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Testing Conditional Tail Independence

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Abstract: Measuring tail dependence structure is crucial in understanding the behavior of bivariate extremes. Common measures of tail dependence include tail dependence index, coefficient of tail dependence, tail dependence function, and so on. However, in practice, there may exist covariates which are related with both variables. Up to our knowledge, there is no measure that focuses on the conditional tail dependence structure. This paper first introduces the concept of conditional tail dependence index, based on which we can distinguish between conditional tail independence and conditional tail dependence. We provide a test statistic named conditional tail quotient correlation coefficient (CTQCC) to test the null hypothesis of conditional tail independence and obtain its asymptotic distribution. Simulation studies are conducted to assess the finite sample performance of the proposed method. We apply CTQCC to investigate conditional tail dependencies of a large-scale problem of daily precipitation and daily average wind speed in the United States, given the daily maximum temperature. The results show that the proposed method is effective in detecting conditional tail dependence structures.

Key words and phrases: Conditional extremes, extreme quantile, tail dependence.

1. Introduction

Modeling extreme events is crucial in many fields, such as environmental sciences, finance, and insurance. Understanding and predicting the occurrence of extreme events allows us to assess risks and make informed decisions. Extreme value theory provides a valuable tool for characterizing multivariate dependencies in extremes, with applications including the analysis of extreme flooding (Engelke and Ivanovs , 2021), risk diversification across stock returns (Poon et al., 2004), and extremal dependence between air pollutants (Heffernan and Tawn , 2004).

Extremal dependence between two random variables can be measured in different ways. A common practice is to classify bivariate random vectors into two regimes—tail independence and tail dependence (also known as asymptotic independence and asymptotic dependence)—based on the tail dependence index λ:

$$\lambda = \lim_{\tau \to 1} \mathbb{P}\left(X > Q_X(\tau) \mid Y > Q_Y(\tau)\right),$$

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where $Q_X(\tau)$ and $Q_Y(\tau)$ are the τth quantiles of X and Y, respectively. If $\lambda = 0$, X and Y are said to be tail independent; if $\lambda > 0$, X and Y are said to be tail dependent. We refer to Ledford and Tawn (1996) for further details on tail independence and tail dependence.

Although most research articles within the bivariate extreme value framework assume a tail dependence model (i.e., $\lambda > 0$), there is increasing evidence that weaker dependence often exists in the bivariate tail region across many applications—for example, significant wave height (Wadsworth and Tawn, 2012), spatial precipitation (Le et al., 2018), and daily stock prices (Lehtomma and Resnick, 2020). Since the largest values of each variable can occur together only when the variables exhibit tail dependence, it is important to distinguish between the two regimes. Several attempts have been made to test for tail independence of a bivariate vector (i.e., $H_0 : \lambda = 0$), including Hüsler and Li (2009), Bacro, et al. (2010), and Zhang et al. (2017).

In practice, confounding variables $\mathbf{Z}$ may exist that are related to both X and Y. It is important to account for the effect of $\mathbf{Z}$ when quantifying the extremal dependence between X and Y. For example, one may be interested in the probability of concurrence of extreme values in X and Y given the values of $\mathbf{Z}$. Traditional notions of tail dependence and tail independence are not sufficient to address the confounding effect of $\mathbf{Z}$.

We introduce the concept of the conditional tail dependence index, denoted by $\lambda(\mathbf{Z})$, which measures the tail dependence of (X, Y) given covariates $\mathbf{Z}$. It is defined as

$$\lambda(\mathbf{Z}) := \lim_{\tau \to 1} \mathbb{P}_{|\mathbf{Z}}\big(X > Q_X(\tau \mid \mathbf{Z}) \mid Y > Q_Y(\tau \mid \mathbf{Z})\big),$$

where $Q_X(\tau \mid \mathbf{Z}) = \inf\{x : F_X(x \mid \mathbf{Z}) \geq \tau\}$ and $Q_Y(\tau \mid \mathbf{Z}) = \inf\{y : F_Y(y \mid \mathbf{Z}) \geq \tau\}$ denote the τth conditional quantiles of X and Y given $\mathbf{Z}$, respectively. Here $F_X(\cdot \mid \mathbf{Z})$ and $F_Y(\cdot \mid \mathbf{Z})$ are the conditional distributions of X and Y given $\mathbf{Z}$, respectively, and $\mathbb{P}_{|\mathbf{Z}}$ denotes the conditional probability given $\mathbf{Z}$. The conditional tail dependence index $\lambda(\mathbf{Z})$ describes the extremal dependence between X and Y given $\mathbf{Z}$.

Suppose $\mathbf{Z}$ has compact support $\mathcal{Z}$. Then $\lambda(\mathbf{Z})$ is a function of $\mathbf{Z} \in \mathcal{Z}$. For a subset $\mathcal{B} \subset \mathcal{Z}$, we say that X and Y are conditional tail independent on $\mathcal{B}$ if $\lambda(\mathbf{Z}) = 0$ for $\mathbf{Z} \in \mathcal{B}$ a.s. We call that X and Y are conditional tail dependent on $\mathcal{B}$ if $\lambda(\mathbf{Z}) > 0$ for $\mathbf{Z} \in \mathcal{B}$ almost surely. Moreover, if $\mathcal{B} = \mathcal{Z}$, we say that X and Y are conditional tail (in)dependent almost surely on the entire support.

While in some cases, X and Y are both tail (in)dependent and conditionally tail (in)dependent, there are examples where X and Y are tail independent but conditionally tail dependent, i.e., $\lambda = 0$ but $\lambda(\mathbf{Z}) > 0$

for $\mathbf{Z} \in \mathcal{Z}$ a.s. (see Example 1 below). In such cases, the classical tail independence tests in the literature may lead to misleading conclusions.

In this paper, we consider the testing of conditional tail independence: $H_0 : \lambda(\mathbf{Z}) = 0$, for $\mathbf{Z} \in \mathcal{Z}$ a.s. We propose the Conditional Tail Quotient Correlation Coefficient (CTQCC), an extension of the Tail Quotient Correlation Coefficient (TQCC; see Zhang et al. (2017)), to test conditional tail independence. We derive the asymptotic distribution of CTQCC and apply it to investigate the conditional tail dependence structure.

The rest of this article is organized as follows. Section 2 illustrates the concept of conditional tail (in)dependence through examples. Section 3 introduces CTQCC as the test statistic and investigates its asymptotic properties. A testing procedure is developed for conditional tail independence. Section 4 presents simulation experiments to assess the finite-sample performance of the proposed method, and Section 5 applies the method to test for conditional tail independence and explore the tail structure of daily precipitation and wind speed from 2011 to 2023, recorded at 317 stations across the United States. A brief conclusion is provided in Section 6. All proofs are relegated to the Appendix.

2. Examples and Equivalency

To further illustrate the concept of conditional tail (in)dependence, we begin with examples that contrast tail and conditional tail dependence structures. Specifically, we present one example where X and Y are tail independent but conditionally tail dependent, and another where they are tail dependent but conditionally tail independent (see Example 1 below).

Denote the Fréchet distribution with parameters (α, s) as Fréchet (α, s), where $\alpha > 0$ is the shape parameter and $s > 0$ is the scale parameter. The probability function is given by

$$F(x) = \exp(-(x/s)^{-\alpha}), x > 0.$$

We call F a standard Fréchet distribution function if $\alpha = s = 1$. The Gumbel copula belongs to the Archimedean family of copulas and is defined by the parametric form:

$$C(u, v) = \exp\left\{-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right\}, \quad \theta \geq 1.$$

The parameter $\theta > 1$ corresponds to the case of tail independence.

Example 1. Let X and Y be two random variables with the following bivariate structure:

$$(X, Y) = (W_1 + V_1, W_2 + V_2)\,,$$

where W_1, W_2, V_1, V_2 are all Fréchet random variables with parameters $(\alpha_1, 1)$, $(\alpha_1, 1)$, $(\alpha_2, 1)$, $(\alpha_2, 1)$, respectively. Suppose (W_1, W_2) follows a Gumbel copula with parameter $\theta > 1$, and assume (W_1, W_2), V_1, and V_2 are mutually independent.

- If $\alpha_1 > \alpha_2$, then V_1 and V_2 have heavier tails than W_1 and W_2, respectively. As a result, X and Y are tail independent. However, due to the dependence structure of (W_1, W_2), X and Y are conditionally tail dependent given $\mathbf{V} = (V_1, V_2)$.

- Conversely, if $\alpha_1 < \alpha_2$, then V_1 and V_2 have lighter tails, and X and Y are tail dependent, but conditionally tail independent given $\mathbf{V}$.

This example highlights that tail dependence and conditional tail dependence capture distinct aspects of extremal dependence and should be evaluated separately. In particular, traditional tail dependence tests may fail to conditional extremal structure, so we need to develop a new testing method.

We now present a second example to demonstrate that the conditional tail dependence index $\lambda(\mathbf{Z})$ can vary with the values of $\mathbf{Z}$.

Example 2. Let X and Y have the bivariate structure:

$$(X, Y) = (ZW_1 + V_1, ZW_2 + V_2),$$

where W_1, W_2, V_1, V_2 are as in Example 1, and Z is a Bernoulli(p) random variable with $p \in (0,1)$. Assume $\alpha_1 < \alpha_2$. Then we have:

- When $Z = 0$, $\lambda(0) = 0$, so X and Y are conditionally tail independent.
- When $Z = 1$, $\lambda(1) > 0$, so X and Y are conditionally tail dependent.

This example illustrates that $\lambda(\mathbf{Z})$ can depend sensitively on the value of $\mathbf{Z}$, emphasizing the importance of clearly specifying the null hypothesis and the domain over which conditional tail independence is being tested.

Remark 1. Heffernan and Tawn (2004) introduced the notion of mutual asymptotic conditional independence for a d-dimensional random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$ with Gumbel marginal distributions, by analyzing the limiting behavior of

$$\mathbb{P}\left(\mathbf{Y}_{-i} \leq \mathbf{y}_{-i} \mid Y_i = y_i\right), \quad \text{and} \quad \mathbb{P}\left(Y_j \leq y_j \mid Y_i = y_i\right), \quad \text{as } y_i \to \infty,$$

for $i = 1, \ldots, d$ and $i \neq j$. Here $\mathbf{Y}_{-i}$ denotes the vector $\mathbf{Y}$ excluding the component Y_i and $\mathbf{y_{-i}}$ has the similar meaning. Their framework differs from ours in that it assumes extreme conditioning values and fixed marginals. In contrast, we do not require $(X, Y, \mathbf{Z})$ to have specific marginal distributions, and we condition on $\mathbf{Z}$, which does not necessarily take extreme values. Furthermore, $\lambda(\mathbf{z})$ is equivalent to $\Lambda(1, 1 \mid \mathbf{z})$, where

$\Lambda(x, y \mid \mathbf{z})$ is the conditional tail copula defined by Gardes and Girard (2015). Thus, $\lambda(\mathbf{z})$ can be estimated using the nonparametric estimator of $\Lambda(1, 1 \mid \mathbf{z})$ proposed in their work, and its asymptotic normality is guaranteed.

To formalize our setup, we now define a regularity condition on the conditional distributions of X and Y. Define the univariate extreme value distribution function as

$$G_\gamma(z) = \exp\{-(1 + \gamma z)^{-1/\gamma}\},$$

for $1 + \gamma z > 0$ and $\gamma \in \mathbb{R}$. Throughout the paper, we assume the following maximum domain of attraction condition, which is standard in the extreme value literature and encompasses a wide range of heavy-tailed distributions. Throughout the paper, we assume the following maximum domain of attraction condition, which is standard in the extreme value literature and encompasses a wide range of heavy-tailed distributions.

Assumption 1. $F_X(\cdot \mid \mathbf{Z})$ and $F_Y(\cdot \mid \mathbf{Z})$ belong to the maximum domains of attraction of the extreme value distributions $G_{\gamma_1}(\cdot)$ and $G_{\gamma_2}(\cdot)$, respectively, for all $\mathbf{Z}$, where $\gamma_1 > 0$ and $\gamma_2 > 0$.

We now state an equivalency result characterizing conditional tail independence under Assumption 1.

Proposition 1. *Let $F_{XY}(\cdot, \cdot \mid \mathbf{Z})$ be the joint conditional distribution of (X, Y) given $\mathbf{Z}$. Suppose Assumption 1 holds, then the following statements are equivalent:*

1. *X and Y are conditional tail independent.*
2. $n\Big(\mathbb{P}_{|\mathbf{Z}}\big(X > Q_X(1 - x/n \mid \mathbf{Z}), Y > Q_Y(1 - y/n \mid \mathbf{Z})\big)\Big) = o_{\mathbb{P}}(1)$, *for all* $0 < x, y < \infty$.
3. *For $\mathbf{Z}$ almost surely, $F_{XY}(\cdot, \cdot \mid \mathbf{Z})$ belongs to the maximum domain of attraction of the bivariate extreme value distribution $G_{\gamma_1}(\cdot)G_{\gamma_2}(\cdot)$.*

Under conditional tail independence, the conditional joint distribution $F_{XY}(\cdot, \cdot \mid \mathbf{Z})$ lies in the maximum domain of attraction of a bivariate extreme value distribution with independent components, specifically, a product of its marginal distributions $G_{\gamma_1}(\cdot)$ and $G_{\gamma_2}(\cdot)$, as shown in Proposition 1. Consequently, the probability of concurrence of extreme values in X and Y, given the value of $\mathbf{Z}$, is asymptotically negligible under conditional tail independence.

3. Proposed Testing Method

3.1 Conditional Tail Quotient Correlation Coefficient

Let $\{(X_i, Y_i, \mathbf{Z}_i), i = 1, \cdots, n\}$ be independent and identically distributed copies of random vector $(X, Y, \mathbf{Z})$. Define

$$\Delta_n = \max_{1\le i\le n}\left\{\frac{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)}\right\}, \quad \Theta_n = \max_{1\le i\le n}\left\{\frac{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)}\right\},$$

where the thresholds $u_n(\mathbf{Z}_i) = \widehat{Q}_X(1 - c/n \mid \mathbf{Z}_i)$ and $v_n(\mathbf{Z}_i) = \widehat{Q}_Y(1 - c/n \mid \mathbf{Z}_i)$ are the estimated conditional $(1 - c/n)$th quantiles of X and Y given $\mathbf{Z}$, respectively (see (3.3) below), and c is a positive constant.

The choice of c involves a trade-off. If $c \to \infty$, the selected quantile level may no longer lie in the tail region, resulting in substantial bias. On the other hand, if $c \to 0$, there may not be enough data to compute the test statistic reliably. The quantile level $1 - c/n$ is commonly referred to as an extreme level in extreme value theory. In practice, we select a finite value of c to balance bias and variance in the tail estimation.

We now define CTQCC as

$$q_n = \frac{\Delta_n + \Theta_n - 2}{\Delta_n \times \Theta_n - 1}. \tag{3.1}$$

The CTQCC q_n in Equation (3.1) is designed to test conditional tail independence between X and Y given all values of $\mathbf{Z}$. If interest lies in a particular subset $\mathcal{B} \subset \mathcal{Z}$, i.e., testing $\tilde{H}_0 : \lambda(\mathbf{Z}) = 0$ for $\mathbf{Z} \in \mathcal{B}$ almost surely, one can restrict the data to the set $\{(X_i, Y_i, \mathbf{Z}_i) : \mathbf{Z}_i \in \mathcal{B}\}$.

To study the properties of CTQCC, we define

$$f(x,y) = \begin{cases} \frac{x+y-2}{xy-1}, & \text{if } x \geq 1, y \geq 1, x+y > 2, \\ 1, & \text{else.} \end{cases}$$

By Theorem 1, we have $\mathbb{P}(\Delta_n > 1) > 0$ and $\mathbb{P}(\Theta_n > 1) > 0$ for any $n \geq 1$, and both Δ_n and Θ_n are almost surely greater than or equal to 1 as $n \to \infty$. Therefore, $q_n = f(\Delta_n, \Theta_n)$ takes values between 0 and 1 almost surely. For a fixed sample size n, the larger Δ_n or Θ_n becomes, the smaller q_n tends to be, reflecting less agreement in tail magnitudes. In this way, q_n serves as a measure of conditional tail dependence between X and Y. A value of q_n close to 1 implies that either Δ_n or Θ_n is close to 1, indicating the presence of co-movement in the tails of X and Y conditional on $\mathbf{Z}$, i.e., nearly complete conditional tail dependence.

Remark 2. CTQCC is inspired by the TQCC proposed in Zhang et al. (2017), but it possesses several distinctive features. Given a random sample of unit Fréchet random variables $\{(X_i, Y_i)\}_{i=1}^n$, TQCC is defined as

$$\frac{\max_{1\leq i\leq n}\{\max(X_i,u_n)/\max(Y_i,u_n)\} + \max_{1\leq i\leq n}\{\max(Y_i,u_n)/\max(X_i,u_n)\} - 2}{\max_{1\leq i\leq n}\{\max(X_i,u_n)/\max(Y_i,u_n)\} \times \max_{1\leq i\leq n}\{\max(Y_i,u_n)/\max(X_i,u_n)\} - 1},$$

where u_n is a random threshold. The asymptotic distribution of TQCC is derived under two classes of thresholds. Unlike TQCC, the threshold selection in CTQCC is based on estimated conditional quantiles at a specified quantile level. Moreover, CTQCC takes maximum over (X_i/u_n) and 1 instead of over X_i and u_n. This construction and the selection of thresholds enable CTQCC to handle variables with differing tail heaviness, and does not require the marginal distributions to be unit Fréchet.

3.2 Asymptotic Distribution of CTQCC

In this subsection, we derive the asymptotic distribution of CTQCC under the null hypothesis of conditional tail independence. Recall that we choose the thresholds u_n and v_n as estimated extreme conditional quantiles, defined as $u_n(\mathbf{Z}_i) = \widehat{Q}_X(1 - c/n \mid \mathbf{Z}_i)$ and $v_n(\mathbf{Z}_i) = \widehat{Q}_Y(1 - c/n \mid \mathbf{Z}_i)$. To estimate thresholds, we consider fitting the following linear quantile regression models:

$$Q_X(\tau \mid \mathbf{Z}) = \mathbf{Z}^T\boldsymbol{\beta}_1(\tau), \quad Q_Y(\tau \mid \mathbf{Z}) = \mathbf{Z}^T\boldsymbol{\beta}_2(\tau), \tag{3.2}$$

where $\tau_0 \in (0, 1)$ is a fixed constant and $\tau \in [\tau_0, 1]$.

Now define a sequence of quantile levels $\mathcal{T} = \{\tilde{\tau}_{n-k} < \tilde{\tau}_{n-k+1} < \cdots < \tilde{\tau}_m\} \subset (\tau_0, 1)$, where $m = n - [n^\eta]$ with $[a]$ denoting the integer part of a, and $\tilde{\tau}_j = j/(n + 1)$. For each $j = n - k, \cdots, m$, we define

$$\widehat{\boldsymbol{\beta}}_1(\tilde{\tau}_j) = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^d} \sum_{i=1}^{n} \rho_{\tilde{\tau}_j}\left(X_i - \mathbf{Z}_i^T\boldsymbol{\beta}\right), \quad \widehat{\boldsymbol{\beta}}_2(\tilde{\tau}_j) = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^d} \sum_{i=1}^{n} \rho_{\tilde{\tau}_j}\left(Y_i - \mathbf{Z}_i^T\boldsymbol{\beta}\right),$$

where $\rho_\tau(u) = (\tau - 1(u \le 0))u$.

Define $q_{1j}(\mathbf{Z}_i) = \mathbf{Z}_i^T \widehat{\boldsymbol{\beta}}_1(\tilde{\tau}_j), q_{2j}(\mathbf{Z}_i) = \mathbf{Z}_i^T \widehat{\boldsymbol{\beta}}_2(\tilde{\tau}_j), i = 1, \cdots, n, j = n - k, \cdots, m$. Consequently, $Q_X(1 - c/n \mid \mathbf{Z}_i)$ and $Q_Y(1 - c/n \mid \mathbf{Z}_i)$ can be estimated by

$$\widehat{Q}_X(1 - c/n \mid \mathbf{Z}_i) = \left(\frac{1 - \tilde{\tau}_{n-k}}{c/n}\right)^{\widehat{\gamma}_1(\mathbf{Z}_i)} q_{1(n-k)}(\mathbf{Z}_i),$$

$$\widehat{Q}_Y(1 - c/n \mid \mathbf{Z}_i) = \left(\frac{1 - \tilde{\tau}_{n-k}}{c/n}\right)^{\widehat{\gamma}_2(\mathbf{Z}_i)} q_{2(n-k)}(\mathbf{Z}_i), \tag{3.3}$$

where the Hill estimators $\widehat{\gamma}_1(\mathbf{Z}_i)$ and $\widehat{\gamma}_2(\mathbf{Z}_i)$ are constructed as

$$\widehat{\gamma}_1(\mathbf{Z}_i) = \frac{1}{k - [n^\eta]} \sum_{j=[n^\eta]}^{k} \log \frac{q_{1(n-j)}(\mathbf{Z}_i)}{q_{1(n-k)}(\mathbf{Z}_i)}, \quad \widehat{\gamma}_2(\mathbf{Z}_i) = \frac{1}{k - [n^\eta]} \sum_{j=[n^\eta]}^{k} \log \frac{q_{2(n-j)}(\mathbf{Z}_i)}{q_{2(n-k)}(\mathbf{Z}_i)}.$$

To establish the asymptotic properties of the estimated extreme quantiles and consequently the proposed CTQCC, we impose the following assumptions. Here, $a_\tau(\mathbf{Z}) \approx b_\tau(\mathbf{Z})$ uniformally for $\mathbf{Z}$ denotes that

$$\sup_{\mathbf{Z} \in \mathcal{Z}} \left| \frac{a_\tau(\mathbf{Z})}{b_\tau(\mathbf{Z})} - 1 \right| \xrightarrow{\mathbb{P}} 0.$$

Assumption 2. The distribution function of $\mathbf{Z}$, $F_{\mathbf{Z}}$ has compact support $\mathcal{Z}$ with $\mathbb{E}\left(\mathbf{Z}\mathbf{Z}^T\right)$ positive definite.

Assumption 3. There exist auxiliary lines $\mathbf{Z} \to \mathbf{Z}^T \boldsymbol{\beta}_{r_1}, \mathbf{Z} \to \mathbf{Z}^T \boldsymbol{\beta}_{r_2}$ such that for $X^* = X - \mathbf{Z}^T \boldsymbol{\beta}_{r_1}, Y^* = Y - \mathbf{Z}^T \boldsymbol{\beta}_{r_2}$ and some heavy-tailed distribution functions $F_1(\cdot), F_2(\cdot)$ with extreme value index $\gamma_1 > 0, \gamma_2 > 0$, respectively, it holds that

$$\frac{1-F_{X^*}(t\mid\mathbf{Z})}{K_1(\mathbf{Z})\left\{1-F_1(t)\right\}}-1=\{1-F_1(t)\}^{\delta_1}\,\widetilde{K_1}(\mathbf{Z})\{1+o_{\mathbb{P}}(1)\},$$

$$\frac{1-F_{Y^*}(t\mid\mathbf{Z})}{K_2(\mathbf{Z})\left\{1-F_2(t)\right\}}-1=\{1-F_2(t)\}^{\delta_2}\,\widetilde{K_2}(\mathbf{Z})\{1+o_{\mathbb{P}}(1)\},$$

uniformly for $\mathbf{Z}\in\mathcal{Z}$ as $t\to\infty$, where $K_1(\cdot)>0, K_2(\cdot)>0, \widetilde{K}_1, \widetilde{K}_2$ are continuous bounded functions and $\delta_1>0, \delta_2>0$ are constants. Note that $K_1(\cdot)$ and $K_2(\cdot)$ satisfy $K_1(\boldsymbol{\mu}_{\mathbf{Z}})=K_2(\boldsymbol{\mu}_{\mathbf{Z}})=1$, where $\boldsymbol{\mu}_{\mathbf{Z}}=\mathbb{E}(\mathbf{Z})$.

Assumption 4. As $\tau\to 1$, it holds that

$$\frac{\partial}{\partial\tau}F_{X^*}^{-1}(\tau\mid\mathbf{Z})\approx\frac{\partial}{\partial\tau}F_1^{-1}\{\tau/K_1(\mathbf{Z})\},\quad \frac{\partial}{\partial\tau}F_{Y^*}^{-1}(\tau\mid\mathbf{Z})\approx\frac{\partial}{\partial\tau}F_2^{-1}\{\tau/K_2(\mathbf{Z})\},$$

uniformally for Z and that $\frac{\partial}{\partial\tau}F_1^{-1}(1-\tau), \frac{\partial}{\partial\tau}F_2^{-1}(1-\tau)$ are regularly varying at zero with index $-\gamma_1-1$ and $-\gamma_2-1$, respectively.

Assumption 5. $U_1(t)=F_1^{-1}(1-1/t)$ and $U_2(t)=F_2^{-1}(1-1/t)$ both satisfy the second-order condition with $\gamma_1>0, \varrho_1<0, A_1(t)=\gamma_1 d_1 t^{\varrho_1}$, and $\gamma_2>0, \varrho_2<0, A_2(t)=\gamma_2 d_2 t^{\varrho_2}$, respectively, with $d_1\neq 0, d_2\neq 0$, i.e. for all $s>0$, as $t\to\infty$,

$$A_1(t)^{-1}\left\{\frac{U_1(ts)}{U_1(t)}-s^{\gamma_1}\right\}\to s^{\gamma_1}\left(s^{\varrho_1}-1\right)/\varrho_1,$$

$$A_2(t)^{-1}\left\{\frac{U_2(ts)}{U_2(t)}-s^{\gamma_2}\right\}\to s^{\gamma_2}\left(s^{\varrho_2}-1\right)/\varrho_2.$$

Assumption 6. As $n \to \infty, k = k_n \to \infty, k/n \to 0, k > n^{\eta}, k^{-1/2} n^{\eta} \log k \to 0, \sqrt{k}(n/k)^{\widetilde{\varrho}} \to 0$, where $\widetilde{\varrho} = \max(\widetilde{\varrho}_1, \widetilde{\varrho}_2), \widetilde{\varrho}_1 = \max(\varrho_1, -\delta_1, -\gamma_1), \widetilde{\varrho}_2 = \max(\varrho_2, -\delta_2, -\gamma_2)$, and $\eta > 0$ is some small constant.

Assumption 3 requires some form of equivalence between $1 - F_{X^*}(t \mid \mathbf{z})$ and $1 - F_1(t)$, and between $1 - F_{Y^*}(t \mid \mathbf{z})$ and $1 - F_2(t)$ at the right tails, respectively. Under Assumptions 2 and 3, Theorem 3.1 of Chernozhukov (2005) shows that $K(\cdot)$ takes the following form: $K(\mathbf{z}) = \left(\mathbf{z}^T \mathbf{w}\right)^{1/\gamma}$ for some $\mathbf{w} \in \mathbb{R}^{p+1}$ such that $\mu_{\mathbf{z}}^T \mathbf{w} = 1$ and $\mathbf{z}^T \mathbf{w} > 0$ for all $\mathbf{z} \in \mathcal{Z}$. Assumption 4 is a von Mises type condition for the distribution belonging to a maximum domain of attraction. Assumption 5 is a second-order condition for the distribution functions F_1 and F_2. Assumption 6 controls the order of k such that the asymptotic bias of the extreme value indices reduce to zero.

The following theorem establishes the limiting joint distribution of Δ_n and Θ_n, as well as the asymptotic distribution of the CTQCC statistic q_n under the null hypothesis of conditional tail independence.

Theorem 1. *Suppose the linear quantile regression model (3.2) and Assumptions 1-6 hold. Under the conditional tail independence, we have*

$$\lim_{n \to \infty} \mathbb{P}(\Delta_n \le x, \Theta_n \le y) = \begin{cases} \exp(-c x^{-\frac{1}{\gamma_1}} - c y^{-\frac{1}{\gamma_2}}), & \text{if } x \ge 1, y \ge 1, x + y > 2, \\ 1, & \text{else.} \end{cases}$$

and as $n \to \infty$,

$$q_n \xrightarrow{d} f(\max(U_1, 1), \max(U_2, 1)),$$

where U_1 *and* U_2 *are independent,* $U_1 \sim$ *Fréchet* $(\gamma_1^{-1}, c^{\gamma_1})$ *and* $U_2 \sim$ *Fréchet*$(\gamma_2^{-1}, c^{\gamma_2})$.

In practice, the tail indices γ_1 and γ_2 are typically unknown and must be estimated in order to evaluate the limiting distribution of $f(\max(U_1, 1),\ \max(U_2, 1))$. Since the estimators $\widehat{\gamma}_1(\mathbf{Z}_i)$ and $\widehat{\gamma}_2(\mathbf{Z}_i)$ are computed for each $i = 1, \ldots, n$, we define the sample averages as plug-in estimates:

$$\widehat{\gamma}_1 = \frac{1}{n} \sum_{i=1}^{n} \widehat{\gamma}_1(\mathbf{Z}_i), \quad \widehat{\gamma}_2 = \frac{1}{n} \sum_{i=1}^{n} \widehat{\gamma}_2(\mathbf{Z}_i). \tag{3.4}$$

These estimates are then substituted into the limiting distribution to approximate the law of q_n under the null.

Remark 3. Note that

$$\lim_{n\to\infty} \mathbb{P}(c^{-\gamma_1}\Delta_n \le x, c^{-\gamma_2}\Theta_n \le y) = \begin{cases} \exp(-x^{-\frac{1}{\gamma_1}} - y^{-\frac{1}{\gamma_2}}), & \text{if } x \ge c^{-\gamma_1}, y \ge c^{-\gamma_2}, \\ 0, & \text{else.} \end{cases}$$

Define the rescaled version of q_n as

$$\tilde{q}_n = \frac{c^{-\gamma_1}\Delta_n + c^{-\gamma_2}\Theta_n - 2}{c^{-\gamma_1}\Delta_n \times c^{-\gamma_2}\Theta_n - 1}.$$

Then, as $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}(\tilde{q}_n \le x) = \mathbb{P}[f\{\max(V_1, c^{-\gamma_1}), \max(V_2, c^{-\gamma_2})\} < x],$$

where V_1 and V_2 are independent Fréchet random variables with parameters $(\gamma_1^{-1}, 1)$ and $(\gamma_2^{-1}, 1)$, respectively. The limiting distribution above only depends on the extreme value indices γ_1 and γ_2, and does not vary with respect to c. However, in numerical experiments, we must estimate γ_1 and γ_2, which leads to inferior performance of $\tilde{q}_n$ compared with q_n. This is why we propose q_n as the test statistic. Moreover, if we set $c = 1$, then $\tilde{q}_n$ equals q_n, and both share a simplified limiting distribution. Nevertheless, we do not recommend using $c = 1$ in practice, as this would imply that only one observation exceeds the threshold for either X or Y, leading to unreliable inference.

We propose a testing procedure based on Theorem 1. For a given significance level $\alpha \in (0, 1)$, the critical value is obtained via Monte Carlo simulation as the upper α percentile of the distribution of $f(\max(U_1, 1), \max(U_2, 1))$. If the observed value q_n exceeds this critical value, we reject the null hypothesis H_0 at level α and conclude that there is evidence of conditional tail dependence between the two variables of interest, given the covariates. Otherwise, we do not have sufficient statistical evidence to reject H_0. Intuitively, a larger value of q_n implies that either Δ_n or Θ_n is smaller (e.g. close to 1). In this case, there exists co-movement in the tails of X and Y conditional on $\mathbf{Z}$, thus providing evidence against the null hypothesis.

4. Simulation

We conduct simulation studies to evaluate the performance of CTQCC in testing conditional tail independence. Data are generated from the following model:

$$X = 1 + Z + e_1, \quad Y = 1 + 2Z + (1 + rZ)e_2, \tag{4.5}$$

where (e_1, e_2) is a random error vector independent of Z, and Z follows a uniform distribution on $[0, 1]$. The parameter r controls the degree of heteroscedasticity in the model. We consider two distinct cases for the joint distribution of (e_1, e_2).

Case 1: (e_1, e_2) has marginal unit Fréchet distributions and a joint dependence structure specified by a normal copula. We vary the correlation parameter ρ of the copula. It is well known that (e_1, e_2) are tail independent when $\rho < 1$.

Case 2: (e_1, e_2) has marginal unit Fréchet distributions and is coupled through a Gumbel copula with dependence parameter θ. In this setting, (e_1, e_2) exhibit tail dependence for $\theta > 1$.

We set the sample size to $n = 2000$ and consider three extreme quantile levels: $1 - c/n = 0.990$, 0.992, and 0.994. Here, c can not be too small because we need sufficient observations exceeding the high threshold for

Table 1: Type I error rates of the proposed test for Case 1 with significance level 5%.

$r = 0$					$r = 2$				
ρ	k/n	$\tau = 0.990$	$\tau = 0.992$	$\tau = 0.994$	ρ	k/n	$\tau = 0.990$	$\tau = 0.992$	$\tau = 0.994$
0.55	2.5%	4.5	3.4	1.8	0.55	2.5%	4.3	3.1	1.7
	5%	4.2	3.5	2.4		5%	4.4	3.3	2.5
	10%	4.8	3.8	2.9		10%	5.1	4.1	3.1
	15%	4.8	4.1	3.0		15%	5.3	4.6	3.5
0.60	2.5%	5.7	3.8	2.0	0.60	2.5%	5.7	3.7	1.9
	5%	5.6	4.2	2.8		5%	5.5	4.0	3.0
	10%	5.9	4.3	3.3		10%	6.1	4.8	3.3
	15%	6.1	4.7	3.3		15%	6.6	5.2	3.9
0.65	2.5%	7.4	4.7	2.9	0.65	2.5%	7.2	4.7	2.7
	5%	7.5	5.3	3.6		5%	7.3	5.1	3.6
	10%	8.0	5.6	3.7		10%	7.9	5.7	4.2
	15%	7.7	5.5	4.0		15%	8.6	6.1	4.4

both X and Y to ensure reliable estimation. For the tail index estimation, we choose $k/n = 2.5\%$,5%, 10%, and 15%. The simulation is repeated 2000 times for each scenario to assess the performance of the proposed test under varying threshold and estimation settings.

Table 1 reports the percentages of rejecting the null hypothesis of conditional tail independence at the significance level $\alpha = 0.05$ for $\rho = 0.55$, 0.60, and 0.65 in Case 1. Since X and Y are conditionally tail independent in this setting, these proportions correspond to the empirical Type I error rates. As shown in Table 1, the Type I error decreases as ρ approaches

Table 2: Empirical power of the proposed test for Case 2 with significance level 5%.

		$r = 0$					$r = 2$		
θ	k/n	$\tau = 0.990$	$\tau = 0.992$	$\tau = 0.994$	θ	k/n	$\tau = 0.990$	$\tau = 0.992$	$\tau = 0.994$
1.5	2.5%	41.4	31.4	21.2	1.5	2.5%	41.4	31.0	21.1
	5%	41.2	32.6	22.6		5%	40.7	32.4	22.5
	10%	41.3	32.5	24.2		10%	41.7	33.2	24.8
	15%	40.7	32.7	22.9		15%	41.8	33.4	24.5
2.5	2.5%	97.5	94.7	87.6	2.5	2.5%	97.3	94.4	87.7
	5%	97.5	94.7	87.6		5%	97.3	94.5	87.3
	10%	97.0	94.2	87.1		10%	97.2	94.5	86.7
	15%	97.1	94.3	86.4		15%	97.3	94.7	87.4
3.5	2.5%	99.9	99.8	99.2	3.5	2.5%	99.9	99.8	99.0
	5%	99.9	99.8	99.2		5%	99.9	99.8	99.1
	10%	99.9	99.8	98.9		10%	99.9	99.8	99.1
	15%	99.9	99.8	99.0		15%	99.9	99.8	99.1

0, which corresponds to the case of independence. In addition, for most combinations of ρ and τ, the empirical Type I error tends to decrease as k decreases. Similarly, for most combinations of ρ and k, the Type I error tends to decrease as τ increases. Based on these observations, we recommend using a smaller value of k and a larger value of τ to improve test performance under the null.

Table 2 reports the percentages of rejecting the null hypothesis at the same significance level $\alpha = 0.05$ for $\theta = 1.5$, 2.5, and 3.5 in Case 2. Since X and Y are conditionally tail dependent given $\mathbf{Z}$, these proportions correspond to the empirical power of the test. As shown in Table 2, the empirical power increases with θ, reflecting stronger tail dependence. The performance of the test is also relatively robust across a wide range of values for both k and τ.

We also plot the empirical distribution of the CTQCC test statistic under H_0, along with the density of its limiting distribution, for Case 1 with $\tau = 0.994$ in Figure 1. The empirical density of CTQCC closely approximates the limiting distribution across a range of values for r, ρ, and τ. This alignment suggests that CTQCC accurately captures the underlying distributional behavior under the null, reinforcing its utility and effectiveness in practical applications.

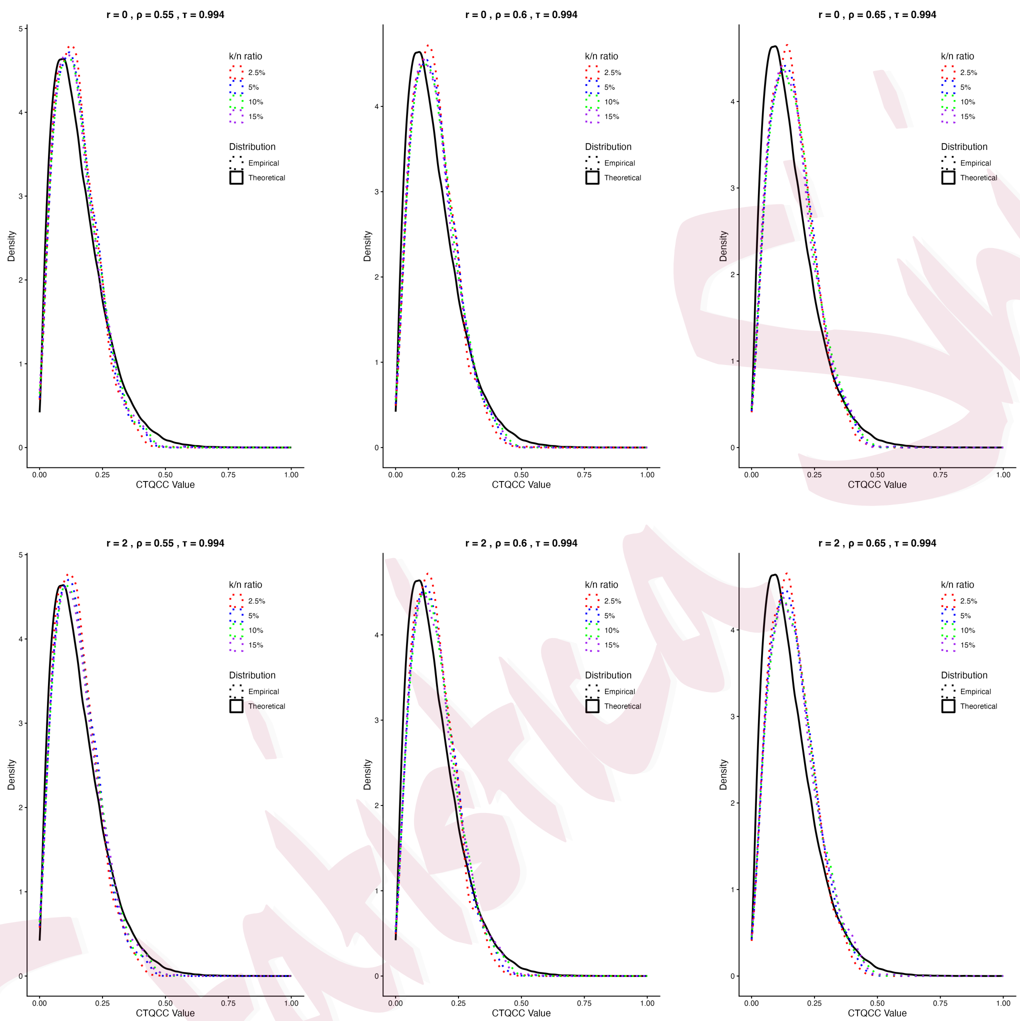


Figure 1: Empirical density of CTQCC for different values of k/n and its limiting distribution under H_0 in Case 1 with $\tau = 0.994$.

5. Real Data Analysis

Understanding the structure of compound climate extremes—such as simultaneous extremes in precipitation and wind speed—is crucial for assessing climate risks. In this section, we apply our proposed method for testing conditional tail independence to the Daily Global Historical Climatology Network (GHCN-Daily) dataset. GHCN-Daily provides daily updates on climatic variables from over 80,000 stations across 180 countries and territories, and is publicly accessible through the NOAA National Climatic Data Center. A detailed description of the dataset can be found in Menne et al. (2012). For our analysis, we use daily data on precipitation (X), average wind speed (Y), and daily maximum temperature (Z) collected from 317 stations across the United States, covering the period from 2011 to 2023.

Before investigating the conditional tail dependence structure, we first examine the unconditional tail dependence between daily precipitation and average wind speed using the TQCC test. The resulting p-values are plotted in Figure 2. As shown, most stations exhibit near-zero p-values, indicating strong evidence against the null hypothesis of tail independence. This suggests that extreme precipitation events are more likely to occur when wind speeds are also extreme at many of these stations.

However, both precipitation and wind speed may be influenced by temperature,a relationship established in several studies, including Trenberth and Shea (2005), Fujibe (2009) and Qu et al. (2012). To further investigate the role of temperature, we stratify the data into two groups: days with maximum temperature above 20°C and days with maximum temperature less than or equal to 20°C. Our goal is to test for conditional tail independence between precipitation and wind speed, given the level of maximum daily temperature.

The analysis is restricted to stations with at least 100 observations in each subgroup after stratification. Figure 5a, 5b and 5c display empirical probability density distribution of p-values of TQCC test and CTQCC tests under the two scenarios, respectively. We note that for all the three cases, the distributions of p-values is heavily concentrated around zero. A closer examination reveals distinct patterns: a majority of stations that are significant in the unconditional TQCC test lose significance under a specific conditional framework. Notably, of all the stations, 55.22% are significant in the unconditional TQCC case but lose significance under the conditional test given daily maximum temperature greater than 20°C. Similarly, 57.83% of the stations are significant in the unconditional case but insignificant under CTQCC given daily maximum temperature less than or equal to 20°C. There is an overlapping of 35.65% significant in the TQCC

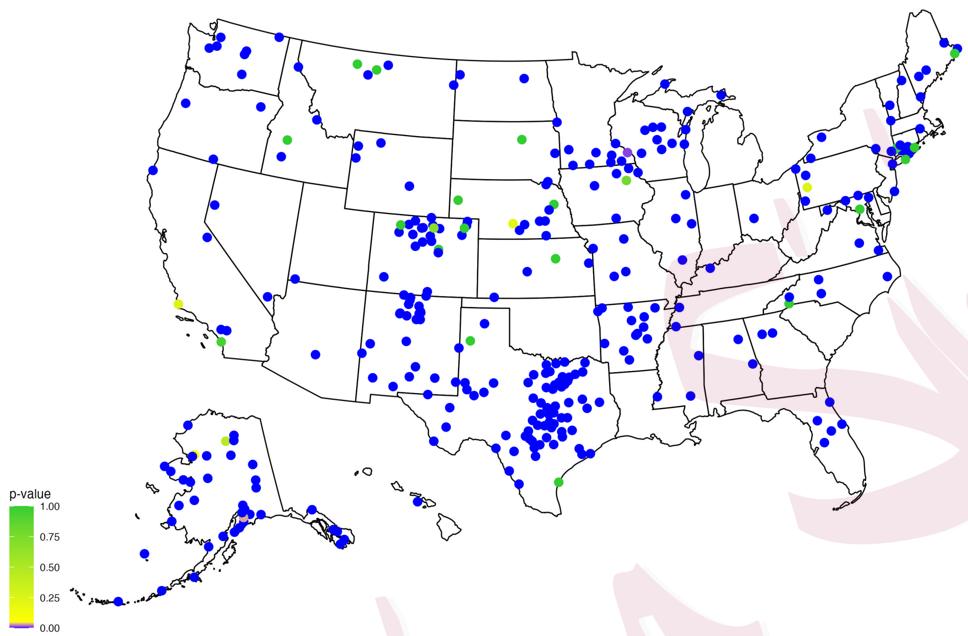


Figure 2: TQCC test for tail independence between daily precipitation and average daily wind speed.

case but not significant under both conditional frameworks.

We also observe interesting regional weather patterns. For example, stations in Alaska exhibit strong tail dependence between precipitation and wind speed in both the unconditional case and when the maximum temperature is at or below 20°C. However, when the temperature exceeds 20°C, the null hypothesis of conditional tail independence is not rejected,

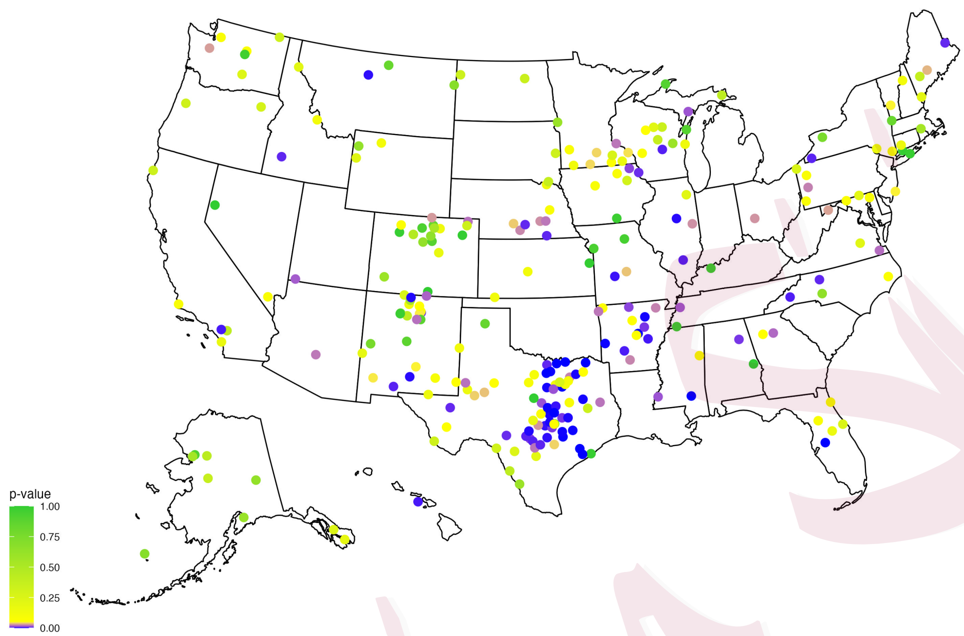


Figure 3: CTQCC test for conditional tail independence given daily maximum temperature greater than 20°C.

suggesting that compound extremes are less likely to occur in Alaska on warmer days. This pattern stands in direct contrast to the weather patterns in Texas. Stations in Texas also show significant unconditional tail dependence, but demonstrate significant conditional tail dependence specifically when temperatures exceed 20°C, while showing insignificance when the maximum temperature is at or below 20°C. This comparison under-

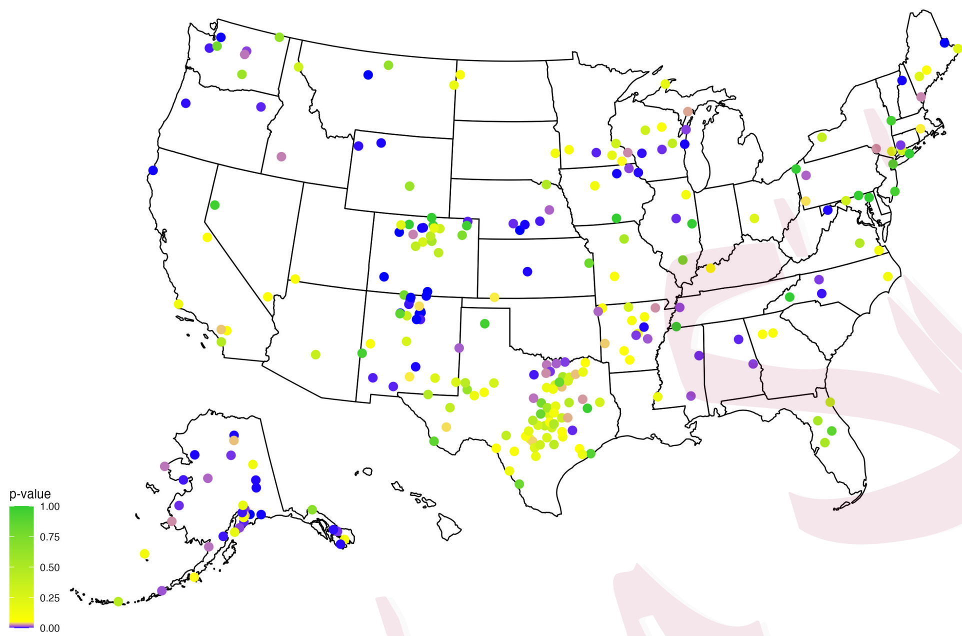
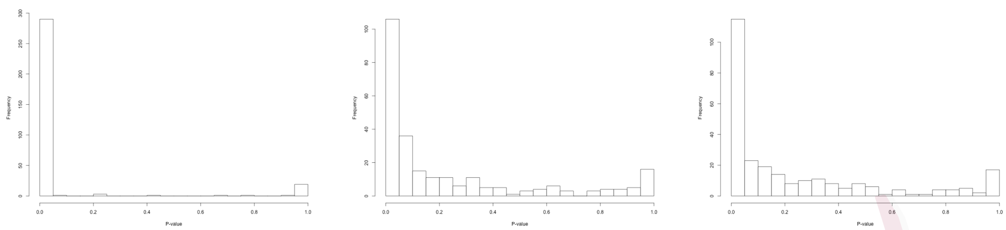


Figure 4: CTQCC test for conditional tail independence given daily maximum temperature less than or equal to 20°C.

scores how the drivers of compound extremes can be highly region-specific and conditional on environmental factors.

6. Conclusion

In this paper, we introduced a new measure and test procedure for characterizing conditional tail dependence, formulated through the conditional



(a) (b) (c)

Figure 5: Empirical pdf of p-values for: (a) TQCC test for tail independence between daily precipitation and average daily wind speed; (b) CTQCC test for conditional tail independence given daily maximum temperature greater than 20°C; (c) CTQCC test for conditional tail independence given daily maximum temperature less than or equal to 20°C.

tail dependence index $\lambda(\mathbf{Z})$, which captures the extremal dependence structure conditional on covariates. Building on this framework, we proposed a statistical testing procedure for conditional tail independence based on a novel test statistic, the Conditional Tail Quotient Correlation Coefficient (CTQCC), which generalizes the previously developed Tail Quotient Correlation Coefficient (TQCC). We derived the asymptotic distribution of CTQCC under the null hypothesis and evaluated its performance through extensive simulation studies. Finally, we applied our method to the GHCN-Daily dataset to examine compound climate extremes. The empirical results demonstrate that the proposed test performs well in detecting conditional tail dependence across a range of settings.

One limitation of the current methodology is its reliance on the assumption that the sample is independent and identically distributed (i.i.d.). In real-world applications, data may exhibit temporal dependence or non-stationarity, which can affect the validity of the test. One way to address this issue is to first fit an ARMA-GARCH model for capturing the evolution of the conditional mean and variance of the underlying stochastic process, assuming a parametric model for innovations (see, e.g. McNeil and Frey (2000) and Nolde et al. (2022)), followed by calculating CTQCC with the sample of realized innovations. Extending the current methodology to accommodate non-i.i.d. data is an interesting direction for future research.

Supplementary Materials

The supplementary material contains the proofs of Proposition 1 and Theorem 1.

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