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Complete List of Authors	Xiaolin Chen, Ruiyun Zhang, Min-Ge Xie and Xiaodong Yan
Corresponding Authors	Xiaolin Chen
E-mails	xlchen@amss.ac.cn
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Upper- and Lower-quantile Functions with an Emphasis on Discrete Populations

Xiaolin Chen*, Ruiyun Zhang*, Minge Xie and Xiaodong Yan

Qufu Normal University, Beijing University of Technology, Rutgers University, Xian Jiaotong University

Abstract: It is well known that the classical quantile enjoys many desirable properties and has solid practical applications under absolutely continuous populations. In the discrete setting, however, most of these properties fail to hold, such as the consistency and asymptotic normality of the sample classical quantile, which in turn may cause inaccurate or even misleading conclusions in practical inferential problems. In this article, we introduce two new definitions of quantile, called the upper- and lower-quantiles, constructed via linear interpolation technique to eliminate discontinuities of classical quantile function under discrete populations. The sample counterparts of upper- and lower-quantiles enjoy appealing theoretical properties, such as consistency and asymptotic normality, for both discrete and absolutely continuous populations. More importantly, these tools could provide valid inferences for statistical problems involving quantiles in discrete populations, which is firmly supported by both theoretical proofs and

*Xiaolin Chen and Ruiyun Zhang equally contributed to this paper, and they are listed in the alphabetic order.

empirical applications. All theoretical results presented in this article are validated through extensive simulation studies, and potential applications of the new concepts are demonstrated with an illustrative example.

Key words and phrases: Confidence or prediction interval; Discrete population; Upper- and lower-quantiles.

1. Introduction

Consider a random variable X with the cumulative distribution function (CDF), say $F(x)$. The classical quantile function is commonly defined as $Q(\alpha) := \inf\{x : F(x) \geq \alpha\}$ for $0 < \alpha < 1$ (Serfling, 1980). The quantile function can describe the statistical properties of random variable X. It is an alternative way to quantify the property of X, similar to CDF, characteristic function, and others. Throughout this article, we use the terms quantile and quantile function interchangeably: the former highlights the case where the level α is fixed, while the latter emphasizes that the quantile is a function of α.

Given an independent and identically distributed sample $X_1, X_2, \cdots, X_n$ from $F(x)$, the sample classical quantile function is then $\hat{Q}_n(\alpha) := \inf\{x : \hat{F}_n(x) \geq \alpha\}$ for $0 < \alpha < 1$, where $\hat{F}_n(x) = n^{-1}\sum_{i=1}^{n}\mathbf{1}(X_i \leq x)$ is the empirical cumulative distribution function (ECDF). It is well known in the literature that $\hat{Q}_n(\alpha)$ has the properties of consistency and asymptotic normality for an absolutely continuous CDF $F(x)$ that is differentiable and has positive density at the quantile value $Q(\alpha)$ (Serfling, 1980). However, these properties no longer hold for discrete populations. The reason is that the CDF of a discrete random variable is a step function and discontinuous. Using Monte Carlo simulations, Genton et al. (2006) demonstrated that the sample classical median is not asymptotic normal, but a discrete distribution. Jentsch and Leucht (2016) provided an interesting coin flip example in which the authors proved that the sample median is not consistent and its limiting distribution is not a normal distribution, but an equally-weighted two-point discrete distribution.

Data from discrete populations are common in practice, for instance, the numbers of airline accidents, earthquakes, doctor visits, etc. Although much progress has been made (for example, Lindsey 1995; Cameron and Trivedi 2013, and the references therein), issues associated with the sample classical quantile function $\hat{Q}_n(\alpha)$ under discrete populations still remain. These issues have promoted several recent works on quantile under discrete populations. For instance, González-Barrios and Rueda (2001) showed that there exists a subsequence $\hat{Q}_{n_k}(\alpha)$ of $\hat{Q}_n(\alpha)$ such that $\hat{Q}_{n_k}(\alpha)$ converges to $Q(\alpha)$ as n_k goes to infinity and suggested an algorithm to seek out this subsequence. Chen and Lazar (2010) proposed a maximum empirical likelihood estimation through the jittering procedure, which is a continuous correction method by adding noise to the sample. The maximum empirical likelihood estimator of Chen and Lazar (2010) is a consistent estimator of the true classical quantile. However, the result from González-Barrios and Rueda (2001) relies on some highly restrictive assumptions and could not apply freely. The approach by Chen and Lazar (2010) restricts values in the support set of the discrete random variable to be consecutive integers. Furthermore, motivated by the mid-CDF (Parzen, 1997, 2004), Ma et al. (2011) proposed a new definition of mid-quantile and rigorously established the asymptotic properties of sample mid-quantile. It is noted that the asymptotic properties of sample mid-quantile hold for both discrete and absolutely continuous populations, and are similar to the asymptotic results of the sample classical quantile. The mid-quantile further enlightens a smooth quantile function for discrete data by the fractional order statistics (Wang and Hutson, 2011). Because its connection to the classical quantile is not clear, especially for absolutely continuous populations, this smooth quantile function is not widely used. Jentsch and Leucht (2016) investigated the bootstrap method for sample classical quantile and mid-quantile under both independent and identically distributed and τ-dependent samples. Recently, Chernozhukov et al. (2020) introduced an inferential procedure for constructing simultaneous confidence bands of quantile and quantile effect functions by inverting simultaneous confidence bands for CDFs, allowing for potentially discrete populations.

Despite these advances, there are still no alternatives to the classical quantile that not only possess favorable asymptotic properties but also could facilitate to developing reliable statistical inference for problems involving quantiles in discrete populations. In this article, we fill this gap by proposing two new definitions of quantile, called upper-quantile and lower-quantile. When viewed as functions on $(0, 1)$, they are continuous piecewise linear interpolation functions. It is precisely this continuity that ensures they enjoy desired properties. The contributions of our work are as follows. First, we reveal that two new concepts, upper- and lower-quantiles, could afford legitimate inference for statistical inference problems under discrete populations. This thoroughly tackles inferential problem involving quantile under discrete populations and offer new and more appropriate statistical analysis instruments. Second, we rigorously establish the large-sample theoretical results for sample upper- and lower-quantiles under both discrete and absolutely continuous populations. Particularly, under the absolutely continuous populations, the same asymptotic distributions hold for sample upper-quantiles, sample lower-quantiles and sample classical quantile. Therefore, upper- and lower-quantiles are generalizations of the classical quantile. Third, the bootstrap theories for sample upper- and lower-quantiles are investigated. And we strictly establish the asymptotic normality and consistency of the m-out-of-n bootstrap procedure. These results deliver practically proper inferences for the upper- and lower-quantiles themselves. Due to space limit, we have deferred this part to the *Supplementary Material*.

The rest of this article is organized as follows. In Section 2, we point out the motivation of this article by presenting an approach to constructing well-behaved alternative definitions of quantile under discrete populations, and briefly review the mid-quantile that is constructed in this way. In Section 3, we propose the concepts of upper- and lower-quantiles following the motivation in Section 2. The large-sample theoretical properties of sample upper- and lower-quantiles are presented in Section 4. Section 5 contains a set of comprehensive simulation studies and an illustrative example. Our concluding remarks and further discussions are provided in Section 6. The *Supplementary Material* includes the bootstrap procedure, further discussion about upper- and lower-quantiles, the technical proofs, and additional numerical experiments.

2. Motivation and Some Insights

The reason why the classical quantile does not enjoy the similar properties in discrete populations as it does in absolutely continuous populations lies in the fact that the CDF of a discrete random variable is not continuous but rather a step function. Hence, a natural way to refine the quantile is to smooth the CDF for a discrete population, which in turn offers an alternative definition, potentially with desirable theoretical properties. In fact, the mid-quantile (Ma et al., 2011) is derived precisely in this manner.

Nevertheless, as pointed out by one reviewer, this kind of continuous modification alters the original population under study. The alternative definition essentially corresponds to another population rather than the original one, which inevitably leads to a mismatch when we attempt to apply it for inference on the original population. As for this point, we have drawn the following conclusion.

Proposition 1. *We follow the notations in Section 1. Besides, let* $G(x)$ *be a CDF that may or may not be the same as* $F(x)$ *and denote by* $Q_G(\alpha) = \inf\{x : G(x) \geq \alpha\}$ *for* $0 < \alpha < 1$*. Then, the following statements hold:*

(1) *If* F *is identical to* G, $F(Q_G(\alpha)-) \leq \alpha \leq F(Q_G(\alpha))$*; otherwise, this inequality may fail to hold, which means that the ordering between*

$Q(\alpha)$ and $Q_G(\alpha)$ is not necessarily determined.

(2) *If $F(x) \geq G(x)$ for all $x \in \mathscr{R}$, then $F(Q_G(\alpha)) \geq \alpha$, or equivalently $Q(\alpha) \leq Q_G(\alpha)$. Conversely, if $F(x) \leq G(x)$ for all $x \in \mathscr{R}$, then $F(Q_G(\alpha)-) \leq \alpha$, or equivalently $Q(\alpha) \geq Q_G(\alpha)$.*

The proof of Proposition 1 is straightforward and therefore omitted. This proposition evaluates the influence of mismatch between the original CDF F and the CDF G on their quantiles, and also provides hints on how to develop alternative definitions of quantiles that could offer valid inferences for statistical problems involving quantiles in discrete populations.

As an alternative definition of quantile, the mid-quantile (Ma et al., 2011; Jentsch and Leucht, 2016) is developed exactly by continuously modifying the CDF of a discrete random variable and then taking its inverse. Suppose that the support of the discrete random variable X has only finite elements and denote by $x_1 < x_2 < \cdots < x_L$ its all possible distinct values with corresponding probability masses $p_l = \Pr(X = x_l)$, $l = 1, 2, \cdots, L$, where L is a finite positive integer. Furthermore, let $p_l^* = \sum_{i=1}^{l} p_i, l = 1, 2, \cdots, L$ and $p_0^* = 0$. Let $p(x)$ denote the probability mass of X at x. Denote the continuous correction of mid-cumulative distribution function-mid (mid-CDF) by $F_M(x)$ here (Parzen, 1997, 2004; Geraci and Farcomeni, 2022). For $x_1 < x < x_L$, $F_M(x)$ is a piecewise linear function passing through $(x_l, F(x_l) - 0.5p(x_l)), l = 1, \cdots, L$. In addition, $F_M(x) = 0$ for $x < x_1$ and $F_M(x) = 1$ for $x \geq x_L$. Then, the mid-quantile function is defined as $Q_M(\alpha) := \inf\{x : F_M(x) \geq \alpha\}$ for $0 < \alpha < 1$. From the construction of $F_M(x)$, we could obtain the explicit formula for $Q_M(\alpha)$ as follows:

$$Q_M(\alpha) = \begin{cases} x_1, & 0 < \alpha < p_1/2; \\ x_l, & \alpha = p^*_{l-1} + p_l/2, l = 1, 2, \cdots, L; \\ \omega x_l + (1-\omega)x_{l+1}, & \alpha = \omega(p^*_{l-1} + p_l/2) + (1-\omega)(p^*_l + p_{l+1}/2), \\ & 0 < \omega < 1, \quad l = 1, \cdots, L-1. \\ x_L, & \alpha > p^*_{L-1} + p_L/2. \end{cases}$$

In Figure 1, we present graphic illustrations of $F_M(x)$ and $Q_M(\alpha)$ together with the original CDF $F(x)$ and the classical quantile function $Q(\alpha)$. From this figure, it is easy to see that the relative magnitudes of $F(x)$ and $F_M(x)$ are not fixed and they intersect in the middle of consecutive supporting points of X. This fact, together with Proposition 1, indicates that the mid-quantile may not be suitable for making inference on the original population due to the mismatch issue. The following proposition elaborates on this issue from the perspective of prediction interval.

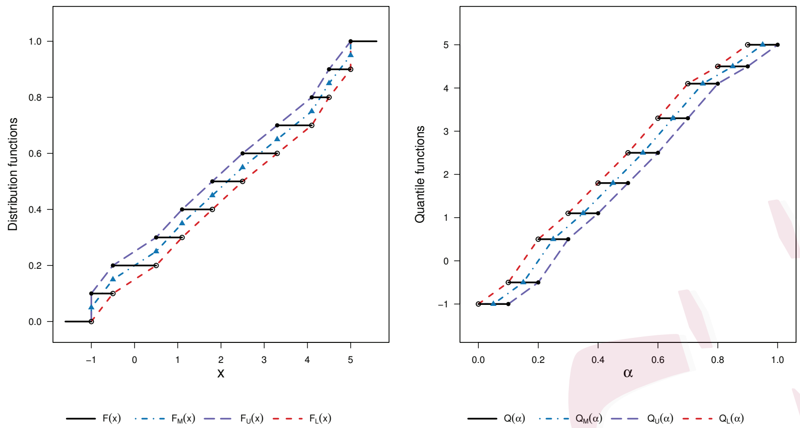

Figure 1: Graphic illustrations of $F(x)$, $F_M(x)$, $F_U(x)$, $F_L(x)$, $Q(\alpha)$, $Q_M(\alpha)$, $Q_U(\alpha)$ and $Q_L(\alpha)$.

Proposition 2. *For $0 < \alpha < 1$, the following results hold:*

(1) *If there exists an integer $0 \le l_1 < L$ such that $p^*_{l_1} \le \alpha \le p^*_{l_1} + p_{l_1+1}/2$, then*

$$\Pr(X < Q_M(\alpha)) \le \alpha.$$

However, if there exists an integer $0 \le l_1 < L - 1$ such that $p^*_{l_1} + p_{l_1+1}/2 < \alpha < p^*_{l_1+1}$, then

$$\Pr(X < Q_M(\alpha)) > \alpha.$$

(2) *If there exists an integer $0 < l_2 \le L$ such that $p^*_{l_2-1} + p_{l_2}/2 \le 1 - \alpha \le p^*_{l_2}$, then*

$$\Pr(X \le Q_M(1 - \alpha)) \ge 1 - \alpha.$$

However, if there exists an integer $1 < l_2 \leq L$ *such that* $p^*_{l_2-1} < 1-\alpha < p^*_{l_2-1} + p_{l_2}/2$, *then*

$$\Pr(X \leq Q_M(1-\alpha)) < 1-\alpha.$$

(3) *If there exist two integers* $0 \leq l_1 < l_2 \leq L$ *such that* $p^*_{l_1} \leq \alpha/2 \leq p^*_{l_1} + p_{l_1+1}/2$ *and* $p^*_{l_2-1} + p_{l_2}/2 \leq 1-\alpha/2 \leq p^*_{l_2}$, *then*

$$\Pr(Q_M(\alpha/2) \leq X \leq Q_M(1-\alpha/2)) \geq 1-\alpha.$$

However, if there exist two integers $0 < l_1 < l_2 < L$ *such that* $p^*_{l_1} + p_{l_1+1}/2 < \alpha/2 < p^*_{l_1+1}$ *and* $p^*_{l_2-1} < 1-\alpha/2 < p^*_{l_2-1} + p_{l_2}/2$, *then*

$$\Pr(Q_M(\alpha/2) \leq X \leq Q_M(1-\alpha/2)) < 1-\alpha.$$

This proposition explicitly identifies the circumstances in which mid-quantile could be used to construct a prediction interval for the original population with guaranteed coverage probability and in which it could not.

3. Upper- and Lower-Quantiles

Inspired by Proposition 1, we propose two new definitions of quantile, called upper- and lower-quantiles by eliminating discontinuities of the classical quantile function under discrete populations. The new definitions are intended to provide valid inference tools for discrete populations. These two terms are associated with two alternative definitions of CDF, say, upper-cumulative distribution function (upper-CDF) and lower-cumulative distribution function (lower-CDF), two continuous corrections of the CDF.

We define the upper-CDF, denoted by $F_U(x)$, as a piecewise linear function passing through $(x_l, F(x_l)), l = 1, \cdots, L$. Let $F_U(x) = 0$ when $x < x_1$ and $F_U(x) = 1$ when $x \geq x_L$. Similarly, we define the lower-CDF, denoted by $F_L(x)$, as a piecewise linear function passing through $(x_l, F(x_l-)), l = 1, \cdots, L$, where $F(x-)$ is the left-hand limit of $F(\cdot)$ at x. Let $F_L(x) = 0$ when $x < x_1$ and $F_L(x) = 1$ when $x \geq x_L$. Then, for $0 < \alpha < 1$, the upper- and lower-quantiles are defined respectively as

$$Q_U(\alpha) := \inf\{x : F_U(x) \geq \alpha\}$$

and

$$Q_L(\alpha) := \inf\{x : F_L(x) \geq \alpha\}.$$

From the definitions of $F_U(x)$, $F_L(x)$, $Q_U(\alpha)$ and $Q_L(\alpha)$, it is easy to see that

$$Q_U(\alpha) = \begin{cases} x_1, & 0 < \alpha < p_1^*; \\ x_l, & \alpha = p_l^*, l = 1, 2, \cdots, L; \\ \omega x_l + (1-\omega) x_{l+1}, & p_l^* < \alpha = \omega p_l^* + (1-\omega) p_{l+1}^* < p_{l+1}^*, 0 < \omega < 1, \end{cases}$$

and

$$Q_L(\alpha) = \begin{cases} x_l, & \alpha = p^*_{l-1}, l = 2, \cdots, L; \\ \omega x_l + (1-\omega)x_{l+1}, & p^*_{l-1} < \alpha = \omega p^*_{l-1} + (1-\omega)p^*_l < p^*_l, 0 < \omega < 1; \\ x_L, & p^*_{L-1} < \alpha < 1. \end{cases}$$

In Figure 1, we provide the graphic illustrations of the functions $F_U(x)$, $Q_U(\alpha)$, $F_L(x)$ and $Q_L(\alpha)$.

As demonstrated in Proposition 2, prediction intervals constructed from mid-quantiles may fail to attain guaranteed coverage probabilities. The following proposition, however, shows that our upper- and lower-quantiles can always deliver prediction intervals with guaranteed coverage probabilities.

Proposition 3. *For any $0 < \alpha < 1$, it holds that*

(1) $\Pr(X < Q_U(\alpha)) \leq \alpha$;

(2) $\Pr(X \leq Q_L(1-\alpha)) \geq 1-\alpha$;

(3) $\Pr(Q_U(\alpha/2) \leq X \leq Q_L(1-\alpha/2)) \geq 1-\alpha$.

Combining this proposition and the consistency of the sample upper- and lower-quantiles (Theorem 1 in Section 4.1), it can be concluded that a conservative $(1-\alpha)$-level prediction interval could be obtained via the tools of sample upper- and lower-quantiles. However, this result does not hold for either the sample classical quantile or the sample mid-quantile. In the *Supplementary Material*, we conduct some numerical studies to evaluate Propositions 2 and 3; see Example S1 there.

Proposition 3 implies that prediction intervals with guaranteed coverage probabilities could be constructed via upper- and lower-quantiles. This desirable property, nevertheless, comes at the expense of wider intervals. As suggested by one reviewer, we examine in detail the width inflation of prediction intervals constructed using upper- and lower-quantiles compared with those based on mid-quantile, and find that the width inflation depends jointly on the interpolation coefficients and the spacing between some adjacent supporting points. The width inflation is small when the support of discrete random variable is densely distributed, i.e. the gaps between adjacent support points are sufficiently narrow. For space constraints, we defer these findings to the *Supplementary Material*, where it appears as Proposition S1.

4. Sample Upper- and Lower-quantiles and Their Asymptotic Properties

In this section, we provide the sample versions of upper- and lower-quantiles, and systematically investigate their large-sample theoretical properties. These new definitions unify the asymptotic theory of sample quantile under discrete and absolutely continuous populations.

4.1 Sample upper- and lower-quantiles

Suppose that $X_1, X_2, \cdots, X_n$ are n independent and identically distributed copies of X. For $l = 1, 2, \cdots, L$, let $\hat{p}_{l,n} = r_{l,n}/n$ with $r_{l,n}$ being the frequency of $X_1, X_2, \cdots, X_n$ taking the value x_l, and $\hat{p}^*_{l,n} = \sum_{i=1}^{l} \hat{p}_{i,n}$. In addition, set $\hat{p}^*_{0,n} = 0$. The sample upper- and lower-quantiles are defined as

$$\hat{Q}_{U,n}(\alpha) = \begin{cases} x_1, & 0 < \alpha < \hat{p}_{1,n}; \\ x_l, & \alpha = \hat{p}^*_{l,n}, l = 1, 2, \cdots, L-1; \\ \omega x_l + (1-\omega)x_{l+1}, & \hat{p}^*_{l,n} < \alpha = \omega \hat{p}^*_{l,n} + (1-\omega)\hat{p}^*_{l+1,n} < \hat{p}^*_{l+1,n}, 0 < \omega < 1, \end{cases}$$

and

$$\hat{Q}_{L,n}(\alpha) = \begin{cases} x_l, & \alpha = \hat{p}^*_{l-1,n}, l = 2, \cdots, L; \\ \omega x_l + (1-\omega)x_{l+1}, & \hat{p}^*_{l-1,n} < \alpha = \omega \hat{p}^*_{l-1,n} + (1-\omega)\hat{p}^*_{l,n} < \hat{p}^*_{l,n}, 0 < \omega < 1; \\ x_L, & \hat{p}^*_{L-1,n} < \alpha < 1. \end{cases}$$

4.2 Asymptotics for discrete populations

In this subsection, we present the asymptotic properties of sample upper- and lower-quantiles for the discrete populations.

The following Theorem 1 states that the sample upper- and lower-quantiles are consistent estimators of their population counterparts.

Theorem 1. *For given* $0 < \alpha < 1$*, as* $n \to \infty$*, it holds that* $\hat{Q}_{U,n}(\alpha)$ *converges in probability to* $Q_U(\alpha)$*, while* $\hat{Q}_{L,n}(\alpha)$ *converges in probability to* $Q_L(\alpha)$*.*

This result, together with Proposition 3, could help acquire prediction intervals for X with guaranteed coverage probabilities; this is formalized in the following corollary.

Corollary 1. *Let* $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)^\top$*. For given* $\alpha \in (0,1)$*, if* p_1 *and* p_L *are no larger than* $\alpha/2$*, it holds that*

$$\Pr_{(\boldsymbol{X},X)}\{X \in [\hat{Q}_{U,n}(\alpha/2), \hat{Q}_{L,n}(1-\alpha/2)]\} \geq 1 - \alpha + o(1),$$

as n *goes to infinity.*

The next theorem describes the limiting distributions of the sample upper-quantile.

Theorem 2. *Define* $f_U(x) := F_U'(x) := p_{l+1}/(x_{l+1} - x_l)$ *for* $x_l < x < x_{l+1}$*,* $f_{U,+}(x_l) := F_{U+}'(x_l) := p_{l+1}/(x_{l+1} - x_l)$ *and* $f_{U,-}(x_l) := F_{U-}'(x_l) := p_l/(x_l - x_{l-1})$*. Moreover, define* $f_U(x, x_l) = f_{U,+}(x_l)$ *if* $x > x_l$ *and* $f_U(x, x_l) = f_{U,-}(x_l)$ *if* $x < x_l$*. Besides, denote* $x_0 = x_1$ *and* $r_{0,n} = 0$*. For* $0 < \alpha < 1$*, the results below are available:*

(1) *If* $\alpha < p_1$, *then* $\hat{Q}_{U,n}(\alpha)$ *converges to* x_1 *in probability as* $n \to \infty$.

(2) *If* $\alpha = \omega p_l^* + (1-\omega)p_{l+1}^*$ *for some* $1 \le l \le L-1$ *and* $0 < \omega < 1$, *then*

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_U[Q_U(\alpha)] \longrightarrow N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]$$

in distribution as $n \to \infty$, *where* $Q_U(\alpha) = \omega x_l + (1-\omega)x_{l+1}$.

(3) *If* $\alpha = p_l^*$ *for some* $2 \le l \le L-1$, *then*

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_U[\hat{Q}_{U,n}(\alpha), Q_U(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$, *where* $Q_U(\alpha) = x_l$.

(4) *If* $\alpha = p_1$, *then*

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_{U,+}(x_1) \longrightarrow ZI(Z > 0)$$

in distribution as $n \to \infty$, *where* $Q_U(\alpha) = x_1$ *and* $Z \sim N[0, \alpha(1-\alpha)]$.

It is noted that $F_U(x) = 0$ on $\{x : F_U(x) < p_1\}$. Thus, $Q_U(\alpha)$ is a constant and is identically equal to x_1 for $0 < \alpha < p_1$. This results in the fact that the limiting distribution of $\hat{Q}_{U,n}(\alpha)$ is a degenerated one. The differences among Cases 2–4 of Theorem 2 are caused by the fact that the upper CDF $F_U(x)$ is differential at $Q_U(\alpha) = \omega x_l + (1-\omega)x_{l+1}$ with $0 < \omega < 1$, while it is only differential from one side at $Q_U(\alpha) = x_l$.

The result in Case 2 is very similar to that in Case 3 of Theorem 2 of Ma et al. (2011). For this situation, the asymptotic properties of the sample upper-quantile and sample mid-quantile are both similar to that of the sample classical quantile. The term $\omega^2 p_{l+1} - \omega p_{l+1}$ appearing in the variance of the limiting distribution is negative, which could be interpreted as the efficiency gain. This arises from the fact that the mass function is more concentrated in the discrete population than in the absolutely continuous one.

The result in Case 3 could be alternatively written as

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)] \longrightarrow N_- \left(0, \sigma^2_{U,-}\right) + N_+ \left(0, \sigma^2_{U,+}\right),$$

where $\sigma^2_{U,-} = \alpha(1-\alpha)/f^2_{U,-}(x_l)$ and $\sigma^2_{U,+} = \alpha(1-\alpha)/f^2_{U,+}(x_l)$. Here, the distribution of $N_- \left(0, \sigma^2_{U,-}\right) + N_+ \left(0, \sigma^2_{U,+}\right)$ denotes an absolutely continuous distribution with a density function being the density function of $N\left(0, \sigma^2_{U,-}\right)$ on the negative half-line and the density function of $N\left(0, \sigma^2_{U,+}\right)$ on the positive half-line. This distribution is called a joined half-Gaussian or two-piece normal distribution (John, 1982; Ma et al., 2011). It is also worth noting that the mean of $N_- \left(0, \sigma^2_{U,-}\right) + N_+ \left(0, \sigma^2_{U,+}\right)$ is not equal to 0, but $(\sigma_{U,+} - \sigma_{U,-})/\sqrt{2\pi}$.

The result in Case 4 is the left boundary counterpart of that in Case 3. When $\alpha = p_1$, the negative side component degenerates to zero (since x_1 is the smallest support point), whereas the positive side component remains nondegenerate. Consequently, $\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_{U,+}(x_1)$ converges in distribution to $ZI(Z > 0)$, where $Z \sim N[0, \alpha(1-\alpha)]$. This limiting distribution places a probability mass of 1/2 at zero, and its density on the positive half-line coincides with that of $N[0, \alpha(1-\alpha)]$.

In a very similar manner, we could obtain the limiting distributions of the sample lower-quantile, which are depicted in the following Theorem 3.

Theorem 3. *Define $f_L(x) := F_L'(x) := p_l/(x_{l+1} - x_l)$ for $x_l < x < x_{l+1}$, $f_{L,+}(x_l) := F_{L+}'(x_l) := p_l/(x_{l+1} - x_l)$ and $f_{L,-}(x_l) := F_{L-}'(x_l) := p_{l-1}/(x_l - x_{l-1})$. Moreover, define $f_L(x, x_l) = f_{L,+}(x_l)$ if $x > x_l$ and $f_L(x, x_l) = f_{L,-}(x_l)$ if $x < x_l$. Besides, denote $x_{L+1} = x_L$, $r_{L+1,n} = 0$ and $\hat{p}_{L+1,n} = 0$, $\hat{p}^*_{L+1,n} = \hat{p}^*_{L,n}$. Let $0 < \alpha < 1$.*

(1) *If $\alpha > p^*_{L-1}$, then $\hat{Q}_{L,n}(\alpha)$ converges to x_L in probability as $n \to \infty$.*

(2) *If $\alpha = \omega p^*_{l-1} + (1-\omega)p^*_l$ for some $1 \le l \le L-1$ and $0 < \omega < 1$, then*

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)]f_L[Q_L(\alpha)] \longrightarrow N\big[0, \alpha(1-\alpha) + \omega^2 p_l - \omega p_l\big]$$

in distribution as $n \to \infty$, where $Q_L(\alpha) = \omega x_l + (1-\omega)x_{l+1}$.

(3) *If $\alpha = p^*_{l-1}$ for some $2 \le l \le L-1$, then*

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)]f_L[\hat{Q}_{L,n}(\alpha), Q_L(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$, *where* $Q_L(\alpha) = x_l$.

(4) *If* $\alpha = p^*_{L-1}$, *then*

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)] f_{L,-}(x_L) \longrightarrow Z I(Z < 0)$$

in distribution as $n \to \infty$, *where* $Q_L(\alpha) = x_L$ *and* $Z \sim N[0, \alpha(1-\alpha)]$.

Theorems 2 and 3 are very similar. So we could acquire similar remarks for Theorem 3 as those for Theorem 2. For brevity, we do not present the remarks extensively anymore. Nevertheless, we want to stress that the result in Case 3 could also be formulated as

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)] \longrightarrow N_-\left(0, \sigma^2_{L,-}\right) + N_+\left(0, \sigma^2_{L,+}\right),$$

where $\sigma^2_{L,-} = \alpha(1-\alpha)/f^2_{L,-}(x_l)$ and $\sigma^2_{L,+} = \alpha(1-\alpha)/f^2_{L,+}(x_l)$. And, the mean of $N_-\left(0, \sigma^2_{L,-}\right) + N_+\left(0, \sigma^2_{L,+}\right)$ is $(\sigma_{L,+} - \sigma_{L,-})/\sqrt{2\pi}$.

4.3 Asymptotics for absolutely continuous populations

This subsection offers the asymptotic properties of the sample upper- and lower-quantiles for absolutely continuous populations, that is, the population X is a random variable with absolutely continuous CDF $F(x)$.

It is worth noting that although the upper- and lower-quantiles are defined only for discrete populations, the sample upper- and lower-quantiles do exist for absolutely continuous populations. Without confusion, for the independent and identically distributed sample $X_1, X_2, \cdots, X_n$ from $F(x)$, we still denote by $x_1 < x_2 < \cdots < x_L$ the distinct values of this sample. Moreover, assume that $x_1, x_2, \cdots, x_L$ are realizations of random variables $Z_1 < Z_2 < \cdots < Z_L$. When $L = n$, $Z_1 < Z_2 < \cdots < Z_L$ are the order statistics of $X_1, X_2, \cdots, X_n$. And there are ties in the sample when $L < n$. For $l = 1, \cdots, L$, $r_{l,n}$ is the frequency of $X_1, X_2, \cdots, X_n$ taking value x_l.

As for the sample upper-quantile $\hat{Q}_{U,n}(\alpha)$ defined in Section 4.1, the following theorem could be achieved.

Theorem 4. *For $0 < \alpha < 1$, assume that the αth quantile $Q(\alpha)$ based on the classical definition satisfies $F[Q(\alpha)] = \alpha$.*

(1) *If $F(x)$ is differentiable and $f(x) := F'(x) > 0$ in a neighborhood of $Q(\alpha)$, then*

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q(\alpha)]f[Q(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$.

(2) *Suppose that at $Q(\alpha)$, $F(x)$ has positive one-sided derivatives $f_+[Q(\alpha)] := \lim_{x \to Q(\alpha)^+}\{F(x) - F[Q(\alpha)]\}/[x - Q(\alpha)]$ and $f_-[Q(\alpha)] := \lim_{x \to Q(\alpha)^-}\{F(x) - F[Q(\alpha)]\}/[x - Q(\alpha)]$. If, in a neighborhood of $Q(\alpha)$, $F(x)$ is differ-*

entiable and $f(x) := F^{'}(x) > 0$ *for* $x \neq Q(\alpha)$*, then*

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q(\alpha)]f[\hat{Q}_{U,n}(\alpha), Q(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$*, where*

$$f[\hat{Q}_{U,n}(\alpha), Q(\alpha)] = \begin{cases} f_+[Q(\alpha)], & \hat{Q}_{U,n}(\alpha) > Q(\alpha); \\ f_-[Q(\alpha)], & \hat{Q}_{U,n}(\alpha) \leq Q(\alpha). \end{cases}$$

The first result in Theorem 4 is precisely the same as that of the sample classical quantile under the same conditions. See Serfling (1980) for reference. The second property is more sophisticated and general than the first. In the second case, the limiting distribution is a two-piece normal distribution. Specifically,

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q(\alpha)] \longrightarrow N_- \left(0, \sigma^2_{U,-}\right) + N_+ \left(0, \sigma^2_{U,+}\right),$$

where $\sigma^2_{U,-} = \alpha(1-\alpha)/f^2_-[Q(\alpha)]$ and $\sigma^2_{U,+} = \alpha(1-\alpha)/f^2_+[Q(\alpha)]$. With a bit of abuse, we simply employ the notations $\sigma^2_{U,-}$ and $\sigma^2_{U,+}$ again, which have been used in the remarks of Theorem 2.

In the case of the sample lower-quantile $\hat{Q}_{L,n}(\alpha)$ defined in Section 4.1, in a similar way, the following conclusions have been established.

Theorem 5. *For* $0 < \alpha < 1$*, assume that the* α*th quantile* $Q(\alpha)$ *based on the classical definition satisfies* $F[Q(\alpha)] = \alpha$*.*

(1) *If* $F(x)$ *is differentiable and* $f(x) := F^{'}(x) > 0$ *in a neighborhood of* $Q(\alpha)$*, then*

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q(\alpha)]f[Q(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$*.*

(2) *Suppose that at* $Q(\alpha)$*,* $F(x)$ *has positive one-sided derivatives* $f_+[Q(\alpha)] := \lim_{x \to Q(\alpha)^+}\{F(x) - F[Q(\alpha)]\}/[x - Q(\alpha)]$ *and* $f_-[Q(\alpha)] := \lim_{x \to Q(\alpha)^-}\{F(x) - F[Q(\alpha)]\}/[x - Q(\alpha)]$*. If, in a neighborhood of* $Q(\alpha)$*,* $F(x)$ *is differentiable and* $f(x) := F^{'}(x) > 0$ *for* $x \neq Q(\alpha)$*, then*

$$\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q(\alpha)]f[\hat{Q}_{L,n}(\alpha), Q(\alpha)] \longrightarrow N[0, \alpha(1-\alpha)]$$

in distribution as $n \to \infty$*, where*

$$f[\hat{Q}_{L,n}(\alpha), Q(\alpha)] = \begin{cases} f_+[Q(\alpha)], & \hat{Q}_{L,n}(\alpha) > Q(\alpha); \\ f_-[Q(\alpha)], & \hat{Q}_{L,n}(\alpha) \leq Q(\alpha). \end{cases}$$

The results in Theorems 4 and 5 are analogous. Hence, similar comments could be afforded.

5. Numerical Studies

In this section, we examine the finite-sample performance of sample upper- and lower-quantiles and evaluate the corresponding theoretical results through extensive Monte Carlo simulations. An illustrative example is also afforded to demonstrate the potential applications of upper- and lower-quantiles.

Example 5.1 In this example, data were generated from a discrete uniform distribution with the supporting set $\{-1.5, -1, 0, 0.7, 2\}$. The upper- and lower-quantiles for nine probabilities $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ were examined. We conducted simulation studies based on 1000 replications with the sample size of 500. For these upper- and lower-quantiles, we report the summary results, including theoretical asymptotic upper- and lower-quantiles, means of sample upper- and lower-quantiles, variances of sample upper- and lower-quantiles, asymptotic variances and coverage probabilities of 95% confidence intervals, where the coverage probabilities refer to the proportions of constructed asymptotic confidence intervals that successfully cover the true unknown upper- and lower-quantiles under repeated data generations. It should be noted that the asymptotic upper- and lower-quantiles may differ from the true upper- and lower-quantiles, which will be clarified below.

For the upper-quantiles, the classification according to Theorem 2 is as follows: Case 1 applies to $\alpha = 0.1$; Case 2 covers $\alpha = 0.3, 0.5, 0.7, 0.9$; Case 3 covers $\alpha = 0.4, 0.6, 0.8$; and Case 4 applies to $\alpha = 0.2$. For $\alpha = 0.1$, which is smaller than $p_1 = 0.2$, the true upper-quantiles is -

1.5. The asymptotic variance for upper-quantiles with $\alpha = 0.1$ is zero. Therefore, the asymptotic confidence interval does not exist. For upper-quantiles with $\alpha = 0.3, 0.5, 0.7, 0.9$, the asymptotic variances and asymptotic confidence intervals could be easily obtained based on the asymptotic results described in Case 2 of Theorem 2. Nevertheless, extra caution is required when α takes values of $0.4, 0.6, 0.8$, since the limiting distributions of sample upper-quantiles with these values are two-piece normal distributions. Specifically, the theoretical asymptotic upper-quantile, denoted by $Q^*_U(\alpha)$, is calculated by $Q^*_U(\alpha) = Q_U(\alpha) + (\sigma_{U,+} - \sigma_{U,-})/\sqrt{2\pi n}$, where $\sigma_{U,+}$ and $\sigma_{U,-}$ are defined in the discussion following Theorem 2. Besides, the asymptotic variances are estimated according to the formula $(\hat{\sigma}^2_{U,+} + \hat{\sigma}^2_{U,-})/(2n) - \{(\hat{\sigma}_{U,+} - \hat{\sigma}_{U,-})/\sqrt{2\pi n}\}^2$ with $\hat{\sigma}_{U,+}$ and $\hat{\sigma}_{U,-}$ being the estimators of $\sigma_{U,+}$ and $\sigma_{U,-}$. For $\alpha = 0.4, 0.6, 0.8$, the 95% confidence intervals are constructed with the aid of 0.025th quantile of $N(0, \hat{\sigma}^2_{U,-})$ and 0.975th quantile of $N(0, \hat{\sigma}^2_{U,+})$, and thus it is asymmetric. For $\alpha = 0.2$, the limiting distribution of $\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]$ assigns probability 0.5 to zero, while its density on the positive half-axis coincides with that of $N[0, \sigma^2_{U,+}]$, where $\sigma^2_{U,+} = \alpha(1-\alpha)/f^2_{U,+}(-1.5)$. Accordingly, the theoretical asymptotic upper-quantile and the estimated asymptotic variance are given by $Q^*_U(\alpha) = Q_U(\alpha) + \sigma_{U,+}/\sqrt{2\pi n}$ and $\hat{\sigma}^2_{U,+}/(2n) - \{\hat{\sigma}_{U,+}/\sqrt{2\pi n}\}^2$, respectively. The corresponding 95% confidence interval is $[\hat{Q}_{U,n}(\alpha) - z_{0.95}\hat{\sigma}_{U,+}/\sqrt{n}, \hat{Q}_{U,n}(\alpha)]$, where $z_{0.95}$ is the 0.95th quantile of the standard normal distribution.

For the lower-quantiles, the cases of Theorem 3 are determined by α as follows: $\alpha = 0.9$ falls into Case 1; $\alpha = 0.1, 0.3, 0.5, 0.7$ into Case 2; $\alpha = 0.2, 0.4, 0.6$ into Case 3; and $\alpha = 0.8$ into Case 4. For $\alpha = 0.9$, which is larger than $p_4^* = 0.8$, the true lower-quantile is 2. The asymptotic variance for the corresponding lower-quantile is zero, and the asymptotic confidence interval does not exist. For lower-quantiles with $\alpha = 0.1, 0.3, 0.5, 0.7$, the asymptotic variances and asymptotic confidence intervals could be easily obtained based on the asymptotic results described in Case 2 of Theorem 3. For the lower-quantiles with $\alpha = 0.2, 0.4, 0.6$, the theoretical asymptotic lower-quantile, denoted by $Q_L^*(\alpha)$, is calculated by $Q_L^*(\alpha) = Q_L(\alpha) + (\sigma_{L,+} - \sigma_{L,-})/\sqrt{2\pi n}$, where $\sigma_{L,+}$ and $\sigma_{L,-}$ are defined in the discussion following Theorem 3. Also, in this case, the asymptotic lower-quantiles are different from the true lower-quantiles. Besides, the asymptotic variance is calculated through $(\hat{\sigma}_{L,+}^2 + \hat{\sigma}_{L,-}^2)/(2n) - \{(\hat{\sigma}_{L,+} - \hat{\sigma}_{L,-})/\sqrt{2\pi n}\}^2$ with $\hat{\sigma}_{L,+}$ and $\hat{\sigma}_{L,-}$ being the estimators of $\sigma_{L,+}$ and $\sigma_{L,-}$. The 95% confidence intervals for the lower-quantiles of $\alpha = 0.2, 0.4, 0.6$ are asymmetric and built based on 0.025th quantile of $N(0, \hat{\sigma}_{L,-}^2)$ and 0.975th quantile of $N(0, \hat{\sigma}_{L,+}^2)$. For $\alpha = 0.8$, the limiting distribution of $\sqrt{n}[\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)]$ has a point mass of 0.5

at zero and agrees with $N[0, \sigma^2_{L,-}]$ on the negative half-axis, where $\sigma^2_{L,-} = \alpha(1-\alpha)/f^2_{L,-}(2)$. Then, the theoretical asymptotic lower-quantile and the estimated asymptotic variance are given by $Q^*_L(\alpha) = Q_L(\alpha) - \sigma_{L,-}/\sqrt{2\pi n}$ and $\hat{\sigma}^2_{L,-}/(2n) - \{\hat{\sigma}_{L,-}/\sqrt{2\pi n}\}^2$. Besides, the 95% confidence interval of $Q_{L,n}(\alpha)$ is constructed as $[\hat{Q}_{L,n}(\alpha), \hat{Q}_{L,n}(\alpha) + z_{0.95}\hat{\sigma}_{L,-}/\sqrt{n}]$.

The simulation results, summarized in Table 1, indicate that the theoretical asymptotic upper- and lower-quantiles agree closely with the averages of the sample upper- and lower-quantiles, and that the sample variances are close to the averages of the asymptotic variances. Moreover, the empirical coverage probabilities are very close to the nominal level of 0.95. These findings confirm the correctness of Theorems 2 and 3.

Example 5.2 In this example, we consider the binomial distribution $\mathrm{Bin}(3, 0.5)$, in which the possible distinct values are $0, 1, 2, 3$ with corresponding probabilities being 0.125, 0.375, 0.375, 0.125. Here, we again examine the upper- and lower-quantiles for probabilities $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. For upper-quantiles, $\alpha = 0.1$ corresponds to Case 1 in Theorem 2, $\alpha = 0.5$ is associated with Case 3 in Theorem 2, and the others correspond to Case 2 in Theorem 2. In addition, for lower-quantiles, $\alpha = 0.9$ belongs to Case 1 in Theorem 3, $\alpha = 0.5$ pertains to Case 3 in Theorem 3, and the others fall into Case 2 in Theorem 3.

Table 1: Results for the discrete uniform distribution in Example 5.1.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					upper-quantiles				
$Q_U^*(\alpha)$	-1.50	-1.48	-1.25	-0.98	-0.50	-0.01	0.35	0.72	1.35
ave$(\hat{Q}_{U,n}(\alpha))$	-1.50	-1.48	-1.25	-0.96	-0.50	-0.01	0.35	0.72	1.35
$n$var$(\hat{Q}_{U,n}(\alpha))$	0	0.33	1.06	3.27	5.14	3.85	2.068	3.66	1.88
$n$ave$(\hat{\sigma}^2_{U,n})$	0	0.35	1.05	3.59	5.18	4.67	2.03	4.21	1.78
CP	N/A	0.95	0.95	0.95	0.96	0.95	0.96	0.95	0.94
					lower-quantiles				
$Q_L^*(\alpha)$	-1.25	-0.98	-0.50	-0.01	0.35	0.73	1.35	1.95	2.00
ave$(\hat{Q}_{L,n}(\alpha))$	-1.25	-0.98	-0.50	-0.01	0.35	0.73	1.35	1.95	2.00
$n$var$(\hat{Q}_{L,n}(\alpha))$	0.27	2.14	4.25	3.86	2.52	5.38	7.13	2.16	0
$n$ave$(\hat{\sigma}^2_{L,n})$	0.265	2.407	4.205	4.499	5.540	6.313	7.011	2.354	0
CP	0.94	0.95	0.95	0.95	0.96	0.95	0.96	0.96	N/A

[1] $Q_U^*(\alpha)$ and $Q_L^*(\alpha)$ represents the theoretical asymptotic upper- and lower-quantiles of $\hat{Q}_{U,n}(\alpha)$ and $\hat{Q}_{L,n}(\alpha)$, respectively; ave$(\hat{Q}_{U,n}(\alpha))$ and ave$(\hat{Q}_{L,n}(\alpha))$ stand for the average of sample upper- and lower-quantiles, respectively; $var(\hat{Q}_{U,n}(\alpha))$ and $var(\hat{Q}_{L,n}(\alpha))$ denote the sample variance of upper- and lower-quantiles, respectively; ave$(\hat{\sigma}^2_{U,n})$ and ave$(\hat{\sigma}^2_{L,n})$ mean the average of asymptotic variances of upper- and lower-quantiles, respectively; CP is the 95% coverage probability.

The simulation results are presented in Table 2, the results of which further verify our theoretical results.

Example 5.3 In this example, we evaluate the results in Theorems 4 and 5 through a continuous random variable X with CDF $F(x)$. Specif-

Table 2: Results for the binomial distribution in Example 5.2.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
				upper-quantiles					
$Q_U^*(\alpha)$	0	0.20	0.47	0.73	1.00	1.27	1.53	1.80	2.20
$\text{ave}(\hat{Q}_{U,n}(\alpha))$	0.00	0.20	0.47	0.74	1.00	1.27	1.53	1.80	2.20
$n\text{var}(\hat{Q}_{U,n}(\alpha))$	0.01	0.70	0.85	1.27	1.65	1.27	0.87	0.72	4.10
$n\text{ave}(\hat{\sigma}^2_{U,n})$	0	0.73	0.85	1.22	1.69	1.21	0.85	0.73	4.77
CP	N/A	0.94	0.94	0.95	0.95	0.94	0.95	0.95	0.96
				lower-quantiles					
$Q_L^*(\alpha)$	0.80	1.20	1.47	1.73	2.00	2.27	2.53	2.80	3.00
$\text{ave}(\hat{Q}_{L,n}(\alpha))$	0.81	1.20	1.47	1.74	2.00	2.27	2.53	2.80	3.00
$n\text{var}(\hat{Q}_{L,n}(\alpha))$	4.10	0.70	0.85	1.27	1.65	1.27	0.87	0.72	0.01
$n\text{ave}(\hat{\sigma}^2_{L,n})$	4.91	0.73	0.85	1.22	1.68	1.21	0.85	0.73	0
CP	0.96	0.94	0.94	0.95	0.95	0.94	0.95	0.95	N/A

[1] The meanings of $Q_U^*(\alpha)$, $Q_L^*(\alpha)$, ave$(\cdot)$, var$(\cdot)$, and CP are the same as those in Table 1.

ically, $F(x) = \Phi(x/2)$ if $x \leq 0$, and $F(x) = \Phi(x/3)$ otherwise, where Φ is the CDF of the standard normal distribution. Under this setting, $F(x)$ is a differentiable function except at 0, which is the 0.5th quantile. In fact, $F(x)$ is one-sided differentiable at 0. We again consider upper- and lower-quantiles at $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. For both upper- and lower-quantiles, $\alpha = 0.5$ is in keeping with Case 2 of Theorems 4 and

5, while the other αs belong to Case 1 of these two theorems. Based on 1000 Monte Carlo replications with the sample size 100, we report the same summary results as those in Examples 5.1 and 5.2.

It is worth noting that sample upper- and lower-quantiles have the same asymptotic results, which have been shown in Theorems 4 and 5. Thus, some elements in summary results for both upper- and lower-quantiles are the same, for example, the theoretical asymptotic upper- and lower-quantiles, and asymptotic variances of sample upper- and lower-quantiles. For upper-quantiles with αs except 0.5, the theoretical asymptotic upper-quantiles, denoted by $Q_U^*(\alpha)$, are the same as the classical quantiles, and can be easily calculated by $Q_U^*(\alpha) = F^{-1}(\alpha)$. However, for $\alpha = 0.5$, the theoretical asymptotic upper-quantile is obtained according to $Q_U^*(\alpha) = Q(\alpha) + (\sigma_+ - \sigma_-)/\sqrt{2\pi n}$, where $Q(\alpha) = F^{-1}(\alpha)$, σ_+ and σ_- are defined similarly as those of $\sigma_{U,+}$ and $\sigma_{L,-}$ for discrete populations. For upper-quantiles with αs except 0.5, the asymptotic variances are easily computed by the asymptotic results in Theorems 4 and 5, while the asymptotic variances are $(\sigma_+^2 + \sigma_-^2)/2n - \{(\sigma_+ - \sigma_-)/\sqrt{2\pi n}\}^2$ for $\alpha = 0.5$. In addition, the 95% confidence interval for the 0.5th upper-quantiles is asymmetric, and constructed with the assistance of 0.025th quantile of $N(0, \sigma_-^2)$ and the 0.975th quantile of $N(0, \sigma_+^2)$. The theoretical asymptotic quantile, asymp-

Table 3: Results for the distribution in Example 5.3.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					upper-quantiles				
$Q^*_U(\alpha)$	-2.56	-1.68	-1.05	-0.51	0.05	0.76	1.57	2.53	3.85
ave($\hat{Q}_{U,n}(\alpha)$)	-2.63	-1.73	-1.08	-0.53	0.03	0.74	1.55	2.49	3.77
$n$var($\hat{Q}_{U,n}(\alpha)$)	11.79	8.30	6.92	6.38	9.52	13.86	15.34	17.18	25.06
$n$ave($\hat{\sigma}^2_n$)	11.69	8.17	6.95	6.43	9.96	14.47	15.63	18.37	26.30
CP	0.94	0.95	0.95	0.96	0.94	0.95	0.94	0.95	0.95
					lower-quantiles				
$Q^*_L(\alpha)$	-2.56	-1.68	-1.05	-0.51	0.05	0.76	1.57	2.53	3.85
ave($\hat{Q}_{L,n}(\alpha)$)	-2.52	-1.66	-1.02	-0.48	0.09	0.81	1.63	2.60	3.95
$n$var($\hat{Q}_{L,n}(\alpha)$)	10.94	8.13	6.81	6.32	10.14	13.91	15.77	17.61	26.75
$n$ave($\hat{\sigma}^2_n$)	11.69	8.17	6.95	6.43	9.96	14.47	15.63	18.37	26.300
CP	0.96	0.950	0.96	0.96	0.95	0.95	0.94	0.95	0.94

[1] The meanings of $Q^*_U(\alpha)$, $Q^*_L(\alpha)$, ave$(\cdot)$, var$(\cdot)$, and CP are the same as those in Table 1.

totic variances and 95% confidence interval could be acquired similarly for lower-quantiles.

Table 3 presents the simulation results for this example, which confirm the theoretical results in Theorems 4 and 5.

We next provide an example to illustrate the potential use of upper- and lower-quantiles in modern data analysis. This example is borrowed from Xie and Wang (2026) and slightly modified for different purposes.

Example 5.4 (Confidence intervals for the Binomial parameter) Suppose that X is a Binomial random variable with n trials and success probability $0 < \theta_0 < 1$, denoted by $\mathrm{Bin}(n, \theta_0)$. For the Binomial model, X can be written as $X = \sum_{i=1}^{n} I(U_i \le \theta_0)$, where $U_1, U_2, \cdots, U_n$ are independent and identically distributed copies of $U(0,1)$. We consider the problem of constructing confidence intervals for θ_0.

Let $X(\boldsymbol{U}, \theta) = \sum_{i=1}^{n} I(U_i \le \theta)$, where $\boldsymbol{U} = (U_1, U_2, \cdots, U_n)^\top$ and $0 < \theta < 1$. Then $X(\boldsymbol{U}, \theta)$ follows the Binomial distribution $\mathrm{Bin}(n, \theta)$. For any θ, suppose that $[a_L(\theta), a_U(\theta)]$ is the $(1-\alpha)$-level prediction interval of X for $0 < \alpha < 1$. According to the Repro sampling methodology (Xie and Wang, 2026), $\Gamma_{1-\alpha}(x) = \{\theta : a_L(\theta) \le x \le a_U(\theta)\}$ is a $(1-\alpha)$-level confidence interval for θ, where x is observed value of X.

In our analysis, we adopt three distinct methods to construct prediction interval $[a_L(\theta), a_U(\theta)]$ for fixed θ: the narrowest interval used in Xie and Wang (2026), the mid-quantile-based interval and the interval derived from upper- and lower-quantiles. For the mid-quantile-based approach, $a_L(\theta)$ and $a_U(\theta)$ correspond to the $\alpha/2$ and $1-\alpha/2$ mid-quantiles, respectively; for the upper- and lower-quantile-based method, $a_L(\theta)$ is the $\alpha/2$ upper-quantile, and $a_U(\theta)$ is the $1-\alpha/2$ lower-quantile. In addition, we also implement the equal-tailed Clopper–Pearson (C-P) approach (Clopper and Pearson, 1934)

Table 4: Results of confidence intervals in Example 5.4.

	CP	width	CP	width	CP	width	CP	width
	$n = 20, \theta_0 = 0.15$		$n = 20, \theta_0 = 0.45$		$n = 20, \theta_0 = 0.58$		$n = 20, \theta_0 = 0.80$	
Narrowest	0.975	0.340	0.957	0.419	0.952	0.410	0.960	0.329
Mid	0.935	0.304	0.957	0.411	0.934	0.408	0.960	0.335
C-P	0.975	0.334	0.957	0.444	0.976	0.441	0.980	0.366
UL	0.975	0.334	0.957	0.444	0.976	0.441	0.980	0.366
	$n = 50, \theta_0 = 0.06$		$n = 50, \theta_0 = 0.31$		$n = 50, \theta_0 = 0.54$		$n = 50, \theta_0 = 0.82$	
Narrowest	0.970	0.161	0.954	0.254	0.947	0.275	0.955	0.209
Mid	0.925	0.134	0.951	0.250	0.928	0.270	0.955	0.209
C-P	0.992	0.147	0.969	0.266	0.964	0.286	0.969	0.225
UL	0.992	0.147	0.969	0.266	0.964	0.286	0.969	0.225

for comparison, constructing confidence intervals for θ directly. Both the mid-quantile-based method and the Clopper-Pearson method are suggested by the reviewer for the comparative purpose.

Numerical results based on 5,000 Monte Carlo replications are summarized in Table 4, in which the abbreviations "Narrowest", "Mid", "UL" and "C–P" refer to the narrowest interval method, the mid-quantile-based approach, the upper- and lower-quantile-based method, the Clopper–Pearson method, respectively. This table includes numerical results corresponding to several different values of n and θ_0. The results in Table 4 lead to two main conclusions. First, Repro sampling procedure equipped with upper- and lower-quantiles could provide confidence intervals with guaranteed coverage probabilities, whereas the mid-quantile-based procedure does not always offer this guarantee. The empirical observations derived from the simulation results indirectly corroborate the conclusions of Propositions 2 and 3. Second, the width inflation caused by the upper- and lower-quantile-based approach is marginal relative to both the mid-quantile-based method and the narrowest interval method. It is noted that UL and C-P approaches yield identical numerical results. Further theoretical analysis reveals that confidence intervals constructed via the Clopper–Pearson approach are equivalent to those derived from the upper- and lower-quantile-based method in this specific example. To keep the manuscript concise, we omit the theoretical analysis.

6. Concluding Remarks

The introduction of upper- and lower-quantiles extends the classical quantile and mid-quantile, providing new and useful tools for quantile-based inference, especially for discrete populations. These two concepts are naturally proposed after dealing with the non-invertibility of CDF of a discrete random variable. As one reviewer pointed out, these quantities can be viewed as one-sided or envelope extensions of classical quantile. From

Figure 1, it is easy to see that the classical quantile lies between upper- and lower-quantiles. By contrast, the mid-quantile can be larger or smaller than the classical quantile, and the relative magnitude is unidentifiable in practice, as the true population distribution is always unknown. Therefore, when practitioners want to have a certainty about the exact relationship between the adopted quantile and the classical quantile, they may opt for upper- or lower-quantiles rather than mid-quantile. In addition, for statistical inference problems involving quantiles in discrete populations, upper- and lower-quantiles enable us to provide definitively conservative or liberal inference. For example, upper- and lower-quantiles enable us to get prediction intervals with guaranteed nominal coverage probabilities as demonstrated in Propositions 2, 3 and Corollary 1, while mid-quantile may fail to do so. A mid-p test, which utilizes the mid-quantile, does not guarantee that the type I error probability is equal to or below the nominal level(Fagerland et al., 2015). However, by adopting the framework of upper- and lower-quantiles, practitioners may obtain definitively conservative or liberal test.

The upper- and lower-quantiles not only contribute to the existing research concerning quantile but also open up several future research opportunities. Here, we only name a few concrete potential research proposals.

Because of the vast applications of count data in economics, it is pregnant to generalize the theory of upper- and lower-quantiles to the time series or integer-valued processes. Motivated by the mid-quantile, Wang and Hutson (2011) developed a different concept of quantile based on fractional order statistics. Thus, similar concepts, inspired by upper- and lower-quantiles, could be put forward and studied. Most recently, Geraci and Farcomeni (2022) extended the mid-quantile to the regression context with discrete response and uncovered some valuable insights when analyzing practical problems. Consequently, it is worthwhile to extend the upper- and lower-quantiles to the regression analysis, although this generalization is challenging. Besides these potential research directions in methodology and theory, our upper- and lower-quantiles could also be helpful for modern statistical inferential procedures under the discrete populations, such as Repro samples method (Xie and Wang, 2026; Chen et al., 2024). We have illustrated this point with a simple illustrative example; see Example 5.4 in Section 5. In fact, the proposed upper- and lower-quantiles could be applied in the Repro samples method for more complex and general inference problems under discrete populations.

Supplementary material

The *Supplementary Material* includes the bootstrap procedure, further discussion about upper- and lower-quantiles, the technical proofs, and additional numerical experiments.

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Xiaolin Chen

School of Statistics and Data Science, Qufu Normal University E-mail: xlchen@amss.ac.cn

Ruiyun Zhang

School of Statistics and Data Science, Qufu Normal University, School of Mathematics, Statistics and Mechanics, Beijing University of Technology E-mail: zhangruiyun0921@163.com

Minge Xie

Department of Statistics, Rutgers University E-mail: mxie@stat.rutgers.edu

Xiaodong Yan

School of Mathematics and Statistics, Xian Jiaotong University E-mail: yanxiaodong@xjtu.edu.cn