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Bubble Modeling and Tagging: A Stochastic Nonlinear Autoregression Approach

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Abstract: Economic and financial time series can feature locally explosive behavior when a bubble is formed. The economic or financial bubble, especially its dynamics, is an intriguing topic that has been attracting longstanding attention. To illustrate the dynamics of the local explosion itself, the paper presents a novel time series model, called the stochastic nonlinear autoregressive model, which is always strictly stationary and geometrically ergodic and can create long swings or persistence observed in many macroeconomic variables. When a nonlinear autoregressive coefficient is outside of a certain range, the model has periodically explosive behaviors and can then be used to portray the bubble dynamics. Further, the quasi-maximum likelihood estimation (QMLE) of our model is considered, and its strong consistency and asymptotic normality are established under minimal assumptions on innovation. A new model diagnostic checking statistic is developed for model fitting adequacy. In addition, two methods for bubble tagging are proposed, one from the residual perspective and the other from the null-state perspective. Monte Carlo simulation studies are conducted to assess the performances of the QMLE and the two bubble tagging methods in finite samples. Finally, the usefulness of the model is illustrated by an empirical application to the monthly Hang Seng Index.

Key words and phrases: Causal process, Financial bubble, Rational expectation, SNAR model, Speculative bubble.

1. Introduction

Financial speculative bubbles have been attracting longstanding attention of economists and financial practitioners as an economic crisis often originates along with a burst of a bubble. In reality, however, economic or financial bubbles cannot be avoided. The presence of bubbles is partially evidenced by that many economic or financial time series possess locally explosive behavior and a subsequent burst, with such a phenomenon appearing periodically. Studying the dynamics of bubble thus becomes important and intriguing.

One classical definition of the bubble is the deviation of the market price from its fundamental value (a sum of discounted future dividends) in rational expectation price models. An important model of the rational bubble is initiated by Blanchard and Watson (1982), where the bubble process is captured via a simple stochastic autoregression (AR) with a fixed explosive rate and an absorbing state zero. Their model is then extended by Evans (1991) via adopting a stochastic rate of explosion. Primarily, the bubble is regarded as an explosive nonstationary process, which motivates to test its presence via unit root and cointegration tests (Diba and Grossman, 1988a,b). Recently, this idea is further developed by Phillips and Yu (2011), Phillips et al. (2011, 2015a,b), Harvey et al. (2019, 2020), Tao et al. (2019), Kurozumi et al. (2023), Esteve and Prats (2023) and references therein. On the other hand, Evans (1991) also notes that periodical collapse of bubbles makes the bubble paths look more like a stationary process. Within a stationary framework, Gouriéroux and Zakoïan (2017) find that noncausal AR(1) models can characterize multiple local explosions in time series. Then this noncausal approach to bubble modelling has been extended to high-order mixed causal-noncausal time series models, see, for example, Gouriéroux and Jasiak (2016), Fries and Zakoïan (2019), Cavaliere et al. (2020), Davis and Song (2020), and Fries (2022). However, one shortcoming of the noncausal approach invites computational challenge and many resampling methods are needed. To bypass this shortcoming, Blasques et al. (2022) propose a new observation driven model with time-varying parameters and study its probabilistic properties and statistical inference. Nevertheless, their estimation heavily depends on the choice of the survival function and the asymptotics can be obtained only for a part of parameters. Motivated by all above facts, we here present a new simple time series model to describe the dynamics of bubbles.

In this paper, a first-order stochastic nonlinear autoregressive (SNAR) model $\{y_t\}$ is defined as

$$y_t = s_t\phi_0|y_{t-1}| + \varepsilon_t, \quad t \in \mathbb{Z} := \{0, \pm 1, \pm 2, ...\}, \tag{1.1}$$

where $\phi_0 \in \mathbb{R}$, $\{\varepsilon_t : t \in \mathbb{Z}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables on some basic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and independent of i.i.d. binary variables $\{s_t : t \in \mathbb{Z}\}$ with $\mathbb{P}(s_t = 1) = p_0 = 1 - \mathbb{P}(s_t = 0)$, $p_0 \in [0, 1)$.

Clearly, when $\phi_0 > 1$, y_t is explosive in the periods where $s_t = 1$ and creates an excursion which stops once $s_t = 0$. Fig. 1 illustrates two simulated paths of the SNAR model (1.1) with $\varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0, 6^2)$. We can observe periodically local explosions followed by bursts. And the 'shape' of a bubble before its burst is similar to a quadratic curve, which conforms to some practitioners' pointview that the accumulation of a bubble before bursting resembles a parabola. Further, when $\mathbb{P}(s_t = 1) = 1$, the SNAR model (1.1) reduces to an absolute AR model or a special threshold AR model with threshold parameter zero, which is studied in Tong (1990), Li and Tong (2020) and references therein.

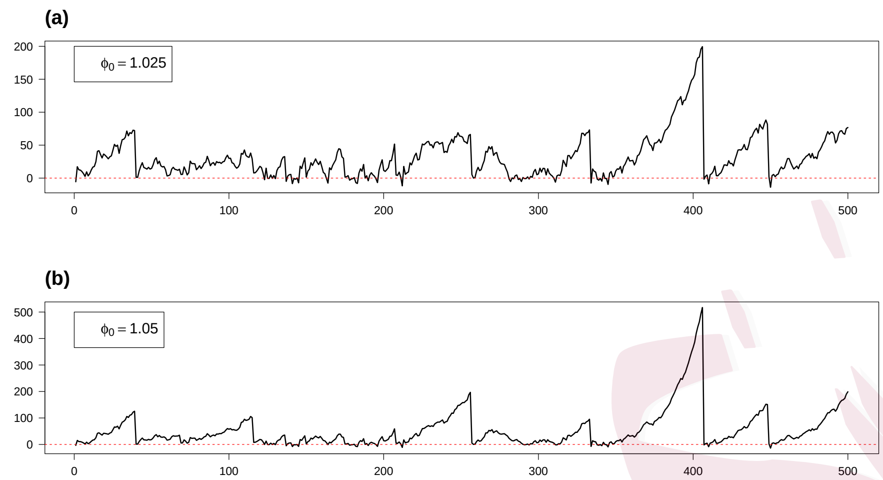

Figure 1: Simulated paths of model (1.1) with $\varepsilon_t \sim \mathcal{N}(0, 6^2)$, $p_0 = 0.977$, and (a) $\phi_0 = 1.025$ and (b) $\phi_0 = 1.05$.

Major contributions of our paper are as follows.

First, we introduce a simple yet useful time series model, the SNAR model, for modeling the dynamics of the bubble process. In particular, it is simple in its first-order autoregressive form as in (1.1) while being useful in addressing certain limitations of existing results on bubble modeling. We then prove that the model is always strictly stationary and geometrically ergodic under minimal assumptions on innovation and the probability p_0. Within a causal and stationary framework, when the parameter $\phi_0 > 1$, our SNAR model still displays local explosions and collapses periodically.

It can create long swings or persistence and can then be used to portray the bubble dynamics. More importantly, our model is always causal in the classical sense of time series. Compared with the noncausal bubble models in the literature, our model facilitates the understanding of the dynamics of bubble and is simpler and more convenient in applications. In particular, our model avoids the computational burden of the noncausal approach and keeps away from the choice of the survival function as in the time-varying parameter model of Blasques et al. (2022).

It is worth mentioning that a related model to ours is a stochastic AR initiated by Blanchard and Watson (1982), which is defined as $y_t = s_t\phi_0 y_{t-1} + \varepsilon_t$, $t \in \mathbb{Z}$, where $\{s_t\}$ and $\{\varepsilon_t\}$ are defined in (1.1). It can also create long swings or persistence (Johansen and Lange, 2013). Unfortunately, it can usually generate many negative local explosions even if $\phi_0 > 1$ and $\varepsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2)$ since within a long swing $\mathbb{E}(\Delta y_t | y_{t-1}, s_t = 1) = (\phi_0 - 1)y_{t-1} < 0$ if $y_{t-1} < 0$, where $\Delta y_t = y_t - y_{t-1}$. In the same setting, for our SNAR model, we always have $\mathbb{E}(\Delta y_t | y_{t-1}, s_t = 1) \geq (\phi_0 - 1)|y_{t-1}| \geq 0$, which implies that there always exists a positive accumulation tendency until bubble burst. This is one of the advantages of introducing a nonlinear mechanism $|y_{t-1}|$.

Second, we consider the quasi-maximum likelihood estimation (QMLE) of the SNAR model and establish its strong consistency and asymptotic normality under minimal assumptions on innovation and probability parameter p_0, regardless of infinite variance or heavy-tailedness of the model.

Third, we develop a new model diagnostic checking statistic since the classical portmanteau test is invalid for our model in view of the absence of the residuals.

Fourth, we consider two methods for tagging the bubbles, one from the residual point of view and the other from the null-state perspective. The problem of bubble detection has been studied in the literature; see for example Phillips and Yu (2011), Phillips et al. (2015a), Phillips et al. (2015b), Blasques et al. (2022), Kurozumi and Skrobotov (2023), and references therein. Existing results in this direction, however, have been mainly developed by viewing the bubble as a separate process occurring on an unknown but deterministic time interval within the observation period. The current paper, on the other hand, aims to consider bubble tagging in the context of a single stationary model that describes the bubble and non-bubble periods along with the generation and burst of bubbles. As a result, unlike existing results that typically assume the bubbles to persist for an adequate duration to achieve their consistent detection, the bubbles in the current model can be transient and thus the problem of bubble tagging can be more challenging in the current setting. For this, we consider two approaches, where the first one utilizes the nonlinear autoregressive residual from the proposed model and the second one is constructed from a hypothesis testing point of view. For both methods, we provide theoretical quantification on the finite-sample probability of correct tagging under reasonably mild conditions. Monte Carlo simulation results are provided to assess the finite-sample performance of the proposed QMLE and bubble tagging methods.

The remainder of the paper is organized as follows. Section 2 investigates strict stationarity and geometric ergodicity of model (1.1). Section 3 considers the QMLE with its asymptotics. Section 4 studies model diagnostic checking. Section 5 considers the problem of bubble tagging, where two approaches are considered with their finite-sample probability bounds studied. Section 6 carries out Monte Carlo simulation studies to assess the finite-sample performances of the QMLE and the two bubble tagging methods. Section 7 gives an empirical application to illustrate the usefulness of the model. Section 8 concludes. Part of simulation results and all technical proofs are provided in the Supplementary Material.

2. Probabilistic Properties of the SNAR Model

The aim of this section is to prove the strict stationarity and geometric ergodicity of model (1.1) under a very mild condition. We will prove the following result, using the approach developed by Meyn and Tweedie (2009) for establishing the geometric ergodicity of Markov chains. This result is important and is a theoretical foundation of inference for model (1.1).

Theorem 1. *Suppose that* (i) $\{\varepsilon_t\}$ *is i.i.d. and independent of i.i.d. binary variables* $\{s_t\}$ *with* $0 \leq p_0 < 1$, *and* (ii) ε_1 *has a positive density on* $\mathbb{R}$ *with* $\mathbb{E}(\log^+ |\varepsilon_1|) < \infty$. *Then there exists a strictly stationary, nonanticipative solution to* $\{y_t\}$ *in model* (1.1) *and the solution is unique and geometrically ergodic.*

We remark that, although Theorem 1 is developed for any choice of ϕ_0 for the purpose of generality, it is typically desired to use model (1.1) with $\phi_0 > 1$ for proper bubble formation. In particular, the choice of $\phi_0 \leq 1$ generally does not lead to positive local explosions with accelerated speed and is thus not considered as suitable for bubble modeling. Also, Theorem 1 is developed under the relatively mild condition $\mathbb{E}(\log^+ |\varepsilon_1|) < \infty$, and as a result it covers the situation when $\mathbb{E}(|\varepsilon_1|) < \infty$ but $\mathbb{E}(\varepsilon_1^2) = \infty$. In such special cases with stronger moment conditions, the proposed model can still continue to generate positive local explosions when $\phi_0 > 1$.

We next consider the existence conditions on moments of y_t. Clearly, when $\mathbb{E}(\varepsilon_t) = 0$, $\mathbb{E}(\varepsilon_t^2) < \infty$, and both ε_t and s_t are independent, then $\mathbb{E}(y_t^2) = \mathbb{E}(\varepsilon_t^2)/(1-p_0\phi_0^2) < \infty$ if $p_0\phi_0^2 < 1$. Fig. 2 plots the strict stationarity region of y_t with $\mathbb{E}(y_t^2) < \infty$. Further, if $\mathbb{E}(\varepsilon_t^3) = 0$, then a simple algebraic calculation gives the kurtosis of y_t:

$$\text{kurtosis}(y_t) = \frac{\{6p_0\phi_0^2 + \text{kurtosis}(\varepsilon_t)(1-p_0\phi_0^2)\}\,(1-p_0\phi_0^2)}{1-p_0\phi_0^4}, \quad \text{if} \quad p_0\phi_0^4 < 1.$$

In particular, when $\varepsilon_t \sim \mathcal{N}(0,1)$, then

$$\text{kurtosis}(y_t) = \frac{3(1-p_0^2\phi_0^4)}{1-p_0\phi_0^4} > 3, \quad \text{if} \quad 0 < p_0\phi_0^4 < 1,$$

which implies that $\{y_t\}$ is heavy-tailed.

3. Quasi-Maximum Likelihood Estimation

Let $\theta_0 = (\phi_0, p_0, \sigma_0^2)'$ be the true parameter with $\sigma_0^2 = \mathbb{E}(\varepsilon_t^2)$. Denote by $\theta = (\phi, p, \sigma^2)'$ be the parameter and by Θ be the parameter space. Assume that the observations $\{y_0, y_1, ..., y_n\}$ are from model (1.1) with the true value θ_0. Clearly, under Assumption 1 below, it follows that $\mathbb{E}(y_t|y_{t-1}) = p\phi|y_{t-1}|$ and $\text{Var}(y_t|y_{t-1}) = p(1-p)\phi^2 y_{t-1}^2 + \sigma^2$. Then the (conditional) log-quasi-likelihood function (omitting a constant) is

$$L_n(\theta) = \sum_{t=1}^{n} \ell_t(\theta) := \sum_{t=1}^{n} \left\{ \log\left[p(1-p)\phi^2 y_{t-1}^2 + \sigma^2\right] + \frac{(y_t - p\phi|y_{t-1}|)^2}{p(1-p)\phi^2 y_{t-1}^2 + \sigma^2} \right\}.$$

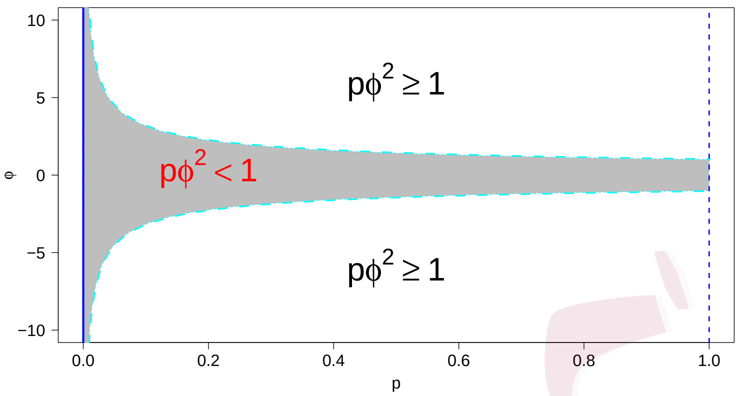

Figure 2: The strict stationarity region $\{(p, \phi) : \phi \in \mathbb{R}, p\phi^2 < 1, 0 \le p < 1\}$ of y_t with finite second moment.

The QMLE of θ_0 is defined as

$$\widehat{\theta}_n = \arg\min_{\theta \in \Theta} L_n(\theta).$$

To study the asymptotics of $\widehat{\theta}_n$, the following assumptions are needed.

Assumption 1. $\{\varepsilon_t\}$ is i.i.d. and independent of i.i.d. binary variables $\{s_t\}$ with $p_0 < 1$. Further, ε_1 has a positive density on $\mathbb{R}$ with zero mean and finite variance.

Assumption 2. The parameter space Θ is a compact subset of $\{\theta = (\phi, p, \sigma^2)' : \phi \neq 0, 0 < p < 1, 0 < \sigma^2 < \infty\}$.

The following two theorems state the strong consistency and the asymptotic normality of $\widehat{\theta}_n$, respectively.

Theorem 2. *If Assumptions 1-2 hold, then* $\widehat{\theta}_n \to \theta_0$ *a.s. as* $n \to \infty$*.*

Theorem 3. *If Assumptions 1-2 hold,* $\mathbb{E}(\varepsilon_t^4) < \infty$*, and* θ_0 *is an interior point of* Θ*, then*

$$\sqrt{n}\big(\widehat{\theta}_n - \theta_0\big) \xrightarrow{d} \mathcal{N}\left(0, \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1}\right), \quad \text{as } n \to \infty,$$

where '$\xrightarrow{d}$*' stands for convergence in distribution,*

$$\mathcal{J} = \mathbb{E}\left\{\frac{\partial^2 \ell_t(\theta_0)}{\partial\theta\partial\theta'}\right\} \quad \text{and} \quad \mathcal{I} = \mathbb{E}\left\{\frac{\partial \ell_t(\theta_0)}{\partial\theta}\frac{\partial \ell_t(\theta_0)}{\partial\theta'}\right\}.$$

Remark 1. The explicit expressions of $\mathcal{I}$ and $\mathcal{J}$ are provided in the Supplementary Material. From their expressions we can see that each element of random matrices within the expectation is bounded and thus it is unnecessary to require moment conditions on y_t for the asymptotics of $\widehat{\theta}_n$. In addition, it is expected that the moment condition $\mathbb{E}(\varepsilon_t^4) < \infty$ can be relaxed via, for example, the quasi-maximum exponential likelihood estimation as considered in Zhu and Ling (2011), for which we shall leave as a future research topic.

Remark 2. In practice, to make statistical inference on θ_0, we must estimate the matrices $\mathcal{I}$ and $\mathcal{J}$. From the proof of Theorem 3, they can be consistently estimated by

$$\widehat{\mathcal{I}}_n = \frac{1}{n}\sum_{t=1}^{n}\frac{\partial \ell_t(\widehat{\theta}_n)}{\partial\theta}\frac{\partial \ell_t(\widehat{\theta}_n)}{\partial\theta'} \quad \text{and} \quad \widehat{\mathcal{J}}_n = \frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2 \ell_t(\widehat{\theta}_n)}{\partial\theta\partial\theta'},$$

respectively. Note that the plug-in method is here invalid since both $\kappa_3 = \mathbb{E}(\varepsilon_t^3)$ and $\kappa_4 = \mathbb{E}(\varepsilon_t^4)$ in matrix $\mathcal{I}$ cannot be estimated from the residuals. Additionally, due to the constraint $p_0 \in (0,1)$, the Delta method may be needed to construct confidence intervals of p_0. If necessary, for example, we can consider the transformation $g(p) = \log[(1-p)/p]$ for $p \in (0,1)$. Note that

$$\sqrt{n}\big(g(\widehat{p}_n) - g(p_0)\big) \xrightarrow{d} \frac{\lambda_p}{p_0(1-p_0)}\mathcal{N}(0,1),$$

if $\sqrt{n}(\widehat{p}_n - p_0) \xrightarrow{d} \mathcal{N}(0, \lambda_p^2)$. Then, for any fixed $\alpha \in (0,1)$, a $100(1-\alpha)\%$ confidence interval of p_0 is $\left[\Big\{1+\exp[g(\widehat{p}_n) - \tilde{z}]\Big\}^{-1}, \Big\{1+\exp[g(\widehat{p}_n) + \tilde{z}]\Big\}^{-1}\right]$, where $\tilde{z} = \lambda_p\, z_{\alpha/2}/\{\sqrt{n}\widehat{p}_n(1-\widehat{p}_n)\}$ with $z_{\alpha/2}$ the lower $\alpha/2$-quantile of the standard normal.

4. Model Diagnostic Checking

Diagnostic checking is important for time series modeling. The most commonly used tool is the portmanteau test, which depends on the autocorrelation of the residuals or the squared residuals, see, e.g., McLeod and Li (1983), Li and Mak (1994), Li (2004), and Chen and Zhu (2015). However, such the portmanteau test fails for the adequacy of model (1.1) since the residuals cannot be obtained. In fact, the residuals should be theoretically calculated by $\widehat{\varepsilon}_t = y_t - s_t\widehat{\phi}_n|y_{t-1}|$ with the initial value y_0 for $i = 1, ..., n$.

Unfortunately, the latent variables $\{s_t : 1 \leq t \leq n\}$ are unknown and prevent us from getting $\{\widehat{\varepsilon}_t\}$.

To check the adequacy of model (1.1), we introduce a new portmanteau test, which is constructed via a transformation of an uncorrelated sequence. Note that the sequence $\{y_t - p_0\phi_0|y_{t-1}| : t \in \mathbb{Z}\}$ is still uncorrelated when $\mathbb{E}(y_t^2) < \infty$ after replacing s_t by its mean p_0 in (1.1). However, substituting this sequence for residual sequence in classic portmanteau test requires an additional assumption $\mathbb{E}(y_t^4) < \infty$ to obtain its asymptotic distribution. To reduce the dependence on the moments of y_t, similar to Ling (2005, 2007), we adopt a self-weight method and then define a new sequence $\{\eta_t\}$ by

$$\begin{aligned}\eta_t :=& \eta_{t,a} = (y_t - p_0\phi_0|y_{t-1}|)I(|y_{t-1}| \leq a) \\ =& \phi_0(s_t - p_0)|y_{t-1}|I(|y_{t-1}| \leq a) + \varepsilon_t I(|y_{t-1}| \leq a), \quad t \in \mathbb{Z},\end{aligned} \tag{4.2}$$

where the constant a is positive and is called a tuning parameter, and $I(\cdot)$ is an indicator function. Clearly, $\{\eta_t\}$ is strictly stationary and ergodic since it is a measurable function of strictly stationary and ergodic sequence $(y_{t-1}, s_t, \varepsilon_t)'$. Further, by the mutually independence among s_t, ε_t, and y_{t-1}, a simple calculation yields that

$$\begin{aligned}&\mathbb{E}(\eta_t) = 0, \qquad \mathbb{E}(\eta_t\eta_{t-k}) = 0, \\ &\sigma_\eta^2 := \mathbb{E}(\eta_t^2) = p_0(1-p_0)\phi_0^2\mathbb{E}\left\{y_1^2 I(|y_1| \leq a)\right\} + \sigma_0^2\mathbb{P}(|y_1| \leq a),\end{aligned} \tag{4.3}$$

for $k \geq 1$. That is, $\{\eta_t\}$ is always a white noise under Assumption 1. Moreover, it is also a martingale difference sequence.

Let $\widehat{\eta}_t = (y_t - \widehat{p}_n\widehat{\phi}_n|y_{t-1}|)I(|y_{t-1}| \leq a)$, $1 \leq t \leq n$. Intuitively, its sample autocorrelation $\widehat{\rho}_{nk}$ should be close to zero if model specification is correct, where

$$\widehat{\rho}_{nk} = \frac{\sum_{t=k+1}^{n}(\widehat{\eta}_t - \bar{\eta})(\widehat{\eta}_{t-k} - \bar{\eta})}{\sum_{t=1}^{n}(\widehat{\eta}_t - \bar{\eta})^2} \quad \text{with} \quad \bar{\eta} = n^{-1}\sum_{t=1}^{n}\widehat{\eta}_t.$$

Denote $\widehat{\boldsymbol{\rho}}_n = (\widehat{\rho}_{n1}, \ldots, \widehat{\rho}_{nM})'$, where $M \geq 1$ is a fixed positive integer. The following theorem gives the limiting distribution of $\widehat{\boldsymbol{\rho}}_n$.

Theorem 4. *Suppose the conditions in Theorem 3 hold. If model (1.1) is correctly specified, then* $\sqrt{n}\widehat{\boldsymbol{\rho}}_n \xrightarrow{d} \mathcal{N}(0, \mathbf{U}\mathbf{G}\mathbf{U}')$, *where* $\mathbf{G} = \mathbb{E}\{v_t v_t'\}$ *with*

$$v_t = \left[\frac{\eta_t\eta_{t-1}}{\sigma_\eta^2}, \ldots, \frac{\eta_t\eta_{t-M}}{\sigma_\eta^2}, \left(-\mathcal{J}^{-1}\frac{\partial \ell_t(\theta_0)}{\partial\theta}\right)'\right]',$$

and $\mathbf{U} = [\mathbf{I}_M, \sigma_\eta^{-2}(u_1, \ldots, u_M)'(p_0, \phi_0, 0)]$ *with* $u_k = -\mathbb{E}\{\eta_{t-k}|y_{t-1}|I(|y_{t-1}| \leq a)\}$ *and* σ_η^2 *being defined in* (4.3).

Based on Theorem 4, our portmanteau test statistic is defined as

$$Q_M = n\widehat{\boldsymbol{\rho}}_n'\left(\widehat{\mathbf{U}}_n\widehat{\mathbf{G}}_n\widehat{\mathbf{U}}_n'\right)^{-1}\widehat{\boldsymbol{\rho}}_n,$$

where $\widehat{\mathbf{U}}_n$ and $\widehat{\mathbf{G}}_n$ are consistently sample counterparts of $\mathbf{U}$ and $\mathbf{G}$, respectively. Under conditions of Theorem 4, we have that $Q_M \xrightarrow{d} \chi_M^2$.

Remark 3. In application, we must choose the tuning parameter a when our test statistic Q_M is used. According to the suggestion in the literature, see, for example, Ling (2005, 2007), we can let a be the 90% or 95% quantile of data $\{|y_1|, ..., |y_n|\}$. Many practical experience shows that this self weight performs well, although it may not be optimal and there exist some other choices. Further, from Section 2, we can see that $\mathbb{E}(y_t^4) < \infty$ if $p_0\phi_0^4 < 1$. In this case, the truncation in (4.2) is not needed and one can simply use the untruncated version with $a = +\infty$.

5. Bubble and Crash Tagging

An important problem in economic or financial data analysis is to tag bubbles and their collapses, which can help the government or financial institutions to alert the abnormal-growth risk or respond timely to resolve a potential financial crisis after a burst. Meanwhile, being able to successfully tag a bubble can also create lucrative trading opportunities. On the other hand, many economic studies can benefit from meaningful tagging of history bubbles and crashes to help understand the economic status and explain the reasoning behind certain economic behaviors in the history. The problem of bubble or crash tagging, however, is often not easy and requires sophisticated statistical modeling and treatment. For this, Phillips and Yu

(2011) considered decomposing the asset price process into a fundamental component determined by expected future dividends and an explosive bubble component, and proposed a recursive testing procedure. Phillips et al. (2015b) modeled the null hypothesis as a random walk with asymptotically negligible drift and studied the limit theory of a dating algorithm for bubble detection; see also Phillips et al. (2015a). We also refer to recent papers by Blasques et al. (2022) and Kurozumi and Skrobotov (2023), and references therein, for additional literature. Existing results on bubble detection, nevertheless, were mainly developed in a nonstationary framework for which the bubble mechanism is not incorporated into the underlying stationary process and treated as a separated period on the timeline.

Compared with the aforementioned results that often concatenate different models at deterministic times to make a nonstationary process, a distinguishable feature of the current paper is to consider bubble tagging using a single stationary model that is able to describe both the bubble and non-bubble periods along with the generation and burst of bubbles. Unlike existing results where bubbles are assumed to persist for an adequate duration to achieve consistent detection, the task of bubble or crash tagging in the current stationary framework can be more challenging as bubbles, especially transient bubbles or bubbles that only last for a very short time, can be easily mixed with large white noise observations. We in the following provide two different tagging methods in the current stationary framework, the first one mainly focuses on the collapse of bubbles and is therefore named crash tagging, while the other aims to detect bubbles as anomalies and is thus named bubble tagging.

5.1 A Residual-Based Method for Crash Tagging

Given model (1.1), we consider the difference

$$r_t := y_t - \phi_0|y_{t-1}| = \begin{cases} \varepsilon_t, & \text{if } s_t = 1, \\ \varepsilon_t - \phi_0|y_{t-1}|, & \text{if } s_t = 0. \end{cases} \tag{5.4}$$

Since $\phi_0 > 1$ is generally assumed in applications with bubbles, intuitively r_t is expected to be smaller at time points when there is no bubble with $s_t = 0$ by a shift from that of the bubble period when $s_t = 1$. Therefore, a natural approach is to tag a crash or more generally $s_t = 0$ if $r_t \leq c_r$ for some threshold c_r.

To provide a theoretical understanding of such a tagging approach, we introduce an auxiliary process $\{z_t\}$, where $z_0 = \epsilon_0$ and

$$z_t = \phi_0|z_{t-1}| + \varepsilon_t, \quad t = 1, 2, \ldots. \tag{5.5}$$

Unlike the full model specified in (1.1) that is stationary for which a bubble can collapse, the process $\{z_t\}$ defined above is a pure bubble process that is explosive and nonstationary. In particular, for any $k \geq 1$, z_k shares the same distribution as y_{t+k} if a bubble forms at time $t+1$ and persists through time $t+k$, which we call a k-th cumulative bubble. This also relates to the excursive period with duration k in financial applications; see for example our data analysis in Section 7. For consistent tagging of bubbles, it is generally required that $k \to \infty$, namely the bubble has to persist for a growing horizon of time; see for example Phillips et al. (2015b) and references therein. The following Proposition 1 provides a theoretical support of the residual-based crash tagging method.

Proposition 1. *For any time t, if the innovation distribution is symmetric, then the conditional probability that the collapse of a k-th cumulative bubble will be correctly tagged by the aforementioned method equals to $\mathbb{P}(z_k \geq -c_r)$, namely the marginal probability that the auxiliary explosive bubble process will exceed the same threshold in the other direction.*

We shall here provide a discussion on the result of Proposition 1. In particular, a threshold of $c_r < 0$ is typically chosen in practice for r_t defined in (5.4), and as a result $-c_r > 0$ will be a positive threshold for z_k. Given the explosive nature of the bubble process $\{z_k\}$, it is expected that $\mathbb{P}(z_k > -c_r) \to 1$ as $k \to \infty$ for any chosen threshold $-c_r > 0$, and as a result the probability that the collapse of a k-th cumulative bubble will be correctly tagged increases to one as $k \to \infty$. This resonates the result of Phillips et al. (2015b) but in very different settings. To be more specific, Phillips et al. (2015b) assumed that the bubble period is a deterministic segment with an increasing number of time points within the whole observation period, while the current setting treats the bubble as an integrated part of an underlying stationary process in (1.1).

In practice, the parameter ϕ_0 in (5.4) is unknown, and we propose to plug in the QMLE and use the residual $\widehat{r}_t = y_t - \widehat{\phi}_n|y_{t-1}|$. In this case, one tags $s_t = 0$ if $\widehat{r}_t < c_r$ for some threshold c_r. We in the following provide a discussion on possible rule of thumb choices of the threshold c_r.

- **Rule 1 (hard threshold).** A natural choice is to set c_r as the $(1-\widehat{p}_n)$-th quantile of $\{\widehat{r}_t\}$, and it can be seen from the simulation results in Section 6 that such a simple choice of c_r seems to perform well and is reasonably robust to different innovation distributions.

In applications where the innovation distribution is believed to be normal with $\varepsilon_t \sim \mathcal{N}(0, \sigma_0^2)$, then additional likelihood-based methods are available for the choice of c_r. In particular, we consider the following approaches that provide time-varying choices of the threshold so that one tags $s_t = 0$ if $\widehat{r}_t < c_{r,t}$ for some threshold $c_{r,t}$ that can depend on time. Following the notation in Section 3, we use $(\widehat{\phi}_n, \widehat{p}_n, \widehat{\sigma}_n^2)$ to denote the QMLE of $(\phi_0, p_0, \sigma_0^2)$.

- **Rule 2 (conditional likelihood)**. Motivated by (5.4) where r_t takes value ε_t or $\varepsilon_t - \phi_0|y_{t-1}|$ depending on whether $s_t = 1$ or $s_t = 0$, we can compare the conditional likelihood of r_t given y_{t-1} to distinguish the two cases. Under the normal innovation distribution, this leads to the data-driven choice of $c_{r,t} = -\widehat{\phi}_n|y_{t-1}|/2$.

- **Rule 3 (time-varying quantile)**. As an alternative to the aforementioned likelihood ratio approach, we may also use the conditional distribution of $r_t|y_{t-1} \sim p_0\mathcal{N}(0,\sigma_0^2) + (1-p_0)\mathcal{N}(-\phi_0|y_{t-1}|,\sigma_0^2)$ to set the threshold $c_{r,t}$ as the conditional $(1-\widehat{p}_n)$-th quantile, i.e.,

$$c_{r,t} = \inf\left\{r \in \mathbb{R} : \widehat{p}_n\Phi\left(r/\widehat{\sigma}_n\right) + (1-\widehat{p}_n)\Phi\big((r + \widehat{\phi}_n|y_{t-1}|)/\widehat{\sigma}_n\big) > 1 - \widehat{p}_n\right\},$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal.

- **Rule 4 (Bayesian)**. We in addition offer a Bayesian approach of choosing the threshold based on the conditional likelihood. In particular, based on the proposed model (1.1) we can tag $s_t = 0$ if $\mathbb{P}(s_t = 1|r_t) < \mathbb{P}(s_t = 0|r_t)$, namely when

$$p_0\psi(r_t/\sigma_0) < (1-p_0)\psi((r_t + \phi_0|y_{t-1}|)/\sigma_0),$$

where $\psi(\cdot)$ denotes the density function of a standard normal. By plugging in the QMLE, we can then tag $s_t = 0$ if

$$\widehat{p}_n \psi\left(\widehat{r}_t/\widehat{\sigma}_n\right) < (1-\widehat{p}_n)\psi\big(\big(\widehat{r}_t + \widehat{\phi}_n|y_{t-1}|\big)/\widehat{\sigma}_n\big).$$

5.2 A Null-Based Method for Bubble Tagging

The method described in Section 5.1 relies on residuals from the one-step ahead recursion specified by model (1.1) to tag the collapse of bubbles. In essence, it treats the explosive bubble alternative as the default and aims at detecting the null of no bubble as an anomaly. We shall here consider its complement which sets the null of no bubble as the baseline and detects the formation of a bubble as an anomaly. To be more specific, when $s_t = 0$ and there is no bubble at time t, we have $y_t = \varepsilon_t$ which forms a stationary white noise sequence. When the bubble starts to form at time t, however, an explosive drift $\phi_0|y_{t-1}|$ will be cumulatively added to the otherwise white noise sequence during the whole bubble period making the observed y_t to cumulatively deviate away from the baseline. Therefore, it becomes natural to tag time t as a bubble if $y_t > c$ for some threshold c.

In contrast to the approach in Section 5.1 which relies exclusively on model (1.1) to compute the residuals $\{\widehat{r}_t\}$, this null-based method directly works on the original observations $\{y_t\}$ and is expected to have a more robust performance when model (1.1) is misspecified. In addition, since y_t

is distributed as a white noise sequence under the null of no bubble, the threshold c can be taken as a uniform constant, which can be a convenient feature that facilitates the decision rule visualization. It can also be more advantageous in situations when bubbles are not prevailing in the observation period. In practical application, the threshold c can be set to the sample quantile of $\{y_t\}$ corresponding to some given quantile level.

Let $\{z_t\}$ be the auxiliary process defined in (5.5), and we in the following provide some theoretical understanding of such a null-based bubble tagging method under the fixed horizon domain.

Proposition 2. *For any time t, the conditional probability that a k-th cumulative bubble will be correctly tagged by the null-based method equals $\mathbb{P}(z_k > c)$, namely the marginal probability that the auxiliary explosive bubble process will exceed the same threshold.*

For bubbles that persist for a growing horizon of time, by the explosive nature of the auxiliary bubble process it is expected that $\mathbb{P}(z_k > c) \to 1$ as $k \to \infty$ for any given threshold c, and as a result the aforementioned null-based bubble tagging method can identify such a persistent bubble with probability tending to one. Phillips et al. (2015b) treated the bubble period as a fixed but unknown deterministic section of the whole observation time, and provided the consistency when the length of the bubble section grows proportionally with the sample size. In contrast, the current paper treats the bubble as an intrinsic feature of a stationary data generating mechanism, which serves as an important step to provide a statistical model to understand the mechanism of an economic phenomenon. We also remark that, unlike the QMLE discussed in Section 3, the aforementioned null-based bubble tagging method and Proposition 2 will continue to hold for situations when the hidden state process $\{s_t\}$ exhibits dependence and forms a stationary or nonstationary process by itself. For example, it can be a stationary Markov chain or a nonstationary Markov chain with time-varying transition matrices. In addition, the proof of Proposition 2 can be readily generalized to handle bubble mechanisms other than the one-step autoregressive recursion specified in (1.1).

We remark that the bubble tagging methods described in Sections 5.1 and 5.2 involve the estimation of certain parameters to determine the tagging region, and as a result they are developed mainly for tagging bubbles and their collapses of a given time series from a retrospective aspect. It is possible, however, to consider an extension to real-time monitoring of bubble formations and collapses, where one can use a set of history data to estimate the underlying model parameters and then perform real-time tagging when new data points arrive. The optimal threshold, however, can be different in the online setting, and we shall leave it as a future research topic.

6. Simulation Studies

To assess the performance of the QMLE of θ_0 and tagging methods in finite samples, we use the sample size $n = 200$, 400, and 800, each with 1000 replications for model (1.1). The error ε_t follows the standard normal, the Laplace, and the standardized Student's t_5 distributions, respectively. Three different true values of $\theta_0 = (\phi_0, p_0, \sigma_0^2)'$ are used, respectively, i.e.,

- Case I: $\theta_0 = (1, 0.9, 1)'$;
- Case II: $\theta_0 = (\sqrt{10/9}, 0.9, 1)'$;
- Case III: $\theta_0 = (1.2, 0.9, 1)'$.

For Case I, since $p_0\phi_0^2 < 1$ we have $\mathbb{E}(y_t^2) < \infty$. In comparison, y_t has an infinite variance in Case III since $p_0\phi_0^2 > 1$. For Case II, θ_0 is on the boundary, i.e., $p_0\phi_0^2 = 1$, which is never considered in the literature.

Table S.1 in the Supplementary Material reports the bias, empirical standard deviation (ESD), and asymptotic standard deviation (ASD) of the QMLE $\widehat{\theta}_n$ for Cases I-III. From the table, we can see that the QMLE performs well irrespective of infinite variance or heavy-tailedness issues.

The biases are small and all the ESDs are close to the corresponding ASDs. To see the overall approximation of the QMLE $\widehat{\phi}_n$, Fig. S.1 in the Supplementary Material displays the histogram of $\sqrt{n}\big(\widehat{\phi}_n - \phi_0\big)$ when the sample size $n = 400$. From the figure, we can see that $\sqrt{n}\big(\widehat{\phi}_n - \phi_0\big)$ is always asymptotically normal irrespective of infinite variance or heavy-tailedness of y_t.

We shall here examine the finite-sample performance of the two tagging methods described in Section 5. For the residual-based tagging method in Section 5.1 with reference rules 1–4 we denote them by RBT_1–RBT_4 respectively in our numerical study, and we abbreviate the null-based tagging method in Section 5.2 as NBT hereafter. For each generated process, let $\{\widehat{s}_t : 1 \leq t \leq n\}$ be the estimated bubble tags and # denote the set cardinality. We consider the following evaluation metrics:

- P: the overall proportion of correct tagging $\#\{t : \widehat{s}_t = s_t\}/n$;
- P0: the proportion of correctly tagged null states $\#\{t : \widehat{s}_t = 0, s_t = 0\}/\#\{t : s_t = 0\}$;
- P1: the proportion of correctly tagged bubbles $\#\{t : \widehat{s}_t = 1, s_t = 1\}/\#\{t : s_t = 1\}$.

The results under normal errors are presented in Table 1 and that for other

Table 1: The values (in percentage) of P, P0, and P1 for RBT_1-RBT_4 and NBT with $\varepsilon_t \sim \mathcal{N}(0,1)$ when $n = 200$.

Method	P	P0	P1	P	P0	P1	P	P0	P1
	$\phi_0 = 1, p_0 = 0.9$			$\phi_0 = \sqrt{10/9}, p_0 = 0.9$			$\phi_0 = 1.2, p_0 = 0.9$		
RBT_1	90.93	53.15	94.73	92.15	59.34	95.37	93.44	65.62	96.12
RBT_2	84.64	21.33	91.25	85.45	25.99	91.67	87.81	38.80	93.01
RBT_3	90.47	51.22	94.48	91.68	57.51	95.11	93.75	67.68	96.25
RBT_4	91.49	56.11	95.04	92.81	62.85	95.73	95.02	74.09	96.99
NBT	87.01	33.20	92.56	87.63	36.48	92.87	89.16	44.62	93.75
	$\phi_0 = 1, p_0 = 0.5$			$\phi_0 = \sqrt{10/9}, p_0 = 0.5$			$\phi_0 = 1.2, p_0 = 0.5$		
RBT_1	66.70	66.79	66.69	67.77	67.87	67.75	69.93	70.05	69.92
RBT_2	68.13	68.24	68.11	69.45	69.56	69.43	72.16	72.32	72.15
RBT_3	68.09	68.20	68.08	69.37	69.48	69.35	71.93	72.09	71.92
RBT_4	68.32	68.43	68.31	69.55	69.66	69.53	72.42	72.58	72.41
NBT	66.85	66.95	66.83	67.97	68.07	67.95	70.00	70.16	69.99

type errors are provided in the Supplementary Material. Each configuration is based on 1000 replications. To provide a fair comparison, we set the thresholds of different tagging methods so their estimated bubble ratios $\#\{t : \widehat{s}_t = 1\}/n$ are controlled at the same level.

From the results, we can observe the followings.

(i) For both the RBT and NBT methods, the results are reasonably close across different error types. This indicates that the bubble tagging methods considered in Sections 5.1 and 5.2 possess a certain degree of robustness with respect to the error distribution.

(ii) For each of the method considered, the performance in general improves when the nonlinear autoregressive coefficient ϕ_0 increases. This is mainly because a larger value of the parameter ϕ_0 in general leads to a stronger degree of explosiveness during the bubble period, making it relatively easier to distinguish between bubbles and null-states. When $p_0 = 0.9$ as in Tables 1 and S.2, the performance of the RBT method can vary depending on which reference rule is used to obtain the threshold. The NBT method, on the other hand, seems to deliver a performance that is between the best and worst performed RBT methods. Note that the RBT method is deigned using the residuals that are more related to the bubble alternative, it meets with our intuition that the RBT method in general outperforms the NBT for most of the threshold choices when the bubble state probability $p_0 = 0.9$ is relatively high.

(iii) When the true underlying bubble state probability p_0 decreases to 0.5 as in Tables 1 and S.3, the bubble state no longer dominates and as a result the difference between the RBT and NBT methods becomes less noticeable and all the methods considered delivered quite similar performance.

7. An empirical example

In this section, we analyze the monthly Hang Seng Index (HSI) from December 1986 to December 2017 with a total of 373 observations. To eliminate the effect of inflation on price, we transform nominal prices into real prices by the consumer price index, which can be obtained from the Federal Reserve Bank of St Louis. Fig. 3 (a) displays the real HSI prices, from which one can see an ascendant linear trend in the time series. Thus, we first subtract such a linear trend from the series. That is, we assume that the HSI real price x_t is decomposed into

$$x_t = b_0 + b_1 t + y_t,$$

where $b_0 + b_1 t$ denotes the linear trend and y_t follows a SNAR model. Note that $b_i, i = 0, 1$ can be seen as unknown parameters and can be estimated jointly. Their estimates are $\widehat{b}_0 = 23.661$ and $\widehat{b}_1 = 0.372$, respectively. The linear time trend is plotted in Fig. 3 (a) by the dotted line and $\{y_t\}$ in Fig. 3 (b). The estimates with standard deviations (SDs) of the SNAR model $\{y_t\}$ are reported in Table 2. All estimates are statistically significant since their corresponding p-values are extremely small which are thus not reported in the table. The estimate of ϕ_0 is larger than one, and its 95% confidence

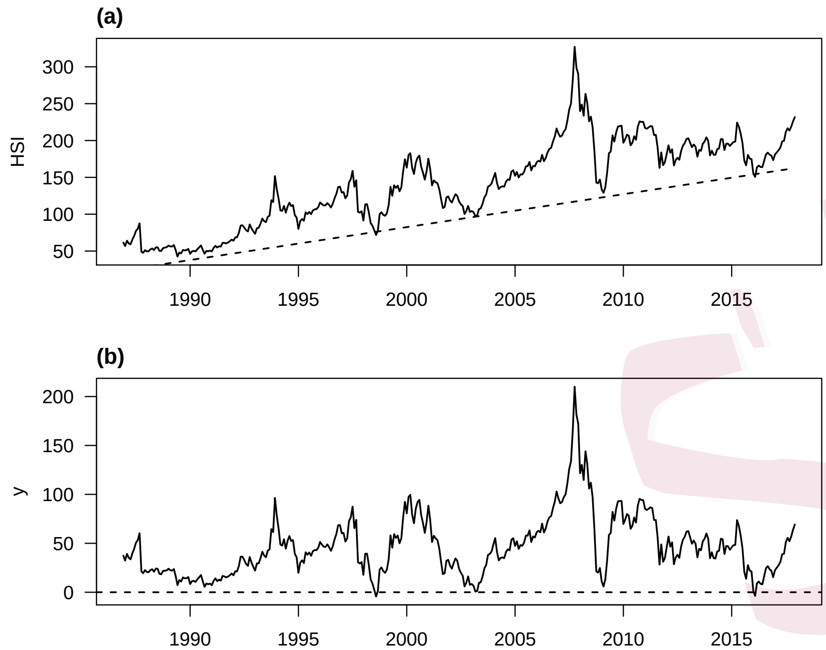

Figure 3: (a) Real HSI prices with the fitted linear trend (the dotted line); (b) $\{y_t\}$ series.

Table 2: The estimate with SDs of the fitted SNAR model.

	ϕ_0	p_0	σ_0^2
Estimate	1.026	0.977	36.314
SD	0.011	0.005	8.649

interval is $(1.005, 1.047)$, conforming to the locally explosive behavior of the series $\{y_t\}$. For the fitting adequacy, we calculate the p-values of the test statistic Q_M with $M = 6, 12, 18$, and 24 when the tuning parameter a is the 90% or 95% quantile of $\{|y_t|, t = 1\ldots, n\}$, respectively. The results are summarized in Table 3, which implies that the fitting is adequate.

Table 3: The p-values of Q_M.

$a\backslash M$	6	12	18	24
90%	0.7099	0.2427	0.3087	0.1549
95%	0.8588	0.7489	0.4898	0.1876

We then apply the tagging methods described in Section 5 to label each time point as either being in a bubble state or being in the null. Since the estimated bubble probability $\widehat{p}_0 = 0.977$ from Table 2 which is very high, in view of the simulation results in Section 6, we shall here consider using the residual-based method in Section 5.1 to tag the collapses of bubbles for the series $\{y_t\}$. In particular, Fig. 4 displays the selected dates of $\widehat{s}_t = 0$ under Rules 1–4. It can be seen from Fig. 4 that the tagging times can vary based on which rule is used, but several important dates are identified simultaneously by at least two rules. Table 4 summarizes such these dates, which coinside with historical financial crises, i.e., the depression started from the Black Monday in 1987, the Asian financial crises in 1997, the global financial turmoil caused by the subprime crisis over 2007-2009, and the Hong Kong stock market plummeting in 2016.

Although the collapse of a bubble can be dated by $\widehat{s}_t = 0$, the emergence and exuberance of a bubble can not be asserted by $\widehat{s}_t = 1$ immediately. Af-

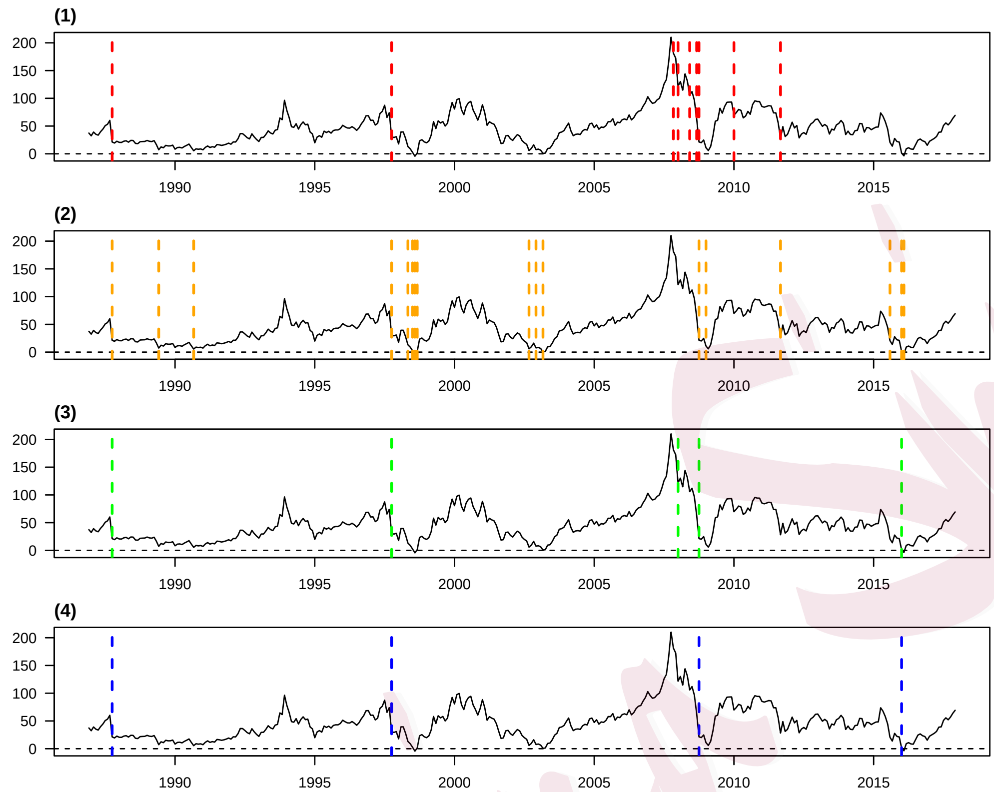

Figure 4: Selected dates of $\widehat{s}_t = 0$ by Rules 1 – 4.

Table 4: The selected important dates of $s_t = 0$.

Date	1987-10	1997-10	2008-01	2008-10	2011-09	2016-01
Rule	{1,2,3,4}	{1,2,3,4}	{1, 3}	{1,2,3,4}	{1,2}	{2,3,4}

ter all, a short-period deviation of the price is reasonable due to the market fluctuations. Of course, a short-period deviation might be regarded as a small bubble in some sense, which bursts quickly by the market adjustment, thus we could pay little attention and ignore them afterwards. What we really need to worry about is the bubble that can trigger tremendous harm, which emerges as the accumulation of long-lasting excursions. Specifically, if $s_t = 0, s_{t+1} = s_{t+2} = \cdots = s_{t+k-1} = 1, s_{t+k} = 0$, then we call it an excursive period that starts from $t+1$ and ends at $t+k$, and define its duration as k. Within an excursive period, the presence of a bubble should be suspected if the duration exceeds some time span, for example, one or two years. For our application, the time span is set to be 18 months. Table 5 summarizes the periods whose durations exceed 18 months, as well as their start and end dates. Fig. 5 plots those periods by gray shadows. We

Table 5: Excursive period with duration exceeding 18 months.

start	end	duration
1987-11	1989-06	20
1990-10	1997-10	85
1998-10	2002-09	48
2003-04	2007-11	56
2010-02	2011-09	20
2011-10	2015-08	47
2016-03	2017-12	22

can see explosive behaviors in most of the periods, indicating the presence and accumulation of bubbles. Note that such periods represent a subset of the bubbles tagged by the method proposed in Section 5 that have persisted for at least 18 months. By Proposition 2 in Section 5.1, the RBT

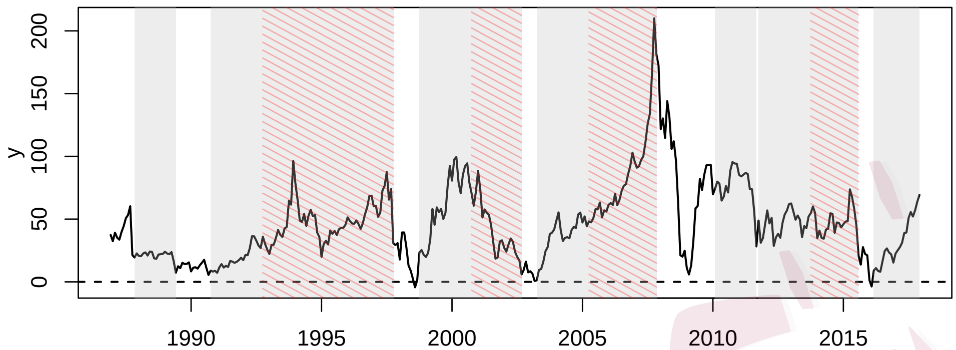

Figure 5: Excursive periods identified by gray shadows; bubbles last over 24 months plotted by the shadow with red backslash.

method is capable of detecting the collapse of an accumulated bubble when its duration $k \to \infty$; see also the same proposition for a probabilistic bound with a finite duration. Another finding is that the magnitude of a bubble is larger as the period lasts longer possibly, for example, the one reaches a value of 210 in October 2007, corresponding to the period from April 2003 to November 2007 with the duration of 56 months. Investors should be alert to such a long-time excursion along with the potential of disastrous bubbles. In the periods where the bubble lasts over 24 months (plotted by the shadow with red backslash in Fig. 5), one should be aware of the false boom in financial markets, and adjust asset allocation to hedge the risk of a potential bubble burst.

Note that certain regions marked in Fig. 5 may exhibit price declines during the bubble period, for which we provide the following discussions. By model (1.1), even when $s_t = 1$ and $\phi_0 > 1$ as in an explosive bubble, it does not exclude the possibility of $y_t < y_{t-1}$ due to the randomness inherited from the innovation. In particular, a price decline of $y_t < y_{t-1}$ with $y_{t-1} > 0$ can be observed during the bubble period within which $s_t = 1$ if the innovation at the time satisfies $\varepsilon_t < -(\phi_0 - 1)|y_{t-1}|$. This may occur more often during the beginning of the bubble accumulation when y_{t-1} is still within a reasonable range or when the coefficient ϕ_0 is close to one. On the other hand, a natural direction to extend model (1.1) is to allow a nonconstant coefficient so that ϕ_0 may change over time to better capture the dynamic behavior of the HSI. For example, during the bubble period when $s_t = 1$, it is possible that the coefficient ϕ_0 may not constantly stay above one in the marked regions in Fig. 5, which results in certain fluctuations and multiple price declines during the bubble periods of the HSI. Although there can be directions to further extend model (1.1) to capture more complicated data-generating mechanisms, the aim of the current paper is to provide a simple model that can better model time series with bubbles than some of the existing models; see also the discussions in Section 1. It can be interesting future research directions to explore possible generalizations of the simple

SNAR model as proposed in the current paper.

8. Conclusions

The paper has introduced a novel stochastic nonlinear autoregressive (SNAR) model to describe the dynamics of economic or financial bubbles within a causal and stationary framework, and discussed its strict stationarity and geometric ergodicity. The paper has further studied the quasi-maximum likelihood estimation of the model and established the asymptotics under minimal assumptions on innovation. Due to the unobservability of the latent variable s_t and the resulting unavailability of the residuals, a new model diagnostic checking tool has been proposed for the adequacy of the fitting. Finally, the paper considers two approaches, one from the residual perspective and the other from the null perspective, for bubble tagging.

Although our new model is useful, the model assumption on the independence between $\{\varepsilon_t\}$ and $\{s_t\}$ seems a little bit stronger from the perspective of empirical pragmatism. To obtain more reasonable interpretation or approximation of the bubble, such an independence assumption can be relaxed. For instance, we can assume that s_t depends on the history of the observed process. Specifically, we can let $\mathbb{P}(s_t = 1|\mathcal{F}_{t-1}) = g(\boldsymbol{\beta}'\mathbf{y}_{t-1})$, where $\mathcal{F}_{t-1} = \sigma(y_{t-j} : j \geq 1)$ be a sigma-field, $\mathbf{y}_{t-1} = (1, y_{t-1}, ..., y_{t-q})'$, and g is a measurable function (e.g. a logistic function). Furthermore, we can also restrict the form of s_t in macroeconomic time series analysis and let $s_t = I(\boldsymbol{\beta}'\mathbf{x}_t > c)$, where $\mathbf{x}_t$ may contain many exogenous macroeconomic variables or indexes and c is a threshold parameter. In addition, it is possible to consider the situation when the hidden state process $\{s_t\}$ exhibits temporal dependence and forms a Markov chain. In this case, the null-based bubble tagging method in Section 5.2 can be more advantageous when bubbles occur in separated but persistent clusters. Moreover, from the empirical study, we find that the extended model with time-varying parameters may enhance practical utility. Another potential topic is to study multivariate SNAR models. We leave these topics for future research.

Supplementary Material

The Supplementary Material contains part of simulation results and all technical proofs of theorems and propositions in the article.

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