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Estimations of Extreme CoVaR and CoES under Asymptotic Independence

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Abstract: The two popular systemic risk measures CoVaR (Conditional Value-at-Risk) and CoES (Conditional Expected Shortfall) have recently been receiving growing attention on applications in economics and finance. In this paper, we study the estimations of extreme CoVaR and CoES when the two random variables are asymptotic independent but positively associated. We propose two types of extrapolative approaches: the first relies on intermediate VaR and extrapolates it to extreme CoVaR/CoES via an adjustment factor; the second directly extrapolates the estimated intermediate CoVaR/CoES to the extreme tails. All estimators, including both intermediate and extreme ones, are shown to be asymptotically normal. Finally, we explore the empirical performances of our methods through conducting a series of Monte Carlo simulations and a real data analysis on S&P500 Index with 12 constituent stocks data.

Key words and phrases: Systemic risk measure; CoVaR; CoES; Asymptotic independence.

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1. Introduction

Systemic risk refers to the widespread crisis of the entire financial system triggered by the distress of a single institution or event through interconnectedness and contagion mechanisms. In the wake of the global financial crisis and the COVID-19 pandemic, modeling and measuring systemic risk have risen to the key priorities in risk management. As extensions to the well-known Value-at-Risk (VaR) and Expected Shortfall (ES), the CoVaR and CoES have become two widely used approaches for measuring systemic risk, introduced by Tobias and Brunnermeier (2016) and Huang and Uryasev (2018), respectively. Actually, Huang and Uryasev (2018) defines a more general conditional expectation measure termed "CoCVaR", which does not exactly coincide with CoES; see Theorem 2.3 of Huang and Uryasev (2018). In this paper, we proceed with CoVaR defined in Girardi and Ergün (2013) and Nolde and Zhang (2020) to facilitate nonparametric inferences, which conditions on the event that the system loss exceeds its VaR. Specifically, let Y denote the loss variable of a system (or market) and X the loss variable of interest. For a given quantile level $\tau \in (0,1)$, the $\mathrm{CoVaR}_{X|Y}(\tau)$ is defined by,

$$\mathbb{P}(X \geq \mathrm{CoVaR}_{X|Y}(\tau) | Y \geq \mathrm{VaR}_Y(\tau)) = 1 - \tau, \tag{1.1}$$

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where $\mathrm{VaR}_Y(\tau)$ denotes the VaR of Y, *i.e.*, $\mathrm{VaR}_Y(\tau) := \inf\{y \in \mathbb{R} : F_Y(y) \geq \tau\}$, with F_Y being $Y's$ marginal distribution. Correspondingly, the definition of $\mathrm{CoES}_{X|Y}(\tau)$ is given by,

$$\mathrm{CoES}_{X|Y}(\tau) := \frac{1}{1-\tau}\int_{\tau}^{1} \mathrm{CoVaR}_{X|Y}(\alpha)\, d\alpha. \tag{1.2}$$

The assumed continuity of $F_{X,Y}(\cdot,\cdot)$ implies that the $\mathrm{CoES}_{X|Y}(\tau)$ coincides with the conditional expectation

$$\mathrm{CoES}_{X|Y}(\tau) = E[X|X \geq \mathrm{CoVaR}_{X|Y}(\tau), Y \geq \mathrm{VaR}_Y(\tau)]. \tag{1.3}$$

A vast body of literature on CoVaR and CoES has emerged, with a primary focus on statistical inferences and their various applications in econometrics. Major directions include: statistical methodologies (Francq and Zakoïan, 2025; Dimitriadis and Hoga, 2026; Huang et al., 2024), backtesting (Fissler and Hoga, 2024), portfolio selection (Capponi and Rubtsov, 2022; Fung et al., 2026), and risk forecasting (Girardi and Ergün, 2013; Bernardi and Catania, 2019), etc.

Most of the aforementioned studies on CoVaR and CoES focus on fixed risk levels. In order to better characterize tail risk, it is both necessary and beneficial to model risk level as a sequence that depends on the sample size n and approaches to 1 as n increases, *i.e.*, *extreme risk*. In the bivariate extreme value theory, a classic approach involves assuming the tail depen-

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dence structure between X and Y, *i.e.*, tail copula is non-degenerate. For example, Cai et al. (2015) and Hou and Wang (2021) respectively developed extrapolative estimators for the Marginal Expected Shortfall (MES, see Acharya et al. (2017)) and the Tail Gini functional at extreme levels within the framework of tail dependence. In the context of extreme CoVaR, two semi-parametric methods have been proposed by Nolde and Zhang (2020) and Nolde et al. (2022). The first approximates the conditional probability in (1.1) via the tail of a skew-elliptical distribution, while the second assumes that the tail copula function follows a particular parametric family.

Although most research in bivariate extreme theory focuses on tail dependence, growing evidence suggests that asymptotic independence may exist in bivariate tail region across various applications, including spatial statistics (Wadsworth and Tawn, 2012; Le et al., 2018; Lalancette et al., 2021), risk management (Kulik and Soulier, 2015; Das and Fasen-Hartmann, 2018; Lalancette and Resnick, 2020). To measure the strength of the extremal dependence asymptotically, Ledford and Tawn (1996) and Ledford and Tawn (1997) proposed a more flexible framework via a coefficient of tail dependence, denoted as η. Namely, we assume that there exists an $\eta \in (0, 1]$ such that the following limit exists and is positive for

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all $(x, y) \in (0, \infty)^2$,

$$\lim_{t\to\infty} t^{\frac{1}{\eta}} \mathbb{P}\left(\overline{F}_X(X) < x/t, \overline{F}_Y(Y) < y/t\right) =: C(x, y) \in (0, \infty). \tag{1.4}$$

Let $C(x, y) = 0$ for either $x = 0$ or $y = 0$. It is worth noting that $C(x, y)$ is a homogeneous function of order $1/\eta$, *i.e.*, $C(ax, ay) = a^{1/\eta}C(x, y)$ for any $a > 0$. The coefficient η describes the strength of tail dependence: $\eta = 1$ implies that X and Y are asymptotically dependent; if $\eta \in (1/2, 1)$, we say that X and Y are asymptotically independent but positively associated; if $\eta \in (0, 1/2)$, we say that X and Y are asymptotically independent but negatively associated. When X and Y are independent, it follows $\eta = 1/2$. We refer to Ledford and Tawn (1996), Draisma et al. (2004) for more interpretations of η. Under the framework of (1.4) with $\eta \in (1/2, 1)$, Cai and Musta (2020), Zhang et al. (2024) and Wang et al. (2026) have re-estimated the MES, Expectile-based MES and Tail Gini functional at extreme levels, respectively, since all tail-dependence methods fall.

The main contribution of this paper lies in addressing the estimations for extreme CoVaR and CoES under asymptotic independence but positive association *i.e.*, $\eta \in (1/2, 1)$. To our best knowledge, there is still no literature considering the estimation issue of CoVaR and CoES within a framework of asymptotic independence. Our work is of great theoretical and practical importance to provide the powerful complement to the

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gaps in existing research. Due to the scarcity of tail observations, we develop two types of approaches to extrapolating extreme CoVaR and CoES. Specifically, the first is built upon the estimated intermediate VaR, with an extrapolation depending on an adjustment factor ξ_τ linking $\mathrm{CoVaR}_{X|Y}(\tau)$ to the marginal $\mathrm{VaR}_X(\tau)$ via (2.7). The idea of the adjustment factor originates from Nolde et al. (2022), but unlike that work, we estimate it using a nonparametric approach. To this end, we extrapolate $\xi_{\tau_n'}$ from $\xi_{1-k/n}$ via a limit relation established in Lemma 1 and estimate $\xi_{1-k/n}$ by defining it as a left-continuous inverse of a non-decreasing empirical function, which serves as the first methodological contribution. The second directly extrapolates intermediate CoVaR/CoES to extreme levels. Unlike other MES-type measures, it is appropriate to directly extrapolate CoES from CoVaR rather than from VaR, since a well-defined limiting relationship between CoVaR and VaR can not be derived. We then estimate $\mathrm{CoVaR}_{X|Y}(1-k/n)$ and $\mathrm{CoES}_{X|Y}(1-k/n)$ nonparametrically using bivariate tail empirical distribution. This constitutes another methodological innovation. Finally, the major technical challenge lies in the absence of asymptotic theoretical guarantee for asymptotic independence, making it intractable to obtain related asymptotic behaviours. Accordingly, we provide the universal asymptotic result for the empirical tail function in Proposition S1. Accounting for

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the intrinsic connection between CoVaR and CoES, we establish the joint bivariate asymptotic normalities of their extrapolative estimators.

The rest of this paper is organized as follows. In Section 2, we begin with describing the basic model setting, and then elaborate in detail on the two proposed classes of extrapolative estimations for CoVaR and CoES under asymptotic independence. We also study the asymptotic behaviors of all estimations therein. The empirical performance of our estimations is evaluated by a simulation study in Section 3, and a real application in Section 4. All technical proofs are collected in the supplementary material.

2. Methodologies

Recall that (X, Y) is a two-dimensional random vector with joint distribution $F_{X,Y}$, and marginal distributions $F_X(\cdot) = F_{X,Y}(\cdot, \infty)$, $F_Y(\cdot) = F_{X,Y}(\infty, \cdot)$. Denote $\overline{F}_X(\cdot) := 1 - F_X(\cdot)$ and $\overline{F}_Y(\cdot) := 1 - F_Y(\cdot)$ as the (upper) tails of $F_X(\cdot)$ and $F_Y(\cdot)$, respectively. Suppose the bivariate samples $\{(X_i, Y_i)\}_{i=1}^n$ are independent copies of (X, Y). Let $k = k(n)$ be an intermediate sequence of integers, such that $k \to \infty$ and $k/n \to 0$ as $n \to \infty$. Accordingly, $1 - k/n$ can serve as an *intermediate* level. Let τ_n' be an *extreme* level, satisfying $\tau_n' \to 1$ and $n(1 - \tau_n') \to c \in [0, \infty)$. Denote $A^\top$ as the transpose of a matrix or vector A. Denote $U_1 = (1/\overline{F}_X)^{\leftarrow}$ as the *tail*

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quantile function with "$\leftarrow$" denoting the left-continuous inverse.

We formulate our basic framework through Assumption 1 (a) and (b), which specify first-order conditions on the tail distribution of X and the asymptotic independence between X and Y.

Assumption 1.

(a) There exists a $\gamma_1 > 0$ such that, for $x > 0$, the following limit exists,

$$\lim_{t\to\infty} \frac{U_1(tx)}{U_1(t)} = x^{\gamma_1}. \tag{2.5}$$

(b) There exists a $\eta \in (1/2, 1)$ such that for all $(x, y) \in (0, \infty)^2$, the following limit exits,

$$\lim_{t\to\infty} t^{\frac{1}{\eta}} \mathbb{P}\left(\overline{F}_X(X) < x/t, \overline{F}_Y(Y) < y/t\right) =: C(x, y) \in (0, \infty).$$

(c) The function $C(x, y)$ is differentiable at each coordinate with partial derivatives

$$C_1(x, y) = \frac{\partial}{\partial x} C(x, y), \text{ and } C_2(x, y) = \frac{\partial}{\partial y} C(x, y).$$

The partial derivatives $C_i(x, y)$ are continuous with respect to x in the neighborhood of $(0, 1)$ and $C_i(0, 1) > 0$ with $i = 1, 2$

Assumption 1 (a) assumes that X follows a heavy-tail distribution, characterized by regular variation (2.5), where γ_1 is the *extreme value index* (EVI), see de Haan and Ferreira (2006). This implies that $\overline{F}_X$ is also

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regularly varying with index $-1/\gamma_1$, *i.e.*, $\lim_{t\to\infty} \frac{\overline{F}_X(tx)}{\overline{F}_X(t)} = x^{-1/\gamma_1}$. Secondly, we impose $\eta \in (1/2, 1)$ in (1.4) throughout this paper. This range of η models tail asymptotic independence alongside a positive association between X and Y, a feature that makes the framework particularly well-suited for modeling financial data. The smoothness condition imposed in Assumption 1 (c) on function $C(\cdot, \cdot)$ and its derivatives ensures the key limit relationships presented in Proposition 1 and Lemma 1 below.

Proposition 1. *Suppose that Assumption 1 holds, we have that, for* $\gamma_1 \in (0, 1)$,

$$\lim_{\tau\uparrow 1} \frac{\mathrm{CoES}_{X|Y}(\tau)}{\mathrm{CoVaR}_{X|Y}(\tau)} = \frac{1}{1-\gamma_1}. \tag{2.6}$$

The restriction $\gamma_1 \in (0, 1)$ guarantees both the existence of the limit and a finite expectation $E[X]$. The resulting limit relationship (2.6) between CoVaR and CoES as $\tau \uparrow 1$ is fundamental to our procedure for extrapolating CoES from CoVaR.

2.1 Estimations based on intermediate VaR

It is the goal of this paper to estimate CoVaR and CoES at an extreme level τ_n' under asymptotic independence specified in Assumption 1 (b). To this end, we consider two types of extrapolative approaches: one based on an estimated intermediate VaR and the other based on an estimated

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intermediate CoVaR/CoES. This section introduces the first method, while the second will be discussed in Section 2.2.

For the VaR-based method, Nolde et al. (2022) proposes an adjustment factor ξ_τ associated with τ, via

$$\mathrm{CoVaR}_{X|Y}(\tau) = \mathrm{VaR}_X(1-(1-\tau)\xi_\tau). \tag{2.7}$$

This implies that $\mathrm{CoVaR}_{X|Y}(\tau)$ reduces to an unconditional quantile evaluated at the adjusted level $1-(1-\tau)\xi_\tau$ and an extrapolation procedure governed by ξ_τ is thereby induced. Under mild distributional conditions, Zhong and Hou (2025) developed the sufficient and necessary condition for the existence and uniqueness of ξ_τ, *i.e.*,

$$\mathbb{P}(X \geq \mathrm{VaR}_X(\tau), Y \geq \mathrm{VaR}_Y(\tau)) > (1-\tau)^2, \tag{2.8}$$

holds on a tail region $\tau \in (\tau_0, 1)$ for some threshold τ_0.

Remark 1. First, condition (2.8) is generally regarded as a weaker condition than the positive quadrant dependence (PQD) used in Nolde et al. (2022) and it restricts τ to a tail region, rather than $(0, 1)$; second, as shown in Zhong and Hou (2025), the existence and uniqueness of ξ_τ do not rely on the specific structure of tail dependence, hence making it compatible with asymptotic independence framework as well; third, ξ_τ indeed admits

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a closed solution,

$$\xi_\tau := \frac{\mathbb{P}(X \geq \mathrm{CoVaR}_{X|Y}(\tau))}{\mathbb{P}(X \geq \mathrm{CoVaR}_{X|Y}(\tau) | Y \geq \mathrm{VaR}_Y(\tau))}.$$

For brevity, we omit the detailed discussions on the existence and uniqueness of ξ_τ as it is identical to that of Zhong and Hou (2025) and just assume it holds by default. We also observe that, the quantile level $1-(1-\tau)\xi_\tau$ is higher than τ as $\xi_\tau \in (0,1)$ deduced by (2.8). Therefore, combining the limit condition (1.4) with (1.1) and (2.7) implies that,

$$\begin{aligned}(1-\tau)^{2-\frac{1}{\eta}} &= (1-\tau)^{-\frac{1}{\eta}} \mathbb{P}(X \geq \mathrm{CoVaR}_{X|Y}(\tau), Y \geq \mathrm{VaR}_Y(\tau)) \\ &= (1-\tau)^{-\frac{1}{\eta}} \mathbb{P}(\overline{F}_X(X) \leq (1-\tau)\xi_\tau, \overline{F}_Y(Y) \leq 1-\tau).\end{aligned} \tag{2.9}$$

By letting $\tau \uparrow 1$, (1.4) and (2.9) suggest a possibility of approximating the true adjustment factor ξ_τ with an approximation, denoted as ξ_τ^*, which is defined implicitly via

$$C(\xi_\tau^*, 1) = (1-\tau)^{2-\frac{1}{\eta}}. \tag{2.10}$$

The assumed continuity of maps: $x \mapsto C(x,1)$ implies that $\xi_\tau^* \to 0$ as $\tau \uparrow 1$. We can further derive the limit relationship between ξ_τ and ξ_τ^* by imposing a second-order condition on (1.4) (see Assumption 2 (b)). For $t > 0$, define

$$C_t(x,y) = t^{\frac{1}{\eta}} \mathbb{P}(\overline{F}_X(X) < x/t, \overline{F}_Y(Y) < y/t), \tag{2.11}$$

which indicates $\lim_{t\to\infty} C_t(x,y) = C(x,y)$ for all $(x,y) \in (0,\infty)^2$ by (1.4).

We present the following regularity conditions used in this paper.

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Assumption 2.

(a) There exists $\gamma_1 > 0$ *such that*

$$\sup_{x>1}\left|x^{-\gamma_1}\frac{U_1(tx)}{U_1(t)} - 1\right| = O(A_1(t)),$$

where A_1 *is an auxiliary function that is regularly varying with index* $\rho_1 \leq 0$ *and satisfies* $\lim_{t\to\infty} A_1(t) = 0$.

(b) There exists $\beta_1 > \gamma_1$ *and* $\alpha < -1$ *such that*

$$\sup_{0<x\leq 1, 1/2\leq y\leq 2} |C_t(x,y) - C(x,y)|x^{-\beta_1} = O(t^{\alpha}).$$

(c) The following integral is finite,

$$\left|\int_0^1 C(x,1)\, dx^{-\gamma_1}\right| < \infty. \tag{2.12}$$

(d) As $n \to \infty$, $k = O(n^{\iota})$ *with* $\iota \in \left(\frac{2}{3}, 1 - \frac{1}{1-2\alpha}\right)$.

(e) Let $d_n = k/(n(1-\tau_n'))$. *As* $n \to \infty$, $\frac{k}{n}\log d_n \to 0$ *when* $n(1-\tau_n') \to 0$.

(f) As $n \to \infty$, *the two sequences* k_j *satisfy* $k_j = O(k)$ *for* $j = 1, 2$.

Note that Assumption 2 (a) and (b) are second-order strengthenings quantifying the rates of convergence in (2.5) and (1.4), respectively. Assumption 2 (c) is a technical condition imposing some integrality on the function $C(\cdot,\cdot)$. We further require Assumption 2 (d) - (f) on the rates of the intermediate sequence k, k_1, k_2 and extreme level τ_n'.

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Lemma 1. *Suppose that Assumption 1 (c) and Assumption 2 (b) hold, we have that,*

$$\lim_{\tau\uparrow 1}\frac{(1-\tau)^{2-\frac{1}{\eta}}}{\xi_\tau^*}=C_1(0,1),\quad \textit{and}\quad \lim_{\tau\uparrow 1}\frac{\xi_\tau}{\xi_\tau^*}=1.$$

This lemma indicates that $\xi_\tau \to 0$ with the same speed as $(1-\tau)^{2-\frac{1}{\eta}}$.

Remark 2. Notably, a loose constraint $\alpha < \frac{1}{\eta} - 2$ is sufficient to ensure Lemma 1 holds. However, in order to study the asymptotic behaviors, we need to impose an additional condition on the growth rate of the intermediate sequence k, namely $k = O(n^\iota)$ with $\iota \in \left(\frac{2}{3}, 1-\frac{1}{1-2\alpha}\right)$ as $n \to \infty$. It is straightforward to verify that the condition $\alpha < -1$ ensures that $\frac{2}{3} < 1-\frac{1}{1-2\alpha}$. Therefore, we formulate Assumption 2 (b) with $\alpha < -1$ (by noting that $-1 < \frac{1}{\eta} - 2$) for the sake of condition consistency.

Next, we consider the VaR-based extrapolative estimation for $\mathrm{CoVaR}_{X|Y}(\tau_n')$. Using (2.7) and regular variation (2.5), we have that,

$$\frac{\mathrm{CoVaR}_{X|Y}(\tau_n')}{\mathrm{VaR}_X(1-k/n)}=\frac{\mathrm{VaR}_X(1-(1-\tau_n')\xi_{\tau_n'})}{\mathrm{VaR}_X(1-k/n)}\sim\left(\frac{k}{n(1-\tau_n')}\right)^{\gamma_1}\xi_{\tau_n'}^{-\gamma_1},$$

which implies that,

$$\mathrm{CoVaR}_{X|Y}(\tau_n')\sim\left(\frac{k}{n(1-\tau_n')}\right)^{\gamma_1}\xi_{\tau_n'}^{-\gamma_1}\mathrm{VaR}_X(1-k/n). \tag{2.13}$$

Moreover, by Lemma 1, we can also extrapolate $\xi_{\tau_n'}$ from an intermediate

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$\xi_{1-k/n}$,

$$\xi_{\tau_n'} \sim \left(\frac{n(1-\tau_n')}{k}\right)^{2-\frac{1}{\eta}} \xi_{1-k/n}. \tag{2.14}$$

Then, combining (2.13) and (2.14) yields,

$$\mathrm{CoVaR}_{X|Y}(\tau_n') \sim \left(\frac{k}{n(1-\tau_n')}\right)^{\gamma_1\left(3-\frac{1}{\eta}\right)} \xi_{1-k/n}^{-\gamma_1} \mathrm{VaR}_X(1-k/n). \tag{2.15}$$

From (2.15), there involves four quantities, $\mathrm{VaR}_X(1-k/n)$, γ_1, η, and $\xi_{1-k/n}$, supposed to estimate. Denote $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ are the order statistics of $\{X_1, ..., X_n\}$. First, we estimate $\mathrm{VaR}_X(1-k/n)$ by using the $(n-k)$-th order statistic,

$$\widehat{\mathrm{VaR}}_X(1-k/n) := X_{n-k,n}. \tag{2.16}$$

Let k_1 and k_2 be two intermediate sequences, satisfying, $k_j = k_j(n) \to \infty$, $k_j/n \to 0$, as $n \to \infty$, for $j = 1, 2$. We then estimate γ_1 by the Hill estimator (Hill (1975))

$$\hat{\gamma}_1 = \frac{1}{k_1} \sum_{i=1}^{k_1} \log X_{n-i+1,n} - \log X_{n-k_1,n}, \tag{2.17}$$

and estimate η by the method proposed in Draisma et al. (2004),

$$\hat{\eta} = \frac{1}{k_2} \sum_{i=1}^{k_2} \log T_{n-i+1,n}^{(n)} - \log T_{n-k_2,n}^{(n)}, \tag{2.18}$$

where $T_{1,n}^{(n)} \le T_{2,n}^{(n)} \le \cdots \le T_{n,n}^{(n)}$ are the order statistics of $\{T_1^{(n)}, T_2^{(n)}, ..., T_n^{(n)}\}$ with $T_i^{(n)} = \frac{n+1}{n+1-R_i^X} \wedge \frac{n+1}{n+1-R_i^Y}$ and R_i^X, R_i^Y denoting the ranks of X_i, Y_i in

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their respective samples. The choice of k_1 and k_2 might be different from k. To ensure a uniform rate of convergence throughout the extrapolations, k_1 and k_2 should be the same order as k. Thus, Assumption 2 (f) implies,

$$\sqrt{k}(\hat{\gamma}_1 - \gamma_1) = O_{\mathbb{P}}(1), \quad \text{and} \quad \sqrt{k}(\hat{\eta} - \eta) = O_{\mathbb{P}}(1). \tag{2.19}$$

The associating asymptotic behaviours of $\hat{\gamma}_1$ and $\hat{\eta}$ are summarized in Theorem 3.2.5 and Theorem 7.6.1 in de Haan and Ferreira (2006).

In addition, according to (2.9), $\xi_{1-k/n}$ can be estimated by,

$$\hat{\xi}_{1-k/n} := \inf\left\{\xi \in (0,1),\ \widehat{C}_{n/k}(\xi, 1) \geq (k/n)^{2-\frac{1}{\hat{\eta}}}\right\}, \tag{2.20}$$

where

$$\widehat{C}_{n/k}(x,y) = \left(\frac{n}{k}\right)^{\frac{1}{\hat{\eta}}} \frac{1}{n}\sum_{i=1}^{n} I\left(1 - \widehat{F}_X(X_i) \leq \frac{kx}{n}, 1 - \widehat{F}_Y(Y_i) \leq \frac{ky}{n}\right), \tag{2.21}$$

serves as an empirical version of the joint probability on the right-hand side of (2.9) by embedding the well-estimated $\hat{\eta}$ and the marginal empirical distributions $\widehat{F}_X(\cdot) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq \cdot)$, $\widehat{F}_Y(\cdot) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \leq \cdot)$. As $\widehat{C}_{n/k}(x,y)$ is non-decreasing on x given any y, $\hat{\xi}_{1-k/n}$ can be regarded as the generalized inverse function of $\widehat{C}_{n/k}(\xi, 1)$ at an estimated level $(k/n)^{2-\frac{1}{\hat{\eta}}}$. A brief algorithm for computing $\hat{\xi}_{1-k/n}$ is provided in Section S3 of the supplementary material.

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The following proposition presents the asymptotic property for $\hat{\xi}_{1-k/n}$ characterized by a centered Gaussian process, $W(\cdot)$, with covariance structure,

$$E[W(x)W(y)] = x \wedge y.$$

Proposition 2. *Suppose that Assumption 1 (c) and Assumption 2 (b), (d), (f) hold with* $\eta \in \left(\frac{7+\sqrt{17}}{16}, 1\right)$. *We have that, as* $n \to \infty$,

$$\frac{k^{3/2}}{n}\left(\frac{\hat{\xi}_{1-k/n}}{\xi_{1-k/n}} - 1\right) \xrightarrow{d} \left(\frac{1}{\eta} - 2\right) C_2(0,1)W(1). \tag{2.22}$$

Remark 3. The narrower range $\eta \in \left(\frac{7+\sqrt{17}}{16}, 1\right)$ is technically required to ensure the asymptotic normality, which can be traced back to the asymptotic analysis of Lemma S3, where the key step is solving the inequality $\frac{1}{4}\left(\frac{1}{2\eta-1} - 1\right) < \frac{1}{4}\left(3 - \frac{1}{\eta}\right)$, *i.e.*, $8\eta^2 - 7\eta + 1 > 0$. Note that the consistency does not require this restriction; by taking $\hat{\xi}_{1-k/n}$ as an example, we provide the analysis of its consistency in Section S4 of the supplementary material.

Therefore, by recalling the extrapolative expression given in (2.15), we can estimate $\mathrm{CoVaR}_{X|Y}(\tau_n')$ by plugging in the estimators $\widehat{\mathrm{VaR}}_X(1-k/n)$, $\hat{\gamma}_1$, $\hat{\eta}$ and $\hat{\xi}_{1-k/n}$,

$$\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n') = \left(\frac{k}{n(1-\tau_n')}\right)^{\hat{\gamma}_1\left(3-\frac{1}{\hat{\eta}}\right)} \hat{\xi}_{1-k/n}^{-\hat{\gamma}_1} \widehat{\mathrm{VaR}}_X(1-k/n). \tag{2.23}$$

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Then, combining (2.23) and (2.6), we can extrapolate the $\mathrm{CoES}_{X|Y}(\tau_n')$ by

$$\begin{aligned}\widetilde{\mathrm{CoES}}^{(1)}_{X|Y}(\tau_n') &= \frac{1}{1-\hat{\gamma}_1}\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n') \\ &= \frac{1}{1-\hat{\gamma}_1}\left(\frac{k}{n(1-\tau_n')}\right)^{\hat{\gamma}_1\left(3-\frac{1}{\hat{\eta}}\right)}\hat{\xi}^{-\hat{\gamma}_1}_{1-k/n}\widehat{\mathrm{VaR}}_X(1-k/n).\end{aligned} \tag{2.24}$$

Under the second-order conditions specified in Assumption 2 (a) and (b), we establish the joint asymptotic properties of $\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n')$ and $\widetilde{\mathrm{CoES}}^{(1)}_{X|Y}(\tau_n')$ in the following theorem.

Theorem 1. *Suppose that Assumption 1 (c) and Assumption 2 (a), (b), (d), (e), (f) hold with* $\eta \in \left(\frac{7+\sqrt{17}}{16}, 1\right)$. *We have that, as* $n \to \infty$,

$$\begin{aligned}&\frac{k^{3/2}}{n}\left(\frac{\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(\tau_n')}-1, \frac{\widetilde{\mathrm{CoES}}^{(1)}_{X|Y}(\tau_n')}{\mathrm{CoES}_{X|Y}(\tau_n')}-1\right)^{\top} \\ &\xrightarrow{d}(1,1)^{\top}\gamma_1\left(2-\frac{1}{\eta}\right)C_2(0,1)W(1).\end{aligned} \tag{2.25}$$

2.2 Estimations based on intermediate CoVaR and CoES

Now, we turn our attention to the second approach of extrapolation for $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$, namely, the intermediate-CoVaR/CoES approach. This approach necessitates the estimators of $\mathrm{CoVaR}_{X|Y}(1-k/n)$ and $\mathrm{CoES}_{X|Y}(1-k/n)$, denoted as $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ and $\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)$, respectively. First, according to (1.1), $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ can be defined by

$$\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n) := \sup\left\{s \in (0,\infty),\ P_n(s) \geq (k/n)^2\right\}, \tag{2.26}$$

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where

$$P_n(s) := \frac{1}{n}\sum_{i=1}^{n} I(X_i \geq s, Y_i \geq \widehat{\mathrm{VaR}}_Y(1-k/n)),$$

and $\widehat{\mathrm{VaR}}_Y(1-k/n) := Y_{n-k,n}$ corresponding to the $(n-k)$-th order statistic of $Y_1, ..., Y_n$. Then, (2.26) indicates that $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ can be treated as the generalized inverse function of $P_n(\cdot)$ at the quantile level $1-k/n$. We also provide an algorithm for computing $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ in Section S3 of the supplementary material. Furthermore, according to the expected form of CoES given in (1.3), $\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)$ can be defined by

$$\begin{aligned}&\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)\\ =:&\frac{n}{k^2}\sum_{i=1}^{n} X_i I\left(X_i \geq \widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n), Y_i \geq \widehat{\mathrm{VaR}}_Y(1-k/n)\right).\end{aligned} \tag{2.27}$$

The following proposition establishes the joint asymptotic normality for $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ and $\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)$.

Proposition 3. *Suppose that Assumption 1 (c) and Assumption 2 (b), (c), (d), (f) hold with* $\eta \in \left(\frac{7+\sqrt{17}}{16}, 1\right)$. *Then, we have that, for any* $0 < \gamma_1 < \frac{1}{4}\left(3-\frac{1}{\eta}\right)$, *as* $n \to \infty$,

$$\begin{aligned}&\frac{k^{3/2}}{n}\left(\frac{\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)}{\mathrm{CoVaR}_{X|Y}(1-k/n)}-1, \frac{\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)}{\mathrm{CoES}_{X|Y}(1-k/n)}-1\right)^{\top}\\ \xrightarrow{d}&(1,2)^{\top}\gamma_1\left(2-\frac{1}{\eta}\right)C_2(0,1)W(1).\end{aligned} \tag{2.28}$$

Note that the narrower constraint $0 < \gamma_1 < \frac{1}{4}\left(3-\frac{1}{\eta}\right) < \frac{1}{2}$ is imposed to ensure the asymptotic behavior of $\widehat{\mathrm{CoES}}_{X|Y}(1-k/n)$. Then, using (2.5),

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(2.6), and (2.14) once again, we obtain the asymptotic relationship between $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoVaR}_{X|Y}(1-k/n)$,

$$\frac{\mathrm{CoVaR}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(1-k/n)} = \frac{\mathrm{VaR}_X(1-(1-\tau_n')\xi_{\tau_n'})}{\mathrm{VaR}_X(1-k\xi_{1-k/n}/n)} \sim \left(\frac{k}{n(1-\tau_n')}\right)^{\gamma_1\left(3-\frac{1}{\eta}\right)},$$

suggesting the second intermediate-CoVaR-based extrapolations for $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$ given by,

$$\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n') = \left(\frac{k}{n(1-\tau_n')}\right)^{\hat{\gamma}_1\left(3-\frac{1}{\hat{\eta}}\right)} \widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n), \tag{2.29}$$

and

$$\begin{aligned}\widetilde{\mathrm{CoES}}^{(2)}_{X|Y}(\tau_n') &= \frac{1}{1-\hat{\gamma}_1}\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n') \\ &= \frac{1}{1-\hat{\gamma}_1}\left(\frac{k}{n(1-\tau_n')}\right)^{\hat{\gamma}_1\left(3-\frac{1}{\hat{\eta}}\right)} \widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n).\end{aligned} \tag{2.30}$$

Finally, the intermediate-CoES-based extrapolation for $\mathrm{CoES}_{X|Y}(\tau_n')$ is given by

$$\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau_n') = \left(\frac{k}{n(1-\tau_n')}\right)^{\hat{\gamma}_1\left(3-\frac{1}{\hat{\eta}}\right)} \widehat{\mathrm{CoES}}_{X|Y}(1-k/n), \tag{2.31}$$

which follows the asymptotic relationship,

$$\begin{aligned}\frac{\mathrm{CoES}_{X|Y}(\tau_n')}{\mathrm{CoES}_{X|Y}(1-k/n)} &= \frac{\mathrm{CoES}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(\tau_n')}\frac{\mathrm{CoVaR}_{X|Y}(1-k/n)}{\mathrm{CoES}_{X|Y}(1-k/n)}\frac{\mathrm{CoVaR}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(1-k/n)} \\ &\sim \left(\frac{k}{n(1-\tau_n')}\right)^{\gamma_1\left(3-\frac{1}{\eta}\right)}.\end{aligned}$$

The following theorem establishes the joint asymptotic normalities of $\widetilde{\mathrm{CoES}}^{(2)}_{X|Y}(\tau_n')/\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau_n')$ and $\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n')$. For the result concerning

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$\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau_n')$, which depends on $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$, the EVI γ_1 should be restricted to the interval $\left(0, \frac{1}{4}\left(3-\frac{1}{\eta}\right)\right)$ as well.

Theorem 2. *Suppose that Assumption 1 (c) and Assumption 2 hold with* $\eta \in \left(\frac{7+\sqrt{17}}{16}, 1\right)$. *Then, we have that, for any* $\gamma_1 \in (0,1)$, *as* $n \to \infty$,

$$\begin{aligned} &\frac{k^{3/2}}{n}\left(\frac{\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(\tau_n')}-1, \frac{\widetilde{\mathrm{CoES}}^{(2)}_{X|Y}(\tau_n')}{\mathrm{CoES}_{X|Y}(\tau_n')}-1\right)^{\top} \\ &\xrightarrow{d}(1,1)^{\top}\gamma_1\left(2-\frac{1}{\eta}\right)C_2(0,1)W(1), \end{aligned} \tag{2.32}$$

and moreover, for any $\gamma_1 \in \left(0, \frac{1}{4}\left(3-\frac{1}{\eta}\right)\right)$,

$$\begin{aligned} &\frac{k^{3/2}}{n}\left(\frac{\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n')}{\mathrm{CoVaR}_{X|Y}(\tau_n')}-1, \frac{\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau_n')}{\mathrm{CoES}_{X|Y}(\tau_n')}-1\right)^{\top} \\ &\xrightarrow{d}(1,2)^{\top}\gamma_1\left(2-\frac{1}{\eta}\right)C_2(0,1)W(1). \end{aligned} \tag{2.33}$$

We further provide a feasible moving block bootstrap (MBB) procedure for the variances approximations in Section S6, along with bootstrap-based confidence intervals for practical inference. These confidence intervals are also implemented in the empirical practice (see Section S8.3).

3. Simulation Study

In this section, we implement some Monte Carlo simulations to investigate the finite sample performance of our extrapolative methods. We consider the following bivariate models:

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- **Model 1.** Define

$$(X, Y) = (Z_1, Z_2),$$

where (Z_1, Z_2) has identical Pareto marginal distributions $\Psi_a(x) = 1 - x^{-a}$ with common parameter $a > 0$ and a dependence structure given by Marshall-Olkin survival copula, defined by $\overline{C}_{MO}(u, v) = uv \min\{u^{-a_1}, v^{-a_2}\}$ for all $u, v \in (0, 1)^2$ and some $a_1, a_2 \in (0, 1)$. For this model, we have that, $\gamma_1 = 1/a$, $\eta = \frac{1}{2-a_2}$, $A_1(t) = 0$, and

$$C(x, y) = \begin{cases} x^{1-a_1} y, & \text{if } a_1 < a_2, \\ xy \max\{x, y\}^{-a_1}, & \text{if } a_1 = a_2, \\ xy^{1-a_2}, & \text{if } a_1 > a_2. \end{cases}$$

In this simulation, we set $a = 3$, $a_1 = 5/6$, $a_2 = 2/3$, implying $\gamma_1 = 1/3$, $\eta = 3/4$ and $C(x, y) = xy^{1/3}$. We refer to Example 4 of Das and Fasen-Hartmann (2018) for more details about this model.

- **Model 2.** Continue using **Model 1.** with parameters $a = 3$, $a_1 = a_2 = 7/10$, implying $\gamma_1 = 1/3$, $\eta = 10/13$ and $C(x, y) = xy \max\{x, y\}^{-7/10}$.

- **Model 3.** Define

$$(X, Y) = B(Z_1, Z_3) + (1 - B)(Z_2, Z_2),$$

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where Z_1, Z_2, Z_3 are independent Pareto random variables with parameters a, b, a $(b > a)$, respectively, and B is a Bernoulli(1/2) random variable independent of $Z_i's$. For this model, we have $\gamma_1 = 1/a$, $\eta = a/b$, $A(t) \sim \frac{1}{1+t^{b-a}}$, $\alpha = 1 - \frac{2b}{a}$, and $C(x, y) = 2^{b/a-1}(x \wedge y)^{b/a}$. In this simulation, we set $a = 3$, $b = 4$, implying $\gamma_1 = 1/3$, $\eta = 3/4$, and $C(x, y) = 2^{1/3}(x \wedge y)^{4/3}$. This model is also used in Cai and Musta (2020) and Wang et al. (2026).

All three models satisfy the two key second-order conditions involved in Assumption 2 (a) - (c). Model 3 captures mixture-type tail-dependence feature, and its distribution can be viewed as a weighted mixture of two basic distributions. Models 2 and 3 are distributionally symmetric, whereas Model 1 exhibits the asymmetry. Moreover, serving as a potential counterexample in finite-sample analysis, Model 3 indeed admits a violation of $C_1(0, 1) > 0$, which plays a more crucial role in ensuring the validity of our estimations. Specifically, when $C_1(0, 1) = 0$, the first limit in Lemma 1 vanishes, so that the convergence $\xi^*_\tau \to 0$ as $\tau \uparrow 1$ can no longer be inferred.

In this simulation, we compare the following methods:

- CoVaR-TD: estimator of $\text{CoVaR}_{X|Y}(\tau'_n)$ under tail dependence proposed by Nolde et al. (2022);

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- CoVaR-WE: empirical estimator $\widehat{\text{CoVaR}}_{X|Y}(\tau_n')$ without extrapolation, given in (2.26);
- CoVaR-I: estimator $\widetilde{\text{CoVaR}}^{(1)}_{X|Y}(\tau_n')$, given in (2.23);
- CoVaR-II: estimator $\widetilde{\text{CoVaR}}^{(2)}_{X|Y}(\tau_n')$, given in (2.29);
- CoES-WE: empirical estimator $\widehat{\text{CoES}}_{X|Y}(\tau_n')$ without extrapolation, $\widehat{\text{CoES}}_{X|Y}(\tau_n') = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i I\left(X_i \geq \widehat{\text{CoVaR}}_{X|Y}(\tau_n'), Y_i \geq \widehat{\text{VaR}}_Y(\tau_n')\right)}{\frac{1}{n}\sum_{i=1}^{n} I\left(X_i \geq \widehat{\text{CoVaR}}_{X|Y}(\tau_n'), Y_i \geq \widehat{\text{VaR}}_Y(\tau_n')\right)}$;
- CoES-I: estimator $\widetilde{\text{CoES}}^{(1)}_{X|Y}(\tau_n')$, given in (2.24);
- CoES-II: estimator $\widetilde{\text{CoES}}^{(2)}_{X|Y}(\tau_n')$, given in (2.30);
- CoES-III: estimator $\widetilde{\text{CoES}}^{(3)}_{X|Y}(\tau_n')$, given in (2.31).

We implement CoVaR-WE and CoES-WE as baseline methods without extrapolation, and include CoVaR-TD as a misspecified benchmark. For all simulated data, we set sample size $n = 500, 1000, 2000$, and 5000, extreme levels $\tau_n' = 0.99, 0.999$, and repeat the simulation $N = 1000$ times. To evaluate the performance of all estimators, we calculate the *Mean Square Relative Error* (MSRE) based on N replications, defined as,

$$\text{MSRE} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{\theta}_n^{(i)}}{\theta} - 1\right)^2,$$

where $\hat{\theta}_n^{(i)}$ is the estimator we are interested in given the simulated data of the i-th replication, and θ is the true value.

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Another delicate problem in the extreme value theory is the proper choice of three hyperparameters k, k_1 and k_2, *that is*, the numbers of tail observations used in the estimations of $\mathrm{CoVaR}_{X|Y}(1-k/n)$ (or $\mathrm{CoES}_{X|Y}(1-k/n)$), γ_1, and η. Following the principle of computational simplicity, we set $k_1 = k_2$ and examine the sensitivity of the MSREs with respect to the values of k and k_1. Specifically, we take the estimator $\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n')$ as an example, consider a range of values for k and k_1 according to different sample size, and compute the MSRE for each pair (k, k_1) based on $N = 100$ replications, then select the pair (k, k_1) that minimizes the MSRE. We observe that larger values of k and k_1 will result in a lower MSRE values. Tables 1 - 2 summarize the selected values of k and k_1 across the three models at $\tau_n' = 0.99, 0.999$.

Moreover, Tables 1 - 2 also report the specific MSRE values for the three models under different risk levels and sample sizes. It can be directly observed that all extrapolative estimators exhibit satisfactory MSRE values across the three models, which decrease as the sample size increases and the risk level decreases. In more detail, the performance of the two CoVaR extrapolations, namely, $\widetilde{\mathrm{CoVaR}}^{(i)}_{X|Y}(\tau_n')$ $(i = 1, 2)$ is similar, as evidenced by only marginal differences in their MSRE across varying sample sizes and risk levels. As a result, this close performance is also mirrored in the first two

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extrapolations of CoES, namely, $\widetilde{\text{CoES}}^{(i)}_{X|Y}(\tau'_n)$ $(i = 1, 2)$. This is because their expressions all take the form of $\widetilde{\text{CoVaR}}^{(i)}_{X|Y}(\tau'_n)$ with $i = 1, 2$ multiplied by a $\hat{\gamma}_1$-dependent term, which implies that the disparities in their empirical performances are entirely determined by $\widetilde{\text{CoVaR}}^{(i)}_{X|Y}(\tau'_n)$ $(i = 1, 2)$. However, the third extrapolation $\widetilde{\text{CoES}}^{(3)}_{X|Y}(\tau'_n)$ is observed to yield little larger MSRE values across most scenarios, which depends on $\widehat{\text{CoES}}_{X|Y}(1 - k/n)$.

We further present a series of boxplots of the ratios between these estimations and their true values in Figures S1 - S3 of the supplementary material, which reflect an intuitive comparison of consistency among these methods. From a global perspective, all extrapolative methods demonstrate a more concentrated pattern as sample size grows, characterized by lower variation. Furthermore, the behaviors of our proposed extrapolations remains highly consistent and stable. A somewhat surprising finding is that the extrapolations for Model 3 empirically outperform the first two models in certain scenarios, particularly with small samples. However, it shows a less robust profile with larger sample sizes, yielding higher MSRE values and more upward skewed boxplots than its predecessors, which possibly stems from the fact that Assumption 1 (c) is not satisfied for Model 3.

In contrast, under all scenarios, the two non-extrapolated empirical estimators CoVaR-WE and CoES-WE exhibit relatively poor MSRE perfor-

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mance. Even in large samples, their MSRE values are substantially larger than those of the extrapolation-based estimators. This highlights the necessity of extrapolation when dealing with extreme risk. More notably, the worst-performing method among all approaches is CoVaR-TD, which exhibits the largest MSRE values. This indicates the potential harm caused by model misspecification and illustrates the value of the proposed methods under asymptotic independence. The poor performance is also reflected in the boxplots. All three benchmarks (CoVaR-TD, CoVaR-WE, CoES-WE) exhibit substantial bias; the latter two estimators tend to significantly underestimate risk, while the CoVaR-TD method tends to substantially overestimate it, especially at more extreme risk levels.

In summary, all proposed extrapolations outperform the three benchmarks and there are no scenarios in which one extrapolation is significantly superior to another, provided that the slightly worse performance of CoES-III in terms of MSRE can be neglected. If not, one can either increase the sample size or prioritize the fist two extrapolations for CoES.

4. Real Data Analysis

In this section, we further examine the finite-sample performance of the proposed extrapolative methods by conducting a real-world analysis on stock

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Table 1: The MSREs of the estimators for $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$ under **Models 1-3** with $\tau_n' = 0.99$.

	(k, k_1)	CoVaR-TD	CoVaR-WE	CoVaR-I	CoVaR-II	CoES-WE	CoES-I	CoES-II	CoES-III
					$n = 500$				
Model 1	$(137, 143)$	0.66460	0.33102	0.04118	0.04117	0.48105	0.06155	0.06146	0.06520
Model 2	$(84, 150)$	0.47252	0.34409	0.04611	0.04674	0.49331	0.06538	0.06568	0.09129
Model 3	$(71, 137)$	0.64805	0.31882	0.03838	0.03907	0.42746	0.06532	0.06592	0.08949
					$n = 1000$				
Model 1	$(287, 274)$	0.53411	0.29099	0.02043	0.02017	0.40696	0.02937	0.02899	0.03273
Model 2	$(182, 274)$	0.39864	0.25152	0.02491	0.02537	0.39208	0.03507	0.03552	0.04136
Model 3	$(155, 287)$	0.56553	0.21988	0.01945	0.02026	0.30552	0.03578	0.03694	0.04587
					$n = 2000$				
Model 1	$(384, 400)$	0.47987	0.27814	0.01386	0.01423	0.33234	0.01987	0.02028	0.02251
Model 2	$(274, 384)$	0.35309	0.22104	0.01596	0.01629	0.31091	0.02237	0.02269	0.02830
Model 3	$(274, 384)$	0.57715	0.16474	0.01650	0.01635	0.22857	0.03475	0.03425	0.03193
					$n = 5000$				
Model 1	$(671, 724)$	0.43473	0.51029	0.00708	0.00727	0.30282	0.01015	0.01034	0.01572
Model 2	$(697, 697)$	0.32531	0.51844	0.00755	0.00778	0.31603	0.01074	0.01098	0.01377
Model 3	$(382, 724)$	0.46046	0.25515	0.00947	0.01020	0.19678	0.02247	0.02368	0.02445

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Table 2: The MSREs of the estimators for $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$ under **Models 1-3** with $\tau_n' = 0.999$.

	(k, k_1)	CoVaR-TD	CoVaR-WE	CoVaR-I	CoVaR-II	CoES-WE	CoES-I	CoES-II	CoES-III
					$n = 500$				
Model 1	$(137, 143)$	2.51235	0.82348	0.11382	0.11341	0.87882	0.15278	0.15231	0.15578
Model 2	$(84, 150)$	1.82263	0.85995	0.12010	0.11969	0.90409	0.15608	0.15492	0.19471
Model 3	$(124, 150)$	2.60226	0.83811	0.09767	0.09747	0.87574	0.15209	0.15099	0.14587
					$n = 1000$				
Model 1	$(287, 274)$	1.80829	0.81825	0.05298	0.05227	0.87494	0.06803	0.06714	0.07246
Model 2	$(182, 274)$	1.41791	0.85562	0.06311	0.06375	0.90072	0.08059	0.08129	0.08826
Model 3	$(116, 300)$	1.82486	0.83725	0.04618	0.04839	0.87407	0.07214	0.07534	0.08829
					$n = 2000$				
Model 1	$(195, 400)$	1.59926	0.72985	0.03878	0.03908	0.81261	0.04873	0.04896	0.06612
Model 2	$(305, 384)$	1.27963	0.78496	0.04026	0.04087	0.85094	0.05087	0.05149	0.05639
Model 3	$(274, 384)$	2.02911	0.77204	0.04263	0.04225	0.82216	0.07509	0.07424	0.06861
					$n = 5000$				
Model 1	$(671, 724)$	1.51710	0.61569	0.01886	0.01908	0.72696	0.02396	0.02418	0.03072
Model 2	$(592, 750)$	1.12094	0.66405	0.01906	0.01940	0.76262	0.02391	0.02425	0.02933
Model 3	$(250, 750)$	1.59587	0.64011	0.02325	0.02569	0.71412	0.04380	0.04815	0.05687

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loss data. In our analysis, we consider 12 constituent stocks from 4 major sectors of the S&P 500 Index, namely,

- IT sector: IBM (International Business Machines), INTC (Intel), MSFT (Microsoft);
- Energy sector: XOM (Exxon Mobil), CVX (Chevron), COP (ConocoPhillips);
- Consumer Discretionary sector: DIS (The Walt Disney), MCD (McDonald's), F (Ford Motor);
- Industrials sector: CAT (Caterpillar), BA (The Boeing), GE (GE Aerospace).

We treat the S&P500 Index as the system variable (Y) and the 12 firms as the individual variables ($X_i, i = 1, ..., 12$). As the theories for statistical methodologies are derived for independent and identically distributed samples, we utilize the weekly historical adjusted closing prices of their stocks to reduce the potential serial dependence. The data spans from January 1, 1995, to December 29, 2024, consisting of 1566 trading records. The weekly losses were calculated as negative log returns. Figure 1 shows the time series plots of weekly losses for the 12 stocks, illustrating pronounced

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volatility clustering, particularly during the global financial crisis and the COVID-19 pandemic.

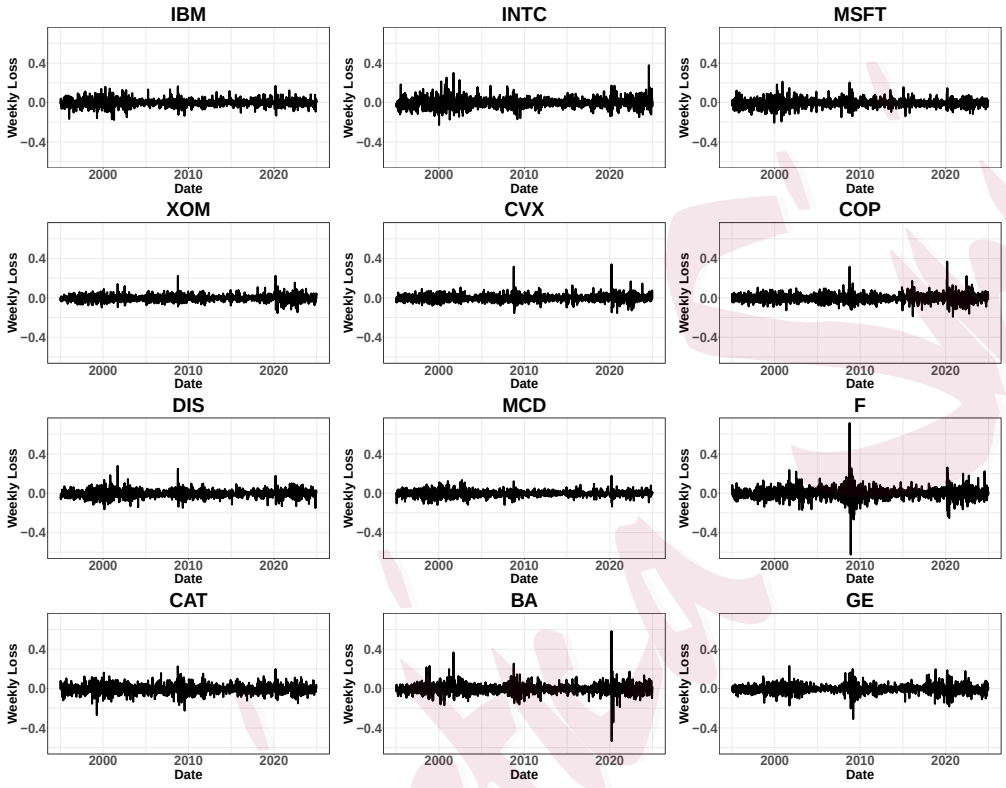

Figure 1: Time series plots of weekly losses for the 12 constituent stocks.

Before investigating the extreme losses measured by CoVaR and CoES of individual stocks conditional on S&P500 Index, we would like to check the asymptotic independence between each pair of (X_i, Y) for $i = 1, ..., 12$ and choose suitable values for three hyperparameters k, k_1 and k_2. The estimations for γ_1 and η against varying levels of k_1 and k_2 are represented in

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Figure S4. It can be observed that the estimations of γ_1 generally fall within the range of $(0, 0.5)$, the estimations of η lie within the interval $(0.78, 0.87)$ for most k_2 and approach around 0.8 as k_2 grows. These results provide significant evidence that X_i and Y are asymptotically independent for all i, as well as with positive association between their tails. The values of k_1 and k_2 can be chosen such that the $\hat{\gamma}_1$ and $\hat{\eta}$ stabilize. We summarize the specific values of k_1 and k_2 and corresponding $\hat{\gamma}_1$ and $\hat{\eta}$ in Table 3. To more formally diagnose tail independence, we also employ the more systematic (Tail Quotient Correlation Coefficient) TQCC test in Zhang et al. (2017) to examine whether there is significant tail independence between the 12 stocks and the S&P 500 Index. We present the detailed test procedures, the values of TQCC statistics and p-values in Section S8.2. For the selection of k, as shown in Figures S5 and S6, we plot the estimations $\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau'_n)$, $\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau'_n)$ and $\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau'_n)$ with $\tau'_n = 0.99$, 0.999 against varying k. As one can observe that all the three estimations exhibit consistent patterns across different τ'_n and tend to be stable as k increase; therefore, we select the values of k for which the estimations achieve stability. These chosen values of k are also collected in Table 3.

For the given values of k, k_1 and k_2, we calculate the $\widetilde{\mathrm{CoVaR}}^{(i)}_{X|Y}(\tau'_n)$ $(i = 1, 2)$ and $\widetilde{\mathrm{CoES}}^{(i)}_{X|Y}(\tau'_n)$ $(i = 1, 2, 3)$ with extreme levels $\tau'_n = 0.99$ and 0.999,

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Table 3: Estimations for $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$ of 12 individual stocks conditional on S&P500 Index, with $\tau_n' = 0.99$ and 0.999 (in brackets).

Ticker	(k, k_1, k_2)	$\hat{\gamma}_1$	$\hat{\eta}$	CoVaR-I	CoVaR-II	CoES-I	CoES-II	CoES-III
Information Technology Sector								
IBM	$(120, 75, 100)$	0.35785	0.82820	0.37879 (1.65912)	0.41245 (1.80652)	0.58989 (2.58371)	0.64230 (2.81326)	0.56368 (2.46895)
INTC	$(125, 75, 200)$	0.37186	0.80667	0.50422 (2.27626)	0.53746 (2.42634)	0.80272 (3.62382)	0.85564 (3.86274)	0.73419 (3.31445)
MSFT	$(100, 75, 100)$	0.33510	0.79462	0.28707 (1.10047)	0.30003 (1.15016)	0.43175 (1.65509)	0.45124 (1.72982)	0.45597 (1.74794)
Energy Sector								
XOM	$(120, 75, 275)$	0.35499	0.85807	0.39466 (1.76799)	0.39174 (1.75493)	0.61186 (2.74102)	0.60734 (2.72078)	0.60774 (2.72254)
CVX	$(125, 75, 280)$	0.31764	0.83270	0.31093 (1.15907)	0.29959 (1.11682)	0.45566 (1.69861)	0.43905 (1.63669)	0.53785 (2.00500)
COP	$(120, 75, 250)$	0.35798	0.80985	0.46190 (1.97902)	0.46588 (1.99605)	0.71945 (3.08247)	0.72564 (3.10900)	0.77490 (3.32006)
Consumer Discretionary Sector								
DIS	$(135, 75, 300)$	0.33940	0.86086	0.38434 (1.61683)	0.37703 (1.58610)	0.58180 (2.44753)	0.57075 (2.40102)	0.58671 (2.46819)
MCD	$(100, 75, 150)$	0.33096	0.79819	0.26894 (1.01836)	0.30968 (1.17262)	0.40198 (1.52214)	0.46287 (1.75271)	0.40241 (1.52381)
F	$(105, 75, 300)$	0.38167	0.84876	0.63425 (3.14479)	0.67251 (3.33453)	1.02575 (5.08597)	1.08764 (5.39283)	1.18816 (5.89127)
Industrials Sector								
CAT	$(120, 75, 175)$	0.29419	0.84092	0.32756 (1.11689)	0.34397 (1.17285)	0.46409 (1.58243)	0.48734 (1.66170)	0.49316 (1.68154)
BA	$(130, 75, 175)$	0.41321	0.83014	0.59652 (3.29230)	0.60089 (3.31645)	1.01658 (5.61073)	1.02404 (5.65188)	1.25659 (6.93541)
GE	$(100, 75, 175)$	0.38565	0.83589	0.59528 (2.95330)	0.59290 (2.94148)	0.96896 (4.80717)	0.96508 (4.78793)	0.75045 (3.72307)

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based on 1565 weekly losses of 12 individual stocks conditional on S&P500 Index losses from January 8, 1995, to December 29, 2024. The results are also reported in Table 3, which represent the average weekly losses of the 12 selected firms under a market crisis within a period of nearly 30 years. A straightforward observation from Tables 3 gives that, as risk level becomes more extreme, all extrapolative estimations of CoVaR and CoES grow, which accords with conventional intuition. Moreover, the estimations $\widetilde{\mathrm{CoVaR}}^{(1)}_{X|Y}(\tau_n')$ and $\widetilde{\mathrm{CoVaR}}^{(2)}_{X|Y}(\tau_n')$ are generally very close in most cases, though their discrepancy increases with the level τ_n', implying that the extrapolations based on estimated $\hat{\xi}_{1-k/n}$ and $\widehat{\mathrm{CoVaR}}_{X|Y}(1-k/n)$ exhibit comparable empirical performance. Note that estimations $\widetilde{\mathrm{CoES}}^{(i)}_{X|Y}(\tau_n')$ $(i=1,2)$ follow the same pattern as the corresponding $\widetilde{\mathrm{CoVaR}}^{(i)}_{X|Y}(\tau_n')$, since $\widetilde{\mathrm{CoES}}^{(i)}_{X|Y}(\tau_n')$ are obtained by multiplying $\widetilde{\mathrm{CoVaR}}^{(i)}_{X|Y}(\tau_n')$ by a $\hat{\gamma}_1$-dependent term. Except rare cases such as BA and GE, $\widetilde{\mathrm{CoES}}^{(3)}_{X|Y}(\tau_n')$ yields results comparable to those of the first two.

In order to effectively capture the time-varying dynamics of extreme risk measured by CoVaR and CoES, we accordingly apply a rolling-window approach to generate monthly estimates of $\mathrm{CoVaR}_{X|Y}(\tau_n')$ and $\mathrm{CoES}_{X|Y}(\tau_n')$. The sample is enlarged to daily observations from January 3, 1995 to December 31, 2024, yielding 7552 observations, and we implement the pro-

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cedure with a 1500-day moving window. For instance, the estimates on October 1st, 2014 are calculated by using 1500 observations from October 15, 2008 to September 30, 2014. This long window captures the evolution of tail-risk interdependence over time. Figures 2 and 3 display the dynamics rolling paths of estimations $\widetilde{\text{CoVaR}}^{(i)}_{X|Y}(\tau_n')$ and $\widetilde{\text{CoES}}^{(i)}_{X|Y}(\tau_n')$ with $\tau_n' = 0.99$ and 0.999, respectively. We observe that all extrapolations move in lock-step, corroborating their similar empirical behaviours. In addition, different industries have been affected by the prevailing economic environment during various periods. For example, almost all firms were affected by the COVID-19 pandemic, with particularly pronounced impacts on IBM, DIS, and BA, all of which exhibit sharp spike-like behaviors. In contrast, the impact of the global financial crisis appears to be much minor: MCD was barely affected by the crisis, and firms in IT sector were also only mildly influenced, whereas GE experienced the most severe impact. Moreover, the Oil excess capacity bubble around 2014 – 2016 led to a significant increase in tail risk for three energy companies: XOM, CVX, COP.

5. Conclusion

In this paper, we investigate the estimation methods of extreme CoVaR and CoES within the framework of asymptotic independence but positively asso-

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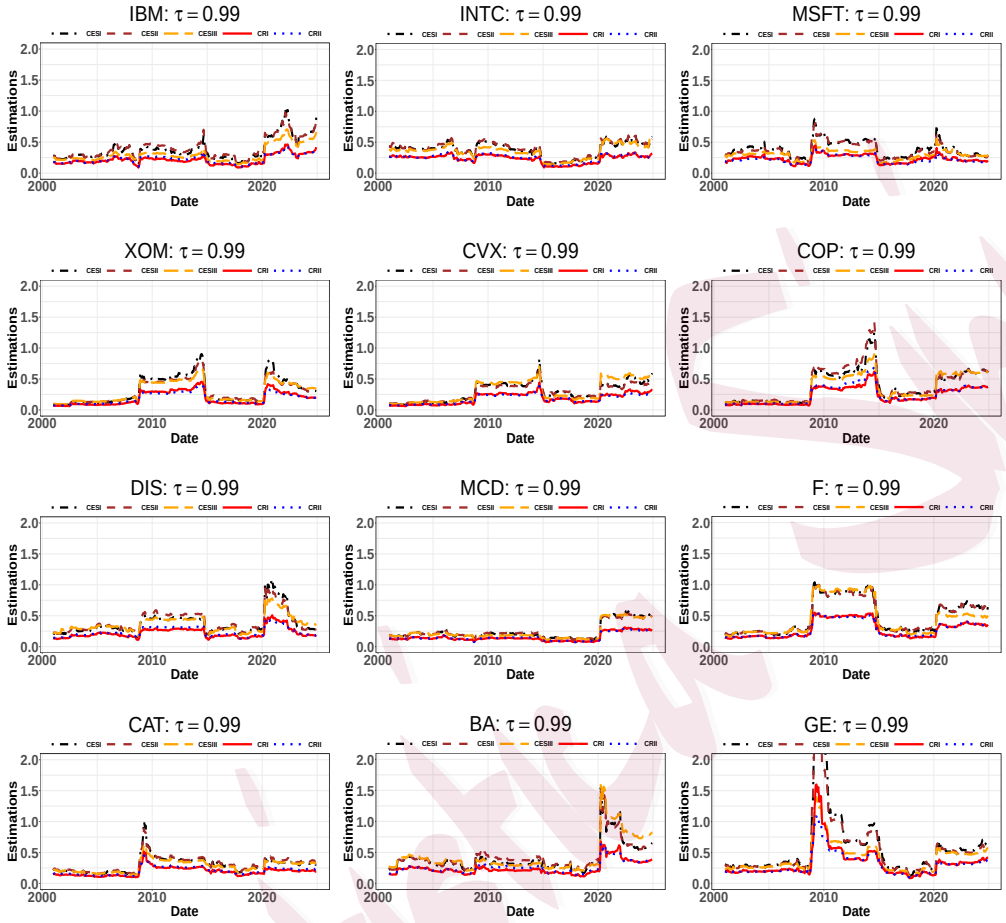

Figure 2: The rolling estimations $\widetilde{\text{CoVaR}}^{(1)}_{X|Y}(\tau_n')$ (CRI in red solid lines), $\widetilde{\text{CoVaR}}^{(2)}_{X|Y}(\tau_n')$ (CRII in blue dotted lines), $\widetilde{\text{CoES}}^{(1)}_{X|Y}(\tau_n')$ (CESI in black dot-dash lines), $\widetilde{\text{CoES}}^{(2)}_{X|Y}(\tau_n')$ (CESII in brown dashed lines), and $\widetilde{\text{CoES}}^{(3)}_{X|Y}(\tau_n')$ (CESIII in orange longdash lines) for 12 individual stocks conditional on S&P500 Index, with $\tau_n' = 0.99$.

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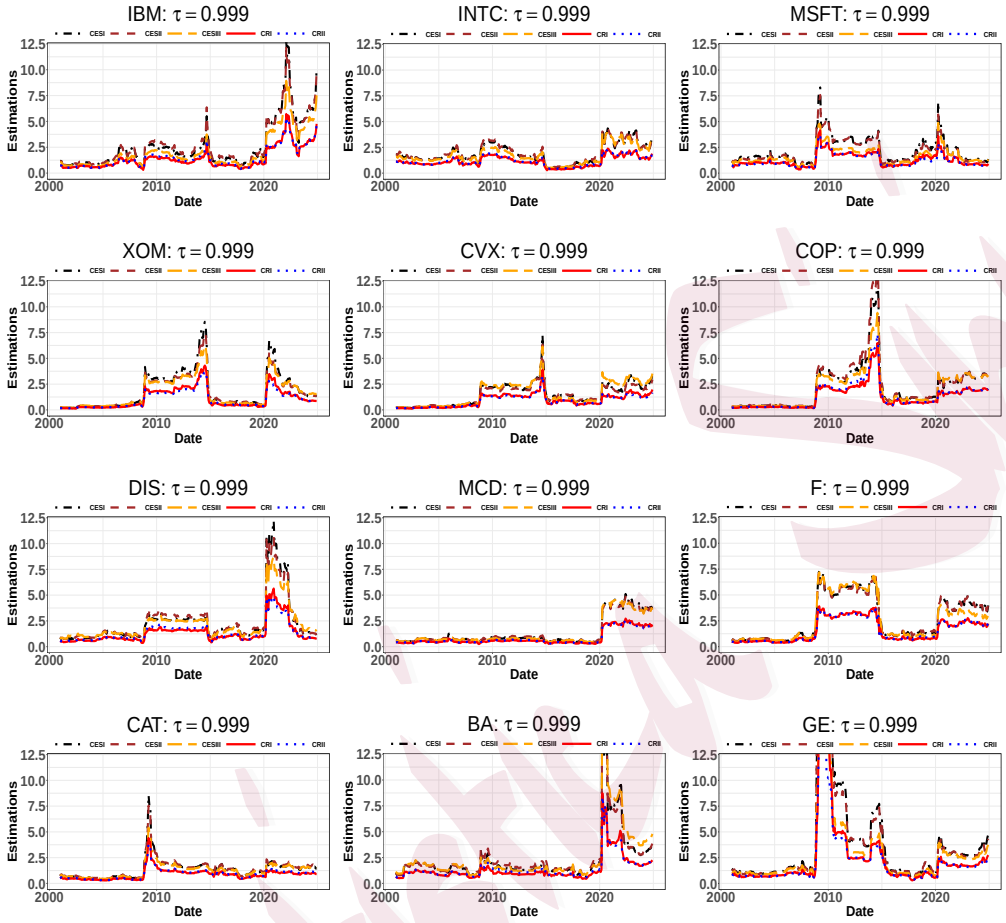

Figure 3: The rolling estimations $\widetilde{\text{CoVaR}}^{(1)}_{X|Y}(\tau_n')$ (CRI in red solid lines), $\widetilde{\text{CoVaR}}^{(2)}_{X|Y}(\tau_n')$ (CRII in blue dotted lines), $\widetilde{\text{CoES}}^{(1)}_{X|Y}(\tau_n')$ (CESI in black dot-dash lines), $\widetilde{\text{CoES}}^{(2)}_{X|Y}(\tau_n')$ (CESII in brown dashed lines), and $\widetilde{\text{CoES}}^{(3)}_{X|Y}(\tau_n')$ (CESIII in orange longdash lines) for 12 individual stocks conditional on S&P500 Index, with $\tau_n' = 0.999$.

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ciation, namely, the coefficient of tail dependence $\eta \in (1/2, 1)$. We develop two extrapolative approaches, where the first relies on intermediate VaR and an adjustment factor, while the second exploits intermediate CoVaR and CoES. Simulation study demonstrates that the proposed estimations are robust and perform well in finite samples. The case of asymptotic independence but negatively association, *i.e.*, $\eta \in (0, 1/2)$, is technically more challenging and is therefore left for future research.

Supplementary Materials

The supplementary material contains all technical proofs, also provides additional simulations and empirical analyses.

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