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Spatial subsample bootstraps for high-dimensional spatial data with irregular spacing

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Abstract: We develop a class of spatial subsample bootstrap (SSB) methods for high-dimensional spatial data observed at irregular locations. The proposed procedures approximate the distributions of high-dimensional spatial statistics under general spatial dependence without requiring regular sampling designs or explicit estimation of dependence structures. Two implementations are studied: the spatial multiplier subsample bootstrap (SMSB) and the spatial empirical subsample bootstrap (SESB), both constructed from overlapping spatial subsample statistics. We establish asymptotic validity of the proposed methods for joint distributional approximation of high-dimensional statistics, including max-type and studentized statistics, under mild moment and weak spatial dependence conditions. The dimension of the parameter is allowed to diverge with the sample size and may exceed it. The proposed framework avoids estimation of high-dimensional covariance matrices and remains computationally efficient in high-dimensional regimes. We further derive an asymptotically optimal subsample size by balancing the leading bias and variance terms of the bootstrap approximation and propose a data-driven plug-in selector with optimality guarantees. The results provide a general theoretical foundation for bootstrap-based inference in high-dimensional spatial problems with irregular designs.

Key words and phrases: Bootstrap, Gaussian approximation, high-dimensional statistical inference, irregularly spaced spatial data, resampling methods, spatial statistics.

1. Introduction

Spatial data with increasing dimension p are ubiquitous in climatology, environmental monitoring, remote sensing, and geosciences, where observations are obtained over large spatial domains with irregular spatial layouts. Statistical inference in such settings is challenged by spatial dependence, nonstationarity, and the absence of regular sampling structures. These challenges motivate the development of flexible bootstrap methodologies that can provide valid uncertainty quantification for irregularly spaced spatial data.

In high-dimensional dependent data settings, bootstrap approaches have proved effective. For example, Zhang and Wu (2017) proposed a Gaussian multiplier bootstrap procedure based on estimated long-run covariance matrices. Chernozhukov et al. (2019) introduced the block multiplier bootstrap in their online supplement and demonstrated its validity for dependent data under general mixing conditions. Chang et al. (2024) proposed a parametric bootstrap that constructs a data-dependent Gaussian analogue of the target statistic. More recently, to approximate the distribution of maxima, Ma and Zhang (2026) developed two novel methods – the multiplier subsample bootstrap and the empirical subsample bootstrap – based on blockwise subsample statistics.

The theoretical development and applications of these bootstrap methods are largely confined to high-dimensional time series, which constitute a canonical example of regularly spaced data. Extending these approaches to high-dimensional spatial data with irregular spacing poses substantial challenges due to the interplay among high dimensionality, spatial dependence, and complex spatial configurations. To the best of our knowledge, the

spatially dependent wild bootstrap (SDWB) of Kurisu et al. (2024) is the only existing bootstrap procedure developed specifically for high-dimensional spatial data with irregular spacing. By combining recent advances in high-dimensional Gaussian approximation (Chernozhukov et al., 2017, 2019; Zhang and Wu, 2017; Zhang and Cheng, 2018) with the stochastic sampling framework of Lahiri (2003), the SDWB extends the dependent wild bootstrap (DWB; Shao, 2010) from time series to high-dimensional spatial settings.

Despite its theoretical appeal, the SDWB has several limitations. First, it relies on a prespecified spatial weighting scheme, which requires generating high-dimensional dependent random weights and can be computationally demanding when the sample size is large. Second, it is inherently designed to mimic covariance structures and can not directly reproduce higher-order dependence features. These limitations motivate the development of alternative bootstrap approaches that are both computationally efficient and better suited to capturing the covariance structure of the statistics or pivotal quantities of interest. To this end, we propose a new class of spatial subsample bootstraps (SSBs), including the spatial multiplier subsample bootstrap (SMSB) and the spatial empirical subsample bootstrap (SESB).

The proposed SSBs construct subsample-based statistics using overlapping spatial blocks and treat each subsample as a local replica of the full dataset. In particular, the SMSB assigns independent and identically distributed (i.i.d.) random weights to the collection of subsample-based statistics, whereas the SESB resamples these statistics in an i.i.d. manner. Unlike the SDWB of Kurisu et al. (2024), our approaches capture spa-

tial dependence implicitly through overlapping subsamples rather than through explicitly constructed dependent weights. Under mild regularity conditions, both methods yield valid distributional approximations for high-dimensional spatial statistics. The dimension p is allowed to grow with the sample size n and may even exceed n , while avoiding explicit estimation of high-dimensional covariance structures. However, the theoretical validity of the proposed SSBs relies on the variance structure of the subsample statistics mimicking that of the statistic of interest, which in turn requires an asymptotically nearly uniform sampling design. In contrast, bootstrap procedures based on reconstructing pseudo-sampling observations can be theoretically valid under weaker requirements on the sampling design (e.g. Lahiri and Zhu, 2006; Kurisu et al., 2024).

As with most bootstrap procedures for dependent data (e.g. Lahiri and Zhu, 2006; Kurisu et al., 2024; Zhang, 2025; Ma and Zhang, 2026), the SSBs require selecting a subsample size b , which is also referred to in the literature as a block size, bandwidth parameter or localization scale, despite serving the same fundamental role. We investigate the choice of subsample size and propose a data-driven selection rule that is both theoretically justified and practically feasible. Our results extend classical optimal block-size selection theory from the finite dimensional setting (Nordman and Lahiri, 2004; Lahiri et al., 2007) to the high-dimensional regime.

The main contributions of this paper are summarized as follows. First, we introduce a general class of spatial subsample bootstrap methods for high-dimensional spatial data with irregular spacing. Second, we establish theoretical validity of the proposed proce-

dures under increasing dimension and mild regularity conditions on spatial dependence, nonstationarity, and sampling design. Third, we derive an optimal subsample size and propose a data-driven plug-in selection rule, enabling adaptive implementation of the SSBs. Finally, we demonstrate the superior finite-sample performance of the proposed methods through extensive simulations and a real-data application to climate data.

The remainder of the paper is organized as follows. Section 2 introduces the problem formulation and the sampling design. Section 3 presents the construction of spatial subsample statistics and details the SMSB and SESB procedures. Section 4 establishes the main theoretical results on variance estimation and distributional approximation. Section 5 derives the optimal subsample size and introduces a practical selection procedure. Section 6 reports simulation results for mean and covariance inference and presents a change-point analysis using real climate data.

Notation. For any vector $\mathbf{x} = (x_1, \dots, x_p)^\top \in \mathbb{R}^p$, define $\|\mathbf{x}\|_1 = \sum_{i=1}^p |x_i|$ and $\|\mathbf{x}\|_\infty = \max_{1 \leq j \leq p} |x_j|$. For any $p \times p$ matrix $A = (a_{ij})$, let $\|A\|_{\max} = \max_{1 \leq i, j \leq p} |a_{ij}|$. For a symmetric matrix, $\text{vech}(\cdot)$ denotes the vectorization operator that stacks the elements on and below the main diagonal. Let $\mathbb{I}_A(\cdot)$ denote the indicator function. For positive sequences a_n, b_n , we write $a_n \asymp b_n$ if $C_1 \leq a_n/b_n \leq C_2$ for some constants $C_1, C_2 > 0$. For any $a, b \in \mathbb{R}$, let $a \wedge b = \min\{a, b\}$. Define $\rho_2(t) = C (\mathbb{I}_{[0,1]}(|t|) + |t|^{-2} \mathbb{I}_{(1,\infty]}(|t|))$ for a constant $C > 0$, and extend it to $\mathbf{t} \in \mathbb{R}^d$ by setting $\rho_2(\mathbf{t}) = \prod_{j=1}^d \rho_2(t_j)$. For a random variable X , define the ψ_1 -norm as $\|X\|_{\psi_1} = \inf\{c > 0 : \mathbb{E}[\exp(|X|/c)] \leq 2\}$. Finally, let $P_{\cdot|\mathbf{s}}$ denote probability conditional on the sampling locations $\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$, and let $P_{\cdot|\mathbf{s}, \mathbf{y}}$

and $\text{Var}_{|\mathbf{s}, \mathbf{Y}}$ denote conditional probability and variance given both the sampling locations $\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ and the observations $\{\mathbf{Y}(\mathbf{s}_1), \dots, \mathbf{Y}(\mathbf{s}_n)\}$.

2. Problem description and sampling design

2.1 Problem description

Let $\{\mathbf{Y}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ be a p -dimensional random field defined on a probability space $(\Omega, \mathcal{F}, P)$, which can be decomposed as

$$\mathbf{Y}(\mathbf{u}) = \mathbf{X}(\mathbf{u}) + \mathbf{\Upsilon}(\mathbf{u}), \quad (2.1)$$

where $\{\mathbf{X}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ is a strictly stationary random field that serves as the main structural component, and $\{\mathbf{\Upsilon}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ represents a mean-zero perturbation that is asymptotically negligible (Kurusu et al., 2024). The model (2.1) provides a flexible framework that accommodates a wide range of spatial processes, allowing for both structured dependence through $\mathbf{X}(\mathbf{u})$ and small, possibly heterogeneous perturbations via $\mathbf{\Upsilon}(\mathbf{u})$. Suppose that observations are available at irregularly spaced locations $\mathbf{s}_1, \dots, \mathbf{s}_n$ within a region $R_\lambda \subset \mathbb{R}^d$, where $\lambda > 1$ gauges the size of the observation domain. The dimension p is allowed to increase with the sample size n .

Assume that the mean of $\{\mathbf{Y}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ is spatially invariant, that is, $\boldsymbol{\mu} = \mathbb{E}[\mathbf{Y}(\mathbf{u})]$ for all $\mathbf{u} \in \mathbb{R}^d$. In high-dimensional spatial data analysis of fields such as environmental science (He et al., 2021), climatology (Sommerfeld et al., 2018) and neuroscience (Bowring et al., 2019), various inferential problems concerning $\boldsymbol{\mu}$ – including the construction of

2.2 The sampling design

simultaneous confidence intervals and the testing of joint hypotheses – arise naturally. In these settings, our primary object of interest is the scaled deviation of the sample mean, defined as

$$\mathbf{T}_{\lambda,n} = \lambda^{d/2}(\bar{\mathbf{Y}}_{\lambda,n} - \boldsymbol{\mu}), \quad (2.2)$$

where the scaling $\lambda^{d/2}$ reflects the square root of the volume of R_λ , and $\bar{\mathbf{Y}}_{\lambda,n} = n^{-1} \sum_{i=1}^n \mathbf{Y}(\mathbf{s}_i)$.

Distributional approximations of the statistic (2.2) have been extensively investigated in the contexts of high-dimensional independent observations and time series (e.g. Chernozhukov et al., 2017; Koike, 2021; Chernozhukov et al., 2023). However, when data are collected over irregularly spaced spatial locations, the intricate dependence structure and the sampling design introduce substantial theoretical and practical challenges. We will focus on developing bootstrap procedures for approximating the distribution of (2.2) in such spatial settings.

2.2 The sampling design

We specify the spatial domain as $R_\lambda = \lambda R_0$, where $\lambda > 1$ is a scaling factor and R_0 serves as the prototype sampling region. Under this scaling, R_λ expands uniformly in all directions while preserving the geometric characteristics of R_0 . We assume that $R_0 \subset \mathbb{R}^d$ is a bounded Borel set containing the origin in its interior, with unit volume $|R_0| = 1$. Its boundary is assumed to satisfy the regularity condition $N(\partial R_0, a) = \mathcal{O}(a^{-d+1})$ for all sufficiently small $a > 0$, where $N(\partial R_0, a)$ denotes the number of cubes $\{\mathbf{i} + [0, 1]^d\}$, $\mathbf{i} \in \mathbb{Z}^d$ intersecting both R_0 and its complement (cf. Lahiri, 2003; Lahiri and

Zhu, 2006; Kurisu et al., 2024). This condition is satisfied by a broad class of convex and non-convex sampling regions and excludes pathological geometries. For the sampling locations $\mathbf{s}_1, \dots, \mathbf{s}_n$, we assume they are i.i.d. random vectors in R_λ . This stochastic design accommodates both the pure increasing domain regime ($n/\lambda^d \rightarrow \Delta \in (0, \infty)$) and the mixing increasing domain regime ($n/\lambda^d \rightarrow \infty$). Such random sampling schemes for irregularly spaced spatial data have been widely employed in the literature (e.g. Lahiri and Zhu, 2006; Bandyopadhyay and Subba Rao, 2017; Lahiri, 2018; Zhang, 2024).

3. Methodology

3.1 Design of spatial subsample statistics

We will propose two spatial subsample bootstrap approaches for approximating the distribution of (2.2): the spatial multiplier subsample bootstrap (SMSB) and the spatial empirical subsample bootstrap (SESB). Both procedures rely on spatial subsample statistics constructed through the following three steps.

Step 1 (Scaled-down regions). We construct a family of scaled-down versions of R_λ , denoted by $\{R_b(k), k = 1, \dots, N\}$, where each $R_b(k)$ is a subset of R_λ that has the same shape as R_λ but is scaled by a factor $b < \lambda$. More specifically, let $m = \lceil \lambda \rceil - 1$ and set $\delta = (\lambda - b)/m$, where $\lceil \lambda \rceil$ denotes the smallest integer greater than λ . Define $R_{\lambda-b} = (\lambda - b) R_0$ and let $\mathbb{Z}_\delta = \{\delta \mathbf{i} : \mathbf{i} \in \mathbb{Z}^d\}$. Then the set $\mathbb{Z}_\delta \cap R_{\lambda-b}$ is finite. Let N be its cardinality and denote its elements by $\mathbf{c}_1, \dots, \mathbf{c}_N$. Note that $N \asymp \lambda^d$ as $b/\lambda \rightarrow 0$. For each $k = 1, \dots, N$, define $R_b(k) = \{\mathbf{s} : \mathbf{s} = \mathbf{c}_k + b \mathbf{u}, \mathbf{u} \in R_0\}$.

3.1 Design of spatial subsample statistics

Step 2 (Localized sample means). For each $k = 1, \dots, N$, define the localized sample mean over $R_b(k)$ as

$$\bar{\mathbf{Y}}_{b,n}(k) = n_k^{-1} \sum_{i=1}^n \mathbf{Y}(\mathbf{s}_i) \mathbb{I}_{R_b(k)}(\mathbf{s}_i),$$

where $n_k = \sum_{i=1}^n \mathbb{I}_{R_b(k)}(\mathbf{s}_i)$ is the number of sampling locations in $R_b(k)$.

Step 3 (Subsample statistics). Define the subsample statistics

$$\hat{\mathbf{T}}_{\lambda,b,n}(k) = (\lambda^d/n)^{1/2} n_k^{1/2} (\bar{\mathbf{Y}}_{b,n}(k) - \bar{\mathbf{Y}}_{\lambda,n}), \quad k = 1, \dots, N. \quad (3.1)$$

These statistics will serve as the building blocks of our spatial subsample bootstrap procedures.

Let $\sigma_{ij}(\mathbf{h}) = \text{Cov}(X_i(\mathbf{0}), X_j(\mathbf{h}))$ denote the covariance function, where $X_i(\mathbf{u})$ denotes the i th component of $\mathbf{X}(\mathbf{u})$. Let $\text{Cov}_{\cdot|\mathbf{s}}(\cdot, \cdot)$ denote the conditional covariance given the sampling locations $\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$. Define

$$\Xi_{\lambda,n} = (\Xi_{\lambda,n}^{(i,j)})_{p \times p} = \text{Cov}_{\cdot|\mathbf{s}}(\mathbf{T}_{\lambda,n}, \mathbf{T}_{\lambda,n}), \quad (3.2)$$

$$\Xi_{\lambda,b,n}(k) = (\Xi_{\lambda,b,n}^{(i,j)}(k))_{p \times p} = \text{Cov}_{\cdot|\mathbf{s}}(\hat{\mathbf{T}}_{\lambda,b,n}(k), \hat{\mathbf{T}}_{\lambda,b,n}(k)). \quad (3.3)$$

The rationale for introducing the normalization factor $(\lambda^d/n)^{1/2} n_k^{1/2}$ in (3.1) is justified by the following lemma.

Lemma 1. Let $\{\mathbf{Y}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ be a p -dimensional random field admitting the decomposition (2.1). Under suitable regularity conditions (Assumptions 1 (a), 2(a) and 3), the following results hold:

3.2 Spatial multiplier and empirical subsample bootstraps

(i) As $n \rightarrow \infty$, $\lambda \rightarrow \infty$ and $n/\lambda^d \rightarrow \Delta \in (0, \infty]$,

$$\Xi_{\lambda,n}^{(i,j)} = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) d\mathbf{h} + \frac{\lambda^d}{n} \sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\left(\frac{\log \lambda}{\lambda}\right), \quad (3.4)$$

where $\mathcal{O}_p\left(\frac{\log \lambda}{\lambda}\right)$ holds uniformly for $i, j = 1, \dots, p$.

(ii) As $n \rightarrow \infty$, $\lambda \rightarrow \infty$, $b \rightarrow \infty$, $b/\lambda \rightarrow 0$ and $n/\lambda^d \rightarrow \Delta \in (0, \infty]$,

$$\Xi_{\lambda,b,n}^{(i,j)}(k) = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) d\mathbf{h} + \frac{\lambda^d}{n} \sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\left(\frac{\log \lambda}{\lambda}\right) + \mathcal{O}_p\left(\frac{1}{b^{\min(1,d/2)}}\right) + \mathcal{O}_p\left(\frac{b^{d/2}}{\lambda^{d/2}}\right), \quad (3.5)$$

where all remainder terms are uniform over $i, j = 1, \dots, p$ and $k = 1, \dots, N$.

Proof. See Section C of the online supplementary material. $\square$

The normalization factor $(\lambda^d/n)^{1/2} n_k^{1/2}$ in (3.1) differs not only from the $n_k^{1/2}$ typically used in constructing subsample statistics for time series (Ma and Zhang, 2026), but also from $b^{d/2}$, which could be the natural analogue of $\lambda^{d/2}$ in $\mathbf{T}_{\lambda,n}$. According to Lemma 1, this factor is deliberately chosen to balance the variance structure, enabling $\hat{\mathbf{T}}_{\lambda,b,n}(k)$ to more accurately mimic the variance behavior of $\mathbf{T}_{\lambda,n}$.

3.2 Spatial multiplier and empirical subsample bootstraps

The first proposed approach is the spatial multiplier subsample bootstrap (SMSB). It approximates the distribution of (2.2) by the SMSB statistic $\mathbf{T}_{\lambda,n}^*$, which applies random weights to the subsample statistics $\{\hat{\mathbf{T}}_{\lambda,b,n}(k)\}_{k=1}^N$ (defined in (3.1)) as follows:

$$\mathbf{T}_{\lambda,n}^* = N^{-1/2} \sum_{k=1}^N \hat{\mathbf{T}}_{\lambda,b,n}(k) e_k, \quad (3.6)$$

where $\{e_k\}_{k=1}^N$ are i.i.d. random multipliers satisfying $E(e_k) = 0$ and $\text{Var}(e_k) = 1$.

3.2 Spatial multiplier and empirical subsample bootstraps

The choice of multipliers $\{e_k\}_{k=1}^N$ in (3.6) follows Chernozhukov et al. (2022). Specifically, we take $e_k = e_{k,1} + e_{k,2}$, where $e_{k,1}$ and $e_{k,2}$ are independent, $e_{k,1} \sim N(0, \sigma_e^2)$ for some $\sigma_e \geq 0$, and $|e_{k,2}| \leq 3$. Typical examples include: Gaussian multipliers: $e_{k,1} \sim N(0, 1)$ and $e_{k,2} = 0$; Rademacher multipliers: $e_{k,1} = 0$ and $P(e_{k,2} = \pm 1) = 1/2$; and Mammen's multipliers: $e_{k,1} = 0$ and $P(e_{k,2} = (1 \pm \sqrt{5})/2) = (\sqrt{5} \mp 1)/(2\sqrt{5})$.

The second proposed approach is the spatial empirical subsample bootstrap (SESB). It approximates the distribution of (2.2) by the SESB statistic $\mathbf{T}_{\lambda,n}^\circ$. The statistic $\mathbf{T}_{\lambda,n}^\circ$ provides a resampled version of $\mathbf{T}_{\lambda,n}$ by first drawing $\{\mathbf{T}_{\lambda,b,n}^\circ(1), \dots, \mathbf{T}_{\lambda,b,n}^\circ(N)\}$ from the empirical distribution of the subsample statistics $\{\hat{\mathbf{T}}_{\lambda,b,n}(k), k = 1, \dots, N\}$, and then computing

$$\mathbf{T}_{\lambda,n}^\circ = N^{-1/2} \sum_{k=1}^N (\mathbf{T}_{\lambda,b,n}^\circ(k) - \tilde{\mathbf{T}}_{\lambda,b,n}), \quad (3.7)$$

where $\tilde{\mathbf{T}}_{\lambda,b,n} = N^{-1} \sum_{k=1}^N \hat{\mathbf{T}}_{\lambda,b,n}(k)$.

By randomly weighting or resampling the set $\{\hat{\mathbf{T}}_{\lambda,b,n}(k), k = 1, \dots, N\}$, both proposed approaches reproduce the sampling variability of $\mathbf{T}_{\lambda,n}$ by introducing stochastic perturbations analogous to classical bootstraps for independent data (cf. Chernozhukov et al., 2013; Deng and Zhang, 2020; Chernozhukov et al., 2022). These methods enable flexible inference for high-dimensional spatial data, accommodating irregular sampling designs and complex dependence structures.

Since $N \asymp \lambda^d$ and, under the approximately uniform sampling design, $n_k \asymp n(b/\lambda)^d$, the total number of observations involved in constructing the N localized sample means is of order $\sum_{k=1}^N n_k = \mathcal{O}(nb^d)$. Once the observations belonging to each subsample are

3.3 Connection to existing dependent-data bootstrap approaches

identified, constructing all subsample statistics for a given b requires $\mathcal{O}(nb^d p)$ operations, while generating each SMSB or SESB bootstrap statistic requires an additional $\mathcal{O}(Np) = \mathcal{O}(\lambda^d p)$ operations. Thus, B bootstrap replications require $\mathcal{O}((nb^d + B\lambda^d)p)$ operations.

3.3 Connection to existing dependent-data bootstrap approaches

Our proposed methods are related to several existing dependent-data bootstrap approaches, though they differ fundamentally in both construction and applicability.

First, the proposed SMSB and SESB are conceptually connected to the grid-based block bootstrap (GBBB) of Lahiri and Zhu (2006), in that both approximate the sampling variability of spatial statistics through localized blocks. However, while the GBBB operates on spatial data blocks embedded in a regular grid to reconstruct pseudo-observations, our approaches work directly with subsample statistics computed from irregularly spaced locations. This statistic-level resampling, combined with multiplier or empirical subsampling schemes, enables the SMSB and SESB to efficiently approximate the distribution of high-dimensional spatial statistics without requiring grid embedding or data-level resampling.

Next, the proposed SMSB is related to the spatial dependent wild bootstrap (SDWB) of Kurisu et al. (2024) in that the SMSB statistic can be expressed as

$$\mathbf{T}_{\lambda,n}^* = \lambda^{d/2} n^{-1/2} \sum_{i=1}^n (\mathbf{Y}(\mathbf{s}_i) - \bar{\mathbf{Y}}_{\lambda,n}) w_b(\mathbf{s}_i), \quad \text{with} \quad w_b(\mathbf{s}_i) = N^{-1/2} \sum_{k \in \{k: R_b(k) \ni \mathbf{s}_i\}} n_k^{-1/2} e_k.$$

This representation shows that the SMSB can be viewed as an SDWB with b -dependent random weights $\{w_b(\mathbf{s}_i)\}$.

Furthermore, the proposed SMSB and SESB are connected to the Wiener-type integral approximation (WIA) of Zhang (2025), as all three aim to capture the second-order characteristics of spatial statistics. However, unlike the WIA, our methods explicitly compute subsample statistics and generate bootstrap replicates via multiplier weights or empirical resampling. In contrast, the WIA approach is more analytical in nature, focusing on integral approximations.

Finally, the proposed SMSB and SESB are closely related to the multiplier subsample bootstrap (MSB) and empirical subsample bootstrap (ESB) of Ma and Zhang (2026), as they extend these ideas from high-dimensional time series to spatial random fields. While both frameworks approximate the distribution of high-dimensional statistics using localized subsample statistics, the SMSB and SESB are specifically adapted to handle spatial dependence and irregular sampling. In particular, spatial blocks replace temporal ones, and the normalization factor $(\lambda^d/n)^{1/2}n_k^{1/2}$ adjusts for irregularly spaced spatial sampling. Moreover, the spatial setting lacks natural ordering, requiring geometrically defined overlapping blocks and a domain-expanding asymptotic regime.

4. Theoretical validity

4.1 Assumptions

Recall that the irregularly spaced spatial data are observed from $\{\mathbf{Y}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$, a p -dimensional random field admitting the decomposition (2.1) with components $\{\mathbf{X}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ and $\{\boldsymbol{\Upsilon}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$.

4.1 Assumptions

To concisely present the regularity conditions on $\{\mathbf{X}(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$, we introduce the following notation. Let $c_4^{(i,j,l,m)}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \text{cum}(X_i(\mathbf{0}), X_j(\mathbf{h}_1), X_l(\mathbf{h}_2), X_m(\mathbf{h}_3))$ denote the fourth-order cumulant (cf. Brillinger, 2001, pp. 19) for $1 \leq i, j, l, m \leq p$. For sets $E_1, E_2 \subset \mathbb{R}^d$, define $\delta(E_1, E_2) = \inf\{\|\mathbf{u}_1 - \mathbf{u}_2\|_1 : \mathbf{u}_i \in E_i, i = 1, 2\}$. Let $\sigma_X(E)$ denote the σ -algebra generated by the random variables $\{\mathbf{X}(\mathbf{u}) : \mathbf{u} \in E\}$. The β -mixing coefficient of the random field $\{\mathbf{X}(\cdot)\}$ is defined as $\check{\beta}(E_1, E_2) = \frac{1}{2} \sup \sum_{j=1}^J \sum_{k=1}^K |\mathbb{P}(A_j \cap B_k) - \mathbb{P}(A_j)\mathbb{P}(B_k)|$, where the supremum is taken over all finite partitions $\{A_j\}_{j=1}^J \subset \sigma_X(E_1)$ and $\{B_k\}_{k=1}^K \subset \sigma_X(E_2)$ of $\mathbb{R}^d$ (e.g. Kurisu et al., 2024). For $a, b > 0$, we define $\beta(a; b) = \sup\{\check{\beta}(E_1, E_2) : \delta(E_1, E_2) \geq a, E_1, E_2 \in \mathbb{C}_b\}$, where $\mathbb{C}_b$ is the collection of d -dimensional sets with volume b or less.

Assumption 1 (Conditions on $\mathbf{X}(\cdot)$). (a) The covariance functions satisfy $\max_{1 \leq j \leq p} |\sigma_{jj}(\mathbf{h})| \leq \rho_2(\mathbf{h})$, $\max_{1 \leq j \leq p} \int_{\mathbb{R}^d} \|\mathbf{h}\|_1 |\sigma_{jj}(\mathbf{h})| d\mathbf{h} < \infty$ and $\min_{1 \leq j \leq p} \int_{\mathbb{R}^d} \sigma_{jj}(\mathbf{h}) d\mathbf{h} \geq c$, where $c > 0$ is a constant. Each component of $\mathbf{X}(\mathbf{u})$ has a finite fourth moment, and the fourth-order joint cumulants satisfy $|c_4^{(i,j,l,m)}(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)| \leq \prod_{j=1}^3 \rho_2(\mathbf{h}_j)$, for all $1 \leq i, j, l, m \leq p$.

(b) The β -mixing coefficient satisfies $\beta(a; b) \leq \beta_1(a) \beta_2(b)$ for $a, b \in (0, \infty)$, where $\beta_1 : (0, \infty) \rightarrow [0, \infty)$ is a left continuous, non-increasing function satisfying $\beta_1(a) \leq Ka^{-\eta}$ for some constants $K > 0$ and $\eta > 1$. Moreover, $\beta_2 : (0, \infty) \rightarrow (0, \infty)$ is a right-continuous, non-decreasing function such that $(\lambda/\lambda_1)^d \lambda_2^{-\eta} \beta_2(\lambda^d) \rightarrow 0$ as $\lambda \rightarrow \infty$, where $\lambda_1, \lambda_2 \rightarrow \infty$ satisfy $\lambda_2/\lambda_1 = o(1)$, and $\lambda_1/\lambda = o(1)$.

(c) There exists a sequence of positive constants $\{D_n\}_{n \geq 1}$ with $D_n \geq 1$ such that

$$\max_{1 \leq j \leq p} \|X_j(\mathbf{u})\|_{\psi_1} \leq D_n.$$

4.1 Assumptions

Assumption 1 imposes moment, covariance, cumulant, and β -mixing conditions on $\mathbf{X}(\cdot)$. Part (a) controls the dependence and moments needed for Lemma 1 (cf. Zhang, 2024, 2025), while part (b) gives a β -mixing condition used in the large-block–small-block argument (cf. Lahiri, 2003; Kurisu et al., 2024). Part (c) imposes a sub-exponential tail condition standard in high-dimensional statistics (cf. Chernozhukov et al., 2017; Deng and Zhang, 2020; Chang et al., 2024).

Assumption 2 (Conditions on $\Upsilon(\cdot)$). (a) The random field $\{\Upsilon(\mathbf{u}), \mathbf{u} \in \mathbb{R}^d\}$ satisfies

$$\mathbb{E}\left[\sup_{\mathbf{u} \in R_\lambda} \max_{1 \leq j \leq p} |\Upsilon_j(\mathbf{u})|^2\right] = \mathcal{O}\left(\frac{\log^2 \lambda}{\lambda^{2d+2}}\right).$$

(b) There exists some $\varsigma > 0$ such that

$$\mathbb{P}_{\cdot|\mathbf{s}}\left(\max_{1 \leq j \leq p} \left|\lambda^{d/2} n^{-1} \sum_{i=1}^n \Upsilon_j(\mathbf{s}_i)\right| > n^{-\varsigma} \log^{-1/2} p\right) = \mathcal{O}(n^{-\varsigma})$$

holds almost surely.

Assumption 2 ensures that the perturbation field $\Upsilon(\cdot)$ is asymptotically negligible. Part (a) uses a weaker second-order condition than Condition (4.2) of Kurisu et al. (2024), while part (b) parallels their Condition (4.1). An illustrative example of a mildly non-stationary spatial field satisfying Assumption 2 is provided in Section B.2 of the online supplementary material.

Assumption 3 (Condition on sampling locations). The sampling locations $\mathbf{s}_1, \dots, \mathbf{s}_n$ are i.i.d. random vectors, independent of the random field, with pdf $f(\mathbf{u})$ supported on R_λ satisfying $|f(\mathbf{u}) - 1/\lambda^d| \leq B_0 \log \lambda / \lambda^{d+1}$, where $B_0 > 0$ is a constant.

4.1 Assumptions

Assumption 3 allows asymptotically vanishing spatial heterogeneity in the sampling design while requiring the sampling density to be sufficiently close to uniform (cf. Zhang, 2025). For example, $f_\lambda(\mathbf{u}) \propto \lambda^{-d} (1 + \epsilon_\lambda q(\mathbf{u}/\lambda))$ with bounded $q(\cdot)$ and $\epsilon_\lambda = \mathcal{O}(\log \lambda/\lambda)$ satisfies the assumption. Designs with persistent spatial heterogeneity, such as the case $\epsilon_\lambda = \mathcal{O}(1)$ in the above construction or $f_\lambda(\mathbf{u}) = \lambda^{-d} g(\mathbf{u}/\lambda)$ for a nonconstant bounded density g , are excluded.

Assumption 4. Let $\{\lambda_1\}$ and $\{\lambda_2\}$ be the sequences of positive real numbers as given in Assumption 1(b). Let $\{D_n\}$ be the sequence given in Assumption 1(c).

$$(a) \text{ The equality } \max \left\{ \frac{b^{d/2}}{\lambda^{d/2}}, \sqrt{\frac{\lambda_2 D_n \log n}{\lambda_1}}, \frac{b^{d/2} \lambda_1^{d-1} \lambda_2 D_n \log n}{\lambda^{d/2}}, \frac{\lambda_1^{2d} b^{2d}}{n^{1/2}}, \frac{\lambda_1^{3d/2} D_n^2 \log^2 n}{\lambda^{d/2}} \right\} = \mathcal{O}(n^{-c_1})$$

holds for some constant $0 < c_1 < 1/2$.

$$(b) \text{ The equality } \max \left\{ \frac{1}{\sqrt{\lambda_2}}, D_n \left(\frac{\lambda^d \log n}{n} + 1 \right) \sqrt{\frac{\bar{\beta}_q \lambda_2}{\lambda_1}}, \frac{\lambda_1^{d/2} \lambda_2^d D_n^2 + \lambda_1^d D_n}{\lambda^{d/2}}, \frac{\lambda^d D_n^6 \lambda_1^{3d}}{n^2} \right\} = \mathcal{O}(n^{-c_2})$$

holds for some constant $0 < c_2 < 1/2$, where $\bar{\beta}_q = \bar{\beta}_q(n) := 1 + \sum_{k=1}^{\lambda_1} k^{d-1} \beta_1^{1-2/q}(k)$, $q > 2$.

Assumption 4 collects the technical rate conditions required for the Gaussian approximation and the validity of the SSB. In particular, part (a) controls the approximation error in Proposition 1, whereas part (b) ensures the central limit theorem required for the spatial statistics (Kurisu et al., 2024, Assumption 4.1(iv)). For illustration, suppose $p \asymp n^\iota$, $\lambda \asymp n^{\alpha_0}$, $b \asymp n^{\alpha_b}$, $\lambda_1 \asymp n^{\alpha_1}$, $\lambda_2 \asymp n^{\alpha_2}$, and $D_n = \mathcal{O}(1)$. The rate conditions are satisfied, for example, when $d = 2$, $\iota = 1/2$, $\eta = 4$, $q = 8$, and $(\alpha_0, \alpha_1, \alpha_2, \alpha_b) = (1/4, 1/16, 1/40, 1/20)$. Thus, the conditions permit power-rate growing spatial domains, subsampling regions, and dimension, with $\lambda_2 \ll \lambda_1 \ll \lambda$.

4.2 Main results

Let

$$\Xi_{n/\lambda^d} = (\Xi_{n/\lambda^d}^{(i,j)})_{p \times p}, \quad \Xi_{n/\lambda^d}^{(i,j)} = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) d\mathbf{h} + \frac{\lambda^d}{n} \sigma_{ij}(\mathbf{0}), \quad i, j = 1, \dots, p. \quad (4.1)$$

To establish the theoretical validity of the SSBs, we impose the following condition:

Condition (C) As $n \rightarrow \infty$, $\lambda \rightarrow \infty$, $b \rightarrow \infty$, $b/\lambda \rightarrow 0$, $n/\lambda^d \rightarrow \Delta \in (0, \infty]$ and $D_n \log n / b^{d/4} \rightarrow 0$.

The following theorem shows that the bootstrap statistics $\mathbf{T}_{\lambda,n}^*$ and $\mathbf{T}_{\lambda,n}^\circ$ admit uniformly consistent variance estimators across all coordinates.

Theorem 1. Suppose that $p = \mathcal{O}(n^\iota)$ for some $\iota \in (0, (\eta-1)/5]$ and that Assumptions 1–3 are satisfied. Under Condition (C), the following results hold:

- (i) $\|\text{Var}_{|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda,n}^*) - \Xi_{n/\lambda^d}\|_{\max} = \mathcal{O}_p(\mathcal{U}_{\lambda,b}) + \mathcal{O}_p(\mathcal{V}_{\lambda,b,p}),$
- (ii) $\|\text{Var}_{|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda,n}^\circ) - \Xi_{n/\lambda^d}\|_{\max} = \mathcal{O}_p(\mathcal{U}_{\lambda,b}) + \mathcal{O}_p(\mathcal{V}_{\lambda,b,p}),$

where $\mathcal{U}_{\lambda,b} = \log \lambda / \lambda + 1/b^{\min(1,d/2)}$ and $\mathcal{V}_{\lambda,b,p} = \sqrt{\log p} (b^{d/2}/\lambda^{d/2} + (\log b)^{d/2}/b^{d/2})$.

Proof. See Section D the online supplementary material. □

Remark 1. Both bootstrap statistics estimate Ξ_{n/λ^d} with the same asymptotic accuracy. The term $\mathcal{O}_p(\mathcal{U}_{\lambda,b})$ reflects bias incurred when approximating Ξ_{n/λ^d} via expectations of subsample statistics, while $\mathcal{O}_p(\mathcal{V}_{\lambda,b,p})$ accounts for random fluctuations of the covariances of these subsample statistics around their means.

Let $\Xi_\Delta = (\Xi_\Delta^{(i,j)})_{p \times p}$, where $\Xi_\Delta^{(i,j)} = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) d\mathbf{h} + \sigma_{ij}(\mathbf{0})/\Delta$, $i, j = 1, \dots, p$. Theorem 1 implies the following corollary.

Corollary 1. Under the conditions of Theorem 1, if, in addition $b^d \log p / \lambda^d \rightarrow 0$, then

$$\left\| \text{Var}_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^*) - \Xi_{\Delta} \right\|_{\max} = o_p(1), \quad \left\| \text{Var}_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^{\circ}) - \Xi_{\Delta} \right\|_{\max} = o_p(1).$$

Theorem 1 establishes consistency of the bootstrap covariance estimators. We next establish a high-dimensional Gaussian approximation for the bootstrap statistics.

Proposition 1. Suppose that $p = \mathcal{O}(n^{\iota})$ for some $\iota \in (0, (\eta - 1)/5]$ and that Assumptions 1–3 and 4(a) hold. Under Condition (C), if $\mathcal{U}_{\lambda, b} \log^2 p = \mathcal{O}(n^{-c_1})$, then the following results hold with $P_{\cdot|\mathbf{s}}$ -probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}(\mathcal{W}_{\lambda, \lambda_1, \lambda_2})$:

- (i) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^* \in A) - P_{\cdot|\mathbf{s}}(\mathbf{Z}_{\lambda, n} \in A)| = \mathcal{O}(n^{-c_1/4});$
- (ii) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^{\circ} \in A) - P_{\cdot|\mathbf{s}}(\mathbf{Z}_{\lambda, n} \in A)| = \mathcal{O}(n^{-c_1/4}).$

Here, $\mathcal{W}_{\lambda, \lambda_1, \lambda_2} = (\lambda/\lambda_1)^d \beta(\lambda_2; \lambda^d)$, $\mathcal{A}$ denotes the class of closed rectangles in $\mathbb{R}^p$, and $\mathbf{Z}_{\lambda, n} = (Z_{\lambda, n}^{(1)}, \dots, Z_{\lambda, n}^{(p)})^{\top}$ is a centered Gaussian random vector under $P_{\cdot|\mathbf{s}}$ with covariance matrix Ξ_{n/λ^d} .

Proof. See Sections E.1 and E.2 of the online supplementary material. $\square$

Combining Proposition 1 with the Gaussian approximation for $\mathbf{T}_{\lambda, n}$ yields the bootstrap validity result.

Theorem 2. Suppose that $p = \mathcal{O}(n^{\iota})$ for some $\iota \in (0, (\eta - 1)/5]$ and that Assumptions 1–4 hold. Under Condition (C), if $\mathcal{U}_{\lambda, b} \log^2 p = \mathcal{O}(n^{-c_1})$, then the following statements hold with $P_{\cdot|\mathbf{s}}$ -probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}(\mathcal{W}_{\lambda, \lambda_1, \lambda_2})$:

- (i) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^* \in A) - P_{\cdot|\mathbf{s}}(\mathbf{T}_{\lambda, n} \in A)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda, \lambda_1, \lambda_2});$
- (ii) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda, n}^{\circ} \in A) - P_{\cdot|\mathbf{s}}(\mathbf{T}_{\lambda, n} \in A)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda, \lambda_1, \lambda_2}).$

Proof. See Section E.3 of the online supplementary material. $\square$

For the construction of simultaneous confidence intervals and the testing of joint hypotheses for μ , we are interested in the sampling distribution of the L^∞ -norm “statistic” $\|\mathbf{T}_{\lambda,n}\|_\infty$. Theorem 2 immediately yields validity for $\|\mathbf{T}_{\lambda,n}\|_\infty$.

Corollary 2. Under the same conditions as in Theorem 2, the following statements hold with $P_{\cdot|\mathbf{s}}$ -probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2})$:

- (i) $\sup_{x \in \mathbb{R}} |P_{\cdot|\mathbf{s},\mathbf{Y}}(\|\mathbf{T}_{\lambda,n}^*\|_\infty \leq x) - P_{\cdot|\mathbf{s}}(\|\mathbf{T}_{\lambda,n}\|_\infty \leq x)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2});$
- (ii) $\sup_{x \in \mathbb{R}} |P_{\cdot|\mathbf{s},\mathbf{Y}}(\|\mathbf{T}_{\lambda,n}^\circ\|_\infty \leq x) - P_{\cdot|\mathbf{s}}(\|\mathbf{T}_{\lambda,n}\|_\infty \leq x)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}).$

Define $D_{\lambda,n}^* = \text{diag}(\Xi_{\lambda,n}^*)^{1/2}$ and $D_{\lambda,n}^\circ = \text{diag}(\Xi_{\lambda,n}^\circ)^{1/2}$, where $\Xi_{\lambda,n}^* = \text{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}_{\lambda,n}^*)$ and $\Xi_{\lambda,n}^\circ = \text{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}_{\lambda,n}^\circ)$. Since the components of $\mathbf{T}_{\lambda,n}$ may have heterogeneous variances, we consider the studentized statistics based on the diagonal covariance estimators $D_{\lambda,n}^*$ and $D_{\lambda,n}^\circ$. Finally, we establish validity after studentization.

Theorem 3. Under the same conditions as in Theorem 2, the following statements hold with $P_{\cdot|\mathbf{s}}$ -probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2})$:

- (i) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s},\mathbf{Y}}(D_{\lambda,n}^{*-1} \mathbf{T}_{\lambda,n}^* \in A) - P_{\cdot|\mathbf{s}}(D_{\lambda,n}^{*-1} \mathbf{T}_{\lambda,n} \in A)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2});$
- (ii) $\sup_{A \in \mathcal{A}} |P_{\cdot|\mathbf{s},\mathbf{Y}}(D_{\lambda,n}^{\circ-1} \mathbf{T}_{\lambda,n}^\circ \in A) - P_{\cdot|\mathbf{s}}(D_{\lambda,n}^{\circ-1} \mathbf{T}_{\lambda,n} \in A)| = \mathcal{O}(n^{-\frac{c_1}{4}}) + \mathcal{O}(n^{-\frac{c_2}{6} \wedge \varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}).$

Proof. See Section E.4 of the online supplementary material. $\square$

5. Choice of the optimal subsample size

5.1 Theoretical choice of the optimal subsample size

To select an optimal subsample size, we consider the criterion of minimizing the asymptotic mean squared error (MSE). Recall that $\Xi_{\lambda,n}^{(i,j)}$ is defined in (3.2). We choose the subsample size b by minimizing the sum of the diagonal MSEs

$$\sum_{j=1}^p \mathbb{E}[(\Xi_{\lambda,n}^{*(j,j)} - \Xi_{\lambda,n}^{(j,j)})^2] \quad \text{or} \quad \sum_{j=1}^p \mathbb{E}[(\Xi_{\lambda,n}^{\circ(j,j)} - \Xi_{\lambda,n}^{(j,j)})^2],$$

corresponding to the applications of the SMSB or the SESB, respectively. Here, $\Xi_{\lambda,n}^{*(i,j)}$ and $\Xi_{\lambda,n}^{\circ(i,j)}$ denote the (i, j) th entries of $\Xi_{\lambda,n}^*$ and $\Xi_{\lambda,n}^\circ$, respectively. This criterion provides a direct measure of variance estimation accuracy while avoiding the additional computational complexity associated with off-diagonal terms when p is large.

For a generic estimator $\hat{\theta}$, it holds that $\mathbb{E}[(\hat{\theta} - \theta)^2] = \text{Var}(\hat{\theta}) + (\mathbb{E}(\hat{\theta}) - \theta)^2$. To derive a theoretically optimal subsample size, we first establish the asymptotic bias and variance of the (i, j) th entries, as stated in the following proposition.

Proposition 2. Suppose that Assumptions 1(a), 2(a) and 3 hold. Suppose further that, as $n \rightarrow \infty$, $\lambda \rightarrow \infty$, $b \rightarrow \infty$, $b \log \lambda / \lambda \rightarrow 0$, $b^{d+1} / \lambda^d \rightarrow 0$, $\lambda^d / (nb^{d-1}) \rightarrow 0$ and $n / \lambda^d \rightarrow \Delta \in (0, \infty]$. If $\lim_{b \rightarrow \infty} \frac{b^d - |R_b \cap (R_b + \mathbf{h})|}{b^{d-1}} = V(\mathbf{h})$ exists for all $\mathbf{h} \in \mathbb{R}^d$, then

- (i) $\mathbb{E}(\Xi_{\lambda,n}^{*(i,j)} - \Xi_{\lambda,n}^{(i,j)}) = (-\Phi_{ij}/b)(1 + o(1))$,
- (ii) $\mathbb{E}(\Xi_{\lambda,n}^{\circ(i,j)} - \Xi_{\lambda,n}^{(i,j)}) = (-\Phi_{ij}/b)(1 + o(1))$,
- (iii) $\text{Var}(\Xi_{\lambda,n}^{*(i,j)}) = K_0 b^d \lambda^{-d} (\Xi_{n/\lambda^d}^{(i,i)} \Xi_{n/\lambda^d}^{(j,j)} + (\Xi_{n/\lambda^d}^{(i,j)})^2)(1 + o(1))$,
- (iv) $\text{Var}(\Xi_{\lambda,n}^{\circ(i,j)}) = K_0 b^d \lambda^{-d} (\Xi_{n/\lambda^d}^{(i,i)} \Xi_{n/\lambda^d}^{(j,j)} + (\Xi_{n/\lambda^d}^{(i,j)})^2)(1 + o(1))$,

5.1 Theoretical choice of the optimal subsample size

where $\Phi_{ij} = \int_{\mathbb{R}^d} V(\mathbf{h}) \sigma_{ij}(\mathbf{h}) d\mathbf{h} \neq 0$, $K_0 = \int_{\mathbb{R}^d} |R_0 \cap (\mathbf{h} + R_0)|^2 d\mathbf{h}$, and $o(1)$ is uniform over $i, j = 1, \dots, p$. The quantities $\Xi_{\lambda, n}^{(i, j)}$ and $\Xi_{n/\lambda^d}^{(i, j)}$ are defined in (3.2) and (4.1), respectively.

Proof. See Section F of the online supplementary material. $\square$

Remark 2. In Proposition 2, K_0 is a finite positive constant that depends only on the shape of the prototype sampling region R_0 . If $R_0 = (-1/2, 1/2]^d \subset \mathbb{R}^d$, then $K_0 = (2/3)^d$. Additional examples can be found in Section 6 of Nordman and Lahiri (2004).

Proposition 2 shows that the asymptotic bias of $\Xi_{\lambda, n}^{*(i, j)}$ is identical to that of $\Xi_{\lambda, n}^{o(i, j)}$ when estimating $\Xi_{\lambda, n}^{(i, j)}$, and the same holds for their asymptotic variances. Consequently, the SMSB and SESB share a common optimal subsample size, which is characterized in the following theorem.

Theorem 4. *Under the assumptions of Proposition 2, the optimal subsample size b^{opt} satisfies*

$$b^{\text{opt}} = \left(\frac{S_1 \lambda^d}{d K_0 S_2} \right)^{1/(d+2)} (1 + o(1)),$$

where $S_1 = \sum_{j=1}^p \Phi_{jj}^2$ and $S_2 = \sum_{j=1}^p (\Xi_{\lambda, n}^{(j, j)})^2$.

Proof. See Section F of the online supplementary material. $\square$

5.2 Practical choice of the optimal subsample size

Motivated by the optimal rate established in Theorem 4, we propose a data-driven plug-in procedure for selecting the subsample size b in practice. Specifically, we define

$$\hat{b}^{\text{opt}} = \left(\frac{\hat{S}_1 \lambda^d}{d K_0 \hat{S}_2} \right)^{1/(d+2)},$$

where $\hat{S}_1$ and $\hat{S}_2$ are consistent estimators of S_1 and S_2 , respectively.

We now describe the construction of $\hat{S}_1$ and $\hat{S}_2$. First, we choose a pilot subsample size $b^{(1)} = \chi_1 \lambda^{d/(d+2)}$, where $\chi_1 > 0$ is a fixed constant, and compute the subsampling variance estimator $\Xi_{\lambda,n}^{*(j,j)}(b^{(1)})$ (or $\Xi_{\lambda,n}^{\circ(j,j)}(b^{(1)})$) for each diagonal entry $\Xi_{\lambda,n}^{(j,j)}$. Next, following Nordman and Lahiri (2004) and Lahiri et al. (2007), we introduce a secondary scale $b^{(2)} = \chi_2 \lambda^{d/(d+4)}$ with $\chi_2 > 0$, and estimate the bias-related quantity Φ_{jj} via

$$\hat{\Phi}_{jj} = \frac{3}{2} b^{(2)} \left(\Xi_{\lambda,n}^{*(j,j)}(3b^{(2)}) - \Xi_{\lambda,n}^{*(j,j)}(b^{(2)}) \right) \quad \left(\text{or} \quad \frac{3}{2} b^{(2)} \left(\Xi_{\lambda,n}^{\circ(j,j)}(3b^{(2)}) - \Xi_{\lambda,n}^{\circ(j,j)}(b^{(2)}) \right) \right).$$

This estimator exploits the difference between variance estimators evaluated at two nearby scales to isolate the leading bias term. Compared with Nordman and Lahiri (2004), we employ a larger spacing between the two scales, which improves the ability to capture the asymptotic bias behavior while maintaining numerical stability. Finally, substituting the estimators of $\hat{\Phi}_{jj}$ and $\Xi_{\lambda,n}^{(j,j)}$ into the defining expressions of S_1 and S_2 yields the plug-in estimators $\hat{S}_1$ and $\hat{S}_2$. The resulting procedure is theoretically well justified and straightforward to implement. In all numerical experiments, we set $\chi_1 = 1$ and $\chi_2 = 0.5$ when computing $\hat{b}^{\text{opt}}$.

The data-driven selection of b requires the construction of the diagonal subsampling variance estimators at only three scales, namely $b^{(1)}$, $b^{(2)}$, and $3b^{(2)}$. The corresponding computational cost is of order $\mathcal{O}(n[(b^{(1)})^d + (b^{(2)})^d + (3b^{(2)})^d]p)$, apart from the cost of identifying the observations within the overlapping subsamples. With $b^{(1)} = \chi_1 \lambda^{d/(d+2)}$ and $b^{(2)} = \chi_2 \lambda^{d/(d+4)}$, this is of order $\mathcal{O}(n\lambda^{d^2/(d+2)}p)$. The remaining calculations of $\hat{\Phi}_{jj}$, $\hat{S}_1$, $\hat{S}_2$, and $\hat{b}^{\text{opt}}$ require only $\mathcal{O}(p)$ operations. Thus, the data-driven selection of b requires $\mathcal{O}(n\lambda^{d^2/(d+2)}p + p)$ operations.

6. Numerical examples

6.1 Approximating the distribution of the sample mean

We generate irregularly spaced spatial data from a p -dimensional random field $\{\mathbf{Y}(\mathbf{u})\}$ on $\mathbb{R}^2$, defined by

$$\mathbf{Y}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{Z}^2} \exp(-\|\mathbf{u} - \mathbf{x}\|/\rho) \boldsymbol{\eta}_{\mathbf{x}}, \quad \boldsymbol{\eta}_{\mathbf{x}} = (\eta_{\mathbf{x},1}, \dots, \eta_{\mathbf{x},p})^\top, \quad (6.1)$$

with $\eta_{\mathbf{x},j} = \xi_{\mathbf{x},j}(\varepsilon_{\mathbf{x},j} + \gamma \xi_{\mathbf{x},j}^2)$ and $\xi_{\mathbf{x},j}^2 = (1 - \zeta + \zeta \varepsilon_{\mathbf{x},j}^2)$, where $\zeta \in [0, 1]$ and $\varepsilon_{\mathbf{x},j}$'s are i.i.d. standard normal random variables. Model (6.1) allows flexible control of spatial dependence through the range parameter ρ , non-Gaussianity through ζ , and marginal skewness through γ . By varying the parameter triple (ρ, ζ, γ) , the model provides a versatile framework for evaluating the finite-sample performance of bootstrap methods. In the simulation study, we set $\rho = 1/3$ and $1/2$ to represent weaker and stronger spatial dependence, respectively; $\zeta = 0$ and $1/2$ to generate Gaussian and non-Gaussian distributions;

6.1 Approximating the distribution of the sample mean

and $\gamma = 0$ and 0.05 to produce symmetric and skewed marginals.

We set $\lambda = 12$ and 24. Over the sampling region $R_\lambda = \lambda(-1/2, 1/2]^2$, sampling locations $\mathbf{s}_1, \dots, \mathbf{s}_n$ are drawn i.i.d. according to two types of sampling designs (cf. Bandyopadhyay et al., 2015):

(UL) The locations satisfy $\mathbf{s}_i = \lambda \mathbf{z}_i$, where $\mathbf{z}_i$ is uniformly distributed on $(-1/2, 1/2]^2$;

(NUL) The locations satisfy $\mathbf{s}_i = \lambda \mathbf{z}_i$, where $\mathbf{z}_i$ follows a truncated bivariate mixture normal distribution $0.5N_2(\mathbf{0}, I_2) + 0.5N_2((-1/4, -1/4)^\top, 2I_2)$, restricted to $(-1/2, 1/2]^2$.

Under both the (UL) and (NUL) designs, we examine the finite-sample performance of the proposed SMSB and SESB by computing empirical probabilities of the event $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{1-\alpha}\}$, where $\mathbf{T}_{\lambda,n}$ is defined in (2.2), and $q_{1-\alpha}$ denotes the $(1-\alpha)$ -quantile of the distribution of $\|\mathbf{T}_{\lambda,n}\|_\infty$. The (NUL) design violates the asymptotically nearly uniform sampling design condition in Assumption 3. We include simulation results for this case to demonstrate the robustness of our approaches beyond their theoretical scope. Empirical probabilities obtained using the SMSB (with Gaussian, Rademacher and Mammen multipliers, denoted by SMSB-G, SMSB-R and SMSB-M, respectively) and the SESB are compared with three competing methods: the grid-based block bootstrap (GBBB; Lahiri and Zhu, 2006), the spatial dependent wild bootstrap (SDWB; Kurisu et al., 2024) and the Wiener-type integral approximation (WIA; Zhang, 2025). For the SMSB, SESB, GBBB, and SDWB, the bootstrap quantile $q_{1-\alpha}$ is computed using $B = 1000$ bootstrap resamples. In contrast, the WIA does not require resampling, as it is based on a Gaussian approximation. For all approaches, each empirical probability is estimated using 1000

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Monte Carlo replications.

Table 1 reports the empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_{\infty} \leq q_{0.95}\}$ for seven methods (SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA) under the (UL) design with $(n, p) = (100, 50)$, a graphical summary for the case $\lambda = 24$ is provided in Figure A-1 of the online supplementary material. Corresponding results for $(n, p) = (50, 50)$, $(n, p) = (50, 400)$ and $(n, p) = (100, 400)$ are reported in Table A.1 – A.3 of the online supplementary material. Results under the (NUL) design are presented in Tables A.4 – A.7 of the online supplementary material. The reported empirical probabilities correspond to six combinations of $\{\rho, \zeta, \gamma\}$. For $\lambda = 12$, subsample sizes $b \in \{4, 5, 6\}$ are considered, whereas for $\lambda = 24$ we examine $b \in \{6, 8, 10\}$, together with the data-driven choice $\hat{b}^{\text{opt}}$ for the SMSB-G, SMSB-R, SMSB-M and SESB.

From Table 1 and Tables A.1, A.2 and A.3 in the online supplementary material, several common patterns emerge among the competing approaches. First, across all combinations of (ρ, ζ, γ) , with either λ or n held fixed and the other increases, no method exhibits a systematic improvement in terms of the empirical probability approaching the nominal level of 0.95. This phenomenon is not unexpected for distributional approximation methods under irregularly spaced sampling designs. In such settings, the convergence rate of the bootstrap-approximated distribution to the target distribution depends intricately on both the sampling design and the choice of tuning parameters; see Theorem 2 and Corollary 2. Similar behavior has also been reported in the numerical study of Zhang (2025). Second, for each fixed pair (λ, n) , the sensitivity of empirical probability to the

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Table 1: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_{\infty} \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (UL), with $n = 100$ and $p = 50$.

(ρ, ζ, γ)	Method	$\lambda = 12$				$\lambda = 24$			
		$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
(1/3, 0, 0)	SMSB-G	93.6	95.6	94.0	93.8	95.4	96.9	97.2	96.1
	SMSB-R	93.3	95.5	94.3	93.4	95.9	97.0	97.6	95.6
	SMSB-M	93.2	95.9	93.5	93.4	95.9	97.2	97.2	95.7
	SESB	92.5	91.7	80.4	93.4	95.2	96.1	93.9	95.3
	SDWB	92.2	92.0	90.5		95.7	96.1	96.1	
	GBBB	90.3	98.1	76.0		95.3	94.6	98.8	
	WIA	95.5	93.6	88.3		96.5	97.4	96.0	
(1/2, 0, 0)	SMSB-G	83.2	88.3	85.6	82.4	92.4	95.9	96.1	93.7
	SMSB-R	81.7	87.1	84.8	81.4	92.9	95.0	96.4	93.6
	SMSB-M	82.7	87.6	85.4	81.4	93.4	95.3	95.9	93.2
	SESB	79.3	79.6	64.3	79.2	92.9	94.9	91.9	93.2
	SDWB	78.5	82.0	82.0		93.6	94.8	94.5	
	GBBB	79.9	96.0	66.5		93.5	93.5	98.5	
	WIA	87.4	86.7	82.8		93.7	96.0	94.5	
(1/3, 1/2, 0)	SMSB-G	94.1	96.7	95.4	95.3	97.3	98.1	97.7	97.6
	SMSB-R	94.5	96.6	95.0	94.7	97.4	97.8	98.2	97.2
	SMSB-M	94.2	96.5	94.9	95.4	97.8	98.1	97.8	97.7
	SESB	93.6	93.3	85.3	94.4	97.0	97.6	95.4	97.3
	SDWB	93.5	93.2	92.6		97.6	96.8	96.5	
	GBBB	92.1	98.5	78.3		97.2	95.7	99.1	
	WIA	97.8	96.1	92.1		98.2	98.7	97.6	
(1/2, 1/2, 0)	SMSB-G	86.4	91.4	88.8	86.6	93.5	96.0	96.5	94.3
	SMSB-R	86.5	90.6	88.9	85.2	93.4	96.5	96.7	94.3
	SMSB-M	86.4	91.0	88.4	86.1	93.9	96.8	96.8	93.8
	SESB	83.9	84.3	69.8	83.8	93.2	95.3	92.8	93.8
	SDWB	82.9	85.1	84.7		94.2	95.3	94.4	
	GBBB	83.3	96.2	68.6		94.8	94.0	98.8	
	WIA	90.8	88.9	83.2		95.3	97.2	95.0	
(1/3, 1/2, 0.05)	SMSB-G	90.1	93.1	90.7	90.9	91.3	92.7	91.3	91.8
	SMSB-R	89.9	92.4	90.6	90.8	91.2	92.4	92.0	91.8
	SMSB-M	89.8	92.5	90.0	91.2	91.4	92.9	91.7	91.8
	SESB	88.3	86.9	73.6	90.0	91.1	91.6	85.2	91.2
	SDWB	88.6	87.6	84.9		89.7	89.6	87.4	
	GBBB	85.5	95.5	65.8		89.1	87.4	95.6	
	WIA	95.3	91.6	84.7		92.0	94.6	90.0	
(1/2, 1/2, 0.05)	SMSB-G	71.6	78.9	74.5	70.0	70.5	79.1	80.3	72.1
	SMSB-R	70.2	78.9	74.5	69.1	69.7	78.6	80.1	71.7
	SMSB-M	70.5	78.4	74.0	69.7	70.8	78.7	80.5	71.7
	SESB	67.3	68.1	48.8	67.0	68.5	74.8	68.1	70.8
	SDWB	66.2	69.2	66.9		71.5	74.9	73.9	
	GBBB	68.0	90.0	50.6		75.7	74.2	91.2	
	WIA	79.1	78.8	68.9		76.6	82.9	78.6	

6.1 Approximating the distribution of the sample mean

subsample size b varies substantially across methods, highlighting the practical importance of bandwidth selection. Among the competing approaches, the SDWB and WIA exhibit relatively stable performance over a range of values of b , although their accuracy may decrease under stronger spatial dependence or non-Gaussian designs. In contrast, the GBBB appears somewhat sensitive to the choice of b in certain settings and can be slightly conservative, occasionally producing coverage above the nominal level. Third, some under-coverage is observed under strong spatial dependence and skewness, possibly reflecting a reduced effective sample size and challenges in tail approximation. These challenges are shared across the methods considered and highlight the inherent difficulty of high-dimensional spatial inference under irregular sampling designs.

The proposed data-driven subsample size $\hat{b}^{\text{opt}}$ offers an adaptive alternative to fixed bandwidth choices. Across a broad range of scenarios, $\hat{b}^{\text{opt}}$ yields empirical probabilities that are nearly comparable to those achieved by the best-performing fixed values of b . Overall, the SMSB family demonstrates robust performance across diverse model settings. The SESB also performs competitively, often outperforming the SDWB, WIA, and GBBB. Moreover, the proposed SSBs exhibit favorable computational performance across the considered simulation settings, while the use of the data-driven subsample size $\hat{b}^{\text{opt}}$ incurs only a modest additional computational cost; see Table A.8 in the online supplementary material. In conclusion, the SMSB and SESB, when equipped with data-driven bandwidth selection, strike a favorable balance among accuracy, robustness, and computational efficiency.

6.2 Approximating the distribution of the covariance comparison statistic

6.2 Approximating the distribution of the covariance comparison statistic

Let $\{\mathbf{Y}(\mathbf{u})\}$ be a p -variate random field on $\mathbb{R}^2$ defined from a spatial factor model

$$\mathbf{Y}(\mathbf{u}) = A\mathbf{f}(\mathbf{u}) + \mathbf{e}(\mathbf{u}), \quad \text{with } A = (1 + \theta)E,$$

where $\theta \geq 0$ is a scalar parameter, E is a $p \times 5$ matrix whose entries are independently drawn from the uniform distribution $U[-1, 1]$, and the noise process $\mathbf{e}(\mathbf{u}) \sim N(0, I_p)$ is independent across $\mathbf{u} \in \mathbb{R}^2$. The latent factor process $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \dots, f_5(\mathbf{u}))^\top$ is a mean-zero Gaussian random field with mutually independent components. Each component $f_j(\mathbf{u})$ follows a Matérn covariance function given by

$$\text{Cov}(f_j(\mathbf{0}), f_j(\mathbf{h})) = \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} \|\mathbf{h}\|}{a} \right)^\nu K_\nu \left(\frac{\sqrt{2\nu} \|\mathbf{h}\|}{a} \right), \quad j = 1, \dots, 5,$$

where the smoothness parameter is $\nu = 3/2$ and the range parameter is $a = 1/\sqrt{3}$. Here, $\Gamma(\cdot)$ denotes the gamma function and $K_\nu(\cdot)$ is the modified Bessel function of the second kind. We further assume that $\{\mathbf{Y}'(\mathbf{u})\}$ is another p -variate random field, independent of $\{\mathbf{Y}(\mathbf{u})\}$, which follows the same model specification as $\mathbf{Y}(\mathbf{u})$ except that the parameter θ is replaced by θ' .

We consider testing the hypothesis

$$H_0 : \Sigma = \Sigma' \quad \text{versus} \quad H_1 : \Sigma \neq \Sigma', \quad (6.2)$$

where $\Sigma = \text{Var}(\mathbf{Y}(\mathbf{u}))$ and $\Sigma' = \text{Var}(\mathbf{Y}'(\mathbf{u}))$ denote the marginal covariance matrices. Given sampling locations $\{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ and the corresponding observations, we use the test statistic

$$\|\mathbf{Q}_{\lambda,n}\|_\infty \quad \text{where} \quad \mathbf{Q}_{\lambda,n} = \lambda^{d/2} \text{vech}(\hat{\Sigma} - \hat{\Sigma}').$$

6.2 Approximating the distribution of the covariance comparison statistic

Here, $\hat{\Sigma} = n^{-1} \sum_{i=1}^n \mathbf{Y}(\mathbf{s}_i) \mathbf{Y}(\mathbf{s}_i)^\top$ and $\hat{\Sigma}' = n^{-1} \sum_{i=1}^n \mathbf{Y}'(\mathbf{s}_i) \mathbf{Y}'(\mathbf{s}_i)^\top$ are empirical covariance matrices. For a nominal significance level $\alpha \in (0, 1)$, the null hypothesis H_0 is rejected whenever the observed value of $\|\mathbf{Q}_{\lambda,n}\|_\infty$ exceeds the $(1 - \alpha)$ -quantile of its null distribution. Note that $\mathbf{Q}_{\lambda,n}$ is a statistic of dimension $p(p+1)/2$ and can be written as $\mathbf{Q}_{\lambda,n} = \lambda^{d/2} n^{-1} \sum_{i=1}^n \mathbf{Z}(\mathbf{s}_i)$, where $\mathbf{Z}(\mathbf{u}) = \text{vech}(\mathbf{Y}(\mathbf{u}) \mathbf{Y}(\mathbf{u})^\top - \mathbf{Y}'(\mathbf{u}) \mathbf{Y}'(\mathbf{u})^\top)$. Thus, $\mathbf{Q}_{\lambda,n}$ can fall within the scope of the theoretical results established in Section 4.

To compare the finite-sample performance of the seven approaches considered in Section 6.1, we compute empirical probabilities under the (UL) sampling design, using the same simulation settings as those in Section 6.1.

Table 2 reports the empirical rejection rates (in %) at the nominal level $\alpha = 0.05$. We fix $p = 30$ and consider sample sizes $n \in \{50, 100\}$. Two scenarios are examined: $\theta = \theta' = 0$, corresponding to the null hypothesis, and $\theta = 0$ with $\theta' = 0.5$, corresponding to the alternative hypothesis.

Several key patterns emerge from Table 2. Under the null hypothesis, the three SMSB variants maintain rejection rates close to the nominal 5% level, demonstrating stable size control across a range of subsample sizes. In particular, the data-driven choice $\hat{b}^{\text{opt}}$ yields especially reliable control of Type I error. The SESB method is less robust to the choice of b ; however, when equipped with $\hat{b}^{\text{opt}}$, its rejection rates also approach the nominal level, resulting in performance comparable to that of the SMSB variants. In contrast, the SDWB, WIA and GBBB methods exhibit greater variability in Type I error control. The rejection rates for SDWB and WIA occasionally deviate from the nominal 5% level,

6.2 Approximating the distribution of the covariance comparison statistic

Table 2: Empirical rejection probabilities (in %) for the covariance matrix comparison test.

b	$\lambda = 12, n = 50$				$\lambda = 24, n = 50$				$\lambda = 12, n = 100$				$\lambda = 24, n = 100$			
	4	5	6	$\hat{b}^{\text{opt}}$	6	8	10	$\hat{b}^{\text{opt}}$	4	5	6	$\hat{b}^{\text{opt}}$	6	8	10	$\hat{b}^{\text{opt}}$
$\theta = \theta' = 0$																
SMSB-G	4.1	5.4	9.3	4.5	5.9	2.9	4.0	4.8	6.5	7.3	9.6	6.2	3.7	4.3	5.4	3.5
SMSB-R	4.7	5.8	10.3	4.9	6.3	2.9	4.2	4.3	7.1	8.0	10.1	6.7	4.0	4.4	5.1	3.7
SMSB-M	3.7	5.5	9.8	4.4	5.9	3.2	4.2	4.2	6.8	7.2	10.3	6.2	4.0	4.4	5.2	3.8
SESB	5.0	7.9	20.9	5.1	6.4	4.2	6.1	5.0	7.7	11.0	19.1	6.8	3.9	5.4	7.8	3.9
SDWB	6.5	7.8	11.2		3.3	4.1	6.3		8.1	8.9	10.1		4.0	5.2	5.8	
GBBB	5.6	2.1	15.1		3.4	4.6	1.9		5.3	18.0	9.3		2.8	3.0	1.7	
WIA	2.0	4.1	11.1		1.8	1.4	4.8		6.3	8.5	10.3		4.0	3.7	5.8	
$\theta = 0, \theta' = 0.5$																
SMSB-G	64.2	68.3	77.6	65.5	81.7	70.3	74.6	76.9	93.0	91.8	94.5	93.1	98.6	97.7	98.0	98.5
SMSB-R	64.8	68.5	78.7	66.5	82.0	70.1	74.5	77.3	93.3	92.7	94.3	93.2	98.7	97.9	97.9	98.7
SMSB-M	64.7	68.7	78.1	65.9	82.3	71.0	74.6	77.0	93.6	92.7	94.7	93.1	98.6	97.6	97.9	98.7
SESB	67.4	75.4	88.3	68.1	82.4	72.3	81.1	78.5	94.2	95.2	98.0	94.2	98.9	98.2	98.8	98.7
SDWB	75.1	77.8	80.0		76.2	77.4	80.6		96.1	96.5	96.8		98.9	98.7	98.8	
GBBB	69.0	51.6	81.7		73.3	76.5	61.6		90.9	77.1	92.6		97.2	96.1	88.3	
WIA	46.4	50.3	65.0		65.0	54.3	59.0		88.4	89.0	92.2		97.0	95.7	95.2	

while GBBB shows more pronounced size distortions, with either under-rejection or over-rejection depending on the configuration. Under the alternative hypothesis, all methods exhibit high power, which increases with both the sample size n and the scaling parameter λ . The SMSB and SESB methods achieve power levels comparable to SDWB and, in several scenarios – particularly when the sample size is small – outperform GBBB and WIA. Overall, these results further demonstrate the robustness and practical effectiveness of the proposed SSB procedures for high-dimensional covariance comparison tests.

6.3 A real data study

We illustrate the practical performance of the SMSB and SESB through a mean change-point analysis of long-term precipitation records. The data are obtained from the Science Data Bank (<https://doi.org/10.57760/sciencedb.01720>). The original dataset consists of monthly and daily precipitation observations collected at 366 meteorological stations over the period 1960–2021. Annual precipitation totals are aggregated from these records and used in the subsequent analysis. To focus on a climatically relevant region, we restrict attention to stations located within the domain 105–120°E and 20–35°N. This selection yields 149 stations, which are irregularly spaced over the study region. The domain approximately covers the core area of the East Asian monsoon system, including the Yangtze River Basin and much of southern China. Precipitation variability and extremes in this region are known to have substantial climatic and hydrological impacts (Wang and Zhou, 2005; Li et al., 2022). Similar spatial domains have also been adopted in previous hydroclimate modeling studies as representative windows for analyzing monsoon precipitation and isotopic variability (Hu et al., 2019).

Figure 1 displays the spatial distribution of the selected stations. Annual precipitation series are well known to exhibit heavy-tailed behavior. To mitigate this effect, we apply a logarithmic transformation and define the spatio-temporal process

$$Z(\mathbf{u}, t) = \log\{1 + M(\mathbf{u}, t)\},$$

where $M(\mathbf{u}, t)$ denotes the annual precipitation total at location $\mathbf{u}$ in year t . Let $\theta =$

6.3 A real data study

$(\theta_1, \dots, \theta_p)^\top \in \mathbb{R}^p$ denote the vector of spatial means, where $\theta_j = E[Z(\mathbf{u}, t_j)]$ represents the common mean across space at time t_j , $j = 1, \dots, p$. Our objective is to detect temporal change points in the mean structure, that is, to identify years t_{j+1} at which the mean precipitation level changes relative to the preceding year t_j . This task is formulated as a high-dimensional multiple testing problem for the vector of successive mean differences.

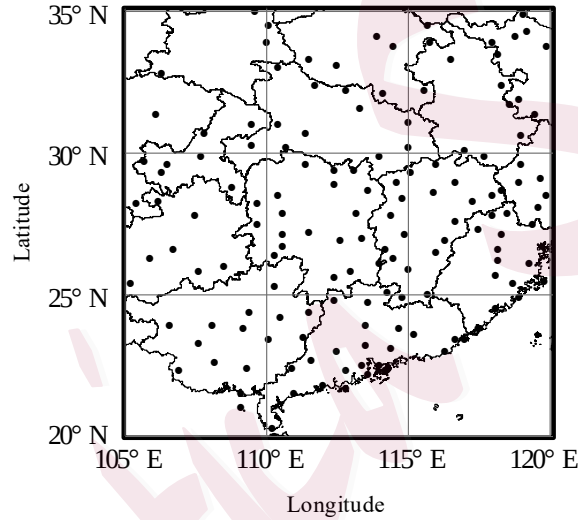


Figure 1: Spatial locations of the 149 selected meteorological stations.

Specifically, we define the $(p-1)$ -dimensional difference field $\mathbf{Y}(\mathbf{u}) = (Y_1(\mathbf{u}), \dots, Y_{p-1}(\mathbf{u}))^\top$, where $Y_j(\mathbf{u}) = Z(\mathbf{u}, t_{j+1}) - Z(\mathbf{u}, t_j)$, $j = 1, \dots, p-1$. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_{p-1})^\top$, with $\mu_j = \theta_{j+1} - \theta_j$. Detecting change points then amounts to simultaneously testing the null hypotheses

$$H_{0j} : \mu_j = 0, \quad j = 1, \dots, p-1.$$

In this framework, the $p-1$ candidate change points are treated jointly rather than through a sequence of independent tests. The approximate stationarity assumption for

6.3 A real data study

the difference field $\{\mathbf{Y}(\mathbf{u})\}$ is a reasonable modeling approximation when the dependence structure of year-to-year precipitation changes remains relatively stable over time, even though the mean precipitation level may exhibit temporal shifts. Under this formulation, the detected change points correspond to years at which the mean level of annual precipitation changes relative to the preceding year.

We model the spatial domain as $R_\lambda = [-\lambda/2, \lambda/2]^2$ with $\lambda = 15$. For all proposed methods (SMSB-G, SMSB-R, SMSB-M, and SESB), we use the data-driven optimal subsample size $\hat{b}^{\text{opt}}$ developed in Section 5.2; for the competing methods (SDWB, GBBB, and WIA), we use the data-driven optimal subsample size $\hat{b}^{\text{opt}}$ obtained for SMSB. To conduct simultaneous inference, we adopt the stepdown procedure of Romano and Wolf (2005). Let $w(1) = \{1, \dots, p-1\}$ denote the initial candidate set. In Step 1, we consider all candidate hypotheses in $w(1)$ and estimate the critical value $\hat{q}_{1-\tau}$, with $\tau = 0.01$, as the $(1-\tau)$ quantile of the bootstrap distribution of the max-type statistic $\|D_{\lambda,n}^{-1}\lambda^{d/2}\bar{\mathbf{Y}}\|_\infty$, where $\bar{\mathbf{Y}} = n^{-1} \sum_{i=1}^n \mathbf{Y}(\mathbf{s}_i)$, $D_{\lambda,n} = \text{diag}(\Xi_{n/\lambda^d})^{1/2}$. Each bootstrap quantile $\hat{q}_{1-\alpha}$ is computed from 1500 bootstrap replicates. A null hypothesis H_{0j} is rejected if $\lambda^{d/2}|D_{\lambda,n}^{*(j,j)-1}\bar{Y}_j| > \hat{q}_{1-\tau}$ or $\lambda^{d/2}|D_{\lambda,n}^{o(j,j)-1}\bar{Y}_j| > \hat{q}_{1-\tau}$ depending on the bootstrap scheme. In this case, t_{j+1} is identified as a significant change point. If no null hypothesis is rejected, the procedure terminates. Otherwise, the rejected indices are removed from the candidate set to form $w(2) = w(1) \setminus \{j : H_{0j} \text{ is rejected in Step 1}\}$. In Step 2, the max-type statistic and its bootstrap critical value are recalculated using only the hypotheses indexed by $w(2)$, with $\tau = 0.05$. The hypotheses remaining in $w(2)$ are then tested against this updated crit-

6.3 A real data study

ical value. The candidate set is updated in the same manner after each step, and the procedure continues until no further hypotheses are rejected.

Figure 2 presents the results of the stepdown change-point analysis for annual precipitation differences from 1960 to 2021 using SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB, and WIA. The vertical lines indicate the final change points identified after completion of the stepdown procedure, while the shaded regions represent the joint 95% and 99% confidence intervals for μ constructed in Steps 1 and 2. Figure 2 shows that SMSB-G, SMSB-R, and SDWB yield broadly consistent change-point patterns, with several common upward shifts in 1987, 1988, 2010, and 2012 and downward shifts in 1966, 1971, 1976, 1984, and 2011. SMSB-M and SESB detect several additional changes, whereas GBBB and WIA recover fewer of the changes identified by the other methods. The agreement in the timing and directions of the major change points suggests that the main conclusions are robust across methods, while the differences in the number of detected change points may reflect their differences in their sensitivity to weaker signals.

From a climatological perspective, several of the detected shifts are temporally consistent with documented precipitation variability and regime changes over the Yangtze River Basin and eastern China. The shifts in 1966, 1971, and 1976 occur around periods associated with major changes in the East Asian summer monsoon and precipitation regime (Shen et al., 2007; Wang, 2001; Ding et al., 2008), while those in 1984, 1987, and 1988 are consistent with documented abrupt precipitation changes over the Yangtze River Delta (Hu et al., 2016). The shifts around 2010–2012 occur during a period character-

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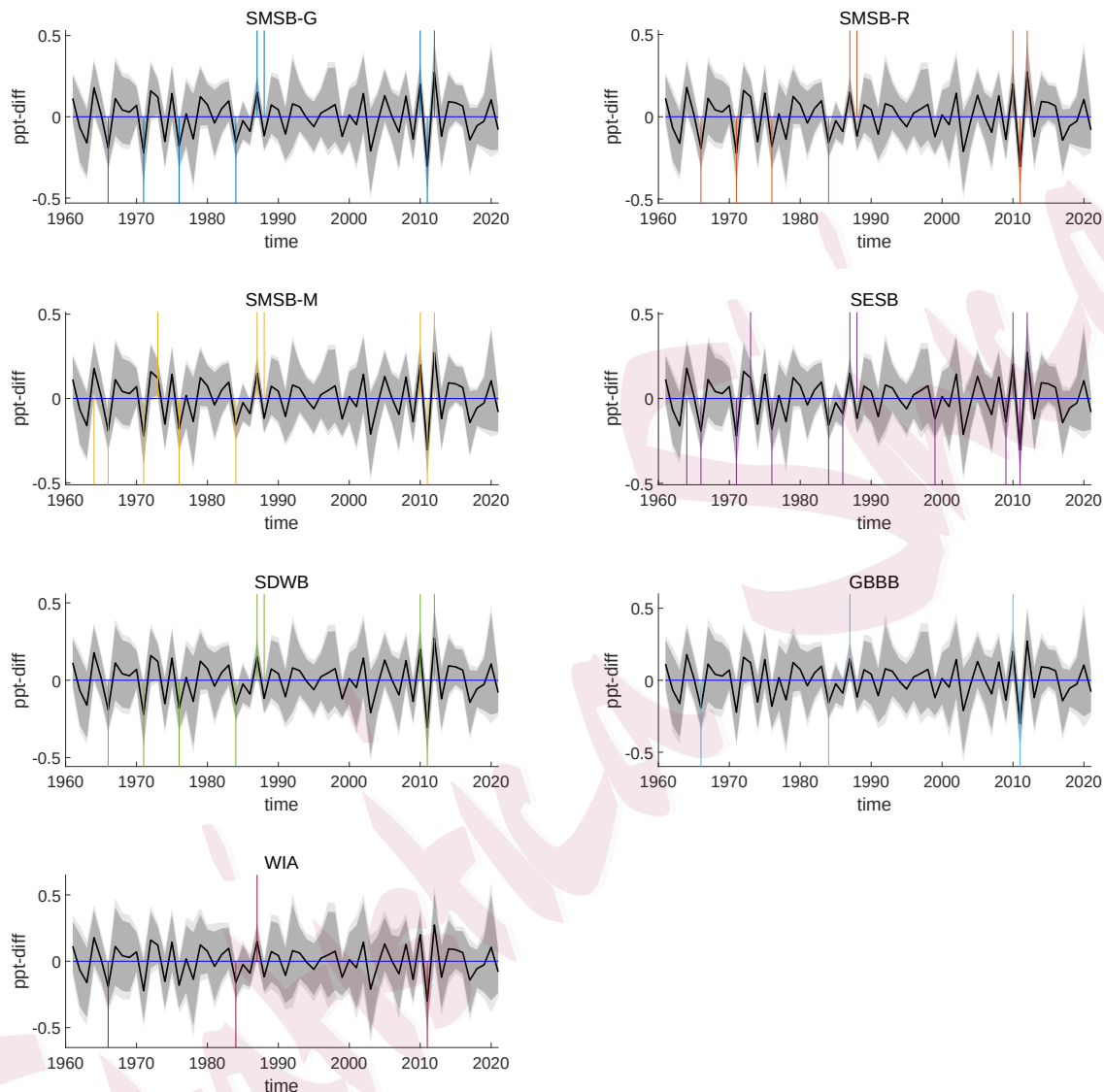


Figure 2: Results of the stepdown change-point analysis for annual precipitation differences (1960–2021). Shaded regions show the joint confidence intervals for μ obtained in Step 1 at the 99% (light gray) and in Step 2 at the 95% (dark gray) confidence levels. Vertical lines indicate detected change points. Results are shown for seven spatial bootstrap schemes: SMSB-G (line 1, left), SMSB-R (line 1, right), SMSB-M (line 2, left), SESB (line 2, right), SDWB (line 3, left), GBBB (line 3, right), and WIA (line 4, left).

ized by notable hydroclimatic anomalies in the Yangtze Basin, including severe flooding in 2010 and drought in 2011; previous studies have linked these anomalies to large-scale climate variability such as ENSO (Lu et al., 2014; Shi et al., 2015; Guo et al., 2020). Overall, the results indicate substantial temporal nonstationarity in annual precipitation and are broadly consistent with documented decadal variability and major hydroclimatic anomalies.

7. Concluding remarks

A distinctive structural feature of the proposed SSB procedures is that the bootstrap statistic is constructed directly from localized subsample statistics, rather than by reconstructing pseudo-sampling observations. This feature provides a natural framework for considering nonlinear statistics. Specifically, let $\theta = \theta(P)$ be a q -dimensional parameter of interest and let $\hat{\theta}_{\lambda,n} = \hat{\theta}(\mathbf{Y}(\mathbf{s}_i), i = 1, \dots, n)$ be its estimator. Suppose that $\mathbf{T}_{\lambda,n} = \lambda^{d/2} (\hat{\theta}_{\lambda,n} - \theta)$ admits a distributional approximation. The SSB statistics in (3.6) and (3.7) can then be constructed by replacing the localized subsample statistics with $\hat{\mathbf{T}}_{\lambda,b,n}(k) = (\lambda^d/n)^{1/2} n_k^{1/2} (\hat{\theta}_{b,n}(k) - \hat{\theta}_{\lambda,n})$, $k = 1, \dots, N$, where $\hat{\theta}_{b,n}(k) = \hat{\theta}(\mathbf{Y}(\mathbf{s}_i), \mathbf{s}_i \in R_b(k))$.

More generally, extending the theoretical validity of the SSB procedures to a new class of statistics would require three ingredients. First, the target statistic needs to admit a suitable Gaussian approximation,

$$\sup_{A \in \mathcal{A}} |\mathbb{P}_{\cdot|\mathbf{s}}(\mathbf{T}_{\lambda,n} \in A) - \mathbb{P}_{\cdot|\mathbf{s}}(\mathbf{Z}_{\lambda,n} \in A)| = o_p(1), \quad (7.1)$$

where $\mathbf{Z}_{\lambda,n} \sim N(\mathbf{0}, \Xi_{\lambda,n})$ and $\Xi_{\lambda,n}$ is defined as in (3.2). Second, the covariance matrix estimated from the localized subsample statistics should consistently estimate the target covariance matrix, that is, $\|\hat{\Xi}_{\lambda,n} - \Xi_{\lambda,n}\|_{\max} = o_p(\log^{-2} q)$, where $\hat{\Xi}_{\lambda,n}$ is the SSB covariance matrix estimator constructed from the subsample statistics. Third, the bootstrap statistic should admit a conditional Gaussian approximation,

$$\sup_{A \in \mathcal{A}} \left| P_{\cdot|\mathbf{s}, \mathbf{Y}}(\mathbf{T}_{\lambda,n}^\dagger \in A) - P_{\cdot|\mathbf{s}}(\mathbf{G}_{\lambda,n}^\dagger \in A) \right| = o_p(1),$$

where $\mathbf{T}_{\lambda,n}^\dagger$ denotes the SSB statistic, and $\mathbf{G}_{\lambda,n}^\dagger | \mathbf{s} \sim N(\mathbf{0}, \hat{\Xi}_{\lambda,n})$. A Gaussian comparison argument can then yield the bootstrap analogue of Proposition 1, which, together with (7.1), establishes the corresponding analogue of Theorem 2. Thus, the extension of the SSB theory to a new class of statistics amounts to verifying these three ingredients.

For smooth function models with a fixed-dimensional parameter of interest, these requirements can be obtained under standard regularity conditions by the delta method. Let $\boldsymbol{\theta} = H(\boldsymbol{\mu})$, $\hat{\boldsymbol{\theta}}_{\lambda,n} = H(\bar{\mathbf{Y}}_{\lambda,n})$, $\hat{\boldsymbol{\theta}}_{b,n}(k) = H(\bar{\mathbf{Y}}_{b,n}(k))$, where H is continuously differentiable in a neighborhood of $\boldsymbol{\mu}$. Then the localized statistics take the natural form $\hat{\mathbf{T}}_{\lambda,b,n}(k) = (\lambda^d/n)^{1/2} n_k^{1/2} (H(\bar{\mathbf{Y}}_{b,n}(k)) - H(\bar{\mathbf{Y}}_{\lambda,n}))$. The delta method shows that both the target statistic and the localized statistics are, to first order, determined by the same Jacobian matrix $\nabla H(\boldsymbol{\mu})$. Consequently, the Gaussian approximation and covariance-consistency requirements reduce to the corresponding results for the linearized mean statistics, and the conditional Gaussian approximation follows regularity conditions analogous to those in the case $\boldsymbol{\theta} = \boldsymbol{\mu}$. For more general high-dimensional nonlinear statistics, the above argument would require appropriate high-dimensional Gaussian approximation

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and bootstrap validity results, which are currently unavailable in our setting. Establishing such results is an interesting direction for future research and is beyond the scope of the present paper.

Supplementary Materials

The online Supplementary Material contains additional simulation results and proofs of the main results.

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