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A VARIATION-RATIO TEST FOR VOLATILITY JUMPS USING NOISY HIGH FREQUENCY DATA

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Abstract: This paper proposes a novel variation-ratio test for the presence of volatility jumps using high frequency data with microstructure noise. Under the null hypothesis that there are no volatility jumps, the test statistic is asymptotically normal. Under the alternative hypothesis that volatility jumps are present, the test statistic diverges at a rate of $n^{1/4-\iota}$ (for arbitrarily small $\iota > 0$), which is faster than the best rate available in the literature (approximately $n^{1/8}$), where n denotes the number of observed returns per unit time. Simulation results corroborate our theoretical findings. Empirical results show that, while modeling volatility as a continuous semimartingale is appropriate for a substantial subset of the 90 U.S. stocks analyzed, a notable portion exhibits features indicative of volatility jumps.

Key words and phrases: Volatility Jumps, Central Limit Theorem, High Frequency Data, Microstructure Noise, Semimartingale.

1. Introduction

Continuous-time stochastic models are widely used in finance and economics to describe the dynamics of asset prices. In the high frequency financial econometrics literature, there is a large body of research on testing the specification of asset price processes. An important aspect of this research involves testing for the presence and characteristics of price jumps. Seminal papers in this regard include, among others, Aït-Sahalia (2002) whose test is based on comparing the transition density and its nonparametric estimator; Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen and Shephard (2006) and Dovonon et al. (2019) who build tests upon the difference between power and multipower variations; Aït-Sahalia and Jacod (2009) and Fan and Fan (2011) who construct tests using the ratio of variations computed at different sampling frequencies; Lee and Mykland (2008) whose test is built on standardized high frequency returns; and the test for constancy of jump activity index of Todorov (2017). Understanding price jumps is crucial, as demonstrated by the seminal work of Todorov (2010), which highlights their key role in explaining the dynamics of the market variance risk premium. However, high frequency data are observed with microstructure noise due to price discretization, bid-ask spread, etc. In the presence of microstructure noise, Aït-Sahalia et al. (2012) and Liu et al. (2020) test for the presence of price jumps using the ratio of power variations constructed from increments of preaveraged prices. Dumitru and Urga (2014) and Maneesoonthorn et al. (2020) provide comprehensive surveys and compare the performance of popular price jump testing methods. For more tests related to price jumps, see, among others, Aït-Sahalia and Jacod (2011), Jing et al. (2012), Aït-Sahalia and Jacod (2012), Kong et al. (2015), Caporin et al. (2017), Aït-Sahalia and Jacod (2018), Dungey et al. (2018), and Kong (2019).

There has also been a sizeable literature on volatility specification test. Under the null hypothesis where the volatility process is of a parametric form, Corradi and White (1999) propose a test statistic that exploits the difference between nonparametric spot volatility estimates and their parametric counterparts. Tests that improve upon the methodology of Corradi and White (1999) include Dette et al. (2006), Vetter (2015), Bull (2017), and Chen et al. (2019). Note that the aforementioned methods do not take microstructure noise into account. In the presence of microstructure noise, Vetter and Dette (2012), Papanicolaou and Giesecke (2016), and Liu, Liu, and Zhang (2022) propose various specification tests for volatility processes. These testing methods are only applicable to null hypothesis where volatility processes are of particular parametric forms.

However, a key concern is whether the parametric specifications of volatility processes are appropriate in the first place. This paper seeks to test for the pres-

Among others, Jacod and Todorov (2010) proposes a novel methodology for testing the presence of co-jumps in price and volatility processes.

ence of jumps in volatility processes before advancing to the volatility specification tests outlined earlier. Testing for volatility jumps, in contrast to price jumps, has received significantly less attention in the literature. Nonetheless, there is compelling empirical evidence for the presence of volatility jumps, as indicated by the findings of Todorov and Tauchen (2011). In the absence of microstructure noise, Bibinger et al. (2017) is the first to develop theoretical results for testing volatility jumps using high frequency data. The authors construct a Gumbel-type test statistic based on spot volatility estimators. Liu, Liu, and Lin (2022) build their test on power variations of spot volatility estimates, again assuming the absence of noise. Bibinger and Madensoy (2019) later extend the Gumbel-type test of Bibinger et al. (2017) to the case where prices are contaminated by noise. However, as evidenced by our simulation and empirical studies, the Gumbel-type test proposed by Bibinger and Madensoy (2019) tends to be less powerful than our method. Note that, under the semimartingale setup, the divergence rate of the Gumbel-type test statistic proposed by Bibinger and Madensoy (2019) is approximately $n^{1/8}$, which we aim to improve upon in this paper, where n denotes the number of observed returns per unit time.

In this paper, we propose a novel volatility jump test in the presence of mi-

More recent nonparametric stationarity and/or change-point tests addressing intraday periodicity in volatility include, among others, Christensen et al. (2018), Andersen et al. (2019), Andersen et al. (2024), Todorov and Zhang (2024), and Kokoszka et al. (2024).

crostructure noise. The test statistic is constructed as a ratio of two power variations, computed at different time scales, based on spot volatilities estimated using the threshold pre-averaging method. Under the null hypothesis that there are no volatility jumps, the feasible standardized test statistic converges to a standard normal random variable. Under the alternative hypothesis that volatility jumps are present, the test statistic diverges at rate $n^{1/4-\iota}$, where n is the sample size and ι is an arbitrarily small positive number. Compared with the Gumbel-type test of Bibinger and Madensoy (2019), our variation-ratio test statistic diverges at a faster rate under the alternative hypothesis. Simulation results corroborate our theoretical findings, demonstrating that the proposed test exhibits desirable size and power properties, and outperforms the test of Bibinger and Madensoy (2019) in terms of power.

The remainder of this paper is organized as follows. Section 2 introduces the model setup and assumptions. Theoretical results on the estimation of spot volatilities and the variations of spot volatility estimates are presented in Sections 3.1 and 3.2. The proposed testing procedure and its associated asymptotic theory are developed in Section 3.3, followed by a comparison with an existing method in Section 3.4. Choices of tuning parameters for implementing the test are discussed in Section 3.5. Sections 4 and 5 present a simulation study and an empirical application, respectively. We conclude in Section 6. Additional theoretical results and all proofs are provided in the supplementary material.

2. Setup and Assumptions

Let the latent log price process and its volatility process be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and suppose they follow the jump-diffusion type model below:

$$\begin{cases} dX_t = b_t\, dt + \sigma_t\, dW_t + dJ_t, \\ d\nu_t = \widetilde{b}_t\, dt + \widetilde{\sigma}_t\, d\widetilde{W}_t + d\widetilde{J}_t, \quad \nu_t = \sigma_t^2, \end{cases} \tag{2.1}$$

where the drift processes $b, \widetilde{b}$ and the volatility processes $\sigma, \widetilde{\sigma}$ are all adapted, locally bounded, and càdlàg; W and $\widetilde{W}$ are (correlated) standard Brownian motions; and the pure jump processes J and $\widetilde{J}$ are given by $J_t = \delta * p_t$ and $\widetilde{J}_t = \widetilde{\delta} * \widetilde{p}_t$, where p and $\widetilde{p}$ are two Poisson random measures on $\mathbb{R}_+ \times \mathbb{E}$ with $(\mathbb{E}, \mathcal{E})$ being an auxiliary Polish space. The compensators of the two Poisson random measures are denoted by $dt \otimes q(dz)$ and $dt \otimes \widetilde{q}(dz)$, respectively. δ and $\widetilde{\delta}$ are two predictable functions on $\Omega \times \mathbb{R}_+ \times \mathbb{E}$. The observed prices $(Y_{t_i})_{i\geq 1}$ are given as follows:

$$Y_{t_i} = X_{t_i} + \varepsilon_i, \tag{2.2}$$

where t_i's are observation times over a finite time interval $[0, T]$, and ε_i represents the time-t_i market microstructure noise induced by the bid-ask spread, discreteness of price changes etc. Equation (2.2) specifies that the observed prices equal the latent prices X_{t_i} contaminated by microstructure noise, which is assumed to occur only at the observation times t_i and to be independent of the latent log-price and volatility processes. In this paper, we only consider the equidistant case, i.e.,

$t_i = i/n$ for $i = 0, 1, \cdots [nT]$, where $[x]$ denotes the integer part of x. We shall test whether a sample path of the latent volatility process σ in (2.1) is continuous over the finite horizon $(0, T]$. Since neither σ nor X is directly observable, the test is conducted based on the observed prices Y_{t_i} in (2.2).

We formally state the assumptions regarding the latent price process, the volatility process, and the microstructure noise as follows.

Assumption 1. The observed log price process and the latent log price process are given by (2.2) and (2.1), respectively.

(i) $\sigma_t > 0$ and $\widetilde{\sigma}_t > 0$ for all $t \geq 0$.

(ii) For some $0 < \beta_1, \beta_2 < 1$, $|\delta(\omega, t, z)|^{\beta_1} \leq \overline{\delta}_n(z)$ and $|\widetilde{\delta}(\omega, t, z)|^{\beta_2} \leq \overline{\delta}_n(z)$ whenever $t \leq \tau_n(\omega)$, where τ_n is a sequence of stopping times increasing to ∞, and $\overline{\delta}_n(z)$ is a sequence of deterministic functions satisfying $\int \overline{\delta}_n(z) q(dz) < \infty$ and $\int \overline{\delta}_n(z) \widetilde{q}(dz) < \infty$.

(iii) The noise variables ε_i, $i = 0, 1, \cdots [nT]$ are independent and identically distributed with mean zero, variance σ_ϵ^2 and finite moments of order $\vartheta > 2$.

Assumptions similar to our Assumption 1 are commonly adopted in the high-frequency financial econometrics literature (see, e.g., Wang and Mykland, 2014; Aït-Sahalia and Jacod, 2014). In particular, the assumptions imposed on the jump components J and $\widetilde{J}$ imply that they are of finite variation.

To formally state the hypothesis tested in this paper, we begin by defining two complementary sets for any fixed $t \in (0, T]$:

$$\begin{cases} \Omega_{t,0} := \{\omega : \omega \in \Omega, \sigma_s(\omega) \text{ is continuous on } (0,t]\}; \\ \Omega_{t,1} := \{\omega : \omega \in \Omega, \sigma_s(\omega) \text{ is discontinuous on } (0,t]\}. \end{cases} \tag{2.3}$$

With the notation in (2.3), we are now prepared to state our hypotheses as follows:

$$H_0 : \omega \in \Omega_{T,0} \quad vs. \quad H_1 : \omega \in \Omega_{T,1}. \tag{2.4}$$

That is, under the null hypothesis H_0, the volatility path up to time T is continuous, whereas under the alternative H_1, it jumps. Throughout, we focus only on the nontrivial case that $0 < P(\Omega_{T,0}) < 1$. See Aït-Sahalia and Jacod (2009) for a related discussion on the trivial case where, under the null, the probability is 0 or 1 in testing for price jumps. We construct a nonparametric test statistic for testing the pair of hypotheses defined in (2.4).

3. Testing Procedure and Theoretical Results

This section is organized into five subsections. The first two provide preliminary results on spot volatility estimation and the theoretical foundations of power variations based on these estimates. In Subsection 3.3, we introduce the proposed test along with its asymptotic theory. Subsection 3.4 presents a comparison between our method and that of Bibinger and Madensoy (2019). Finally, Subsection 3.5 discusses the selection of tuning parameters.

3.1 Estimating Spot Volatilities

The building blocks of our test method are estimates of spot volatilities. We adopt the pre-averaging method (Jacod et al., 2009) for estimating spot volatilities in the presence of microstructure noise. We next briefly introduce this method which involves a kernel function and a moving window. Let $g(x), x \in [0,1]$ be the kernel function and k_n the number of observations within the moving window. The kernel function g satisfies: g is continuous, piecewise C^1 with a piecewise Lipschitz derivative g', $g(0) = g(1) = 0$, and $\int_0^1 g(s)^2 ds > 0$. We need some additional notation associated with the weight function g:

$$\begin{cases} \phi_1(s) = \int_s^1 g'(u)g'(u-s)du, \quad \phi_2(s) = \int_s^1 g(u)g(u-s)du, \ for \ s \in [0,1] \\ \phi_1(s) = \phi_2(s) = 0, \ for \ s > 1 \\ \Phi_{ij} = \int_0^1 \phi_i(s)\phi_j(s)ds, \quad \psi_i = \phi_i(0), \ for \ i,j = 1,2. \end{cases}$$

For any process V, set

$$\overline{V}_i^n = \sum_{j=1}^{k_n-1} g_j^n \Delta_{i+j}^n V, \quad g_j^n = g(j/k_n), \quad \Delta_i^n V = V_{i/n} - V_{(i-1)/n}.$$

Note that $\overline{V}_i^n$ is a local weighted average of increments of V. We shall choose $g(x) = x \wedge (1-x)$ in our numerical studies. The local weighted average of increments of V, i.e., $\overline{V}_i^n$, associated with this particular choice of g takes the following form

$$\overline{V}_i^n = \frac{1}{2}\left(\frac{1}{k_n'}\sum_{j=0}^{k_n'-1} V_{i+j+k_n'}^n - \frac{1}{k_n'}\sum_{j=0}^{k_n'-1} V_{i+j}^n\right),$$

for $k_n = 2k_n'$ where k_n' is an integer, and

$$\psi_1 = 1, \quad \psi_2 = \frac{1}{12}, \quad \Phi_{11} = \frac{1}{6}, \quad \Phi_{12} = \frac{1}{96}, \quad \Phi_{22} = \frac{151}{80640}.$$

In our theoretical derivations, we shall always treat g as a general kernel function satisfying the aforementioned conditions.

Our spot volatility estimator is approximately given by differentiating the threshold pre-averaging integrated volatility estimator of Jing et al. (2014) with respect to time. Let l_n be the number of price increments (with time step $1/n$) in a local window that is wider than the pre-averaging moving window (i.e., $l_n > k_n$). The threshold pre-averaging integrated volatility estimator over the time interval $[i/n, (i + l_n)/n]$ is given by

$$IV_{i,l_n}^{PA} = \frac{1}{\psi_2 k_n} \sum_{j=0}^{l_n - k_n + 1} (\overline{Y}_{i+j}^n)^2 1_{|\overline{Y}_{i+j}^n| < u_{1,n}} - \frac{\psi_1}{2\psi_2 k_n^2} \sum_{j=0}^{l_n} (\Delta_{i+j}^n Y)^2,$$

where $u_{1,n}$ is the threshold for removing the impact of price jumps. Under mild conditions, $IV_{0,[nt]}^{PA}$ is a consistent estimator of the integrated variance $\int_0^t \nu_s ds$. A natural estimator for the spot variance ν_t is then given by

$$\widehat{\nu}_t = \frac{1}{l_n/n} IV_{i,l_n}^{PA}, \quad \text{for } t \in ((i-1)/n, i/n].$$

Note that the observations over the time interval $[i/n, (i + l_n)/n]$ are used to estimate ν_t for $t \in ((i-1)/n, i/n]$. Throughout the paper, we set

$$l_n = [c_1 n^{\theta_1}],\ u_{1,n} = c_2 n^{-\theta_2},\ k_n = [c_3 n^{\theta_3}],\ \frac{1}{2} < \theta_1 < 1,\ 0 < \theta_2 < \frac{1}{4},\ c_i > 0, \quad (3.5)$$

where $c_i, i = 1, 2, 3$, are fixed constants, and $\theta_3 = 1/2$. In what follows, when volatility is continuous, we consider three subregions of θ_1, namely $(1/2, 3/4)$, $\{3/4\}$, and $(3/4, 1)$, since each subregion leads to different asymptotic behavior for the power variations of spot volatility estimates. Define

$$\gamma_t^2 = \frac{4}{\psi_2^2}\left[\Phi_{22} c_3 \sigma_t^4 + \frac{2\Phi_{12}}{c_3}\sigma_\epsilon^2 \sigma_t^2 + \frac{\Phi_{11}}{c_3^3}\sigma_\epsilon^4\right].$$

By Jacod et al. (2009), $\int_0^t \gamma_s^2 ds$ is the asymptotic variance of the pre-averaging estimator of the integrated variance $\int_0^t \nu_s ds$.

As is commonplace in the high frequency financial econometrics literature, we also need *stable convergence* in law. Suppose that a sequence of variables (Y_n) is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sub-sigma-field $\mathcal{G} \subseteq \mathcal{F}$, and that the limiting variable Y is defined on an extended probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. We say that Y_n converges $\mathcal{G}$-stably in law to Y if, for all bounded $\mathcal{G}$-measurable variables Z and all bounded functions g, $Eg(Y_n)Z \to \widetilde{E}g(Y)Z$. If $\mathcal{G} = \mathcal{F}$, we simply say that Y_n converges stably in law to Y and write $Y_n \xrightarrow{L-s} Y$. For more details, see, for example, Jacod and Protter (2012).

3.2 Power Variations of Spot Volatility Estimates

In this subsection, we shall provide preliminary theoretical results on power variations of spot volatility estimates. To this end, for any fixed $t \in (0, T]$, we define

$$V_p(\nu; c_1, \theta_1)_t^n = \sum_{i=0}^{[nt]-2l_n} |\widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n}|^p, \quad p > 0, \tag{3.6}$$

$$TV_p(\nu; c_1, \theta_1)_t^n = \sum_{i=0}^{[nt]-2l_n} |\widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n}|^p 1_{|\widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n}| < u_{2,n}}, \tag{3.7}$$

where $u_{2,n} = c_4 n^{-\theta_4}$ is the threshold for truncating volatility jumps and $c_4, \theta_4 > 0$ are two constants. When the volatility σ is continuous, the asymptotic properties of the variation in (3.6) are summarized in the following proposition.

Proposition 1. *Suppose that Assumption 1 with $\vartheta > p$ and $0 < \theta_2 < \frac{1}{4}$ hold, and, in addition, that the volatility σ is continuous.*

(1) When $\frac{1}{2} < \theta_1 < \frac{3}{4}$, suppose that $p > 0$ and $\frac{2\theta_1 - 1}{4(2-\beta_1)} < \theta_2$ hold. Then, for any fixed $t \in (0, T]$,

$$n^{(\theta_1 - \frac{1}{2})\frac{p}{2} - 1} c_1^{\frac{p}{2}} V_p(\nu; c_1, \theta_1)_t^n \xrightarrow{P} \mu_p 2^{\frac{p}{2}} \int_0^t \gamma_s^p ds, \quad n \to \infty, \tag{3.8}$$

where $\mu_p = E|N(0,1)|^p$.

(2) When $\theta_1 = \frac{3}{4}$, suppose that $p > 0$ and $\frac{1}{8(2-\beta_1)} < \theta_2$ hold. Then, for any fixed $t \in (0, T]$,

$$n^{\frac{p-8}{8}} c_1^{-\frac{p}{2}} V_p(\nu; c_1, \theta_1)_t^n \xrightarrow{P} \mu_p \int_0^t \left(\frac{2}{3}\widetilde{\sigma}_s^2 + \frac{2}{c_1^2}\gamma_s^2 \right)^{\frac{p}{2}} ds. \tag{3.9}$$

(3) When $\frac{3}{4} < \theta_1 < 1$, suppose that $p > 0$ and $\frac{1-\theta_1}{2(2-\beta_1)} < \theta_2$ hold. Then, for any fixed $t \in (0, T]$,

$$n^{(1-\theta_1)\frac{p}{2}-1} c_1^{-\frac{p}{2}} V_p(\nu; c_1, \theta_1)_t^n \xrightarrow{P} \mu_p \left(\frac{2}{3}\right)^{\frac{p}{2}} \int_0^t \widetilde{\sigma}_s^p ds. \tag{3.10}$$

The next proposition establishes the asymptotic properties of the variation in (3.6) when the volatility σ contains a jump component.

Proposition 2. *Suppose that Assumption 1 with $\vartheta > p$, $\frac{1}{2} < \theta_1 < 1$, $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1 - 1}\}$, and $\max\{\frac{1-\theta_1}{p(2-\beta_1)}, \frac{(1-2\theta_1)p+1}{2(2p-\beta_1)}\} < \theta_2 < \frac{1}{4}$ hold. Then, for any fixed $t \in (0, T]$,*

$$c_1^{-1} n^{-\theta_1} V_p(\nu; c_1, \theta_1)_t^n \xrightarrow{P} \sum_{0 \le s \le t} |\Delta \widetilde{J}_s|^p, \tag{3.11}$$

where $\Delta \widetilde{J}_s = \widetilde{J}_s - \widetilde{J}_{s-}$ is the jump size at time s.

The asymptotic properties of the variation in (3.7) are summarized in the following proposition.

Proposition 3. *Suppose that Assumption 1 with $\vartheta > p$, $\frac{1}{2} < \theta_1 < 1$, $0 < \theta_2 < \frac{1}{4}$ and $0 < \theta_4 < \min\{\frac{1-\theta_1}{2}, \frac{2\theta_1 - 1}{4}, (2-\beta_1)\theta_2\}$ hold.*

(1) When $\frac{1}{2} < \theta_1 < \frac{3}{4}$, suppose that either one of the following two conditions holds: (i) $0 < p \le \beta_2$ and $\frac{2\theta_1 - 1}{4(2-\beta_1)} < \theta_2$; (ii) $\beta_2 \le p$, $\beta_2 < \frac{1-\theta_1}{2\theta_1 - 1}$, $\frac{2\theta_1 - 1}{4(2-\beta_1)} < \theta_2$ and $\frac{p(2\theta_1 - 1) + 4(\theta_1 - 1)}{4(p-\beta_2)} < \theta_4$. Then, the result in (3.8) holds with $V_p(\nu; c_1, \theta_1)_t^n$ being replaced with $TV_p(\nu; c_1, \theta_1)_t^n$ for any fixed $t \in (0, T]$.

(2) When $\theta_1 = \frac{3}{4}$*, suppose that either one of the following two conditions holds: (i)* $0 < p \leq \beta_2$ *and* $\frac{1}{8(2-\beta_1)} < \theta_2$*; (ii)* $\beta_2 \leq p$, $\frac{1}{8(2-\beta_1)} < \theta_2$ *and* $\theta_4 > \frac{p+8\theta_1-8}{8(p-\beta_2)}$*. Then, the result in (3.9) holds with* $V_p(\nu; c_1, \theta_1)_t^n$ *being replaced with* $TV_p(\nu; c_1, \theta_1)_t^n$ *for any fixed* $t \in (0, T]$*.*

(3) When $\frac{3}{4} < \theta_1 < 1$*, suppose that either one of the following two conditions holds: (i)* $0 < p \leq \beta_2$ *and* $\frac{1-\theta_1}{2(2-\beta_1)} < \theta_2$*; (ii)* $\beta_2 \leq p$, $\frac{1-\theta_1}{2(2-\beta_1)} < \theta_2$ *and* $\frac{(1-\theta_1)(p-2)}{2(p-\beta_2)} < \theta_4$*. Then, the result in (3.10) holds with* $V_p(\nu; c_1, \theta_1)_t^n$ *being replaced with* $TV_p(\nu; c_1, \theta_1)_t^n$ *for any fixed* $t \in (0, T]$*.*

Remark 1. In the case where the volatility process does not jump, Li et al. (2022) obtained the result in (3.9) for $p = 2$ and $p = 4$.

Remark 2. When the price process X and the volatility process σ are continuous, we have $\widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n} = \eta(1)_i^n + \eta(2)_i^n$, where

$$\begin{aligned} \eta(1)_i^n &= \frac{1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [c_{i+l_n+j}^n - c_{i+j}^n], \quad c_i^n = \sum_{j=1}^{k_n-1} (g_j^n)^2 \int_{(i+j-1)/n}^{(i+j)/n} \sigma_s^2 ds, \\ \eta(2)_i^n &= \widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n} - \eta(1)_i^n. \end{aligned}$$

Then, by Lemma 1 and (S2.21) in supplementary material, we have

$$E_i^n(\eta(1)_i^n)^2 = \frac{2}{3}\widetilde{\sigma}_{i\delta_n}^2(1 + o_P(1)) l_n \delta_n, \quad E_i^n(\eta(2)_i^n)^2 = 2\gamma_{i\delta_n}^2(1 + o_P(1))\frac{n^{1/2}}{l_n},$$

where E_i^n denotes the conditional expectation with respect to the σ-field $\mathcal{F}_{i\delta_n}$. When $\frac{1}{2} < \theta_1 < \frac{3}{4}$, because $\eta(2)_i^n$, which is related to the error $\gamma_{i\delta_n}^2$ arising from estimating the integrated volatility by the preaveraging method, is the dominant term in the increment $\widehat{\nu}_{(i+l_n)/n} - \widehat{\nu}_{i/n}$, γ_t^2 determines the limiting behaviour of $V_p(\nu; c_1, \theta_1)_t^n$. When $\frac{3}{4} < \theta_1 < 1$, because $\eta(1)_i^n$, which is related to the volatility of volatility $\widetilde{\sigma}$, is the dominant term in the increment, $\widetilde{\sigma}$ determines the limiting behaviour of $V_p(\nu; c_1, \theta_1)_t^n$. When $\theta_1 = \frac{3}{4}$, because the two terms have the same order of magnitude, they both appear in the limit.

When the volatility process contains a jump component and we set $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1 - 1}, \frac{2(1-\theta_1)}{1-2\beta_1\theta_2}\}$, the jump component of volatility dominates the power variation in (3.6) and thus determines the limit.

3.3 A Variation-Ratio Test

To test for the hypotheses in (2.4), we define our (infeasible) variation-ratio statistic as follows,

$$\breve{T}_n := \frac{V_p(\nu; c_1, \theta_1)_T^n}{V_p(\nu; 2c_1, \theta_1)_T^n}. \tag{3.12}$$

When $\frac{1}{2} < \theta_1 < \frac{3}{4}$ and $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1 - 1}\}$, we have $\frac{(1-2\theta_1)p+1}{2(2p-\beta_1)} < 0$. Suppose further that $\max\left\{\frac{2\theta_1 - 1}{4(2-\beta_1)}, \frac{1-\theta_1}{p(2-\beta_1)}\right\} < \theta_2 < \frac{1}{4}$, by the results in (3.8) and (3.11), we readily obtain

$$\breve{T}_n \xrightarrow{P} \begin{cases} 2^{\frac{p}{2}}, & \text{conditional on the set } \Omega_{T,0}; \\ 2^{-1}, & \text{conditional on the set } \Omega_{T,1}. \end{cases} \tag{3.13}$$

Hence, the (infeasible) test statistic $\breve{T}_n$ is capable of effectively distinguishing between the null and alternative hypotheses.

To present asymptotic properties of $\breve{T}_n$, we need some additional notation. First, for $\mathcal{H}_1, \mathcal{H}_2 \sim N(0,1)$ with the correlation coefficient $\rho = \text{Corr}(\mathcal{H}_1, \mathcal{H}_2) = E\mathcal{H}_1\mathcal{H}_2$, we define

$$h_p(\rho) := \text{cov}(|\mathcal{H}_1|^p, |\mathcal{H}_2|^p). \tag{3.14}$$

Second, we define

$$\begin{aligned}
\rho_1(s) &:= (1-s)1_{0\le s<1} + \left(\frac{s}{2}-1\right)1_{1\le s\le 2},\\
\rho_2(s) &:= 2^{-\frac{3}{2}}\left[-(4+s)1_{-4\le s<-3} + (2+s)1_{-3\le s<-2} + (4+2s)1_{-2\le s<-1}\right.\\
&\qquad \left. -2s1_{-1\le s<0} - s1_{0\le s<1} + (s-2)1_{1\le s<2}\right],
\end{aligned}$$

where $\rho_1(s) = 0$ for $s \notin [0,2]$ and $\rho_2(s) = 0$ for $s \notin [-4,2]$. Note that $\rho_k(s)$ is the limit of the correlation between $\sum_{j=0}^{l_n-k_n-1}((\bar{\epsilon}^n_{j+l_n})^2 - (\bar{\epsilon}^n_j)^2)$ and $\sum_{j=0}^{kl_n-k_n-1}((\bar{\epsilon}^n_{i+j+kl_n})^2 - (\bar{\epsilon}^n_{i+j})^2)$ for $k=1,2$ as $i/l_n \longrightarrow s$, i.e.,

$$\begin{aligned}
\rho_k(s) &:= \lim_{n\to+\infty} \rho_k\left(\frac{i}{l_n}\right)\\
&= \lim_{n\to+\infty} \text{Corr}\left(\sum_{j=0}^{l_n-k_n-1}((\bar{\epsilon}^n_{j+l_n})^2 - (\bar{\epsilon}^n_j)^2),\ \sum_{j=0}^{kl_n-k_n-1}((\bar{\epsilon}^n_{i+j+kl_n})^2 - (\bar{\epsilon}^n_{i+j})^2)\right).
\end{aligned}$$

This is a direct consequence of the fact that the sequences

$$k_n^{1/2} l_n^{-1/2} \sum_{j=0}^{l_n-k_n-1}((\bar{\epsilon}^n_{j+l_n})^2 - (\bar{\epsilon}^n_j)^2) \quad \text{and} \quad k_n^{1/2} l_n^{-1/2} \sum_{j=0}^{kl_n-k_n-1}((\bar{\epsilon}^n_{i+j+kl_n})^2 - (\bar{\epsilon}^n_{i+j})^2)$$

converge jointly in distribution to a bivariate normal random vector with correlation coefficient $\rho_k(s)$, together with the uniform integrability of the two sequences.

The following theorem presents the asymptotic behavior of the (infeasible) test statistic $\breve{T}_n$ under the null hypothesis.

Theorem 1. *Suppose that Assumption 1 with $\vartheta > 2p$, $p > 1$, $\frac{1}{2} < \theta_1 < \frac{3}{4}$ and*

$$\max\left\{\frac{2(p-1)\theta_1 + 2 - p}{4p(2-\beta_1)}, \frac{p - 2\theta_1(p-1)}{4(2p-\beta_1)}\right\} < \theta_2 < \frac{1}{4}$$

hold, and, in addition, that the volatility σ is continuous. Then,

$$n^{\frac{1-\theta_1}{2}}\left(\breve{T}_n - 2^{\frac{p}{2}}\right) \xrightarrow{L-s} \int_0^T \Sigma_s dB_s,$$

where B is a standard Brownian motion independent of $\mathcal{F}$ and

$$\int_0^T \Sigma_s^2 ds = c_1 d_p \frac{\int_0^T \gamma_s^{2p} ds}{(\int_0^T \gamma_s^p ds)^2},\ d_p = \frac{2^{p+1}\left[3\int_0^2 h_p(\rho_1(s))ds - \int_{-4}^2 h_p(\rho_2(s))ds\right]}{\mu_p^2}.$$

Furthermore,

$$n^{\frac{1-\theta_1}{2}} \frac{\left(\breve{T}_n - 2^{\frac{p}{2}}\right)}{\sqrt{\int_0^T \Sigma_s^2 ds}} \xrightarrow{L-s} N(0,1). \quad (3.15)$$

To make the above theory feasible, we need to provide a consistent estimator for the asymptotic variance in (3.15). When $\frac{1}{2} < \theta_1 < \frac{3}{4}$ and $\frac{2\theta_1-1}{4(2-\beta_1)} < \theta_2 < \frac{1}{4}$, then $\frac{2\theta_1-1}{4} < \frac{1-\theta_1}{2}$ and $\frac{2\theta_1-1}{4} < (2-\beta_1)\theta_2$. By Proposition 3, suppose that $\frac{1}{2} < \theta_1 < \frac{3}{4}$, $\frac{2\theta_1-1}{4(2-\beta_1)} < \theta_2 < \frac{1}{4}$, $\beta_2 < \frac{1-\theta_1}{2\theta_1-1}$ and $\frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)} < \theta_4 < \frac{2\theta_1-1}{4}$, for $p \geq 1$, $\int_0^T \gamma_s^p ds$ can be consistently estimated by $\widehat{\int_0^T \gamma_s^p ds} = \frac{l_n^{\frac{p}{2}} TV_p(\nu; c_1, \theta_1)_T^n}{n^{1+\frac{p}{4}} \mu_p 2^{\frac{p}{2}}}$. Then, we obtain a consistent estimator of $\int_0^t \Sigma_s^2 ds$ as follows

$$\widehat{\int_0^T \Sigma_s^2 ds} = c_1 n \overline{d}_p \frac{TV_{2p}(\nu; c_1, \theta_1)_T^n}{(TV_p(\nu; c_1, \theta_1)_T^n)^2}, \quad (3.16)$$

where

$$\overline{d}_p = d_p \frac{\mu_p^2}{\mu_{2p}} = \frac{2^{p+1}\left[3\int_0^2 h_p(\rho_1(s))ds - \int_{-4}^2 h_p(\rho_2(s))ds\right]}{\mu_{2p}}. \quad (3.17)$$

We now present the final feasible test statistic, as given below:

$$T_n = n^{\frac{1-\theta_1}{2}} \frac{\left(\breve{T}_n - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}}.$$

Let α denote the significance level and z_α the α-quantile of the standard normal distribution. The asymptotic properties of the above feasible test are summarized in the following theorem.

Theorem 2. *Suppose that Assumption 1 with* $\vartheta > 2p$, $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1-1}\}$, $\frac{1}{2} < \theta_1 < \frac{3}{4}$, $\beta_2 < \frac{1-\theta_1}{2\theta_1-1}$, $\max\{\frac{2(p-1)\theta_1+2-p}{4p(2-\beta_1)}, \frac{p-2\theta_1(p-1)}{4(2p-\beta_1)}\} < \theta_2 < \frac{1}{4}$ *and* $\frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)} < \theta_4 < \frac{2\theta_1-1}{4}$ *hold. Then,* $\lim_{n\to\infty} P(T_n < z_\alpha|\Omega_{T,0}) = \alpha$; *and conditional on the set* $\Omega_{T,1}$,

$$T_n \asymp_P n^{\frac{1-\theta_1}{2}}, \qquad T_n \xrightarrow{P} -\infty, \tag{3.18}$$

The identification of volatility jump times becomes of interest once the null hypothesis is rejected. A heuristic approach to this problem may rely on a maximum-type statistic, loosely of the form

$$\arg\max_{t\in(0,T]} T_n(t),$$

to localize the occurrence of a dominant volatility jump over the interval $(0, T]$, where $T_n(t)$ denotes the feasible test statistic constructed over the time window $(0, t]$, emphasizing its dependence on the terminal time point t. Additional volatility jumps, if present, could in principle be detected by applying standard multiple change-point procedures, such as binary segmentation (cf. Fryzlewicz (2014) among others), suitably adapted to the framework of the current paper. A formal detection theory, however, is beyond the scope of this paper and is therefore left for future research.

and hence $\lim_{n\to\infty} P(T_n < z_\alpha|\Omega_{T,1}) = 1$*, where* $a_n \asymp_P b_n$ *stands for* $a_n/b_n = O_P(1)$ *and* $b_n/a_n = O_P(1)$*.*

The following remark further clarifies the assumptions on the power p and the constants θ_l, $l \in \{1, 2, 4\}$, in Theorems 1 and 2, as well as their suitable choices.

Remark 3. The result (3.18) still holds when the condition

$$\max\left\{\frac{2(p-1)\theta_1 + 2 - p}{4p(2-\beta_1)}, \frac{p - 2\theta_1(p-1)}{4(2p-\beta_1)}\right\} < \theta_2 < \frac{1}{4}$$

is relaxed to $\max\{\frac{(1-2\theta_1)p+1}{2(2p-\beta_1)}, \frac{2\theta_1-1}{4(2-\beta_1)}\} < \theta_2 < \frac{1}{4}$. Note that, under the conditions $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1-1}\}$ and $\frac{1}{2} < \theta_1 < \frac{3}{4}$, $\max\{\frac{2(p-1)\theta_1+2-p}{4p(2-\beta_1)}, \frac{p-2\theta_1(p-1)}{4(2p-\beta_1)}\} \geq \max\{\frac{(1-2\theta_1)p+1}{2(2p-\beta_1)}, \frac{2\theta_1-1}{4(2-\beta_1)}\}$. Both $\frac{2(p-1)\theta_1+2-p}{4p(2-\beta_1)}$ and $\frac{p-2\theta_1(p-1)}{4(2p-\beta_1)}$ are decreasing functions in p for fixed $1/2 < \theta_1 < 3/4$ and $0 < \beta_1 < 1$. Hence, the lower bound (in terms of θ_1 and β_1) for the smallest feasible θ_2 under the null hypothesis is obtained, as $p \to \infty$, from the following equations:

$$\lim_{p\to+\infty} \frac{2(p-1)\theta_1 + 2 - p}{4p(2-\beta_1)} = \frac{2\theta_1 - 1}{4(2-\beta_1)}, \quad \lim_{p\to+\infty} \frac{p - 2\theta_1(p-1)}{4(2p-\beta_1)} = \frac{1 - 2\theta_1}{8}.$$

For fixed $\frac{1}{2} < \theta_1 < \frac{3}{4}$ and $0 < \beta_2 < \min\{1, \frac{1-\theta_1}{2\theta_1-1}\}$, the lower bound $\frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)}$ for θ_4 is increasing in p. Since $\lim_{p\to+\infty} \frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)} = \frac{2\theta_1-1}{4}$, it follows that $\frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)} < \frac{2\theta_1-1}{4}$ for $0 < p < \infty$, which implies that the feasible region for θ_4 cannot be empty.

As to the choice of p, the closer θ_1 is to $1/2$, the larger p should be chosen. Even when $p = 4$, the lower bound of the feasible region for θ_1 is $2/3$, which yields the fastest possible divergence rate of $n^{1/6}$ under the alternative. This rate is still faster than the rate $n^{1/8}$ obtained in Bibinger and Madensoy (2019).

Remark 4. The evaluation of $h_p(\rho)$ defined in (3.14) is complicated in general. However, when p is even, by (3.17), we have $h_2(\rho) = 2\rho^2, h_4(\rho) = 72\rho^2 + 24\rho^4, h_6(\rho) = 4050\rho^2 + 5400\rho^4 + 720\rho^6$, and $\overline{d}_2 = 4$, $\overline{d}_4 = \frac{3552}{175}$, $\overline{d}_6 = \frac{5840}{77}$. For instance, when $p = 4$, we have

$$T_n = \frac{\left(\breve{T}_n - 2^2\right)}{\sqrt{3552 l_n TV_8(\nu; c_1, \theta_1)^n_T / (175(TV_4(\nu; c_1, \theta_1)^n_T)^2)}}, \tag{3.19}$$

which is implemented in both the simulations and the empirical study.

A decision rule is provided below for clarity:

$$\Im(n) = \Im(n,p) := \begin{cases} \text{Not to reject the null hypothesis } H_0 \text{ in (2.4)}, & \text{if } T_n \geq z_\alpha, \\ \text{Reject the null hypothesis } H_0 \text{ in (2.4)}, & \text{if } T_n < z_\alpha. \end{cases}$$

3.4 Comparison with an Existing Method

Bibinger and Madensoy (2019) constructed a Gumbel-type statistic to test the structural breaks in the spot volatility process using data with noise. They first used local Fourier method of moments to estimate spot volatilities, and then used the maximum of the standardized differences of the adjacent spot volatility estimators as their test statistic. That is, their statistic takes the following form

$$V_n = \max_{i=\alpha_n,\ldots,h_n^{-1}-\alpha_n} \left| \frac{\overline{RV}^{tr}_{n,i} - \overline{RV}^{tr}_{n,i+\alpha_n}}{\sqrt{8\hat{\eta}} \left| \overline{RV}^{tr}_{n,i+\alpha_n} \right|^{3/4}} \right|, \tag{3.20}$$

where $\overline{RV}^{tr}_{n,i} = \frac{1}{\alpha_n}\sum_{\ell=i-\alpha_n+1}^{i} \hat{\sigma}^{2,ad}_{(\ell-1)h_n} 1_{\left\{\left|\hat{\sigma}^{2,ad}_{(\ell-1)h_n}\right| \leq h_n^{\tau-1}\right\}}$ is a consistent estimator for spot volatility $\sigma^2_{i\alpha_n h_n}$, $i = \alpha_n, \ldots, h_n^{-1}$, $\hat{\sigma}^{2,ad}_{(\ell-1)h_n}$ is a biased estimator of spot volatility, h_n is the bandwidth and α_n is an integer sequence, $\tau \in (0, 1)$ is a positive constant, and $\hat{\eta}$ is an estimator of the variance for the noise. Under the null hypothesis where the spot volatility is continuous, their maximum test statistic V_n converges to a Gumbel-type extreme-value limiting distribution. Under the alternative hypothesis where the volatility process jumps, V_n diverges to infinity at a rate close to $n^{1/8}$. In contrast, under the alternative hypothesis, our test statistic diverges at a rate of $n^{\frac{1}{4}-\iota}$ for arbitrarily small $\iota > 0$, as shown in (3.18). This means that our test compares favorably with that of Bibinger and Madensoy (2019) in terms of power.

For the choices of h_n, α_n, and τ, the theory of Bibinger and Madensoy (2019) suggests the settings: $h_n \propto n^{-1/2}\log(n)$ and $\alpha_n \propto n^{1/4}/\log(n)$, in the case where the volatility process follows a semimartingale. In their simulations, Bibinger and Madensoy (2019) simply sets $h_n^{-1} = 120, \alpha_n = 15$, and $\tau = 3/4$. We will adopt this setting in our own simulations and empirical studies when comparing our method with that of Bibinger and Madensoy (2019).

We also implement the settings $h_n \propto n^{-1/2}\log(n)$ and $\alpha_n \propto n^{1/4}/\log(n)$, as suggested by the theory of Bibinger and Madensoy (2019), for selecting h_n and α_n. However, this results in rather unstable test performance, and thus the corresponding results are omitted from the present paper.

3.5 Choices of Tuning Parameters

We choose $p = 4$ as suggested by Remarks 3 and 4. Following the discussion in Li et al. (2022), we adopt $c_3 = 0.1$, which amounts to using a 4-minute window to mitigate the impact of microstructure noise by pre-averaging in the case of the second-by-second data used in our numerical studies. As to the choices of c_2 and θ_2, which appear in the threshold $u_{1,n} = c_2 n^{-\theta_2}$ (see (3.5)) for truncating price jumps, recalling the condition $\theta_2 < \frac{1}{4}$ in Theorem 2 and by taking Aït-Sahalia et al. (2013) as reference, we set $u_{1,n} = c_2\sqrt{\widehat{\sigma^2}}n^{-1/4}$ with $c_2 = 4$, where $\widehat{\sigma^2}$ is an estimator of the spot variance (i.e., the realized volatility computed at the 5-minute frequency using data that span one year). For the choices of c_4 and θ_4, which appear in the threshold $u_{2,n} = c_4 n^{-\theta_4}$ for truncating volatility jumps when we estimate the asymptotic variance of $\breve{T}_n$, similar to the choice of $u_{1,n}$, we set $u_{2,n} = c_4\sqrt{\frac{\widehat{\int_0^T \gamma_s^2 ds}}{T}}\left(\frac{l_n}{n}\right)^{1/2}$ with $c_4 = 4$, where

$$\begin{aligned}\widehat{\int_0^T \gamma_s^2 ds} &= \frac{4\Phi_{22}}{3c_3\psi_2^4}V(Y,4)_T^n + \frac{8}{c_3^2}\left(\frac{\Phi_{12}}{\psi_2^3} - \frac{\Phi_{22}\psi_1}{\psi_2^4}\right)V(Y,2)_T^n\widehat{\sigma_\epsilon^2} \\ &\quad + \frac{4T}{c_3^3}\left(\frac{\Phi_{11}}{\psi_2^2} - 2\frac{\Phi_{12}\psi_1}{\psi_2^3} + \frac{\Phi_{22}\psi_1^2}{\psi_2^4}\right)(\widehat{\sigma_\epsilon^2})^2,\end{aligned}$$

and

$$V(Y,p)_T^n = n^{p/4-1}\sum_{i=0}^{[nT]-k_n+1}|\overline{Y}_i^n|^p 1_{|\overline{Y}_i^n|<u_{1,n}} \quad \text{and} \quad \widehat{\sigma_\epsilon^2} = \frac{1}{2nT}\sum_{i=0}^{[nT]-1}(\Delta_i^n Y)^2.$$

By Lemma A.1 of Jing et al. (2014), suppose that $\frac{1}{8-2\beta_1} < \theta_2 < \frac{1}{4}$, then $\widehat{\int_0^T \gamma_s^2 ds}$ is a consistent estimator of $\int_0^T \gamma_s^2 ds$. As to $l_n = c_1 n^{\theta_1}$ that controls the size of blocks over which spot volatilities are estimated, by the condition $\frac{1}{2} < \theta_1 < \frac{3}{4}$ of Theorem 2, we set $\theta_1 = 0.6$ and $c_1 = 1$.

4. Simulation Study

In this section, we conduct simulation studies to investigate the performance of our testing method. The results show that our method performs reasonably well in finite sample, supporting the asymptotic theories developed in the previous section.

4.1 Simulation Design

The data generating process is given by a log-variance stochastic volatility model of Andersen et al. (2002). To be specific, the log price process X_t and volatility process ν_t follow the following stochastic differential equations,

$$\begin{cases} dX_t = (\mu + \kappa\nu_t)dt + \nu_t^{1/2}\, dW_t + dJ_t, \\ d\log\nu_t = (\alpha - \beta\log\nu_t)dt + \eta\, d\widetilde{W}_t + d\widetilde{J}_t, \end{cases} \tag{4.21}$$

where $J_t = \sum_{i=1}^{N_t} z_i$ and $\widetilde{J}_t = \sum_{i=1}^{\widetilde{N}_t} \widetilde{z}_i$ are independent jump processes; (z_i) and $(\widetilde{z}_i)$ are independent and i.i.d. normal with mean zero and variances σ_J^2 and $\widetilde{\sigma}_J^2$, respectively; N_t and $\widetilde{N}_t$ are independent Poisson processes with intensities λ and $\widetilde{\lambda}$, respectively; W and $\widetilde{W}$ are standard Brownian motions with quadratic covariation given by $\langle W, \widetilde{W}\rangle_t = \ddot{\rho}\, t$; and $(\mu, \kappa, \alpha, \beta, \eta, \ddot{\rho}, \lambda, \sigma_J, \widetilde{\lambda}, \widetilde{\sigma}_J)$ are model parameters. This model features stochastic volatility, leverage effect, and jumps in prices and volatilities. These stylized empirical facts are widely reported in the high-frequency financial econometrics literature.

Taking Andersen et al. (2002) as reference, we set model parameters under the null hypothesis as follows:

$$(\mu, \kappa, \alpha, \beta, \eta, \ddot{\rho}, \lambda, \sigma_J) = (0.05, -0.5, \log(0.2), 1, 0.1, -0.5, 1, 0.5).$$

Note that, under the null hypothesis, the volatility process is continuous and $(\widetilde{\lambda}, \widetilde{\sigma}_J) \equiv (0, 0)$. The values of parameters $(\widetilde{\lambda}, \widetilde{\sigma}_J)$ are specified in the subsequent subsections when results under the alternative hypothesis are presented. The noise process ϵ_i is taken to be i.i.d. normal with mean zero and variance $\sigma_\epsilon^2 = 0.0001^2$. The terminal time point is chosen to be $T = 1$ year. We further set the sampling interval to five seconds, with each sample path containing $1,179,361$ price observations. The number of simulation trials is 1000.

4.2 Test Results

We first investigate the finite sample size property of the test statistic T_n defined in (3.19). Figure 1 reports the empirical distribution of the test statistic T_n using five-second data generated from the log-volatility model (4.21) in the absence of volatility jumps ($\widetilde{J}_t \equiv 0$ for all $t \geq 0$). The results show that the empirical distribution of the test statistic is quite close to its limiting distribution $N(0, 1)$.

In Table 1, we report the sizes (rejection percentages) at various significance levels for both test statistics T_n defined in (3.19) and V_n defined in (3.20). The

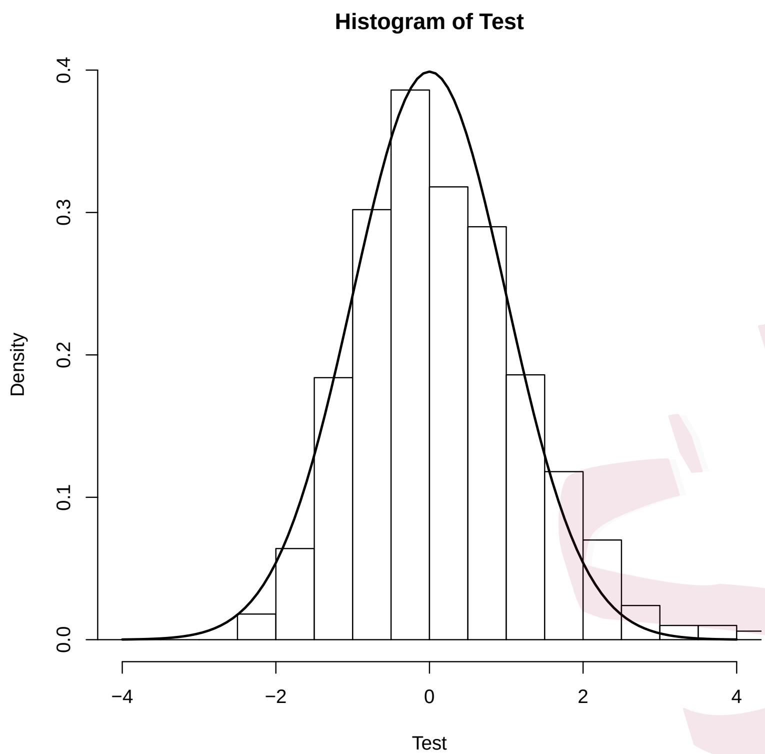

Figure 1: *The empirical distribution of the test statistic T_n, as defined in (3.19), based on data generated from the log-volatility model (4.21) in the absence of volatility jumps. The results show that the empirical distribution of the test statistic is quite close to its limiting distribution $N(0,1)$.*

tuning parameters associated with the test V_n are set as those suggested in Section 3.5. To check the robustness of our test performance to the choices of tuning parameter c_1, we also report results for different values of c_1. From Table 1, it is evident that both our test T_n and the test V_n of Bibinger and Madensoy (2019) control the type I error reasonably well. Both tests tend to exhibit undersizing.

Our test T_n, with $c_1 = 1$ or higher, appears to mitigate this issue and compares favorably with V_n at the 10% nominal level. For nominal levels below 5%, the size properties of T_n, with $c_1 = 1$ or higher, and V_n are comparable.

Table 1: *Sizes of the tests T_n in (3.19) and V_n, as defined in* (3.20)*, of Bibinger and Madensoy (2019) under the null hypothesis H_0 where the volatility processes are continuous and follow (4.21) with $\widetilde{J}_t \equiv 0$ for all $t \geq 0$. The data are generated at the five-second frequency.*

Significance	T_n			V_n
level	$c_1 = 0.9$	$c_1 = 1$	$c_1 = 1.1$	
10%	6.90%	7.10%	7.90%	5.9%
5%	2.20%	2.60%	3.90%	3.1%
2.5%	0.50%	1.10%	1.80%	2.1%
1%	0.10%	0.40%	0.70%	1.2%

Figure 2 displays the empirical distributions of the unstandardized test statistic $\hat{T}_n$ defined in (3.12) and the standardized test statistic T_n defined in (3.19) using data generated at the five-second frequency from model (4.21) with fixed $\widetilde{\lambda} = 4$ and different values of $\widetilde{\sigma}_J$. Under the null hypothesis H_0 where the volatility process is continuous, the empirical distributions (solid curves) are quite close to the standard normal distribution as predicted by Theorem 2. Under the al-

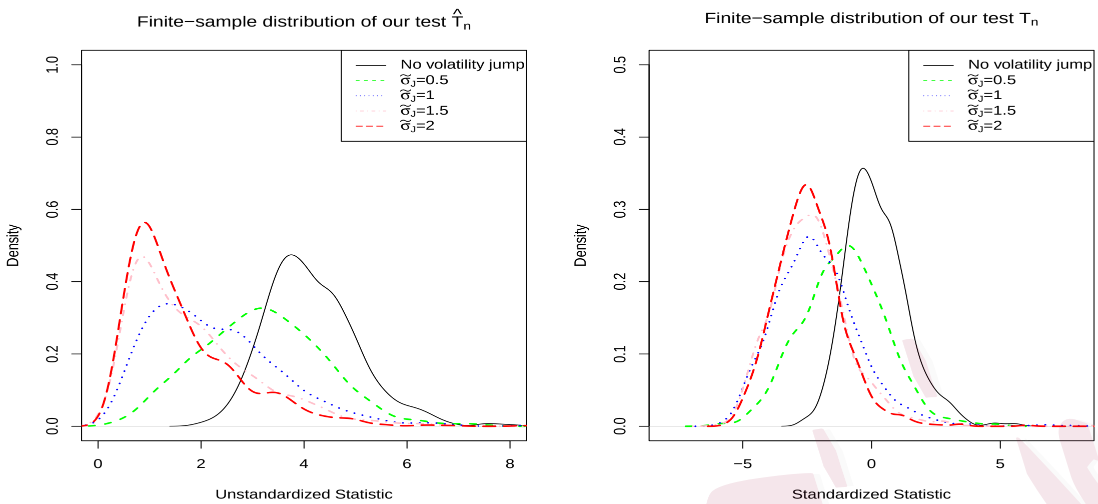

Figure 2: *Empirical distributions of the unstandardized test statistic $\widehat{T}_n$ (left panel) and the standardized test T_n (right panel). The results are based on data generated from the model (4.21) with fixed $\widetilde{\lambda} = 4$ and different values of $\widetilde{\sigma}_J$. The solid curves correspond to the continuous case $\widetilde{\sigma}_J \equiv 0$. Under the alternative hypothesis where the volatility process jumps, we consider the following cases: $\widetilde{\sigma}_J = 2$ (short-dashed curves), $\widetilde{\sigma}_J = 1.5$ (dotted curves), $\widetilde{\sigma}_J = 1$ (dash-dotted curves) and $\widetilde{\sigma}_J = 0.5$ (long-dashed curves). In all cases, we set the tuning parameter $c_1 = 1$.*

ternative hypothesis where the volatility process jumps, one readily sees that the differences between empirical distributions under the alternative and null hypotheses are fairly significant, implying that our test statistic is powerful. Figure 2 also shows that the larger $\widetilde{\sigma}_J$ is, the greater the distance between the empirical distributions under the two hypotheses.

We now turn to investigate the power property of our test statistic under the log-volatility model (4.21) for various values of $\widetilde{\sigma}_J$, $\widetilde{\lambda}$, and c_1. Table 2 provides the empirical powers of our test based on T_n and the test of Bibinger and Madensoy (2019) at the 5% significance level. The results from Table 2 show that our test performs reasonably well in terms of powers. In particular, the more frequently the volatility process jumps and/or the larger the jump size is, the higher the power of our test is. Notably, due to the slower convergence rate of the test statistic V_n of Bibinger and Madensoy (2019) compared to our test statistic T_n, it is not surprising that the power of the test of Bibinger and Madensoy (2019) is considerably lower than that of our test across different settings.

5. Empirical Analysis

In this section, we apply our method to a sample of 90 stocks from the US stock market. These stocks cover ten different stock market sectors, namely, Energy (7 stocks: HAL, SLB, CVX, OXY, XOM, COP, KMI), Industrials (11 stocks: BA, GD, LMT, EMR, GE, HON, MMM, CAT, FDX, UPS, UNP), Consumer Discretionary (9 stocks: F, GM, NKE, MCD, SBUX, AMZN, TGT, HD, LOW), Consumer Staples (11 stocks: WBA, COST, WMT, KO, PEP, KHC, MDLZ, MO, PM, CL, PG), Health Care (15 stocks: ABT, DHR, MDT, CVS, UNH, ABBV, AMGN, BIIB, GILD, AGN, BMY, JNJ, LLY, MRK, PFE), Financials (14 stocks: BAC, C, JPM, USB, WFC, AXP, COF, BK, BLK, GS, MS, MET,

Table 2: *Powers of the tests* T_n *in (3.19) and* V_n*, as defined in* (3.20)*, of Bibinger and Madensoy (2019) under the alternative hypothesis where the volatility processes follow (4.21). The data are generated at the five-second frequency with different values for* $\widetilde{\lambda} \in \{3, 4, 5\}$ *and* $\widetilde{\sigma}_J \in \{0.5, 1, 1.5, 2\}$.

Power		T_n			V_n
	$\widetilde{\sigma}_J$	$c_1 = 0.9$	$c_1 = 1$	$c_1 = 1.1$	
$\widetilde{\lambda} = 3$	0.5	26.1%	27.7%	21.8%	4.6%
	1	54.9%	55.3%	62.2%	9.5%
	1.5	63.5%	66.7%	70.1%	16.9%
	2	66.7%	72.7%	75.0%	20.0%
$\widetilde{\lambda} = 4$	0.5	30.8%	34.1%	39.7%	6.1%
	1	63.8%	68.2%	67.9%	12.1%
	1.5	71.1%	75.9%	76.7%	20.7%
	2	74.7%	77.7%	82.9%	24.4%
$\widetilde{\lambda} = 5$	0.5	38.0%	36.9%	48.1%	6.7 %
	1	70.0%	70.9%	76.0%	15.9%
	1.5	75.8%	78.4%	82.7%	23.5%
	2	77.0%	79.3%	81.4%	26.8%

AIG, ALL), Information Technology (11 stocks: ACN, IBM, MA, PYPL, V, MSFT, ORCL, CSCO, AAPL, QCOM, TXN), Communication Service (7 stocks: T, VZ, DIS, FOX, FOXA, FB, GOOG), Utilities (4 stocks: DUK, EXC, NEE, SO), Real Estates (1 stock: SPG) according to the Global Industry Classification Standard. We study the five-second trade price data of these 90 stocks during the years 2017 and 2019. The data are available in the Wharton Research Data Services (WRDS). There are 251 and 252 trading days in the years 2017 and 2019, respectively. On average, there are about 21 trading days per month. We consider trade data during the usual exchange trading hours from 9:30 to 16:00. Also, overnight returns are excluded from our analysis. The five-second trade price data are constructed from the tick-by-tick data by taking the last transaction price within each five-second interval preceding each time point on a regularly spaced five-second grid. The conventional previous-tick method of Zhang (2011) was adopted to deal with those time stamps where prices are not available.

We summarize test results for years 2017 and 2019 across different values of c_1 in Table 3. For each year, we report the averages of 90 realizations of test statistics $\breve{T}_n$ (defined in (3.12)) and V_n (defined in (3.20)), respectively. Moreover, the rejection percentages (at the 5% significance level) are also reported. For example, as can be seen from Table 3, for the year 2017 and $c_1 = 0.9$, the rejection percentage is 45.56%. That is, the null hypothesis gets firmly rejected for 41 out of 90 stocks at the 5% significance level. We further find that the results are fairly robust to the choices of c_1. The test V_n of Bibinger and Madensoy (2019) is found to reject significantly less frequently than our test T_n, which aligns with its comparatively lower statistical power.

To provide some visual evidence, we plot the two estimated spot volatility trajectories for stocks KHC and UNH in Figure 3. From our testing results, the null hypothesis is firmly rejected for KHC but not for UNH. As illustrated in Figure 3, there are several prominent jumps in the estimated volatility process of KHC, which leads to the rejection of the null hypothesis by our test. In contrast, the changes in the volatility process of UNH appear visually minor, explaining why the test statistic T_n does not reject the null hypothesis in this case.

6. Concluding Remarks

This paper proposes a novel ratio-type test for detecting volatility jumps using high-frequency data in the presence of microstructure noise. We begin by estimating spot volatilities via the threshold pre-averaging method. Subsequently, we construct power variations of the estimated spot volatility processes and rigorously analyze their asymptotic properties. Based on power variations computed at two different time scales, we develop a ratio-type test statistic, and establish its asymptotic behavior under both the null and alternative hypotheses.

Under the null hypothesis that the volatility path is continuous, our feasible test statistic converges in distribution to a standard normal random variable.

Table 3: *Testing results of the test statistics T_n and V_n on 90 stocks in 2017 and 2019. The averages of 90 realizations of $\breve{T}_n$ (defined in (3.12)) and V_n (defined in (3.20)) as well as their rejection rates (Reject.) are reported. Numbers in parentheses refer to the numbers of stocks for which the null hypothesis gets rejected. The significance level is 5%.*

T_n						V_n	
$c_1 = 0.9$	Reject.	$c_1 = 1$	Reject.	$c_1 = 1.1$	Reject.		Reject.
2017							
3.1421	45.56%(41)	3.1869	43.33%(39)	3.1651	42.22%(38)	56.4675	4.44%(4)
2019							
3.4397	42.22%(38)	3.3973	43.33%(39)	3.3815	43.33%(39)	47.4154	2.22%(2)

Under the alternative hypothesis, where the volatility exhibits jumps, our test statistic diverges to infinity at a faster rate of $n^{1/4-\iota}$ for arbitrarily small $\iota > 0$, compared to the test of Bibinger and Madensoy (2019). Simulation studies corroborate the theoretical findings. An empirical analysis suggests that, while modeling volatility as a continuous semimartingale may be suitable for a substantial subset of the 90 U.S. stocks analyzed, a non-negligible portion exhibit characteristics consistent with volatility jumps.

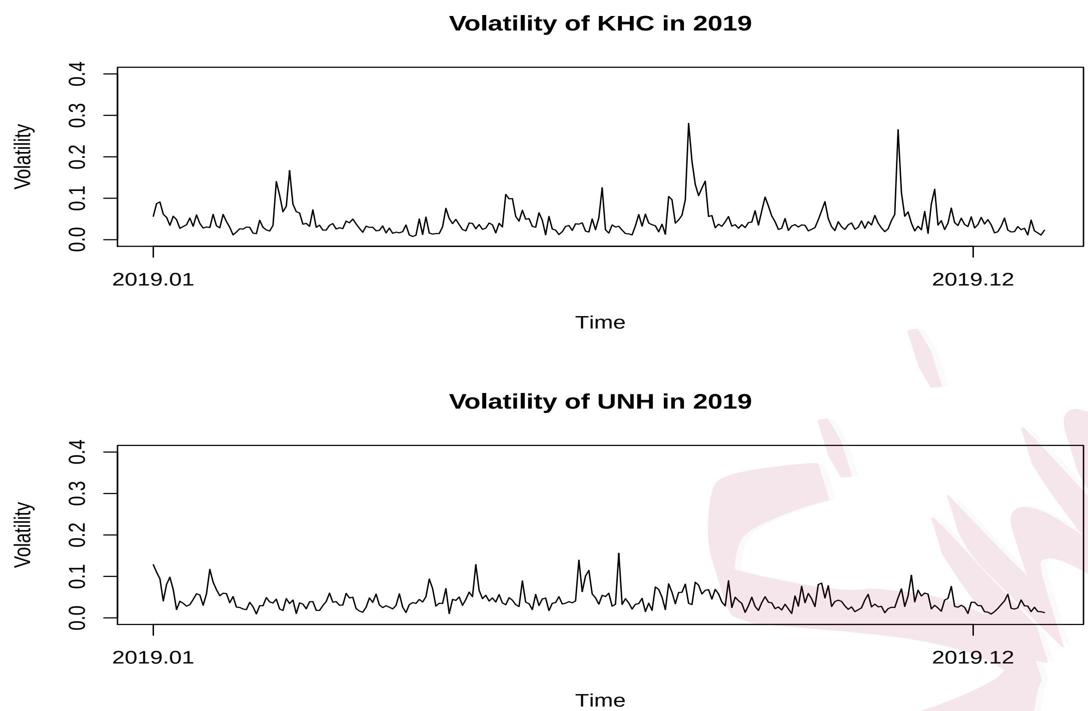

Figure 3: *The estimated spot volatility processes of KHC (top panel) and UNH (bottom panel) in the year 2019. Note that our test rejects the null hypothesis for KHC but not for UNH.*

A promising direction for future research is to enhance the power of the test by incorporating the Estimated-price Realized Volatility approach of Li et al. (2016), which achieves greater efficiency by leveraging trading information and can be applied to ultra-high-frequency data such as tick-by-tick observations. Given the considerable attention that rough volatility has recently received, see, e.g., Chong et al. (2024), Chong and Todorov (2025), Fukasawa et al. (2022), Wang et al.

(2023) and the references therein, another natural direction for future research is to test for volatility jumps in settings where the continuous volatility component is rough. This issue warrants deeper investigation.

Supplementary Materials

The supplementary material provides the proofs of the theoretical results and related lemmas.

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