copulas (Nelsen, 2006, Cor. 4.4.6), we see that PD copulas have an ordering with respect to λ.

Theorem 2. *The family of power-divergence copulas in* (3.4) *is negatively*

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ordered. That is, if $-\infty < \lambda_1 < \lambda_2 < \infty$*, then* $C_{\lambda_1}(u_1, u_2) \geq C_{\lambda_2}(u_1, u_2)$ *for all* $u_1, u_2 \in [0, 1]$.

3.6 Limiting cases

The following proposition shows that the family of bivariate PD copulas includes the FHLB and FHUB as limiting cases. The family does not include the product copula.

Proposition 4. *(i) The limit of* $C_\lambda(u_1, u_2)$ *as* $\lambda \to \infty$ *is the Fréchet-Hoeffding lower bound. (ii) The limit of* $C_\lambda(u_1, u_2)$ *as* $\lambda \to -\infty$ *is the Fréchet-Hoeffding upper bound.*

The limiting cases of the PD copula are consistent with the negative ordering of the family with respect to the parameter λ, per Theorem 2.

3.7 Concordance and dependence measures

In this section, we derive Kendall's tau and the tail-dependence coefficients for the PD copulas. Kendall's tau for an Archimedean copula with generator ψ_θ is defined as $\tau(\theta) \equiv 1 + 4 \int_0^1 \psi_\theta(s)/\psi'_\theta(s)\, \mathrm{d}s$ (e.g., Nelsen, 2006, Cor 5.1.4). Using (3.1) and (3.2) in this expression gives the Kendall's tau for the PD

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copula: That is, for $\lambda \in (-\infty, \infty)$,

$$\tau(\lambda) \equiv \begin{cases} 1 + \frac{2}{\lambda+1} - \frac{4\lambda}{\lambda+1} \int_0^1 \frac{s-1}{s^\lambda - 1} \mathrm{d}s & \lambda \neq -1, 0, \\ 3 - 4\log(2) & \lambda = 0 \\ 7 - 2\pi^2/3 & \lambda = -1, \end{cases} \tag{3.12}$$

where the integral in the first case exists for all $\lambda \neq -1, 0$, though it may not have a closed form. The last two cases correspond to numerical values of approximately $\tau(0) \simeq 0.227$ and $\tau(-1) \simeq 0.420$. Proposition 4 implies that, as λ goes to $-\infty$, $\tau(\lambda)$ goes to 1 and, as λ goes to ∞, $\tau(\lambda)$ goes to -1. Since Kendall's tau obeys the ordering of the copula family (e.g., Nelsen, 2006, Def 5.1.7), $\tau(\lambda)$ is also a monotone decreasing function of λ. The expression in (3.12) enables method-of-moments estimation for λ based on inversion of Kendall's tau (Genest and Rivest, 1993). Let $(X_1, X_2)^\top$ be two continuous random variables with joint CDF F, and let $\bar{\tau}_n$ be the sample version of Kendall's tau for the dataset, $\mathcal{D}_n = \{(x_{1i}, x_{2i})^\top \sim F : i = 1, ..., n\}$. The PD copula can be fitted to the dataset $\mathcal{D}_n$ by using a root-finding algorithm to compute $\hat{\lambda}$ such that $\bar{\tau}_n - \tau(\hat{\lambda}) = 0$. Since $\lim_{\lambda \to \infty} \tau(\lambda) = -1$ and $\lim_{\lambda \to -\infty} \tau(\lambda) = 1$ and (3.12) is a strictly decreasing function of λ, there is a unique value of $\hat{\lambda}$ associated with every value of $\bar{\tau}_n \in [-1, 1]$. This estimator is consistent and asymptotically unbiased for absolutely continuous copulas (Genest and Rivest, 1993; Kojadinovic and Yan, 2010), meaning $\lambda \leq 0$ for

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the PD copulas. Maximum likelihood estimation of λ is also possible when $\lambda \leq 0$, using the copula density in (3.11). König et al. (2015) provide a consistent estimator for Archimedean-copula parameters when the copula has a singular part, which could be used when $\lambda > 0$.

We now calculate the lower and upper tail-dependence coefficients of the PD copula. For continuous X_1 and X_2, the tail-dependence coefficients depend only on the copula of $\mathbf{X} \equiv (X_1, X_2)^\top$. Let $U_1 \equiv F_1(X_1)$ and $U_2 \equiv F_2(X_2)$, where F_1 and F_2 are the marginal CDFs of X_1 and X_2. Then define $\mathbf{U} \equiv (U_1, U_2)^\top$, whose joint CDF is the copula $C(u_1, u_2)$ with $u_1, u_2 \in [0, 1]$. The lower-tail dependence coefficient is $T_L \equiv \lim_{u \downarrow 0} \Pr(U_1 \leq u \mid U_2 \leq u)$; the upper-tail dependence coefficient is $T_U \equiv \lim_{u \uparrow 1} \Pr(U_1 \geq u \mid U_2 \geq u)$. For an Archimedean copula with generator ψ, there are standard expressions for the tail-dependence coefficients in terms of ψ and $\psi^{[-1]}$ (e.g., Nelsen, 1997). A copula is said to exhibit lower-tail dependence if T_L is defined and $T_L > 0$; if $T_L = 0$ then there is no lower-tail dependence. The same can be said for upper-tail dependence.

Using the results of Charpentier and Segers (2009), we show the lower tail-dependence coefficient of the PD copula depends on λ, while the upper tail-dependence coefficient is non-zero but constant for all $\lambda \in (-\infty, \infty)$.

Theorem 3. *(i) The lower tail-dependence coefficient is* $T_L(\lambda) = 2^{1/(\lambda+1)}$

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for $\lambda < -1$ *and* $T_L(\lambda) = 0$ *for* $\lambda \geq -1$. *(ii) The upper tail-dependence coefficient is* $T_U(\lambda) = 2 - \sqrt{2} \simeq 0.5858$ *for all* $\lambda \in (-\infty, \infty)$.

A notable feature of the PD copula family is the constant upper tail-dependence coefficient of $2 - \sqrt{2}$. This can be a limitation when modelling real datasets, since there is no flexibility in terms of upper-tail dependence. One method to alleviate this is to consider the exterior power family associated with ϕ_λ (Nelsen, 1997), which is a two-parameter family of Archimedean copulas with the modified generator $[\phi_\lambda(t)]^\beta$ for $\beta \geq 1$ and the upper tail-dependence coefficient, $2 - 2^{1/2\beta}$.

Note also that $T_U(\lambda) = 2 - \sqrt{2}$ for $\lambda \in (-\infty, \infty)$ does not agree with the upper tail-dependence coefficients in the limiting cases as $\lambda \to -\infty$, where the FHUB has an upper tail-dependence coefficient of 1, or $\lambda \to \infty$, where the FHLB has a upper tail-dependence coefficient of 0. This is not problematic and is observed among other Archimedean-copula families (Nelsen, 2006, Ex. 5.22).

3.8 PD copulas in $d \geq 3$ dimensions

It is possible to define PD copulas in $d \geq 3$ dimensions for some values of λ but not for others. We consider three cases: $\lambda > 0$, $-1 < \lambda \leq 0$, and $\lambda \leq -1$. For the case where $\lambda > 0$, recall that, in order for $\phi_\lambda^{[-1]}(t)$ to

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be 3-monotone on $[0, \infty)$, it is first necessary that the derivative $(\phi_\lambda^{[-1]})'(t)$ exists everywhere on $t \in (0, \infty)$ (McNeil and Nešlehová, 2009). The next result shows it does not. In particular, it fails to exist at $t = 1/(\lambda + 1)$.

Lemma 1. *Let $\lambda > 0$. Then the derivative of $\phi_\lambda^{[-1]}(t)$ does not exist at $t = \phi_\lambda(0) = 1/(\lambda + 1)$.*

This immediately implies the following.

Theorem 4. *For $\lambda > 0$ and $u_1, ..., u_d \in [0, 1]$, $\phi_\lambda^{[-1]}(\phi_\lambda(u_1) + \cdots + \phi_\lambda(u_d))$ is not a valid copula for any $d \geq 3$.*

The situation is more complicated when $-1 < \lambda \leq 0$. Like the $\lambda > 0$ case, the inverse is non-strict, and $\phi_\lambda(0) = 1/(\lambda + 1)$. The following result guarantees that the derivative $(\phi_\lambda^{[-1]})'(t)$ exists for all $t \in (0, \infty)$, even at $t = 1/(\lambda + 1)$, and that $-(\phi_\lambda^{[-1]})'(t) \geq 0$ for all $t \in (0, \infty)$.

Lemma 2. *For $-1 < \lambda \leq 0$, the pseudoinverse $\phi_\lambda^{[-1]}(t)$ is differentiable over all $t \in (0, \infty)$, and the derivative $(\phi_\lambda^{[-1]})'(1/(\lambda + 1)) = 0$. Further, $-(\phi_\lambda^{[-1]})'(t) \geq 0$.*

Yet $\phi_\lambda^{[-1]}(t)$ is not necessarily 3-monotone for all $-1 < \lambda \leq 0$. Per McNeil and Nešlehová (2009), $-(\phi_\lambda^{[-1]})'(t)$ must also be convex over $t \in (0, \infty)$ in order for $\phi_\lambda^{[-1]}(t)$ to be 3-monotone. The next result shows $-(\phi_\lambda^{[-1]})'(t)$ fails to be convex for all $t \in (0, \infty)$ unless $\lambda \leq -0.5$.

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Lemma 3. *For $-1 < \lambda \leq 0$, the function $-(\phi_\lambda^{[-1]})'(t)$ is non-increasing for $t \in (0, \infty)$. However, while the function $-(\phi_\lambda^{[-1]})'(t)$ is convex for all $t \in (0, \infty)$ for $-1 < \lambda \leq -0.5$, it fails to be convex for some values of t when $-0.5 < \lambda \leq 0$.*

Then all pseudoinverses in the set $\{\phi_\lambda^{[-1]} : \lambda \in (-1, -0.5]\}$ are 3-monotone on $[0, \infty)$. The corollary is given below.

Theorem 5. *For $-1 < \lambda \leq -0.5$ and $u_1, u_2, u_3 \in [0, 1]$, $C_\lambda(u_1, u_2, u_3) \equiv \phi_\lambda^{[-1]}(\phi_\lambda(u_1) + \phi_\lambda(u_2) + \phi_\lambda(u_3))$ is a valid copula in three dimensions.*

PD copulas with $\lambda > -0.5$ are restricted to two dimensions only because the associated pseudo-inverses are only 2-monotone, due to Lemma 3.

For $\lambda \leq -1$, the inverses $\{\phi_\lambda^{-1} : \lambda \leq -1\}$ are strict and, in fact, the inverses have derivatives of all orders over $t \in (0, \infty)$. This follows from the Inverse Function Theorem (IFT) since, by inspection, $\phi_\lambda(x)$ in (3.1) has infinitely many derivatives for $x \in (0, \infty)$ when $\lambda \leq -1$. Momentarily setting aside $\lambda = -1$, for which we prove a much stronger result in Lemma 5 below, Lemma 4 shows $\{\phi_\lambda^{-1} : \lambda \in (-\infty, -1)\}$ are 3-monotone.

Lemma 4. *The strict inverses in the set $\{\phi_\lambda^{-1} : \lambda \in (-\infty, -1)\}$ are 3-monotone on $[0, \infty)$.*

This implies Theorem 6, which constructs PD copulas in $d = 3$.

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Theorem 6. *For* $-\infty < \lambda < -1$ *and* $u_1, u_2, u_3 \in [0,1]$, $C_\lambda(u_1, u_2, u_3) \equiv \phi_\lambda^{-1}(\phi_\lambda(u_1) + \phi_\lambda(u_2) + \phi_\lambda(u_3))$ *is a valid copula in three dimensions.*

The next question is whether any strict inverses for $\lambda \leq -1$ are completely monotone on $[0, \infty)$. Here, we show $\phi_{-1}^{-1}(t)$ and $\phi_{-2}^{-1}(t)$ are completely monotone on $[0, \infty)$. For $\lambda = -1$, Lemma 5 follows from properties of the principal branch of the Lambert W function (Kalugin et al., 2012).

Lemma 5. *Let* $\lambda = -1$. *Then* $\phi_{-1}^{-1}(t)$ *is completely monotone on* $[0, \infty)$

For the $\lambda = -2$ case, we must first demonstrate a property of the incomplete exponential Bell polynomials (Bell, 1934), or simply *Bell polynomials*. We take the definition in Comtet (1974, pp. 133-7).

Definition 6. Let k and j be positive integers with $k \geq j$. For variables $x_1, ..., x_{k-j+1}$ and non-negative integers $c_1, c_2, ..., c_{k-j+1}$, the incomplete exponential Bell polynomial $\mathfrak{B}_{k,j} \equiv B_{k,j}(x_1, ..., x_{k-j+1})$ is defined as,

$$\mathfrak{B}_{k,j} \equiv B_{k,j}(x_1, ..., x_{k-j+1}) = \sum \frac{k!}{c_1! c_2! \cdots c_{k-j+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_{k-j+1}}{(k-j+1)!}\right)^{c_{k-j+1}}, \tag{3.13}$$

where the sum is taken over all (non-negative integer) solutions of the system of equations,

$$c_1 + c_2 + \cdots + c_{k-j+1} = j, \tag{3.14}$$

$$1c_1 + 2c_2 + \cdots + (k-j+1)c_{k-j+1} = k. \tag{3.15}$$

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Bell polynomials appear in Faà di Bruno's formula for the k-th derivative of a composition of two functions $h(t) \equiv f(g(t))$: For integer $k \geq 1$, $h^{(k)}(t) = \sum_{j=1}^{k} f^{(k)}(g(t)) \times \mathfrak{B}_{k,j}(t)$, where $\mathfrak{B}_{k,j}(t) \equiv B_{k,j}(g^{(1)}(t), ..., g^{(k-j+1)}(t))$. This formula is needed to study derivatives of the strict inverses $\phi_{-\gamma}^{-1}$ for $\gamma > 1$. In particular, the following result is needed for our analysis of ϕ_{-2}^{-1}.

Lemma 6. *For positive integers k, j where $k \geq j$, let $x_1, ..., x_{k-j+1}$ be variables that satisfy $(-1)^l x_l \geq 0$ for $l = 1, ..., k-j+1$. Define $\mathfrak{B}_{k,j} \equiv B_{k,j}(x_1, ..., x_{k-j+1})$ as in Definition 6. Then, for all $j = 1, ..., k$, $\mathfrak{B}_{k,j} \geq 0$ if k is even, and $\mathfrak{B}_{k,j} \leq 0$ if k is odd.*

The next result makes use of Lemma 6 in an induction on k.

Lemma 7. *Let $\lambda = -2$. Then $\phi_{-2}^{-1}(t)$ is completely monotone on $[0, \infty)$.*

It immediately follows from Lemmas 5 and 7 that the PD copulas with $\lambda = -1$ and $\lambda = -2$ are valid copulas in $d \geq 3$; see the next result.

Theorem 7. *For all $d \geq 3$ and $u_1, ..., u_d \in [0, 1]$, $C_{-1}(u_1, ..., u_d) \equiv \phi_{-1}^{-1}(\phi_{-1}(u_1) + \cdots + \phi_{-1}(u_d))$ and $C_{-2}(u_1, ..., u_d) \equiv \phi_{-2}^{-1}(\phi_{-2}(u_1) + \cdots + \phi_{-2}(u_d))$ are valid copulas.*

We conjecture that all members of $\{\phi_\lambda^{-1} : \lambda \leq -1\}$ are completely monotone on $[0, \infty)$. Responding to this conjecture in an earlier preprint

of this present work, Salazar (2026) uses different strategies than ours to propose a proof of this conjecture in the affirmative. On the other hand, we can also use, for example, Mathematica (Wolfram Research, Inc., 2024) to check whether $\phi_\lambda^{-1}(t)$ is d-monotone on $[0,\infty)$ for given $d \geq 3$ and $\lambda < -1$. Letting $\gamma > 1$ and setting $\lambda = -\gamma$, note that $(\phi_{-\gamma}^{-1})'(t) = -\gamma\phi_{-\gamma}^{-1}(t)^\gamma/(1-\phi_{-\gamma}^{-1}(t)^\gamma)$ for all $t \in (0,\infty)$ by the IFT, and then compute the $(d-2)$-th, $(d-1)$-th, and d-th derivatives of $\phi_{-\gamma}^{-1}(t)$ to verify that $(-1)^{d-2}(\phi_\lambda^{-1})^{(d-2)}(t) \geq 0$ and that $(-1)^{d-2}(\phi_\lambda^{-1})^{(d-2)}(t)$ is non-increasing and convex for $t \in (0,\infty)$. We need only check these three derivatives because McNeil and Nešlehová (2009, Prop. 2.3) show it suffices to check only that $(-1)^{d-2}(\phi_\lambda^{-1})^{(d-2)}(t)$ is non-negative, convex, and non-increasing for a given $d \geq 3$.

4. Computational aspects

4.1 Exact formulas for certain values of λ

The PD copula has no closed form for general $\lambda \neq 0, -1$. A non-exhaustive list of exceptions is $\lambda \in \{-4, -3, -2, -2/3, -1/2, 1, 2, 3\}$. In these cases, (3.9) or a suitable rearrangement reduces to the problem of finding the zeros of a quadratic, cubic, or quartic equation in x or $w \equiv x^{\lambda+1}$. Regardless, the exact solutions can be cumbersome. Therefore, we do not examine every possibility. Supplement S3 presents three illustrative examples, where

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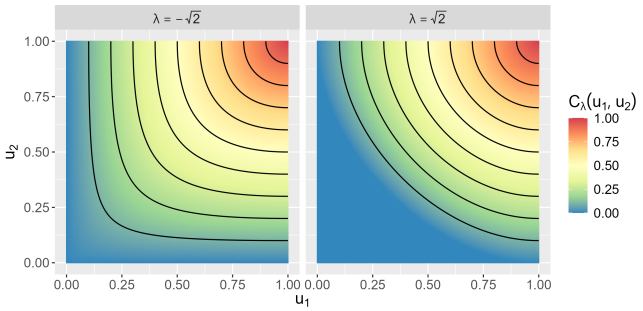

Figure 1: Plots of $C_\lambda(u_1, u_2)$ for $\lambda \in \{-\sqrt{2}, \sqrt{2}\}$ and $u_1, u_2 \in [0, 1]$.

$\lambda \in \{-2, -0.5, 1\}$.

4.2 Computing the PD copula and its density

The analysis in Section 3.2 indicates that the PD copula is well defined and can, in theory, be computed for all $\lambda \in (-\infty, \infty)$, even if it lacks a closed form. Supplement S4.1 presents a numerical routine for computing the PD copula for any $u_1, u_2 \in [0, 1]$ and $\lambda \in (-\infty, \infty)$. Copulas for $\lambda \in \{-\sqrt{2}, \sqrt{2}\}$ (i.e., values of λ that do not admit closed-form representations of the corresponding PD copulas) are plotted in Fig. 1 for illustration.

Computing (3.11) on its domain is straightforward. For any particular $u_1, u_2 \in [0, 1]$ and $\lambda \in (-\infty, \infty)$, check if $\phi_\lambda(u_1) + \phi_\lambda(u_2) < \phi_\lambda(0)$. If so, simply compute the copula $C_\lambda(u_1, u_2)$ by the method in Supplement S4.1, and then substitute this value into (3.11). For $\lambda > 0$, note that (3.11) is only for the absolutely continuous part of the copula. The singular

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component of the copula is supported on the zero curve and, when $\lambda > 0$, has $100 \times \lambda/(\lambda+1)\%$ of the probability mass of the copula. Fig. 2 shows the computed values of $\partial^2 C_\lambda(u_1, u_2)/\partial u_1 \partial u_2$ for $\lambda = -\sqrt{2}$ and $\lambda = \sqrt{2}$ over the relevant domains.

4.3 Simulating from the PD copula

Simulation from the bivariate PD copula is enabled by the well known 'conditional distribution method' (e.g., Nelsen, 2006, Sec 2.9). Supplement S4 gives the details of the algorithm for the PD copulas. Fig. 3 plots data simulated from the PD copulas for selected values of λ, namely $\lambda \in \{-10, -3, -2, -0.5, 1, 2, 3, 10\}$. Some notable features of the plots include the upper tail dependence apparent in the simulated random variates and the heavy zero curves when $\lambda = 1, 2, 3, 10$. This reflects the large probability

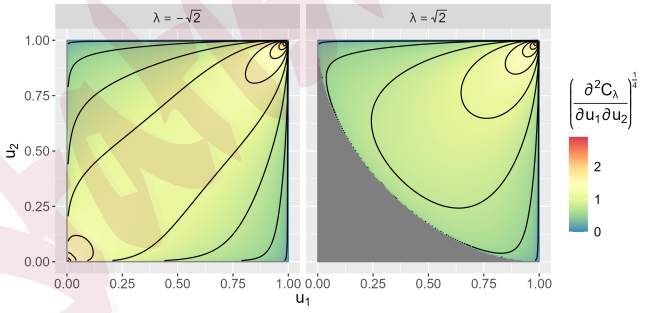

Figure 2: For $\lambda = -\sqrt{2}$ (left panel) and $\lambda = \sqrt{2}$ (right panel), plots of the fourth root of $\partial^2 C_\lambda(u_1, u_2)/\partial u_1 \partial u_2$ for $u_1, u_2 \in [0, 1]$.

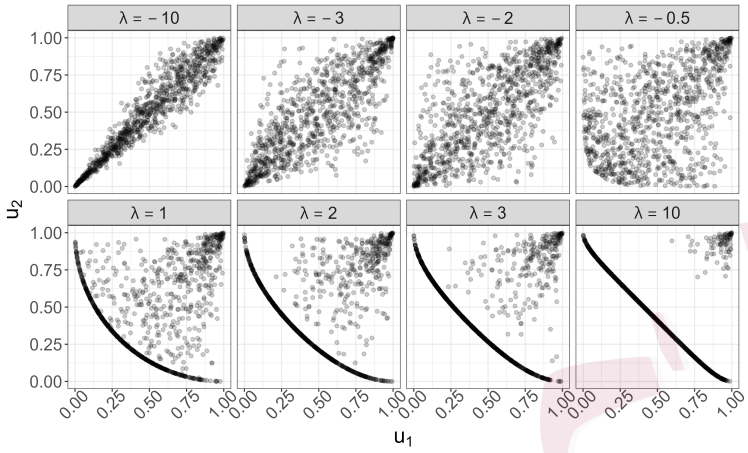

Figure 3: Scatterplots of 1000 bivariate realisations from the power-divergence copulas, with $\lambda \in \{-10, -3, -2, -0.5, 1, 2, 3, 10\}$.

masses of the zero curves (e.g., approximately 0.91 when $\lambda = 10$). For $\lambda = -0.5$, a zero set with positive area is apparent, but the probability mass on the zero curve is 0.

5. Analysis of the dependence between stock-market indices

The Dow Jones Industrial Average (DJIA) and the Standard & Poor's 500 (S&P500) are stock-market indices in the United States. Though they are computed differently, these two indices both track the overall performance of US stock markets, and they are calculated from overlapping stocks. Their movements in response to global events are naturally related. Extreme negative and positive movements in one are likely to correspond to similar

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movements in the other, so it behoves us to investigate several families of copulas to capture the dependence between these two indices, instead of simply assuming, for example, that they are jointly Gaussian. Here, we use the PD copulas (and others) to characterise the dependence between the DJIA and S&P500 for four US fiscal years from 1 October, 2022 to 30 September, 2026 (Fig. 4(a)-(b); FRED, 2026a,b).

We begin by removing the serial dependence in the time series of the DJIA and S&P500 to allow the fitting of copulas based on (approximately) independent realisations. To this end, we obtain the residuals of the ARIMA model selected for each series using the 'auto.arima' function in the 'forecast' R package (Hyndman and Khandakar, 2008) with default settings but conducting a full (not stepwise) search over competing models. In both cases, an ARIMA(4,1,0) model with drift is chosen. Figure S3 in Supplement S5 confirms that the residuals are approximately uncorrelated in time. Subsequently, we use rank-based transformations on the paired residuals to use as 'copula data' (Fig. 4(c)), and we fit a range of copula models from the 'copula' R package (Hofert et al., 2025) via maximum (pseudo) likelihood. Candidate models include the Clayton, Frank, Gumbel, Joe, and PD copulas (Archimedean copulas), the Galambos and Hüsler-Reiss copulas (extreme-value copulas), and the Gaussian copula (an elliptical copula).

The survival copulas of the Archimedean copulas other than the PD copula are also considered, and we denote them as S-Clayton, S-Gumbel, and S-Joe. (The Frank copula is its own survival copula.)

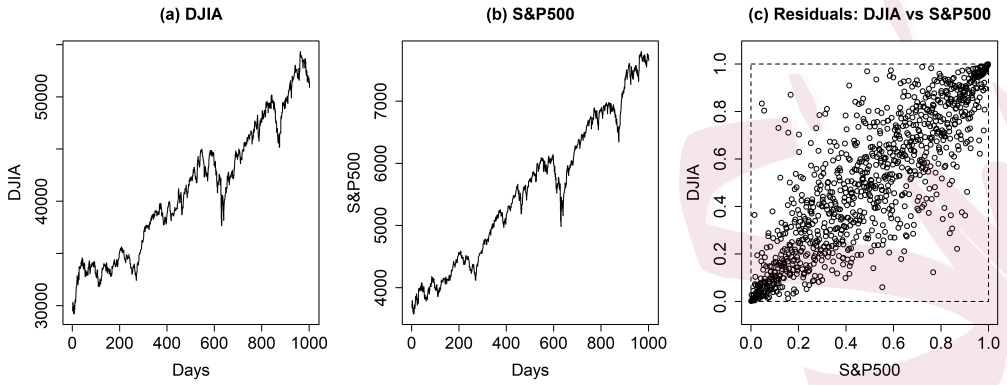

Figure 4: Time series plots of (a) DJIA and (b) S&P500. (c) Bivariate scatter plot of the rank-transformed ARIMA(4,1,0) residuals.

Table 1 shows the log-likelihood values achieved by each candidate model when fitted to the rank-transformed copula data. Since every model only has a single parameter, we compare the models via their log-likelihood values. The PD copula ($\hat{\lambda} = -3.16$) has the highest log-likelihood value, followed by the S-Gumbel copula. The Gaussian copula has the third-highest log-likelihood value, but its tail behaviour is likely inappropriate for capturing the dependence between these indices. These results indicate, firstly,

that the dependence between the DJIA and S&P500 may be non-Gaussian and, secondly, the PD copulas can be practically useful for modelling real datasets. Section S6 of the Supplement presents an insurance example where the PD copulas are also shown to be useful.

Table 1: Several copula models with their fitted parameters, and maximised log-likelihoods for the DJIA vs S&P500.

Model	Fitted parameter	Log-likelihood
Clayton	3.86	543.16
S-Clayton	3.86	446.36
Frank	9.71	609.61
Gumbel	2.80	646.90
S-Gumbel	2.90	681.90
Joe	3.28	517.23
S-Joe	3.57	581.33
Power-divergence	**-3.16**	**700.61**
Galambos	2.08	641.93
Hüsler-Reiss	2.52	615.35
Gaussian	0.86	663.50

6. Discussion and conclusion

We develop the PD copulas, a family of Archimedean copuals associated with the power divergences. Their properties are extensively studied, and we show they can be useful for applied modelling. Beyond the results presented for the PD copulas, Proposition 1 implies there are Archimedean copulas associated with other ϕ divergences. Further, our partial results for extending the PD copulas to arbitrary dimensions have applications outside of copula modelling. For example, Schoenberg's theorem (e.g., Cressie, 1993, p. 86) and Theorem 7 imply that, for length-scale parameter $\theta_r > 0$ and d-dimensional vector $\mathbf{h} \in \mathbb{R}^d$ with magnitude $\|\mathbf{h}\| \geq 0$, the following are two positive-definite, isotropic, stationary correlation functions on $\mathbb{R}^d$ for $d = 1, 2, ...$:

$$\rho_{-1}(\mathbf{h}; \theta_r) = -\mathfrak{W}_0(-\exp\{-(\|\mathbf{h}\|^2 + \theta_r^2)/\theta_r^2\}),$$

$$\rho_{-2}(\mathbf{h}; \theta_r) = 1 + (\|\mathbf{h}\|/\theta_r)^2 - \sqrt{(\|\mathbf{h}\|/\theta_r)^4 + 2(\|\mathbf{h}\|/\theta_r)^2}.$$

If our conjecture in Section 3.8 is true, which appears to be the case (Salazar, 2026), then $\rho_\lambda(\mathbf{h}; \theta_r) = \phi_\lambda^{-1}((\|\mathbf{h}\|/\theta_r)^2)$ defines a whole family of positive-definite correlation functions on $\mathbb{R}^d$ indexed by $\lambda \in (-\infty, -1]$. These may prove useful for spatial, temporal, and spatio-temporal data analysis.

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Supplementary Materials

The Supplement contains additional details, figures, and proofs.

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