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Two Sample Tests for Bivariate Heteroscedastic Extremes with a Changing Tail Copula

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Abstract: We study the two-sample test problem on marginal tail features including the extreme value indices and scedasis functions in the presence of non-stationary tail dependence structures. To address this problem, we introduce a unified bootstrap-based framework for bivariate heteroscedastic extremes, where both margins and tail dependence structures are allowed to evolve over time. Our approach is built upon a bivariate sequential tail empirical process, whose weak convergence and bootstrap counterpart are established. Our simulations validate the robustness and efficiency of the bootstrap-based tests. An empirical analysis is conducted on 8 assets, where two different scedasis functions are identified.

Key words and phrases: extreme value analysis; functional limit theorems; heteroscedastic extremes; two-sample test.

1. Introduction

Recent advancements in extreme value theory (EVT) have significantly expanded its scope, moving beyond the classical setting of independent and identically distributed (IID) observations to accommodate independent but non-identically distributed (IND) data. Within the IND framework, substantial progress has been made in modeling univariate heterogeneous extremes. For instance, Einmahl et al. (2014) introduced the *scedasis function* to characterize local variations in tail probabilities across observations. This concept was further developed by He and Einmahl (2024); Einmahl and He (2023), who extended the framework to incorporate more general forms of marginal heterogeneity. Parallel strands of research have addressed related challenges such as non-stationary trends (Mefleh et al., 2020) and serial dependence in extremes (Bücher and Jennessen, 2024). These studies have demonstrated the promising potential for applying heterogeneous extremes in complex modeling and real-world applications.

In many empirical settings, heterogeneity arises not only in marginal tail behavior but also in the extremal dependence structure. Dynamic dependence has been widely documented in financial markets (Erb et al., 1994; Longin and Solnik, 1995; Engle, 2002), oil markets (Aloui et al., 2013), and climate systems (Sarhadi et al., 2016; Xu et al., 2023). These findings underscore the need for statistical frameworks that accommodate non-identically distributed data in both margins and dependence. However, studies of heteroscedastic extremes in bivariate and multivariate settings have only recently gained attention. Einmahl and Zhou (2024) propose a framework that accommodates marginal heteroscedastic extremes but assumes a constant tail copula. Research addressing heterogeneity in the dependence structure is still relatively new. Notably, Drees (2023) develop a testing procedure for detecting changes in the tail copula, assuming stationary marginals.

While recent studies have begun to investigate heterogeneity in both marginal distributions and dependence structures, there remains a lack of valid two-sample testing procedures for marginal tail estimators under changing dependence structure. When the dependence structure is unknown and potentially non-stationary, standard test statistics may fail to exhibit valid asymptotic behavior. For instance, we show that the Kolmogorov–Smirnov statistic proposed by Einmahl and Zhou (2024), which is designed to test the equality of scedasis functions, does not converge to a standard Brownian bridge when the tail copula varies over time. In practice, many applied problems require comparisons of marginal tail behavior, such as testing the equality of extreme value indices or scedasis functions across different populations. For example, Kinsvater et al. (2016) examine GDP growth and identify differing patterns in upper and lower extreme quantiles, while Adrian et al. (2019) report substantial heterogeneity in extreme value indices across 18 rivers impacted by severe summer floods. Given the prevalence of changing dependence in real-world data, these limitations raise a fundamental methodological question: Can we construct valid two-sample tests for marginal tail features without assuming a stable dependence structure?

To study the problem, we model the survival distribution function $S_{n,i}$ of the bivariate sample $(X_i^{(n)}, Y_i^{(n)})$ for $i = 1, \ldots, n$ by Sklar's theorem, and assume a (survival) copula $C_{n,i}$ satisfying for the two marginal distributions $F_{n,i}^{(j)}$, $j = 1, 2$, such that

$$S_{n,i}(x, y) = C_{n,i}(1 - F_{n,i}^{(1)}(x), 1 - F_{n,i}^{(2)}(y)), \quad (x, y) \in \mathbb{R}^2. \tag{1.1}$$

Moreover, the two marginal distributions $\{F_{n,i}^{(1)}\}_{i=1}^n$ and $\{F_{n,i}^{(2)}\}_{i=1}^n$ follow univariate heteroscedastic extremes. More specifically, there exists a heavy-tailed distribution function G_j and a *scedasis function* c_j such that for all $i = 1, \ldots, n$ and $n \in \mathbb{N}$,

$$\lim_{t \to \infty} \frac{1 - F_{n,i}^{(j)}(t)}{1 - G_j(t)} = c_j\left(\frac{i}{n}\right), \quad j = 1, 2, \tag{1.2}$$

where c_j is positive and continuous subject to the constraint $\int_0^1 c_j(s)ds = 1$

for $j = 1, 2$. $C_j(z) = \int_0^z c_j(s)ds$ is called the *integrated scedasis function*. Without an explicit statement in our paper, the two scedasis functions are not assumed to be identical. For the changing copula structure, we assume a positive function R on $\mathbb{R}^3$ satisfying for all $i = 1, \ldots, n$ and $n \in \mathbb{N}$,

$$\lim_{t\to\infty} tC_{n,i}\left(\frac{x}{t}, \frac{y}{t}\right) = R\left(x, y, \frac{i}{n}\right), \quad (x, y) \in \mathbb{R}^2. \tag{1.3}$$

Here, $R(x, y, z)$ is a tail copula given any z and captures the changing tail dependence across samples, which generalizes the common tail copula in Einmahl and Zhou (2024). We refer to this model as *Bivariate Heteroscedastic Extremes*, which extends the concept of heteroscedastic extremes to incorporate a changing tail dependence structure. More detailed assumptions and discussions about this model are provided in Section 2.

Building on the proposed model, we develop a bootstrap-based framework for conducting two-sample tests under Bivariate Heteroscedastic Extremes. Bootstrap methods are widely used in many problems to bypass the direct estimation of complex variance structures and have significant theoretical value. Recent developments have introduced bootstrap procedures for EVT under both independent and dependent settings, covering estimators such as probability weighted moments (PWM) (de Haan and Zhou, 2024), Hill estimators (Jentsch and Kulik, 2021), and tail copulas (Bücher and Dette, 2013). For instance, Bücher and Dette (2013) apply transformations to the empirical process to enable valid bootstrap inference for tail copulas in IID settings. Similarly, our analysis identifies key statistics as functionals of the *bivariate sequential tail empirical process* (B-STEP), including estimators of the integrated scedasis functions and the marginal EVIs. We prove the weak convergence of the B-STEP and develop a bootstrap B-STEP that jointly resamples margins and copulas, accommodating both sources of heterogeneity. Using the convergence results of B-STEP, we derive the asymptotic statistical properties of the integrated scedasis functions and Hill estimator, establishing the corresponding asymptotic theorems and the properties of their bootstrap counterparts.

Leveraging these results, we develop consistent bootstrap tests for two hypothesis testing problems: (i) equality of EVIs, and (ii) equality of scedasis functions. For each case, we prove the asymptotic validity of the bootstrap test, showing that the rejection probabilities converge to the significance level as both the sample size and number of bootstrap replications increase. These results enable robust marginal inference in the presence of changing dependence, and address practical scenarios where copula constancy cannot be assumed a priori. To illustrate the practical value of our method, we apply our proposed tests for 8 assets, and reveal two distinct clusters of firms based on their scedasis functions, despite similar tail heaviness across stocks. One group shows structural changes in tail risk following the 2008 financial crisis, while the other remains stable over time.

The rest of this paper is organized as follows. In Section 2, we provide details of the assumptions of Bivariate Heteroscedastic Extremes. In Section 3, we introduce the B-STEP and its bootstrap process. Moreover, we develop estimators as functionals of the B-STEP in this section. We also establish the asymptotic convergence results and then provide joint asymptotic properties of the estimators. In Section 4, we examine three hypothesis tests and provide the simulation results. In Section 5, we provide an empirical study and illustrate the practical value of our method.

2. Bivariate Heteroscedastic Extremes

In this section, we introduce the assumptions for the bivariate heteroscedastic extremes model,

$$\begin{cases} X_i^{(n)} \sim F_{n,i}^{(1)}, \quad Y_i^{(n)} \sim F_{n,i}^{(2)}, \text{where } F_{n,i}^{(j)} \text{ satisfies (1.2) for } j=1,2, \\ \left(1-F_{n,i}^{(1)}(X_i^{(n)}), 1-F_{n,i}^{(2)}(Y_i^{(n)})\right) \sim C_{n,i}, \text{with } C_{n,i} \text{ satisfying (1.3).} \end{cases} \tag{2.4}$$

Assumption 1 characterizes the tail behavior of the marginal distributions $F_{n,i}^{(j)}$. These conditions are consistent with those established in Einmahl et al. (2014) for univariate distributions.

Assumption 1 (Marginal Heteroscedastic Extremes)**.** For $j = 1, 2$, the scedasis function $c_j(s)$ is positive, continuous, and bounded away from 0 on $[0, 1]$, satisfying $C_j(1) = 1$. Moreover, there exist a positive, eventually decreasing function A_j with $\lim_{t\to\infty} A_j(t) = 0$, and a distribution function G_j, such that as $t \to \infty$,

$$\sup_{n\in\mathbb{N}} \max_{1\leq i\leq n} \left| \frac{1 - F_{n,i}^{(j)}(t)}{1 - G_j(t)} - c_j\left(\frac{i}{n}\right) \right| = O\left[A_j \left\{ \frac{1}{1 - G_j(t)} \right\} \right].$$

For the reference function G_j, there exist some $\gamma_j > 0, \beta_j < 0$, an eventually positive or negative function B_j, such that

$$\lim_{t\to\infty} \frac{1}{B_j\left(1/(1 - G_j(t))\right)} \left(\frac{1 - G_j(tx)}{1 - G_j(t)} - x^{-1/\gamma_j} \right) = x^{-1/\gamma_j} \frac{x^{\beta_j/\gamma_j} - 1}{\gamma_j \beta_j}, \quad x > 0\,.$$

Remark 1. The parameter γ_j, also known as the *extreme value index (EVI)*, characterizes the tail heaviness of the marginal distribution. In the IID setting, γ_j determines the shape parameter of the limiting generalized extreme value distribution for sample maxima, and a larger $\gamma_j > 0$ corresponds to a heavier tail, implying higher probabilities of extreme events. In IND settings, Einmahl et al. (2014) shows that, under mild regularity conditions, the Hill estimator remains consistent and asymptotically normal for estimating γ_j. Hence, γ_j serves as a key quantity summarizing the extremal behavior of each marginal process.

Assumption 2 specifies the convergence rate of (1.3). This condition extends the assumptions in Einmahl and Zhou (2024) to accommodate the modeling of a changing copula structure. Specifically, the tail copula R is required to be non-degenerate at each $z \in [0,1]$. For every fixed z, the conditions imposed on the derivatives of R are analogous to those discussed in Bücher and Dette (2013).

Assumption 2 (Tail Dependence Structure)**.** $R(x,y,z)$ is continuous on $\mathbb{R}^3$ satisfying $R(1,1,z) > 0$ for all $z \in [0,1]$. The partial derivatives of R exist and satisfy that

$$\partial R(x,y,z)/\partial x \text{ is continuous on } 0 < x < \infty, 0 \leq y < \infty;$$

$$\partial R(x,y,z)/\partial y \text{ is continuous on } 0 < y < \infty, 0 \leq x < \infty.$$

Moreover, it holds for an $\alpha > 0$ and a constant $T \geq 1$, that as $t \to \infty$,

$$\sup_{n\in\mathbb{N}} \sup_{\substack{0<x,y\leq T \\ i=1,\ldots,n}} \left| tC_{n,i}\left(\frac{x}{t}, \frac{y}{t}\right) - R\left(x, y, \frac{i}{n}\right) \right| = O\left(t^{-\alpha}\right). \tag{2.5}$$

Assumption 3 generalizes the smoothing conditions presented in Einmahl et al. (2014) to the bivariate context. Furthermore, to demonstrate the practicality of these assumptions, we provide two examples that meet both Assumptions 2 and 3.

Assumption 3 (Smoothing Conditions)**.** For the scedasis functions and tail copula functions, it holds for $j=1,2$, that

$$\lim_{n\to\infty}\sup_{|u-v|\le 1/n}\sqrt{k}|c_j(u)-c_j(v)|=0,$$

$$\lim_{n\to\infty}\sup_{|u-v|\le 1/n,\, y\in[0,T].}\sqrt{k}\,|R(1,y,u)-R(1,y,v)|=0,$$

$$\lim_{n\to\infty}\sup_{|u-v|\le 1/n,\, x\in[0,T].}\sqrt{k}\,|R(x,1,u)-R(x,1,v)|=0.$$

Example 1 (Mixture Copula)**.** Denote $C_1(u,v)=(u^{-1}+v^{-1}-1)^{-1}$ as a Clayton copula, and $C_2(u,v)=u+v-1+(1-u)(1-v)/(1-uv)$ as a Ali-Mikhail-Haq copula. A changing (survival) copula is given by

$$C_{n,i}(u,v):=p(i/n)C_1(u,v)+(1-p(i/n))C_2(u,v),\quad (u,v)\in[0,1]^2,$$

where $0<p(z)\le 1$ is a function satisfying $\lim_{n\to\infty}\sqrt{k}\sup_{|u-v|\le 1/n}|p(u)-p(v)|=0$. The function R is defined as $R(x,y,z)=p(z)(x^{-1}+y^{-1})^{-1}$.

We then prove that the mixture copula $C_{n,i}$ satisfies Assumption 2. For sufficiently large $t>T$, we observe that

$$\begin{aligned}
&\sup_n \sup_{\substack{0\le x,y\le T\\ 1\le i\le n}}\left|tC_{n,i}(xt^{-1},yt^{-1})-p(i/n)(x^{-1}+y^{-1})^{-1}\right|\\
\le&\sup_n \sup_{\substack{0\le x,y\le T\\ 1\le i\le n}}(1-p(i/n))\left|x+y-t+\frac{t(1-xt^{-1})(1-yt^{-1})}{1-xyt^{-2}}\right|\\
&+\sup_n \sup_{\substack{0\le x,y\le T\\ 1\le i\le n}}\left|\frac{tp(i/n)}{tx^{-1}+ty^{-1}-1}-\frac{p(i/n)}{x^{-1}+y^{-1}}\right|\\
\le&\sup_{0\le x,y\le T}\left|\frac{xy^2+x^2y-2xyt}{xy-t^2}\right|+\left|\frac{1}{x^{-1}+y^{-1}-t(x^{-1}+y^{-1})^2}\right|
\end{aligned}$$

$$\leq \frac{2T^2 t}{t^2 - T^2} + \frac{T^2/4}{t - T/2}$$

$$= O(1/t).$$

Furthermore, Assumption 3 is satisfied if the function p meets the condition:

$$\lim_{n\to\infty} \sqrt{k} \sup_{|u-v|\leq 1/n} |p(u) - p(v)| = 0.$$

In this example, the mixture probability $p(z)$ and the tail copula $(x, y) \mapsto (x^{-1} + y^{-1})^{-1}$ of the Clayton copula govern the tail dependence structure.

Example 2 (Parameterized Copula Sequence)**.** Consider the copula

$$C_{n,i}(u, v) = \left(u^{-\theta(i/n)} + v^{-\theta(i/n)} - 1\right)^{-1/\theta(i/n)}.$$

If θ is Lipschitz continuous, and if $M_1 = \inf_{0\leq z\leq 1} \theta(z) > 0$ and $M_2 = \sup_{0\leq z\leq 1} \theta(z) < \infty$, then the function R is defined as $R(x, y, z) = (x^{-\theta(z)} + y^{-\theta(z)})^{-1/\theta(z)}$.

We first verify that the parameterized copula sequence $C_{n,i}$ satisfies Assumption 2. For $t > T$, the following uniform convergence holds:

$$\sup_{n} \sup_{\substack{0\leq x,y\leq T \\ 1\leq i\leq n}} \left| tC_{n,i}\left(\frac{x}{t}, \frac{y}{t}\right) - \left(x^{-\theta(i/n)} + y^{-\theta(i/n)}\right)^{-1/\theta(i/n)} \right|$$

$$= \sup_{n\in\mathbb{N}} \sup_{\substack{0< x,y\leq T \\ i=1,\dots,n}} \left(x^{-\theta(i/n)} + y^{-\theta(i/n)}\right)^{-1/\theta(i/n)} \frac{t - \left[t^{\theta(i/n)} - \frac{(xy)^{\theta(i/n)}}{x^{\theta(i/n)}+y^{\theta(i/n)}}\right]^{1/\theta(i/n)}}{\left[t^{\theta(i/n)} - \frac{(xy)^{\theta(i/n)}}{x^{\theta(i/n)}+y^{\theta(i/n)}}\right]^{1/\theta(i/n)}}$$

$$\leq T \sup_{n\in\mathbb{N}} \sup_{i=1,\dots,n} \left(1 - \left(1 - \frac{T^{\theta(i/n)}}{2t^{\theta(i/n)}}\right)^{1/\theta(i/n)}\right)$$

$$\leq T \max\left(2^{1-1/M_2}, 1\right) \frac{T^{M_2}}{2M_1 t^{M_1}}.$$

To verify that $C_{n,i}$ satisfies Assumption 3, we derive the derivative of R with respect to θ,

$$\left(y^{-\theta} + x^{-\theta}\right)^{-1/\theta} \left(\frac{\ln\left(y^{-\theta} + x^{-\theta}\right)}{\theta^2} + \frac{(\ln y) \cdot y^{-\theta} + (\ln x) \cdot x^{-\theta}}{\theta\left(y^{-\theta} + x^{-\theta}\right)} \right).$$

Due to the symmetry of the expression in x and y, we set $y = 1$ and obtain

$$\left(1 + x^{-\theta}\right)^{-1/\theta} \left(\frac{\ln\left(1 + x^{-\theta}\right)}{\theta^2} + \frac{\ln(x) \cdot x^{-\theta}}{\theta\left(1 + x^{-\theta}\right)} \right),$$

which is bounded for $x \in [0, 1]$. Thus, by the Lipschitz continuity of $\theta(z)$, Assumption 3 is satisfied. This example corresponds to the 'G-linear' and 't-linear' models used in the simulation study of Drees (2023), where the copula structure is controlled by a changing parameter.

Assumption 4 specifies the conditions for intermediate orders. Notably that we permit distinct intermediate order sequences k_j for $j = 1, 2$.

Assumption 4 (Intermediate Order)**.** The sequences k and k_j satisfy $k/n \to 0$, $k/k_j \to s_j > T^{-1}$, $\sqrt{k}A_j(n/(2Tk)) \to 0$, $\sqrt{k}B_j(n/k) \to 0$, and $\sqrt{k}(n/k)^{-\alpha} \to 0$ as $n \to \infty$ and for $j = 1, 2$.

3. Estimators as Functionals of Tail Processes

This section concentrates on the development of tail estimators and their associated processes under the assumptions in Section 2. We first introduce some necessary notations. Let $X_{k,n}$ denote the k-th smallest order statistic of $X_1^{(n)}, \dots, X_n^{(n)}$, and $Y_{k,n}$ denote the k-th order statistic of $Y_1^{(n)}, \dots, Y_n^{(n)}$. We define $\lfloor x \rfloor := \max\{i \in \mathbb{Z} \mid i \le x\}$, $x \vee y = \max(x, y)$, and $x \wedge y = \min(x, y)$. The inverse of a non-decreasing function f is given by

$$f^{\leftarrow}(x) = \begin{cases} \sup\left\{t \in \mathbb{R}_+ \mid f(t) = 0\right\}, & x = 0, \\ \inf\left\{t \in \mathbb{R}_+ \mid f(t) \ge x\right\}, & 0 < x < \sup \operatorname{ran} f, \\ \inf\left\{t \in \mathbb{R}_+ \mid f(t) = \sup \operatorname{ran} f\right\}, & x \ge \sup \operatorname{ran} f. \end{cases}$$

Specifically, the inverse function of $1/(1 - G_j)$ at t is denoted as $U_j(t)$ for $j = 1, 2$. We introduce a weight function $q(x, y)$ with a constant $0 \le \eta < 1/2$ by

$$q(x, y) = \begin{cases} (x \vee y)^\eta, & \text{if } (x, y) \in \mathbb{R}^2, \\ x^\eta, & \text{if } x \in \mathbb{R},\ y = \infty, \\ y^\eta, & \text{if } y \in \mathbb{R},\ x = \infty. \end{cases} \tag{3.6}$$

For simplicity, we denote $x \mapsto q(x, \infty)$ and $y \mapsto q(\infty, y)$ as $q_1(x)$ and $q_2(y)$, respectively. An important space in our analysis is defined as

$$\begin{aligned} \mathbb{D}_T := &\{(x, y, z) \mid 0 \le z \le 1, 0 \le x, y \le T\} \\ &\cup \{(x, y, z) \mid 0 \le z \le 1, x = \infty, 0 \le y \le T\} \\ &\cup \{(x, y, z) \mid 0 \le z \le 1, y = \infty, 0 \le x \le T\}. \end{aligned}$$

The space $\ell^\infty(\mathbb{D}_T)$ represents the set of bounded functions on $\mathbb{D}_T$.

We use the notation $W_n \rightsquigarrow W$ in $\ell^\infty(\mathbb{D}_T)$ to indicate the weak convergence of the process W_n to a tight process W in the metric space $\ell^\infty(\mathbb{D}_T)$ as $n \to \infty$. For the bootstrap process, we establish the conditional weak convergence, as defined by Kosorok (2003). For an asymptotically measurable process in $\ell^\infty(\mathbb{D}_T)$

$$W_n := W_n\left(X_1^{(n)}, \ldots, X_n^{(n)}, Y_1^{(n)}, \ldots, Y_n^{(n)}, \xi_1, \ldots, \xi_n\right),$$

the conditional weak convergence of W_n to W is defined as

$$\sup_{h \in \mathrm{BL}_1(\ell^\infty(\mathbb{D}_T))} \left|\mathbf{E}_\xi h\left(W_n\right) - \mathbf{E}h(W)\right| = o_{\mathbf{P}}(1), \quad \text{as } n \to \infty, \tag{3.7}$$

where $\mathbf{E}_\xi$ represents the conditional expectation given $(X_1^{(n)}, \ldots, X_n^{(n)}, Y_1^{(n)}, \ldots, Y_n^{(n)})$, and $\mathrm{BL}_1(\ell^\infty(\mathbb{D}_T))$ is the the class of functionals $h : \ell^\infty(\mathbb{D}_T) \to \mathbb{R}$, such that

$$\mathrm{BL}_1(\ell^\infty(\mathbb{D}_T)) = \left\{h \mid \|h\|_\infty \leq 1, |h(f_1) - h(f_2)| \leq \|f_1 - f_2\|, \forall f_1, f_2 \in \ell^\infty(\mathbb{D}_T)\right\}.$$

We denote the conditional weak convergence as $W_n \underset{\xi}{\overset{\mathbb{P}}{\rightsquigarrow}} W$ in $\ell^\infty(\mathbb{D}_T)$. For a review of conditional weak convergence, we refer to Bücher and Kojadinovic (2019) and van der Vaart and Wellner (1996). Equation (3.7) states that the expectation of the unknown distribution $h(W)$ can be approximated by computing the conditional expectation $h(W_n)$.

3.1 Bivariate Sequential Tail Empirical Process

In Einmahl et al. (2014), the STEP process serves as an important theoretical tool, which is defined as:

$$\frac{1}{k}\sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left\{X_i^{(n)} > U_1(n/(kx))\right\} - C_1(z)x. \tag{3.8}$$

A natural extension is to analyze a process whose projection on the two margins is a STEP in the univariate context. Thus, we denote $\tilde{R}'(x,y,z)$

$$\tilde{R}'(x,y,z) := \frac{1}{k}\sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left\{X_i^{(n)} > U_1(n/(kx)), Y_i^{(n)} > U_2(n/(ky))\right\}. \tag{3.9}$$

Suppose $1 - F_{n,i}^{(j)}(t) \approx c_j(i/n)(1-F_j(t))$, the expectation of $\tilde{R}'(x,y,z)$ is,

$$\begin{aligned}\mathbf{E}(\tilde{R}'(x,y,z)) &\approx \frac{1}{n}\frac{n}{k}\sum_{i=1}^{\lfloor nz \rfloor} C_{n,i}\left(c_1\left(\frac{i}{n}\right)\frac{kx}{n}, c_2\left(\frac{i}{n}\right)\frac{ky}{n}\right) \\ &\approx R'(x,y,z) := \int_0^z R(c_1(t)x, c_2(t)y)\,dt, \quad \text{as } n \to \infty.\end{aligned}$$

We then define the B-STEP with the weight function q by

$$\mathbb{F}_n(x,y,z) = \frac{1}{q(x,y)}\left(\tilde{R}'(x,y,z) - R'(x,y,z)\right). \tag{3.10}$$

We now formulate the bootstrap B-STEP. For a fixed index b, let $\{\xi_{bi}\}_{i=1}^n$ be an IID sequence of positive random variables, independent of $\{(X_i^{(n)}, Y_i^{(n)})\}_{i=1}^n$. This sequence is replicated for $b = 1, 2, \ldots, B$. We define the bootstrap empirical distribution functions as

$$F_b^{(1)}(x) = \frac{1}{n}\sum_{i=1}^{n} \xi_{bi}\mathbf{1}\left(X_i^{(n)} \le x\right), \quad \text{and} \quad F_b^{(2)}(y) = \frac{1}{n}\sum_{i=1}^{n} \xi_{bi}\mathbf{1}\left(Y_i^{(n)} \le y\right),$$

and their corresponding generalized inverses $U_b^{(j)} = \left(1/(1-F_b^{(j)})\right)^{\leftarrow}$ for $j = 1, 2$. The bootstrap estimator of (3.9) is given by

$$\tilde{R}^{b\prime}(x, y, z) := \frac{1}{k}\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}\left\{X_i^{(n)} > U_1(n/(kx)), Y_i^{(n)} > U_2(n/(ky))\right\}.$$

The bootstrap B-STEP is then given by

$$\mathbb{F}_n^b(x, y, z) = \frac{1}{q(x, y)}\left(\tilde{R}^{b\prime}(x, y, z) - \tilde{R}'(x, y, z)\right). \tag{3.11}$$

The following assumption is for the bootstrap weights.

Assumption 5 (Bootstrap Weight)**.** The bootstrap weights $\{\xi_{bi}\}_{i=1}^n$ satisfy that for the constant $\eta > 0$ of q in (3.6), we have

$$\mathbf{E}[\xi_{bi}] = 1, \quad \mathbf{E}[(\xi_{bi} - 1)^2] = 1, \quad \text{and} \quad \mathbf{E}[|1 - \xi_{bi}|^{1/\eta}] < \infty.$$

In the following, we assume the process W is a Weiner process on $\mathbb{D}_T$ with covariance function

$$\operatorname{cov}\left(W\left(x_1, y_1, z_1\right), W\left(x_2, y_2, z_2\right)\right) = R'\left(x_1 \wedge x_2, y_1 \wedge y_2, z_1 \wedge z_2\right), \tag{3.12}$$

where $\infty \wedge \infty := \infty$. We now establish the asymptotic property of B-STEP.

Theorem 1. *Under Assumptions 1-5, we have as $n \to \infty$,*

$$\sqrt{k}\mathbb{F}_n \rightsquigarrow W/q \quad \textit{and} \quad \sqrt{k}\mathbb{F}_n^b \underset{\xi_b}{\overset{\mathbb{P}}{\rightsquigarrow}} W/q. \tag{3.13}$$

To prove Theorem 1, we first establish the weak convergence of the *simple bivariate sequential tail empirical process* in Propositions S1 and S2. Then, under Assumptions 1-5, we prove that the ℓ^∞ distance between the B-STEP and the simple B-STEP converges in probability to 0, thereby establishing the weak convergence of the B-STEP. The definition of the simple B-STEP, Propositions S1 and S2, and the detailed proof of Theorem 1 are provided in Section S1 of the supplementary material.

As U_j is typically unknown, empirical quantiles are used to provide a data-driven approximation for tail thresholds. Specifically, we define the empirical version of $\tilde{R}'$ by

$$\hat{R}'(x, y, z) = \frac{1}{k} \sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left(X_i^{(n)} > X_{n-\lfloor kx \rfloor, n}, Y_i^{(n)} > Y_{n-\lfloor ky \rfloor, n} \right). \tag{3.14}$$

The process $\hat{R}'$ can be derived from the process $\tilde{R}'$ by applying a functional Φ and the delta method to the process $\mathbb{F}_n$, such that $\Phi(\tilde{R}'/q) - \Phi\left(R'/q\right) = \hat{R}' - R'$. For a function θ on $\mathbb{D}_T$ satisfying $\theta(0+, \infty, 1) = 0$, $\theta(\infty, 0+, 1) = 0$, where $\theta(x, \infty, 1)$ and $\theta(\infty, y, 1)$ are non-decreasing on $[0, T]$, Φ is given by

$$\Phi(\theta)(x, y, z) = \begin{cases} (q \cdot \theta)\left((q \cdot \theta)^{\leftarrow}(x, \infty, 1), (q \cdot \theta)^{\leftarrow}(\infty, y, 1), z\right) & \text{if } x, y \neq \infty, \\ (q \cdot \theta)\left((q \cdot \theta)^{\leftarrow}(x, \infty, 1), \infty, z\right) & \text{if } y = \infty, \\ (q \cdot \theta)\left(\infty, (q \cdot \theta)^{\leftarrow}(\infty, y, 1), z\right) & \text{if } x = \infty. \end{cases} \tag{3.15}$$

Notice for R', it holds that $R'^{\leftarrow}(x,\infty,1) = x$, and $R'^{\leftarrow}(\infty,y,1) = y$. Moreover, the inverse of $\tilde{R}'(x,\infty,1)$ and $\tilde{R}'(\infty,y,1)$ satisfy that

$$\tilde{R}'^{\leftarrow}(x,\infty,1) = \frac{n}{k}\{1 - G_1(X_{n,n-\lfloor kx \rfloor})\}, \quad \tilde{R}'^{\leftarrow}(\infty,y,1) = \frac{n}{k}\{1 - G_2(Y_{n,n-\lfloor ky \rfloor})\}.$$

Similarly for the bootstrap process $\mathbb{F}_n^b$, we have $\Phi(\tilde{R}^{b\prime}/q) - \Phi(\tilde{R}'/q) = \hat{R}^{b\prime} - \hat{R}'$ on $\mathbb{D}_T$, where $\hat{R}^{b\prime}(x,y,z)$ is defined as

$$\hat{R}^{b\prime}(x,y,z) := \frac{1}{k}\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi} \mathbf{1}\left\{X_i^{(n)} > U_b^{(1)}\left(\frac{n}{kx}\right), Y_i^{(n)} > U_b^{(2)}\left(\frac{n}{ky}\right)\right\}.$$

We then apply the functional delta method to derive the asymptotic result of $\hat{R}'$. We further define W_R as

$$W_R(x,y,z) := W(x,y,z) - R'_1(x,y,z)W(x,\infty,1) - R'_2(x,y,z)W(\infty,y,1),$$

where we denote the partial derivatives of R' as

$$R'_1(x,y,z) = \begin{cases} 0, & \{(x,y,z) \in \mathbb{D}_T \mid x = 0 \text{ or } x = \infty\}, \\ \partial R'(x,y,z)/\partial x, & \{(x,y,z) \in \mathbb{D}_T \mid 0 < x < \infty, 0 \le y \le \infty\}, \end{cases}$$

$$R'_2(x,y,z) = \begin{cases} 0, & \{(x,y,z) \in \mathbb{D}_T \mid y = 0 \text{ or } y = \infty\}, \\ \partial R'(x,y,z)/\partial y, & \{(x,y,z) \in \mathbb{D}_T \mid 0 \le x \le \infty, 0 < y < \infty\}. \end{cases}$$

Theorem 2. *Under Assumptions 1-5, we have as $n \to \infty$*

$$\sqrt{k}\left(\hat{R}' - R'\right) \rightsquigarrow W_R, \quad \textit{and} \quad \sqrt{k}\left(\hat{R}^{b\prime} - \hat{R}'\right) \underset{\xi_b}{\overset{\mathbb{P}}{\rightsquigarrow}} W_R. \tag{3.16}$$

Remark 2. Specifically, we calculate R' on $\mathbb{D}_T$ as:

$$R'(x,y,z) = \begin{cases} \int_0^z R(c_1(t)x, c_2(t)y, t)\,dt, & 0 \le x, y < \infty, 0 \le z \le 1, \\ \int_0^z R(c_1(t)x, \infty, t)\,dt = x\,C_1(z), & y = \infty,\ 0 < x < \infty, 0 \le z \le 1, \\ \int_0^z R(\infty, c_2(t)y, t)\,dt = y\,C_2(z), & x = \infty,\ 0 < y < \infty, 0 \le z \le 1. \end{cases}$$

Since for every fixed z, R is a tail copula, the definition of R' aligns with $\int_0^z R(c_1(t)x, c_2(t)y, t)\,dt$ when extended to $\mathbb{D}_T$.

$R'(x,y,z)$ serves as a key component in the limit theory. For instance, we will see in the later sections that the function is related to the asymptotic covariance of the Hill estimators. The quality also has practical applications in risk modeling. For instance, $R'(x,y,z)$ quantifies the limiting probability that two assets simultaneously exceed intermediate thresholds, $U_1(n/(kx))$ and $U_2(n/(ky))$, during the time interval $[0,z]$ as $n \to \infty$. This characteristic makes $R'(x,y,z)$ a dynamic index of extreme co-movements.

3.2 Estimator for the Integrated Scedasis Functions.

We estimate the integrated scedasis functions $C_j(z)$ for $z \in [0,1]$ by

$$\hat{C}_1(z) := \frac{1}{k_1}\sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left(X_i^{(n)} > X_{n-k_1,n}\right),$$

$$\hat{C}_2(z) := \frac{1}{k_2}\sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left(Y_i^{(n)} > Y_{n-k_2,n}\right).$$

Notice that the intermediate orders for the two margins are different. By plugging $(k_1/k, \infty, z)$ into the process $\hat{R}'$, we have that

$$\hat{R}'(k_1/k, \infty, z) = \frac{1}{k}\sum_{i=1}^{\lfloor nz \rfloor} \mathbf{1}\left(X_i^{(n)} > X_{n-k_1,n}\right) - \frac{k_1}{k} C_1(z).$$

Similarly, by plugging $(k_1/k, \infty, z)$ into the process $\hat{R}^{b\prime}$, we derive $\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}(X_i^{(n)} > U_b^{(1)}(n/k_1))/k_1$, and $\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}(Y_i^{(n)} > U_b^{(2)}(n/k_2))/k_2$. Notice the two processes do not equal 1 when $z = 1$, so we modify and derive the following bootstrap estimators.

$$\hat{C}_1^b(z) := \frac{\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}\left(X_i^{(n)} > U_b^{(1)}(n/k_1)\right)}{\sum_{i=1}^{n} \xi_{bi}\mathbf{1}\left(X_i^{(n)} > U_b^{(1)}(n/k_1)\right)},$$

$$\hat{C}_2^b(z) := \frac{\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}\left(Y_i^{(n)} > U_b^{(2)}(n/k_2)\right)}{\sum_{i=1}^{n} \xi_{bi}\mathbf{1}\left(Y_i^{(n)} > U_b^{(2)}(n/k_2)\right)}.$$

We then establish the asymptotic results for the integrated scedasis function. Define $W_C^{(1)}$ and $W_C^{(2)}$ by $W_C^{(1)}(z) = s_1 W_R(s_1^{-1}, \infty, z)$, and $W_C^{(2)}(z) = s_2 W_R(\infty, s_2^{-1}, z)$.

Theorem 3. *Under Assumptions 1–5, we have for $j = 1, 2$ and as $n \to \infty$,*

$$\sqrt{k}\left(\hat{C}_j - C_j\right) \rightsquigarrow W_C^{(j)} \quad \text{and} \quad \sqrt{k}\left(\hat{C}_j^b - \hat{C}_j\right) \underset{\xi_b}{\overset{\mathbb{P}}{\rightsquigarrow}} W_C^{(j)}. \tag{3.17}$$

3.3 Estimators for the Extreme Value Indices.

The well-known Hill estimators for γ_j are given by

$$\hat{\gamma}_1 = \frac{1}{k_1}\sum_{i=0}^{k_1} \log(X_{n-i,n}) - \log(X_{n-k_1,n}),$$

$$\hat{\gamma}_2 = \frac{1}{k_2}\sum_{i=0}^{k_2} \log(Y_{n-i,n}) - \log(Y_{n-k_2,n}).$$

To study the joint asymptotic properties of $\hat{\gamma}_j$ with other estimators and to propose the bootstrap estimators, it is necessary to rewrite $\hat{\gamma}_j$ as a functional of the tail empirical process. We start by taking $(n(1-G_1(U_1(n/k_1)x^{-\gamma_1}))/k, \infty, 1)$ into the process (3.10) and derive the following tail empirical process

$$\mathbf{F}_{1n}(x) := \frac{1}{q_1(x)k_1}\sum_{i=1}^{n} \mathbf{1}\Big(X_i^{(n)} > x^{-\gamma_1}U_1(n/k_1)\Big).$$

We then define the functional for a non-decreasing function θ on $\mathbb{R}$ with $\theta(0+) = 0$,

$$\Psi(\theta) := \int_0^{(q_1\cdot\theta)^{\leftarrow}(1)} \theta(t)q_1(t)\frac{dt}{t}. \tag{3.18}$$

Notice that for $\Pi(x) = x/q_1(x)$, it holds that $\Psi(\Pi) = 1$. Thus, the Hill estimators can be written as

$$\hat{\gamma}_1 = \gamma_1 \int_0^{\left(X_{n-k_1,n}/U_1(n/k_1)\right)^{-1/\gamma_1}} \frac{1}{k_1}\sum_{i=1}^{n} \mathbf{1}\Big(X_i^{(n)} > s^{-\gamma_1}U_1(n/k_1)\Big)\frac{ds}{s} = \gamma_1\Psi\left(\mathbf{F}_{1n}\right).$$

Based on this functional, the bootstrap Hill estimator for $\hat{\gamma}_1$ is given by

$$\hat{\gamma}_1^b := \gamma_1\,\Psi\left(\mathbf{F}_{1n}^b\right) = \frac{1}{k_1}\sum_{i=1}^{n} \xi_{bi}\left(\log(X_i^{(n)}) - \log(U_b^{(1)}(n/k_1))\right)\mathbf{1}\left(X_i^{(n)} > U_b^{(1)}(n/k_1)\right).$$

with the bootstrap tail empirical process

$$\mathbf{F}_{1n}^b(x) := \frac{1}{q_1(x)k_1}\sum_{i=1}^{n} \xi_{bi}\mathbf{1}\left\{X_i^{(n)} > x^{-\gamma_1}U_1(n/k_1)\right\},$$

Similarly, the bootstrap estimator for $\hat{\gamma}_2$ is given by

$$\hat{\gamma}_2^b := \frac{1}{k_2}\sum_{i=1}^n \xi_{bi}\left(\log(Y_i^{(n)}) - \log(U_b^{(2)}(n/k_2))\right)\mathbf{1}\left(Y_i^{(n)} > U_b^{(2)}(n/k_2)\right).$$

Next, we prove the asymptotic result of the bootstrap Hill estimator. Denote

$$\Gamma_1 := s_1\gamma_1\left(\int_0^1 W\left(ts_1^{-1},\infty,1\right)\frac{dt}{t} - W\left(s_1^{-1},\infty,1\right)\right),$$
$$\Gamma_2 := s_2\gamma_2\left(\int_0^1 W\left(\infty,ts_2^{-1},1\right)\frac{dt}{t} - W\left(\infty,s_2^{-1},1\right)\right).$$

Theorem 4. *Under Assumptions 1-5, we have that for $j = 1,2$ and as $n\to\infty$,*

$$\sqrt{k}(\hat{\gamma}_j - \gamma_j)\rightsquigarrow\Gamma_j, \quad \textit{and} \quad \sqrt{k}(\hat{\gamma}_j^b - \hat{\gamma}_j)\underset{\xi_b}{\overset{\mathbb{P}}{\rightsquigarrow}}\Gamma_j. \tag{3.19}$$

Remark 3. The asymptotic covariance of $(\sqrt{k}(\hat{\gamma}_1 - \gamma_1), \sqrt{k}(\hat{\gamma}_2 - \gamma_2))$ is given by

$$\begin{bmatrix} s_1\gamma_1^2 & R'(s_2,s_1,1)\gamma_1\gamma_2 \\ R'(s_2,s_1,1)\gamma_1\gamma_2 & s_2\gamma_2^2 \end{bmatrix}.$$

In applications, we estimate $R'(s_2,s_1,1)$ by $k^2\hat{R}'(k_1/k,k_2/k,1)/(k_1k_2)$.

4. Bootstrap-Based Tests

In this section, we formally define two hypothesis testing problems and subsequently propose a bootstrap-based approach for conducting these tests.

Following this, we design comprehensive simulation experiments to evaluate the performance of the proposed tests.For each realization of $\{(X_i^{(n)}, Y_i^{(n)})\}_{i=1}^n$, we simulate ξ_b and $\hat{\gamma}_j^b$, $\hat{C}_j'^b$, $\hat{R}^{b\prime}$ for $b = 1, 2, \ldots, B$ as defined in Section 3. We denote the significance level as α in this section.

4.1 Test for Equal Tail Heaviness.

The first test examines whether $\{X_i^{(n)}\}$ and $\{Y_i^{(n)}\}$ exhibit the same tail heaviness without assuming a prior knowledge of the changing dependence structure $C_{n,i}$ and the scedasis functions c_1 and c_2. This hypothesis test is

$$H_{10} : \gamma_1 = \gamma_2 \quad \text{vs.} \quad H_{11} : \gamma_1 \neq \gamma_2. \tag{4.20}$$

To formulate the bootstrap-based test, we define

$$T_{H10} = k\left(\hat{\gamma}_1 - \hat{\gamma}_2\right)^2, \quad T_{H10}^b = k\left(\hat{\gamma}_1^b - \hat{\gamma}_2^b - \hat{\gamma}_1 + \hat{\gamma}_2\right)^2.$$

The empirical distribution of the bootstrap samples is defined as $F_{H10}(x) := \frac{1}{B}\sum_{b=1}^B \mathbf{1}(T_{H10}^b \leq x)$, and its inverse function is defined as $\hat{u}_{10}(\alpha) = F_{H10}^{\leftarrow}(\alpha)$. We reject the null hypothesis if $T_{H10} \geq \hat{u}_{10}(1-\alpha)$.

The bootstrap method can effectively address cases where the two samples exhibit asymptotically tail independence. Given that the covariance between $W(x, \infty, z)$ and $W(\infty, y, z)$ is 0 for $x, y \in [0,1]^2$, the two Hill estimators become asymptotically independent under asymptotically tail independence. Consequently, the functional delta method and continuous mapping theorem remain applicable to the marginal statistics, thereby ensuring the validity of the asymptotic results presented in Theorem 4. This finding indicates the potential effectiveness and conciseness of the bootstrap method as an inference tool under heteroscedastic extremes.

4.2 Test for Equal Scedasis Functions.

The second hypothesis test evaluates whether the two marginal distributions have the same scedasis function, i.e.,

$$H_{20}: c_1 = c_2, \quad \forall\, t \in [0,1] \quad \text{vs.} \quad H_{21}: c_1 \neq c_2, \quad \exists\, t \in [0,1]. \tag{4.21}$$

In financial applications, this test helps to determine whether two assets undergo identical crises. Variations in the scedasis function, which capture the impact of financial crises, play a crucial role in this assessment (Einmahl et al., 2014). Specifically, we define the test statistics,

$$T_{20}(z) = \hat{C}_1(z) - \hat{C}_2(z), \quad T_{20}^b(z) = \hat{C}_1^b(z) - \hat{C}_2^b(z).$$

The KS and Cramér-von Mises (CVM) statistics are defined as

$$\begin{aligned} T_{H20}^{(\mathrm{KS})} &= \sup_{z\in[0,1]} \sqrt{k}\left|T_{20}(z)\right|, & T_{H20}^{b(\mathrm{KS})} &= \sup_{z\in[0,1]} \sqrt{k}\left|T_{20}^b(z) - T_{20}(z)\right|, \\ T_{H20}^{(\mathrm{CVM})} &= k\int_0^1 \left(T_{20}(z)\right)^2 dz, & T_{H20}^{b(\mathrm{CVM})} &= k\int_0^1 \left(T_{20}^b(z) - T_{20}(z)\right)^2 dz, \end{aligned}$$

respectively, with the bootstrap distributions given by

$$F_{H20}^{(\mathrm{KS})}(x) := \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(T_{H20}^{b(\mathrm{KS})} \leq x), \quad F_{H20}^{(\mathrm{CVM})}(x) := \frac{1}{B}\sum_{b=1}^{B}\mathbf{1}(T_{H20}^{b(\mathrm{CVM})} \leq x).$$

The corresponding quantile function is denoted as $\hat{u}_{20}^{(\mathrm{KS})}$, $\hat{u}_{20}^{(\mathrm{CVM})}$. We reject the null hypothesis when $T_{H20}^{(\mathrm{KS})} \geq \hat{u}_{20}^{(\mathrm{KS})}(\alpha)$ and $T_{H20}^{(\mathrm{CVM})} \geq \hat{u}_{20}^{(\mathrm{CVM})}(\alpha)$ for KS and CVM tests, respectively.

Note that when $C_1 = C_2$ and the copula changes across sample, the process $\hat{C}_1 - \hat{C}_2$ does not converge to a Brownian bridge. For example, if we examine the covariance structure of $W_C^{(1)} - W_C^{(2)}$ for $0 \leq z_1 \leq z_2 \leq 1$, by assuming $s_1 = s_2 = 1$, we find that

$$\begin{aligned}
&\operatorname{cov}\left(W_C^{(1)}(z_1) - W_C^{(2)}(z_1), W_C^{(1)}(z_2) - W_C^{(2)}(z_2)\right)\\
=&2C_1(z_1)(1 - C_1(z_2)) + 2\left[C_1(z_2)R'(1,1,z_1) + C_1(z_1)R'(1,1,z_2)\right]\\
&- 2\left[R'(1,1,z_1) + C_1(z_1)C_1(z_2)R'(1,1,1)\right].
\end{aligned}$$

Consequently, the bootstrap method becomes essential for conducting the test. A similar result is also observed in the CVM test for $k\int_0^1 \left(\hat{C}_1(z) - \hat{C}_2(z)\right)^2 dz$.

The asymptotic result for our test statistic is established as follows.

Proposition 1. *Under Assumptions 1-5, as $n \to \infty$ and $B \to \infty$, we have for* $\mathrm{M} \in \{\mathrm{KS}, \mathrm{CVM}\}$,

(a) *If H_{10} holds, then $\mathbf{P}(T_{H10} \geq \hat{u}_{10}(1-\alpha)) \to \alpha$;*

(b) *If* $|\gamma_1 - \gamma_2| = \delta > 0$*, then* $\mathbf{P}(T_{H10} \geq \hat{u}_{10}(1-\alpha)) \to 1$*;*

(c) *If* H_{20} *holds,* $\mathbf{P}(T_{H20}^{(\mathrm{M})} \geq \hat{u}_{20}^{(\mathrm{M})}(1-\alpha)) \to \alpha$*;*

(d) *If there exists a constant* z *such that* $|C_1(z) - C_2(z)| = \delta > 0$*, then* $\mathbf{P}(T_{H20}^{(\mathrm{M})} \geq \hat{u}_{20}^{(\mathrm{M})}(1-\alpha)) \to 1$*.*

4.3 Simulation Results

We now turn to the finite sample performance of the two proposed tests. The simulation study comprises 1000 replications for each setting, with each replicate undergoing 200 bootstrap iterations. We set the sample size $n = 1000$ and the intermediate order $k = 50, 100, 200$. All tests are conducted at $\alpha = 0.05$ and 0.01. The random weights ξ_{bi} are generated from a standard exponential distribution.

The two marginal scedasis functions for $z \in [0,1]$ are chosen from the following options:

$$
\begin{aligned}
&a_1(z) = 0.8 + 0.4z, && a_2(z) = 0.6 + 0.8z,\\
&a_3(z) = 1 + 0.5\sin(2\pi z), && a_4(z) = 1 + 0.2\sin(2\pi z),\\
&a_5(z) = \begin{cases} 0.5 + 2z, & z \in \left[0, \frac{1}{2}\right] \\ 2.5 - 2z, & z \in \left(\frac{1}{2}, 1\right] \end{cases}, && a_6(z) = \begin{cases} 0.25 + 3z, & z \in \left[0, \frac{1}{2}\right] \\ 3.25 - 3z, & z \in \left(\frac{1}{2}, 1\right] \end{cases}.
\end{aligned}
$$

These functions have been utilized in prior research, such as Einmahl and

Zhou (2024) and Drees (2023). Among them, a_1 and a_2 are linear functions, a_3 and a_4 follow sinusoidal patterns, while a_5 and a_6 are piecewise linear functions. For $j = 1, 2, 3$, the even-indexed functions a_{2j} display volatility characteristics akin to their odd-indexed counterparts a_{2j-1}. However, the fluctuation behavior of a_1, a_3, and a_5 differs markedly. In our experiment, we examine how these similarities and differences in scedasis functions impact the two-sample test.

In all cases, the marginal distributions follow a Fréchet distribution such that

$$F_{n,i}^{(j)}(x) = \exp(-c_j(i/n)x^{-\lambda_j}),$$

where c_j is the scedasis function and λ_j is a shape parameter. The Fréchet distribution with shape parameter λ_j has an EVI of $1/\lambda_j$. Since the test statistics of H_{20} is rank-based, the marginal distributions do not influence their results. Therefore, for these tests, we fix $\lambda_j = 2$. For the test assessing the equivalence of EVIs, the values of λ_j are provided in Table 1.

On the other hand, we construct the changing copula by combining the t copula and the Gumbel Copula. Denote the bivariate t-copula as $\mathcal{C}_t(u, v; \nu, \rho)$ with degrees of freedom ν and correlation ρ, and the Gumbel copula with a dependence parameter $\theta \geq 1$ as $\mathcal{C}_g(u, v; \theta)$, which is given by

$$\mathcal{C}_g(u, v; \theta) = \exp\left[-\left\{(-\ln u)^\theta + (-\ln v)^\theta\right\}^{1/\theta}\right], \quad u, v \in \mathbb{R}_+,$$

Based on these two copula families, we define six changing (survival) copula for our tests. For $j = 1, 2, 5$, we define

$$\mathcal{C}_j(x, y, z) = p_j(z) \cdot \mathcal{C}_t(x, y; 2, 0) + (1 - p_j(z)) \cdot \mathcal{C}_g(x, y; 2), \quad x, y \in \mathbb{R}_+, z \in [0, 1],$$

where $p_1(z) = 0.5 + 0.5\cos(2\pi z)$, $p_2(z) = 0.5 + 0.2\cos(2\pi z)$, $p_5 \equiv 0.5$ for $z \in [0, 1]$. These copulas possess the characteristic that the mixture weight $p_j(t)$ changes smoothly across z. For $j = 3, 4, 6$, we define

$$\mathcal{C}_j(x, y, z) = \mathcal{C}_g(x, y; \theta = p_j(z)), \quad x, y \in \mathbb{R}_+, \ z \in [0, 1],$$

where $p_3(z) = 2 + 3z$, $p_4(z) = 2 + z$, and $p_6 \equiv 2$ for $z \in [0, 1]$. These copulas possess the characteristic that the dependence parameter θ of the Gumbel copula varies as a function of z.

Table 1 provides a summary of the simulated rejection frequencies for the test (4.20). The first two columns specify the λ_j; the third and fourth columns represent the scedasis functions selected from a_1, a_3, a_5; the fifth column identifies the changing copula structures. For cases with equivalent EVIs (rows 1-3), the rejection frequencies at $k = 200$ closely match the significance levels (0.05 and 0.10), demonstrating that the test effectively controls Type I errors. When $k = 50$ or $k = 100$, the rejection frequencies are lower than the significance levels. Thus, in our experiment settings, $k = 200$ is necessary to achieve accurate performance. For cases with

Table 1: Simulated rejection frequency for testing H_{10}.

EVIs		Scedasis		Copula	k	50		100		200	
λ_1	λ_2	c_1	c_2		α	0.05	0.1	0.05	0.1	0.05	0.1
2.5	2.5	a_1	a_3	$\mathcal{C}_1$		0.03	0.07	0.04	0.08	0.05	0.10
2	2	a_1	a_5	$\mathcal{C}_2$		0.03	0.07	0.04	0.08	0.05	0.10
3	3	a_3	a_5	$\mathcal{C}_3$		0.02	0.06	0.03	0.07	0.03	0.06
2.5	2	a_1	a_3	$\mathcal{C}_4$		0.34	0.47	0.69	0.79	0.95	1.00
2.5	2.2	a_1	a_5	$\mathcal{C}_1$		0.11	0.19	0.20	0.31	0.38	0.53
2.5	2.4	a_3	a_5	$\mathcal{C}_2$		0.04	0.10	0.06	0.12	0.07	0.13
2.5	2.6	a_1	a_3	$\mathcal{C}_3$		0.04	0.08	0.05	0.09	0.07	0.12
2.5	2.8	a_1	a_5	$\mathcal{C}_4$		0.10	0.19	0.22	0.33	0.41	0.53
2.5	3	a_3	a_5	$\mathcal{C}_1$		0.17	0.28	0.34	0.47	0.64	0.76

differing EVIs (rows 4-9), the rejection frequencies increase with larger k. For example, in row 4 (where $\lambda_1 = 2.5$ and $\lambda_2 = 2$), the rejection frequency reaches 95% at $\alpha = 0.05$ when $k = 200$, compared to only 34% at $k = 50$.

Table 2 summarizes the simulated rejection frequencies for the test (4.21). The first six rows correspond to scenarios under the null hypothesis, where we observe consistently low rejection frequencies. The subsequent six rows assess the power of the proposed methods under alternative hypotheses. When $k = 50, 100$, the rejection frequencies are relatively low for the

Table 2: Simulated rejection frequency for testing H_{20}.

Distribution			k	50				100				200			
scedasis			α	0.05		0.1		0.05		0.1		0.05		0.1	
c_1	c_2		M	KS	CVM	KS	CVM	KS	CVM	KS	CVM	KS	CVM	KS	CVM
a_1	a_1	$\mathcal{C}_1$		0.03	0.04	0.06	0.07	0.04	0.03	0.07	0.07	0.05	0.05	0.09	0.10
a_2	a_2	$\mathcal{C}_2$		0.03	0.04	0.07	0.07	0.03	0.04	0.08	0.09	0.05	0.05	0.10	0.10
a_3	a_3	$\mathcal{C}_3$		0.01	0.01	0.04	0.04	0.01	0.01	0.03	0.04	0.03	0.02	0.06	0.06
a_4	a_4	$\mathcal{C}_4$		0.01	0.02	0.03	0.03	0.01	0.02	0.03	0.04	0.03	0.03	0.06	0.06
a_5	a_5	$\mathcal{C}_5$		0.03	0.03	0.06	0.06	0.04	0.04	0.08	0.08	0.05	0.05	0.11	0.10
a_6	a_6	$\mathcal{C}_6$		0.01	0.02	0.04	0.04	0.02	0.03	0.05	0.06	0.03	0.03	0.07	0.06
a_1	a_2	$\mathcal{C}_1$		0.07	0.09	0.12	0.15	0.10	0.14	0.18	0.22	0.22	0.26	0.32	0.35
a_3	a_4	$\mathcal{C}_2$		0.13	0.16	0.22	0.24	0.30	0.32	0.44	0.42	0.60	0.58	0.70	0.70
a_5	a_6	$\mathcal{C}_3$		0.02	0.02	0.05	0.07	0.05	0.06	0.11	0.15	0.14	0.20	0.27	0.39
a_1	a_3	$\mathcal{C}_4$		0.86	0.90	0.94	0.95	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
a_1	a_5	$\mathcal{C}_5$		0.14	0.13	0.25	0.27	0.32	0.32	0.45	0.47	0.68	0.68	0.79	0.80
a_3	a_5	$\mathcal{C}_6$		0.64	0.68	0.76	0.79	0.96	0.97	0.98	0.99	1.00	1.00	1.00	1.00

seventh and ninth experiments. However, the rejection frequencies improve when $k = 200$, indicating that larger intermediate thresholds enhance the test's ability to detect differences in scedasis functions. Additionally, in most scenarios, the CVM test exhibits slightly higher rejection frequencies than the KS test.

The simulation results are sensitive to the choice of k. When k is small, many studies have shown that the bootstrap may fail to approximate the correct asymptotic distribution; see, for example, Chernozhukov and Fernández-Val (2011). Consequently, the empirical rejection frequencies can fall below α. In a more recent contribution, Zhang (2018) recommend employing the bootstrap procedure of Bickel and Sakov (2008) to better approximate the limiting distribution under extreme quantile settings. Several alternative approaches have also been proposed to improve bootstrap accuracy. For instance, Li et al. (2011) (Figure 1) document that the bootstrap-based coverage tends to be lower than the nominal confidence level when k is small, and they introduce a bootstrap calibration procedure to correct this bias. Enhancing the finite-sample reliability of bootstrap methods for small k remains an open and challenging research problem.

5. Empirical Study

We collect 2516 daily stock return data of 8 companies from the S&P index, from January 4th, 2010 to January 3rd, 2020. We use the negative daily return to indicate the loss for each company, which follows a similar modeling approach in Einmahl et al. (2014). Our data analysis conducts tests on the problems (4.20) and (4.21) for each pair of the 8 companies.

Table 3 lists the basic data information of each stock. We implement the two tests in Einmahl et al. (2014) for each univariate loss; one is the

Table 3: Stock Symbol, company name, k and hill estimator of 8 stocks. A validation test and a test for $c \equiv 1$ are conducted by the methodologies in Einmahl et al. (2014).

				p-value	
	Company Name	k_j	Hill Estimator	Validation Test	Test for $c \equiv 1$
EXPE	Expedia Group, Inc.	173	0.391	0.144	0.002
RMD	ResMed Inc.	191	0.402	0.447	0.000
AKAM	Akamai Technologies, Inc.	248	0.421	0.701	0.000
IT	Gartner, Inc.	159	0.400	0.769	0.000
ILMN	Illumina, Inc.	151	0.385	0.797	0.216
LDOS	Leidos Holdings, Inc.	236	0.422	0.649	0.354
QCOM	Qualcomm Incorporated	151	0.420	0.926	0.236
HSY	The Hershey Company	192	0.352	0.149	0.698

validation test T_4 from Einmahl et al. (2014) and the other is T_1 in Einmahl et al. (2014) to test whether $c \equiv 1$ or not. The p-values of the two tests are summarized in Table 3. We conclude that the heteroscedastic extremes are fit for the marginal distribution of each stock loss data. The tests for the first four stocks reject the $c \equiv 1$, while the tests for the other stocks do not reject the hypothesis that $c_j \equiv 1$.

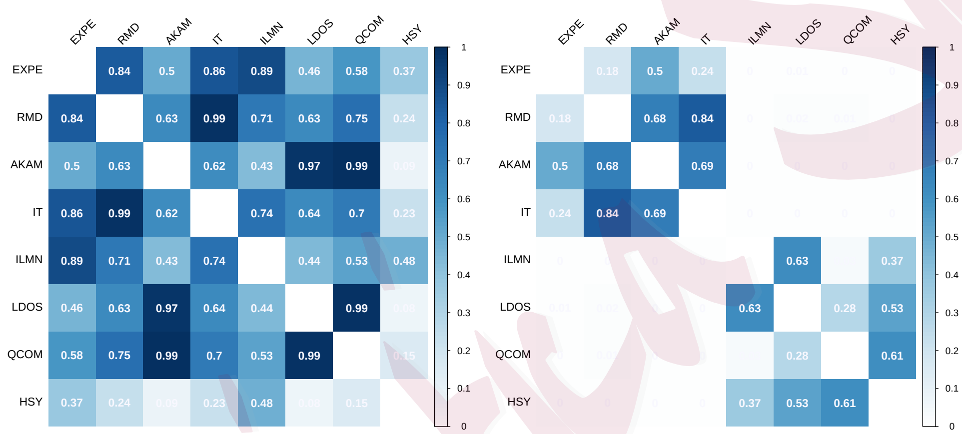
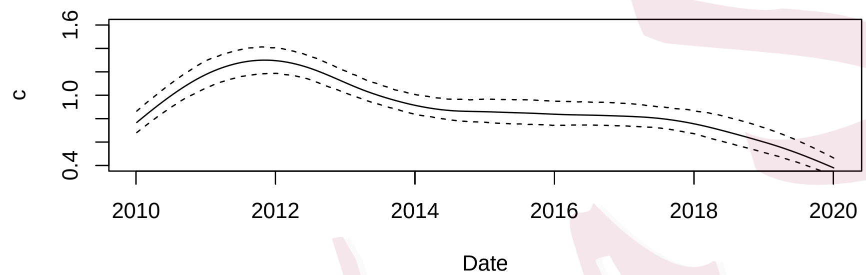


Figure 1: P-values of the pair two sample tests of H_{10}(the equivalence of extreme value index, left) and H_{20}(the equivalence of scedasis functions, right), for 8 stocks from January 4th, 2010, to January 3rd, 2020.

We then apply the testing procedures outlined in Section 4 to each pair of stock returns. For each test, we implement the bootstrap method with $B = 500$ replications. The resulting p-values are presented in Figure 1.

In the test for H_{10}, we find that most stocks exhibit similar tail heaviness, with Hill estimators ranging from 0.34 to 0.43, as reported in Table 3. However, when testing the equality of scedasis functions, the stocks appear to cluster into two distinct groups, suggesting that certain stocks may be subject to common underlying factors and therefore exhibit similar responses to extreme events. This clustering further highlights the presence of co-movement in extreme quantiles across the market, offering empirical evidence for recent developments in quantile factor models (Chen et al., 2021; Ando and Bai, 2020; Ando et al., 2022).

To further investigate the differing trends in scedasis functions across the two clusters, Figure 2 presents the estimated scedasis functions for each group. For each company, we estimate $\hat{c}_j(s)$ using the kernel method proposed by Einmahl et al. (2014), employing a biweight kernel with bandwidth 0.3. The group-level estimator $\hat{c}(s)$ is defined as $\frac{1}{g}\sum j = 1^g \hat{c}_j(s)$, where g denotes the number of companies in the group. The scedasis functions for EXPE, RMD, AKAM, and IT exhibit a pronounced structural change between 2010 and 2012, whereas those for ILMN, LDOS, QCOM, and HSY remain relatively smooth and stable over time. These clustering patterns and differing scedasis trends offer potential insights for risk management. In particular, researchers may cluster the estimated scedasis functions to identify latent factors driving tail risk and investigate the underlying economic sources of their variation. The p-values from our proposed tests can serve as a natural similarity measure for such clustering, highlighting an avenue for future research in identifying and modeling common drivers of tail risk across assets.



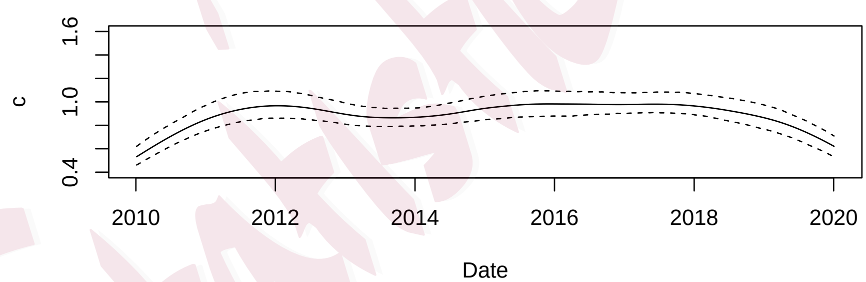


Figure 2: Estimation of $\hat{c}$ of EXPE, RMD, AKAM, IT (top), and ILMN, LDOS, QCOM, HSY(bottom). The dashed lines are bootstraped 90% confidence interval.

6. Conclusion

This paper proposes a copula-based model for independent but not identically distributed data with bivariate heteroscedastic extremes. The model allows both the copula structure and marginal distributions to vary across samples. We develop a comprehensive framework to derive bootstrap estimators via the B-STEP process, using the functional delta method. Simulation results validate the robustness of our bootstrap approach. Other bootstrap techniques (de Haan and Zhou (2024); Jentsch and Kulik (2021)) are not covered in our framework, and their asymptotic validity under bivariate heteroscedastic extremes remains unexplored. Extending our framework to time-series settings with mild dependence (e.g., mixing conditions as in Bücher and Segers (2018); Zou et al. (2021)) is a potential next step, but the asymptotic analysis under such conditions is left as an open problem due to the need for a separate theoretical framework.

Supplementary Materials

This supplement collects the proofs of the theoretical results in the article.

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