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CONSTRUCTION OF (NEARLY) ORTHOGONAL SYMMETRIC LATIN HYPERCUBE DESIGNS

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Abstract: Latin hypercube designs (LHDs) have been widely used as computer experimental designs when a linear model is fitted. The symmetric LHD (SLHD), as a special kind of LHDs, can guarantee that the estimates of second-order effects and main effects are uncorrelated. In this paper, we propose two methods to construct orthogonal SLHDs (OSLHDs) and nearly orthogonal SLHDs (NOSLHDs). The first method can generate new designs based on existing OSLHDs. Some new NOSLHDs with flexible run sizes and good nearly orthogonality can be constructed. Moreover, the resulting OSLHDs of the second construction method have better stratification properties than existing OSLHDs. A case study is provided to highlight the effectiveness of constructed designs in data collection.

Key words and phrases: Computer experiment, Kronecker product, orthogonal array, second-order effect, stratification property.

1. Introduction

Computer experiments play a crucial role in recent decades, especially when the physical experiments are too costly or impractical to be carried out. For designing a

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computer experiment, space-filling property is always required. Space-filling designs aim to spread the design points throughout the design region as uniformly as possible. Latin hypercube designs (LHDs), proposed by McKay, Beckman and Conover (1979), can achieve the maximum stratification when projected onto any one dimension. Thus, LHDs have been widely used as space-filling designs for computer experiments. To make LHDs have good multivariate properties, Owen (1994) and Tang (1998) proposed to use LHDs with small correlations among factors, and Ye (1998) constructed orthogonal LHDs (OLHDs). Thereafter, many deterministic methods and algorithms have been proposed to obtain OLHDs and nearly orthogonal LHDs, see e.g., Steinberg and Lin (2006), Pang, Liu and Lin (2009), Lin, Mukerjee and Tang (2009), Lin et al. (2010), Wang et al. (2015), Sun and Tang (2017) and Sheng, Yang and Liu (2023).

When a linear model is fitted, OLHDs can result in the uncorrelated estimates of main effects. Furthermore, Ye (1998) constructed LHDs satisfying that the elementwise product of every two columns is orthogonal to all columns in the design. Such designs are called second-order orthogonal LHDs, which can guarantee that the estimates of second-order effects and main effects are uncorrelated. Symmetric LHDs (SLHDs) as a special kind of second-order orthogonal LHDs have been widely discussed. Ye, Li and Sudjianto (2000) found that SLHDs perform better than regular LHDs with respect to entropy and distance criteria. Sun, Liu and Lin (2009, 2010) proposed some algebraic constructions for orthogonal SLHDs (OSLHDs). Ai, He and Liu (2012) used the rotation method to construct OLHDs and OSLHDs. Yang and Liu (2012) generalized the construction methods in Sun, Liu and Lin (2009) to obtain nearly orthogonal

SLHDs (NOSLHDs). Georgiou and Stylianou (2011), Georgiou and Efthimiou (2014) and Evangelaras and Koutras (2017) constructed OSLHDs with fewer factors. Wang et al. (2018) summarized existing construction methods, and proposed a method to construct OSLHDs and NOSLHDs with flexible run sizes and high factor-to-run ratios. Su, Wang and Zhou (2020) provided an algorithm to search maximin NOSLHDs.

To evaluate the space-filling property of an LHD, various criteria have been proposed, including distance based criteria (Johnson, Moore and Ylvisaker, 1990), discrepancy based criteria (Fang et al., 2018) and the stratification property (He and Tang, 2013). Compared with the first two kinds of criteria, the stratification property considers the space-filling property in low-dimensional projections. Recently, some methods have been proposed to construct OLHDs with good stratification properties, such as Yang et al. (2021), Li, Yang and Liu (2022) and the references therein. However, the resulting designs of existing methods are not symmetric.

In this paper, we propose two methods to construct SLHDs. The first method can be used to construction OSLHDs and NOSLHDs based on existing OSLHDs. Some new NOSLHDs can be constructed with flexible run sizes and good nearly orthogonality. The second construction method can result in OSLHDs with better stratification properties compared with existing OSLHDs.

The rest of this paper is organized as follows. Section 2 provides some basic definitions and notation. In Section 3, we propose two construction methods for SLHDs. The resulting SLHDs are orthogonal or nearly orthogonal. Some theoretical results and illustrative examples are also given. A case study is provided in Section 4 to evaluate the performance of the OSLHD and NOSLHD generated by the proposed method against that obtained by existing methods. Section 5 provides some concluding remarks. All the proofs are deferred to the Appendix.

2. Preliminaries and Notation

Let $D(n, s^m)$ denote a design of n runs with m factors of s levels. Without loss of generality, the s levels are taken as $-(s-1)/2, -(s-3)/2, \ldots, (s-1)/2$. If the inner product between any two columns of a design is zero, we call this design a column-orthogonal design. An LHD(n, m) is a $D(n, n^m)$, where the n levels are uniformly spaced. Such designs are seen to enjoy good space-filling properties, covering the design space well without repetitions. A column-orthogonal LHD(n, m) is called an orthogonal LHD, denoted by OLHD(n, m). It is easy to verify that the correlation matrix of an OLHD(n, m) is $n(n^2-1)I_m/12$, where I_m is an $m \times m$ identity matrix. Let $\rho_{ij}(L)$ be the correlation between the i-th and j-th columns of a design L. The nearly orthogonality of a design L with m factors can be measured by $\rho_M(L)$ and $\rho^2(L)$, where

$$\rho_M(L) = \max_{i \neq j} |\rho_{ij}(L)| \quad \text{and} \quad \rho^2(L) = \frac{1}{m(m-1)} \sum_{i \neq j} \rho_{ij}^2(L).$$

A design will be said to be symmetric if for any row d, $-d$ is also one of the rows in the design. Let L be an SLHD of n runs with m factors, denoted by SLHD(n, m). Without loss of generality, depending upon n is even or odd, we can write L as $L = (L_0^T, -L_0^T)^T$ for even n and $L = (L_0^T, \mathbf{0}_m, -L_0^T)^T$ for odd n, where $\mathbf{0}_m$ is an $m \times 1$ vector of all zeros, L_0 is a $p \times m$ matrix with $p = \lfloor n/2 \rfloor$, and $\lfloor x \rfloor$ denotes the largest integer not exceeding x. It is easy to verify that the sum of elementwise products of any three columns of a symmetric design is zero. Thus, when a second-order polynomial model is fitted with an OSLHD, the estimates of main effects are uncorrelated with that of other main effects and second-order effects.

A design with n runs and m factors is called an orthogonal array (OA) of strength t, denoted by OA$(n, m, \prod_{j=1}^{m} s_j, t)$, if all possible combinations of symbols appear equally often as rows in every $n \times t$ subarray, where the level size of the j-th factor is s_j for $j = 1, \ldots, m$. If $s_1 = \cdots = s_m = s$, the OA can be simply written as OA(n, m, s, t). A column pair of a design with n runs is said to achieve a stratification on an $s_1 \times s_2$ grid if it can be collapsed into an OA$(n, 2, s_1 \times s_2, 2)$. If any two distinct columns of a design can achieve a stratification on an $s_1 \times s_2$ grid, we call that this design can achieve a stratification on an $s_1 \times s_2$ grid when projected onto any two dimensions.

3. Construction Methods and Results

In this section, we provide two construction methods for SLHDs. All of the resulting designs are orthogonal or nearly orthogonal. Besides, the resulting designs of the second method have better stratification properties than existing OSLHDs.

3.1 Construction of SLHD$(n_1 n_2, m_1 m_2)$'s

Suppose there exist an SLHD(n_1, m_1) and an SLHD(n_2, m_2), then, an SLHD$(n_1 n_2, m_1 m_2)$ can be constructed as follows.

Construction 1.

Step 1. For $u = 1, 2$, let $A_0^{(u)}$ be an $\lfloor n_u/2 \rfloor \times m_u$ matrix with entries from $\{-1, 1\}$, and define A_u as

$$A_u = \begin{cases} \begin{pmatrix} (A_0^{(u)})^T & (A_0^{(u)})^T \end{pmatrix}^T, & \text{if } n_u \text{ is even;} \\ \begin{pmatrix} (A_0^{(u)})^T & \mathbf{e}_u & (A_0^{(u)})^T \end{pmatrix}^T, & \text{if } n_u \text{ is odd,} \end{cases}$$

where $\mathbf{e}_u$ is any $m_u \times 1$ vector with ± 1.

Step 2. For $u = 1, 2$, take an SLHD(n_u, m_u), denoted by L_u, and reorder the rows so that it can be written as

$$L_u = \begin{cases} \begin{pmatrix} (L_0^{(u)})^T & -(L_0^{(u)})^T \end{pmatrix}^T, & \text{if } n_u \text{ is even;} \\ \begin{pmatrix} (L_0^{(u)})^T & \mathbf{0}_{m_u} & -(L_0^{(u)})^T \end{pmatrix}^T, & \text{if } n_u \text{ is odd.} \end{cases}$$

Step 3. A design of $n_1 n_2$ runs and $m_1 m_2$ factors can be obtained as $L = A_1 \otimes L_2 + n_2(L_1 \otimes A_2)$, where $\otimes$ denotes the Kronecker product.

We first demonstrate that the resulting design of Construction 1 is an SLHD, and then discuss the choices of L_u and $A_0^{(u)}$ with $u = 1, 2$ to make L an OSLHD or an NOSLHD.

Lemma 1. *The design L obtained in Construction 1 is an SLHD$(n_1 n_2, m_1 m_2)$.*

It should be noticed that Step 1 only requires that $\mathbf{e}_u$ is an $m_u \times 1$ vector with entries from $\{-1, 1\}$. And the choice of $\mathbf{e}_u$ does not influence the property of L shown in Lemma 1, which can be seen from the proof of it. Besides, after obtaining A_1, A_2, L_1 and L_2 through Steps 1 and 2, the matrix $\breve{L} = A_2 \otimes L_1 + n_1(L_2 \otimes A_1)$ is also an SLHD$(n_1 n_2, m_1 m_2)$. Especially, when $n_1 \neq n_2$, these two LHDs are different. In the rest of this paper, we only discuss the properties of L, and the properties of $\breve{L}$ can be obtained similarly by swapping L_1, L_2 and A_1, A_2 respectively.

We find that when n_1 and n_2 are even, the resulting design of Construction 1 can be an OSLHD if we choose L_1, L_2, A_1 and A_2 appropriately.

Theorem 1. *The design L obtained in Construction 1 is an OSLHD(n_1n_2, m_1m_2), if the following conditions hold:* (i) *both n_1 and n_2 are even integers;* (ii) *both L_1 and L_2 are OSLHDs;* (iii) *both $A_0^{(1)}$ and $A_0^{(2)}$ are column-orthogonal designs.*

We now provide an illustrative example of Theorem 1.

Example 1. Given $n_1 = 4, n_2 = 8, m_1 = 2$ and $m_2 = 4$, an OSLHD$(32, 8)$ can be obtained through Construction 1. From Steps 1 and 2, take L_i and A_i for $i = 1, 2$ as

$$L_1 = \begin{pmatrix} 0.5 & 1.5 & -0.5 & -1.5 \\ 1.5 & -0.5 & -1.5 & 0.5 \end{pmatrix}^T, \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^T,$$

$$L_2 = \begin{pmatrix} 0.5 & 1.5 & 2.5 & 3.5 \\ 1.5 & -0.5 & -3.5 & 2.5 \\ 2.5 & 3.5 & -0.5 & -1.5 \\ 3.5 & -2.5 & 1.5 & -0.5 \\ -0.5 & -1.5 & -2.5 & -3.5 \\ -1.5 & 0.5 & 3.5 & -2.5 \\ -2.5 & -3.5 & 0.5 & 1.5 \\ -3.5 & 2.5 & -1.5 & 0.5 \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Under this configuration, the three conditions in Theorem 1 are all satisfied. Following Step 3, a design with 32 runs and 8 factors can be constructed as shown in Table 1. It is easy to verify that this design is an OSLHD. Note that, an OLHD$(32, 12)$ can be constructed by Lin et al. (2010). Although the design constructed by Lin et al. (2010) can accommodate more factors, it is not symmetric, and up to 4 columns can be extracted from the 12 columns to make the subdesign symmetric. Thus, the resulting

Table 1: The resulting OSLHD(32, 8) obtained by Construction 1 in Example 1.

OSLHD(32, 8)							
4.5	5.5	6.5	7.5	12.5	13.5	14.5	15.5
5.5	−4.5	0.5	−1.5	13.5	−12.5	8.5	−9.5
6.5	7.5	−4.5	−5.5	14.5	15.5	−12.5	−13.5
7.5	−6.5	−2.5	3.5	15.5	−14.5	−10.5	11.5
3.5	2.5	1.5	0.5	11.5	10.5	9.5	8.5
2.5	−3.5	7.5	−6.5	10.5	−11.5	15.5	−14.5
1.5	0.5	−3.5	−2.5	9.5	8.5	−11.5	−10.5
0.5	−1.5	−5.5	4.5	8.5	−9.5	−13.5	12.5
12.5	13.5	14.5	15.5	−4.5	−5.5	−6.5	−7.5
13.5	−12.5	8.5	−9.5	−5.5	4.5	−0.5	1.5
14.5	15.5	−12.5	−13.5	−6.5	−7.5	4.5	5.5
15.5	−14.5	−10.5	11.5	−7.5	6.5	2.5	−3.5
11.5	10.5	9.5	8.5	−3.5	−2.5	−1.5	−0.5
10.5	−11.5	15.5	−14.5	−2.5	3.5	−7.5	6.5
9.5	8.5	−11.5	−10.5	−1.5	−0.5	3.5	2.5
8.5	−9.5	−13.5	12.5	−0.5	1.5	5.5	−4.5
−3.5	−2.5	−1.5	−0.5	−11.5	−10.5	−9.5	−8.5
−2.5	3.5	−7.5	6.5	−10.5	11.5	−15.5	14.5
−1.5	−0.5	3.5	2.5	−9.5	−8.5	11.5	10.5
−0.5	1.5	5.5	−4.5	−8.5	9.5	13.5	−12.5
−4.5	−5.5	−6.5	−7.5	−12.5	−13.5	−14.5	−15.5
−5.5	4.5	−0.5	1.5	−13.5	12.5	−8.5	9.5
−6.5	−7.5	4.5	5.5	−14.5	−15.5	12.5	13.5
−7.5	6.5	2.5	−3.5	−15.5	14.5	10.5	−11.5
−11.5	−10.5	−9.5	−8.5	3.5	2.5	1.5	0.5
−10.5	11.5	−15.5	14.5	2.5	−3.5	7.5	−6.5
−9.5	−8.5	11.5	10.5	1.5	0.5	−3.5	−2.5
−8.5	9.5	13.5	−12.5	0.5	−1.5	−5.5	4.5
−12.5	−13.5	−14.5	−15.5	4.5	5.5	6.5	7.5
−13.5	12.5	−8.5	9.5	5.5	−4.5	0.5	−1.5
−14.5	−15.5	12.5	13.5	6.5	7.5	−4.5	−5.5
−15.5	14.5	10.5	−11.5	7.5	−6.5	−2.5	3.5

design of Construction 1 is more suitable for fitting the second-order polynomial model, since the estimates of main effects are uncorrelated with that of second-order effects.

Furthermore, we find that in certain special cases, an OSLHD can be constructed with twice the number of factors obtained by Construction 1.

Corollary 1. *If $n_1 = n_2 = n_0$ are even integers, choose A_i and L_i according to Theorem 1 for $i = 1$ and 2. The design $(L, \widetilde{L})$ is an OSLHD$(n_0^2, 2m_1m_2)$, where L is obtained by Construction 1 and $\widetilde{L} = L_1 \otimes A_2 - n_0(A_1 \otimes L_2)$.*

Table 2 summarizes the sizes of the resulting OSLHDs derived from Theorem 1 and Corollary 1, where n denotes the run size and m represents the number of factors.

Table 2: The sizes of the resulting OSLHDs derived from Theorem 1 and Corollary 1.

n	m	Requirements
$(c_1 2^{r_1+1})^{d_1}(c_2 2^{r_2+1})^{d_2}$	$\prod_{i=1}^{2} d_i f_1(c_i, r_i, d_i)$	$c_i, r_i \in \mathcal{Z}^+,\ d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(c2^{r+1})^{2d}$	$2^{2r+1} d^2 f_1^2(c, r, d)$	$c, r \in \mathcal{Z}^+,\ d = 2^t$ and $t \in \mathcal{N}$
$(c_1 2^{r+1})^{d_1}(2c_2k)^{d_2}$	$2^r k d_1 d_2 f_1(c_1, r, d_1) g_1(c_2, k, d_2)$	$c_i, r \in \mathcal{Z}^+,\ k \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(2c_1k_1)^{d_1}(2c_2k_2)^{d_2}$	$\prod_{i=1}^{2} k_i d_i f_1(c_i, k_i, d_i)$	$c_i \in \mathcal{Z}^+,\ k_i \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(2ck)^{2d}$	$2k^2 d^2 g_1^2(c, k, d)$	$c \in \mathcal{Z}^+,\ k \in \{12, 20, 24\},\ d = 2^t$ and $t \in \mathcal{N}$

† $f_1(c, r, d) = \lfloor [(c2^{r+1})^d - 1]/(c2^{r+1} - 1)/d \rfloor$, $g_1(c, k, d) = \lfloor [(2ck)^d - 1]/(2ck - 1)/d \rfloor$.

The above results only apply to the case that both n_1 and n_2 are even. If at least one of them is odd, the resulting design of Construction 1 can be an NOSLHD(n_1n_2, m_1m_2)

by choosing L_1, L_2, A_1 and A_2 according to the following theorem.

Theorem 2. *If at least one of n_1 and n_2 is odd, the design L obtained in Construction 1 must not be orthogonal. At this time, L can be an NOSLHD(n_1n_2, m_1m_2), if the following conditions hold:* (i) *both L_1 and L_2 are OSLHDs;* (ii) *both $A_0^{(1)}$ and $A_0^{(2)}$ are column-orthogonal designs. At this time, $\rho_M(L)$ and $\rho^2(L)$ are*

$$\rho_M(L) = \begin{cases} \frac{n_2^2-1}{n_1(n_1^2n_2^2-1)}, & \text{if } n_1 \text{ is odd and } n_2 \text{ is even;} \\ \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}, & \text{if } n_1 \text{ is even and } n_2 \text{ is odd;} \\ \max\left\{\frac{n_2^2-1}{n_1(n_1^2n_2^2-1)}, \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}\right\}, & \text{if both } n_1 \text{ and } n_2 \text{ are odd;} \end{cases}$$

$$\text{and} \quad \rho^2(L) = \begin{cases} \frac{(m_1-1)(n_2^2-1)^2}{(m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2}, & \text{if } n_1 \text{ is odd and } n_2 \text{ is even;} \\ \frac{(m_2-1)(n_1^2-1)^2n_2^2}{(m_1m_2-1)(n_1^2n_2^2-1)^2}, & \text{if } n_1 \text{ is even and } n_2 \text{ is odd;} \\ \frac{(m_1-1)(n_2^2-1)^2+(m_2-1)n_1^2(n_1^2-1)^2n_2^2}{(m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2}, & \text{if both } n_1 \text{ and } n_2 \text{ are odd.} \end{cases}$$

Both $\rho_M(L)$ and $\rho^2(L)$ converge to 0 as $n_1 \to +\infty$ and $n_2 \to +\infty$.

From Theorem 2, we can know that if n_1 is odd and n_2 is even, given L_1, L_2, A_1 and A_2, although both $L = A_1 \otimes L_2 + n_2(L_1 \otimes A_2)$ and $\breve{L} = A_2 \otimes L_1 + n_1(L_2 \otimes A_1)$ are SLHDs, the nearly orthogonalities of them are different. Obviously, $\rho_M(L)$ is always smaller than $\rho_M(\breve{L})$, and $\rho^2(L)$ is always smaller than $\rho^2(\breve{L})$. This result implies that when n_1 is odd and n_2 is even, we should construct an NOSLHD by $A_1 \otimes L_2 + n_2(L_1 \otimes A_2)$, while when n_1 is even and n_2 is odd, the design $A_2 \otimes L_1 + n_1(L_2 \otimes A_1)$ has better nearly orthogonality. Here is an example.

Example 2. Given $n_1 = 5$, $n_2 = 4$ and $m_1 = m_2 = 2$, take L_1, L_2, A_1 and A_2 as

$$L_1 = \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 0 & 0 \\ -1 & 2 \\ -2 & -1 \end{pmatrix}, \ L_2 = \begin{pmatrix} -1.5 & 0.5 \\ -0.5 & -1.5 \\ 1.5 & -0.5 \\ 0.5 & 1.5 \end{pmatrix}, \ A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then, two SLHD$(20,4)$'s can be obtained as $L = A_1 \otimes L_2 + 4(L_1 \otimes A_2)$ and $\breve{L} = A_2 \otimes L_1 + 5(L_2 \otimes A_1)$ respectively, which are shown in Table 3. It is easy to calculate that, $\rho_M(L) = 0.008$, $\rho_M(\breve{L}) = 0.188$, $\rho^2(L) < 0.001$ and $\rho^2(\breve{L}) = 0.012$, which implies that the nearly orthogonality of L is much better than that of $\breve{L}$.

Table 3: Two resulting NOSLHD$(20,4)$'s obtained by Construction 1 in Example 2.

L				$\breve{L}$			
2.5	4.5	−9.5	−7.5	−6.5	−9.5	3.5	0.5
3.5	−5.5	−8.5	6.5	−5.5	8.5	4.5	−1.5
5.5	3.5	−6.5	−8.5	−7.5	−7.5	2.5	2.5
4.5	−2.5	−7.5	9.5	−8.5	−5.5	1.5	4.5
6.5	8.5	5.5	3.5	−9.5	6.5	0.5	−3.5
7.5	−9.5	4.5	−2.5	−1.5	−4.5	−8.5	−5.5
9.5	7.5	2.5	4.5	−0.5	3.5	−9.5	6.5
8.5	−6.5	3.5	−5.5	−2.5	−2.5	−7.5	−7.5
−1.5	0.5	−1.5	0.5	−3.5	−0.5	−6.5	−9.5
−0.5	−1.5	−0.5	−1.5	−4.5	1.5	−5.5	8.5
1.5	−0.5	1.5	−0.5	8.5	5.5	−1.5	−4.5
0.5	1.5	0.5	1.5	9.5	−6.5	−0.5	3.5
−5.5	−3.5	6.5	8.5	7.5	7.5	−2.5	−2.5
−4.5	2.5	7.5	−9.5	6.5	9.5	−3.5	−0.5
−2.5	−4.5	9.5	7.5	5.5	−8.5	−4.5	1.5
−3.5	5.5	8.5	−6.5	3.5	0.5	6.5	9.5
−9.5	−7.5	−2.5	−4.5	4.5	−1.5	5.5	−8.5
−8.5	6.5	−3.5	5.5	2.5	2.5	7.5	7.5
−6.5	−8.5	−5.5	−3.5	1.5	4.5	8.5	5.5
−7.5	9.5	−4.5	2.5	0.5	−3.5	9.5	−6.5

Following a similar approach as in Corollary 1, an NOSLHD can be consructed with twice the number of factors obtained by Construction 1.

Corollary 2. *If at least one of n_1 and n_2 is odd, choose A_i and L_i according to Theorem 2 for $i = 1$ and 2. The design $D = (L, \widetilde{L})$ is an OSLHD$(n_1n_2, 2m_1m_2)$, where L is obtained by Construction 1 and $\widetilde{L} = L_1 \otimes A_2 - n_1(A_1 \otimes L_2)$. At this time, $\rho_M(D)$ and $\rho^2(D)$ are*

$$\rho_M(D) = \begin{cases} \max\left\{\frac{n_1(n_2^2-1)}{n_1^2n_2^2-1}, \frac{n_1-n_2}{n_1n_2-1}\right\}, & \text{if } n_1 \text{ is odd and } n_2 \text{ is even;} \\ \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}, & \text{if } n_1 \text{ is even and } n_2 \text{ is odd;} \\ \max\left\{\frac{n_1(n_2^2-1)}{n_1^2n_2^2-1}, \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}\right\}, & \text{if both } n_1 \text{ and } n_2 \text{ are odd;} \end{cases} \quad \text{and}$$

$$\rho^2(D) = \begin{cases} \frac{c_1+2n_1^2(n_1n_2+1)^2(n_1-n_2)^2}{2(2m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2}, & \text{if } n_1 \text{ is odd and } n_2 \text{ is even;} \\ \frac{c_2+2n_2^2(n_1n_2+1)^2(n_1-n_2)^2}{2(2m_1m_2-1)n_2^2(n_1^2n_2^2-1)^2}, & \text{if } n_1 \text{ is even and } n_2 \text{ is odd;} \\ \frac{c_1}{2n_1^2f} + \frac{c_2}{2n_2^2f} + \frac{(n_1-n_2)^2}{f}, & \text{if both } n_1 \text{ and } n_2 \text{ are odd;} \end{cases}$$

where $c_1 = (m_1 - 1)(n_1^2 + 1)^2(n_2^2 - 1)^2$, $c_2 = (m_2 - 1)(n_1^2 - 1)^2(n_2^2 + 1)^2$ and $f = (2m_1m_2 - 1)(n_1^2n_2^2 - 1)^2$. Both $\rho_M(D)$ and $\rho^2(D)$ converge to 0 as $n_1 \to +\infty$ and $n_2 \to +\infty$.

From Corollary 2, we can know that if n_1 is odd and n_2 is even, given L_1, L_2, A_1 and A_2, two NOSLHD$(n_1n_2, 2m_1m_2)$'s can be obtained, which are $D = (L, \tilde{L})$ with $L = A_1 \otimes L_2 + n_2(L_1 \otimes A_2)$ and $\widetilde{L} = L_1 \otimes A_2 - n_1(A_1 \otimes L_2)$, and $D^* = (L^*, \widetilde{L}^*)$ with $L^* = A_2 \otimes L_1 + n_1(L_2 \otimes A_1)$ and $\widetilde{L}^* = L_2 \otimes A_1 - n_2(A_2 \otimes L_1)$. Although both D and D^* are SLHDs, the nearly orthogonalities of them are different. According to Theorem 2 and Corollary 2, it is easy to verify that

$$\rho_M(L) < \rho_M(L^*) \leq \rho_M(D^*) \leq \rho_M(D),$$

and $\rho^2(D) = \rho^2(D^*)$ always hold. This result implies that when n_1 is odd and n_2 is even, we should construct an NOSLHD$(n_1n_2, 2m_1m_2)$ by D^*, while when n_1 is even and n_2 is odd, the design D has better nearly orthogonality. Here is an example.

Example 3. Given $n_1 = 33$, $n_2 = 4$, $m_1 = 16$ and $m_2 = 2$, take L_1 and L_2 as OSLHD$(33, 16)$ and OSLHD$(4, 2)$ constructed by Sun, Liu and Lin (2009), respectively. Let A_1 and A_2 be

$$A_1 = \left((A_0^{(1)})^T \quad \mathbf{1}_{16} \quad (A_0^{(1)})^T\right)^T, \text{ and } A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^T,$$

where $A_0^{(1)}$ is a Hadamard matrix with order 16, and $\mathbf{1}_{16}$ is a 16×1 vector of all ones. Then, two SLHD$(132, 64)$'s can be obtained as $D = (L, \tilde{L})$ and $D^* = (L^*, \widetilde{L}^*)$ respectively, where $L = A_1 \otimes L_2 + 4(L_1 \otimes A_2)$, $\widetilde{L} = L_1 \otimes_2 -33(A_1 \otimes L_2)$, $L^* = A_2 \otimes L_1 + 33(L_2 \otimes A_1)$, and $\widetilde{L}^* = L_2 \otimes A_1 - 4(A_2 \otimes L_1)$. It is easy to calculate that, $\rho_M(D) = 0.028$, $\rho_M(D^*) = 0.221$, $\rho^2(D) = \rho^2(D^*) < 0.001$, which implies that the nearly orthogonality of D^* is much better than that of D.

With Theorem 2 and Corollary 2, some new NOSLHDs can be constructed based on existing OSLHDs. Table 4 summarizes the sizes of the resulting NOSLHDs derived from Theorem 2, where n denotes the run size and m represents the number of factors. Moreover, Corollary 2 can be used to generate NOSLHDs with numbers of factors that are twice as large as those of the NOSLHDs shown in Table 4.

Table 5 collects some resulting NOSLHDs of Theorem 2 and Corollary 2 with $n < 120$. In this table, n is the run size, m is the number of factors of the constructed NOSLHDs, ρ_M and ρ^2 reflect the nearly orthogonality of the resulting designs, and L_1, L_2 are existing OSLHDs. Among existing OSLHDs, OSLHD$(11, 3)$ and OSLHD$(13, 3)$ are

Table 4: The sizes of the resulting NOSLHDs of Theorem 2.

n	m	Requirements
a_1a_2	9	$a_i \in \{11, 13\}$ for $i = 1, 2$
$a(c2^{r+1})^d$	$3d2^r f_1(c, r, d)$	$a \in \{11, 13\},\ c, r \in \mathcal{Z}^+,\ d = 2^t$, and $t \in \mathcal{N}$
$a(c2^{r+1} + 1)^d$	$3d2^r f_2(c, r, d)$	$a \in \{11, 13\},\ c, r \in \mathcal{Z}^+,\ d = 2^t$, and $t \in \mathcal{N}$
$a(2ck)^d$	$3kdg_1(c, k, d)$	$a \in \{11, 13\},\ c \in \mathcal{Z}^+,\ d = 2^t$, $t \in \mathcal{N}$ and $k \in \{12, 20, 24\}$
$a(2ck + 1)^d$	$3kdg_2(c, k, d)$	$a \in \{11, 13\},\ c \in \mathcal{Z}^+,\ d = 2^t$, $t \in \mathcal{N}$ and $k \in \{12, 20, 24\}$
$(c_1 2^{r_1+1} + 1)^{d_1}(c_2 2^{r_2+1})^{d_2}$	$\prod_{i=1}^2 2^{r_i} d_i f_i(c_i, r_i, d_i)$	$c_i, r_i \in \mathcal{Z}^+,\ d_i = 2^{t_i}$, and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(2c_1k + 1)^{d_1}(c_2 2^{r+1})^{d_2}$	$2^r kd_1d_2g_2(c_1, k, d_1)f_1(c_2, r, d_2)$	$c_i, r \in \mathcal{Z}^+,\ k \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$\prod_{i=1}^2 (c_i 2^{r_i+1} + 1)^{d_i}$	$\prod_{i=1}^2 2^{r_i} d_i f_2(c_i, r_2, d_2)$	$c_i, r_i \in \mathcal{Z}^+,\ d_i = 2^{t_i}$, and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(c_1 2^{r+1} + 1)^{d_1}(2c_2k)^{d_2}$	$2^r kd_1d_2f_2(c_1, r, d_1)g_1(c_2, k, d_2)$	$c_i, r \in \mathcal{Z}^+,\ k \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(c_1 2^{r+1} + 1)^{d_1}(2c_2k + 1)^{d_2}$	$2^r kd_1d_2f_2(c_1, r, d_1)g_2(c_2, k, d_2)$	$c_i, r \in \mathcal{Z}^+,\ k \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$(2c_1k_1)^{d_1}(2c_2k_2 + 1)^{d_2}$	$\prod_{i=1}^2 k_i d_i g_i(c_i, k_i, d_i)$	$c_i \in \mathcal{Z}^+,\ k_i \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$
$\prod_{i=1}^2 (2c_ik_i + 1)^{d_i}$	$\prod_{i=1}^2 k_i d_i g_2(c_i, k_i, d_i)$	$c_i \in \mathcal{Z}^+,\ k_i \in \{12, 20, 24\}$, $d_i = 2^{t_i}$ and $t_i \in \mathcal{N}$ for $i = 1, 2$

† $f_1(c, r, d) = \lfloor [(c2^{r+1})^d - 1]/(c2^{r+1} - 1)/d \rfloor$, $f_2(c, r, d) = \lfloor [(c2^{r+1} + 1)^d - 1]/c/2^{r+1}/d \rfloor$,

$g_1(c, k, d) = \lfloor [(2ck)^d - 1]/(2ck - 1)/d \rfloor$, $g_2(c, k, d) = \lfloor [(2ck + 1)^d - 1]/(2ckd) \rfloor$.

Table 5: Some new NOSLHDs generated by Theorem 2 and Corollary 2 with $n \leq 120$.

n	m	ρ_M	ρ^2	L_1	L_2	Method
20	4	0.0075	<0.0001	OSLHD(5,2)	OSLHD(4,2)	Theorem 2
	8	0.1880	0.0031	OSLHD(4,2)	OSLHD(5,2)	Corollary 2
36	8	0.0013	<0.0001	OSLHD(9,4)	OSLHD(4,2)	Theorem 2
	16	0.1042	0.0025	OSLHD(4,2)	OSLHD(9,4)	Corollary 2
44	6	0.0007	<0.0001	OSLHD(11,3)	OSLHD(4,2)	Theorem 2
	12	0.0853	0.0031	OSLHD(4,2)	OSLHD(11,3)	Corollary 2
45	8	0.1067	0.0049	OSLHD(5,2)	OSLHD(9,4)	Theorem 2
	16	0.1976	0.0031	OSLHD(5,2)	OSLHD(9,4)	Corollary 2
52	6	0.0004	<0.0001	OSLHD(13,3)	OSLHD(4,2)	Theorem 2
	12	0.0721	0.0033	OSLHD(4,2)	OSLHD(13,3)	Corollary 2
55	6	0.1984	0.0079	OSLHD(11,3)	OSLHD(5,2)	Theorem 2
	12	0.1984	0.0026	OSLHD(5,2)	OSLHD(11,3)	Corollary 2
60	4	0.0079	<0.0001	OSLHD(5,2)	OSLHD(12,2)	Theorem 2
	8	0.1987	0.0051	OSLHD(5,2)	OSLHD(12,2)	Corollary 2
68	16	0.0002	<0.0001	OSLHD(17,8)	OSLHD(4,2)	Theorem 2
	32	0.0552	0.0016	OSLHD(4,2)	OSLHD(17,8)	Corollary 2
72	16	0.0014	<0.0001	OSLHD(9,4)	OSLHD(8,4)	Theorem 2
	32	0.1094	0.0006	OSLHD(8,4)	OSLHD(9,4)	Corollary 2
85	16	0.0565	0.0015	OSLHD(5,2)	OSLHD(17,8)	Theorem 2
	32	0.1993	0.0017	OSLHD(5,2)	OSLHD(17,8)	Corollary 2
88	12	0.0007	<0.0001	OSLHD(11,3)	OSLHD(8,4)	Theorem 2
	24	0.0895	0.0004	OSLHD(8,4)	OSLHD(11,3)	Corollary 2
99	12	0.0898	0.0015	OSLHD(9,4)	OSLHD(11,3)	Theorem 2
	24	0.1102	0.0012	OSLHD(9,4)	OSLHD(11,3)	Corollary 2
100	24	0.0001	<0.0001	OSLHD(25,12)	OSLHD(4,2)	Theorem 2
	48	0.0375	0.0011	OSLHD(4,2)	OSLHD(25,12)	Corollary 2
104	12	0.0004	<0.0001	OSLHD(13,3)	OSLHD(8,4)	Theorem 2
	24	0.0757	0.0004	OSLHD(8,4)	OSLHD(13,3)	Corollary 2
108	8	0.0014	<0.0001	OSLHD(9,4)	OSLHD(12,2)	Theorem 2
	16	0.1103	0.0013	OSLHD(9,4)	OSLHD(12,2)	Corollary 2
117	12	0.0760	0.0011	OSLHD(9,4)	OSLHD(13,3)	Theorem 2
	24	0.1105	0.0011	OSLHD(9,4)	OSLHD(13,3)	Corollary 2
120	24	0.0080	<0.0001	OSLHD(5,2)	OSLHD(24,12)	Theorem 2

constructed by Wang et al. (2018), OSLHD$(24, 12)$ and OSLHD$(25, 12)$ are constructed by Georgiou and Stylianou (2011), and the others are all constructed by Sun, Liu and Lin (2009). Note that, the constructed NOSLHDs shown in Table 5 all have new run sizes or perform better than existing NOSLHDs. It can be seen that, the run sizes of the resulting designs are flexible, and both ρ_M and ρ^2 are small. Besides, although the resulting design in Corollary 2 sacrifices the performance of ρ_M compared to the resulting design in Theorem 2 during the process of doubling the number of factors, ρ^2 does not necessarily deteriorate. For example, ρ^2 of NOSLHD$(99, 24)$ is smaller than that of NOSLHD$(99, 12)$.

3.2 Construction of OSLHD$(2^q, 2^{q-1})$'s with an Even q

The application of Construction 1 relies on the existence of two small OSLHDs. In this subsection, we provide another method to construct OSLHD$(2^q, 2^{q-1})$'s for any even $q > 2$. Besides, we theoretically prove that the resulting OSLHDs have better stratification properties than existing designs.

Construction 2.

Step 1. For $k = 1$, let $S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $T_1 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

Step 2. If $q > 4$, for $k = 2, \ldots, q/2 - 1$, define S_k and T_k as

$$S_k = \begin{pmatrix} S_{k-1} & -S^*_{k-1} \\ S_{k-1} & S^*_{k-1} \end{pmatrix} \quad \text{and} \quad T_k = \begin{pmatrix} T_{k-1} & -T^*_{k-1} - 2^{k-1} S^*_{k-1} \\ T_{k-1} + 2^{k-1} S_{k-1} & T^*_{k-1} \end{pmatrix},$$

where S^*_{k-1} and T^*_{k-1} are two $2^{k-1} \times 2^{k-1}$ matrices obtained by multiplying the entries in the first half rows of S_{k-1} and T_{k-1} by -1 while keeping those in the last half of S_{k-1} and T_{k-1} unchanged, respectively.

Step 3. Let A_0 be a Hadamard matrix with order $2^{q/2-1}$, and define

$$D_{q/2} = \begin{pmatrix} T_{q/2-1} - S_{q/2-1}/2 \\ -T_{q/2-1} + S_{q/2-1}/2 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} A_0 \\ A_0 \end{pmatrix}.$$

Step 4. A design of 2^q runs and 2^{q-1} factors can be obtained as $(L, \widetilde{L})$, where

$$L = A_1 \otimes D_{q/2} + 2^{q/2}(D_{q/2} \otimes A_1) \quad \text{and} \quad \widetilde{L} = D_{q/2} \otimes A_1 - 2^{q/2}(A_1 \otimes D_{q/2}).$$

Note that, Sun, Liu and Lin (2009) proved that $D_{q/2}$ in Step 3 is an OSLHD$(2^{q/2}, 2^{q/2-1})$, and an OSLHD$(2^q, 2^{q-1})$ with $q \geq 2$ can be constructed as

$$D_q = \begin{pmatrix} T_{q-1} - S_{q-1}/2 \\ -T_{q-1} + S_{q-1}/2 \end{pmatrix}.$$

The following example compares two OSLHDs generated by Construction 2 and Sun, Liu and Lin (2009).

Example 4. An OSLHD$(16, 8)$ can be generated by Construction 2 as follows. Obviously, there is $q = 4$ when the run size equals 16. With Steps 1–3, we have

$$D_2 = \begin{pmatrix} 0.5 & 1.5 & -0.5 & -1.5 \\ 1.5 & -0.5 & -1.5 & 0.5 \end{pmatrix}^T \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^T.$$

Thus, $L = A_1 \otimes D_2 + 4(D_2 \otimes A_1)$ and $\widetilde{L} = D_2 \otimes A_2 - 4(A_1 \otimes D_2)$ can be obtained as shown in Table 6. At this time, $(L, \widetilde{L})$ is an OSLHD$(16, 8)$. Note that, a different OSLHD$(16, 8)$ shown in Table 7 can be constructed with the method proposed by Sun, Liu and Lin (2009). Although both of them are OSLHDs, the design obtained by Construction 2 has better stratification property.

Figure 1 shows the two-dimensional projections of these two OSLHD$(16, 8)$'s. It can be seen that in the OSLHD$(16, 8)$ obtained by Construction 2, there are 16 pairs

Table 6: The resulting OSLHD$(16, 8)$ obtained by Construction 2 in Example 4.

L				$\widetilde{L}$			
2.5	3.5	6.5	7.5	−1.5	−5.5	−0.5	−4.5
3.5	−2.5	7.5	−6.5	−5.5	1.5	−4.5	0.5
1.5	0.5	5.5	4.5	2.5	6.5	3.5	7.5
0.5	−1.5	4.5	−5.5	6.5	−2.5	7.5	−3.5
6.5	7.5	−2.5	−3.5	−0.5	−4.5	1.5	5.5
7.5	−6.5	−3.5	2.5	−4.5	0.5	5.5	−1.5
5.5	4.5	−1.5	−0.5	3.5	7.5	−2.5	−6.5
4.5	−5.5	−0.5	1.5	7.5	−3.5	−6.5	2.5
−1.5	−0.5	−5.5	−4.5	−2.5	−6.5	−3.5	−7.5
−0.5	1.5	−4.5	5.5	−6.5	2.5	−7.5	3.5
−2.5	−3.5	−6.5	−7.5	1.5	5.5	0.5	4.5
−3.5	2.5	−7.5	6.5	5.5	−1.5	4.5	−0.5
−5.5	−4.5	1.5	0.5	−3.5	−7.5	2.5	6.5
−4.5	5.5	0.5	−1.5	−7.5	3.5	6.5	−2.5
−6.5	−7.5	2.5	3.5	0.5	4.5	−1.5	−5.5
−7.5	6.5	3.5	−2.5	4.5	−0.5	−5.5	1.5

Table 7: The OSLHD$(16, 8)$ constructed by Sun, Liu and Lin (2009).

OSLHD$(16, 8)$							
0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
1.5	−0.5	−3.5	2.5	5.5	−4.5	−7.5	6.5
2.5	3.5	−0.5	−1.5	−6.5	−7.5	4.5	5.5
3.5	−2.5	1.5	−0.5	−7.5	6.5	−5.5	4.5
4.5	5.5	6.5	7.5	−0.5	−1.5	−2.5	−3.5
5.5	−4.5	−7.5	6.5	−1.5	0.5	3.5	−2.5
6.5	7.5	−4.5	−5.5	2.5	3.5	−0.5	−1.5
7.5	−6.5	5.5	−4.5	3.5	−2.5	1.5	−0.5
−0.5	−1.5	−2.5	−3.5	−4.5	−5.5	−6.5	−7.5
−1.5	0.5	3.5	−2.5	−5.5	4.5	7.5	−6.5
−2.5	−3.5	0.5	1.5	6.5	7.5	−4.5	−5.5
−3.5	2.5	−1.5	0.5	7.5	−6.5	5.5	−4.5
−4.5	−5.5	−6.5	−7.5	0.5	1.5	2.5	3.5
−5.5	4.5	7.5	−6.5	1.5	−0.5	−3.5	2.5
−6.5	−7.5	4.5	5.5	−2.5	−3.5	0.5	1.5
−7.5	6.5	−5.5	4.5	−3.5	2.5	−1.5	0.5

of columns achieving stratifications on 4×4 grids. However, none of column pairs of the OSLHD$(16, 8)$ obtained by Sun, Liu and Lin (2009) can achieve a stratification on a 4×4 grid.

Actually, the fact that the stratification properties of the resulting designs in Construction 2 are better than that of designs constructed by Sun, Liu and Lin (2009) can be proved theoretically. First, we demonstrate the stratification properties of OSLHD$(2^q, 2^{q-1})$'s obtained by Sun, Liu and Lin (2009) as follows.

Proposition 1. *For design D_q with $q \geq 2$, the following stratification properties hold:*

(i) *this design achieves a stratification on a 2×2 grid when projected onto any two dimensions;*

(ii) *there are $2^{q-1}(2^{q-2} - 1)$ column pairs achieving stratifications on 2×4 and 4×2 grids;*

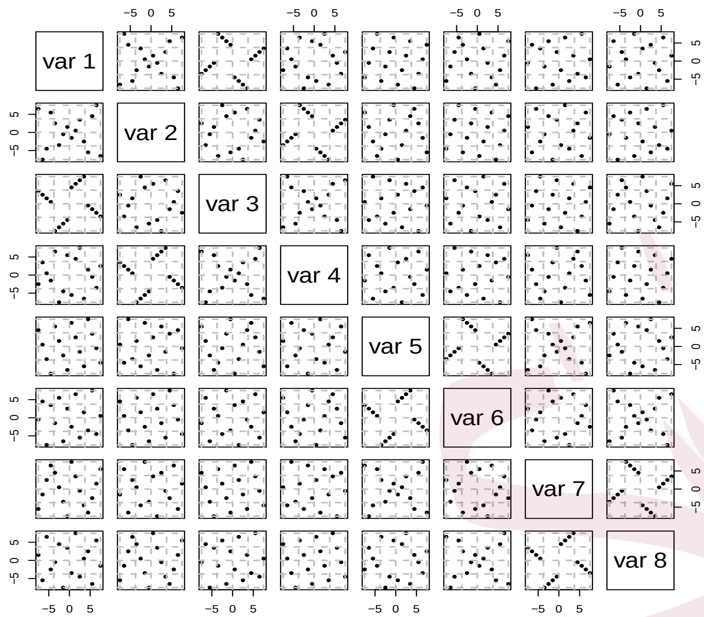
(iii) *none of column pairs can achieve a stratification on a 4×4 grid.*

Next, we show the stratification properties of the resulting designs of Construction 2 as follows.

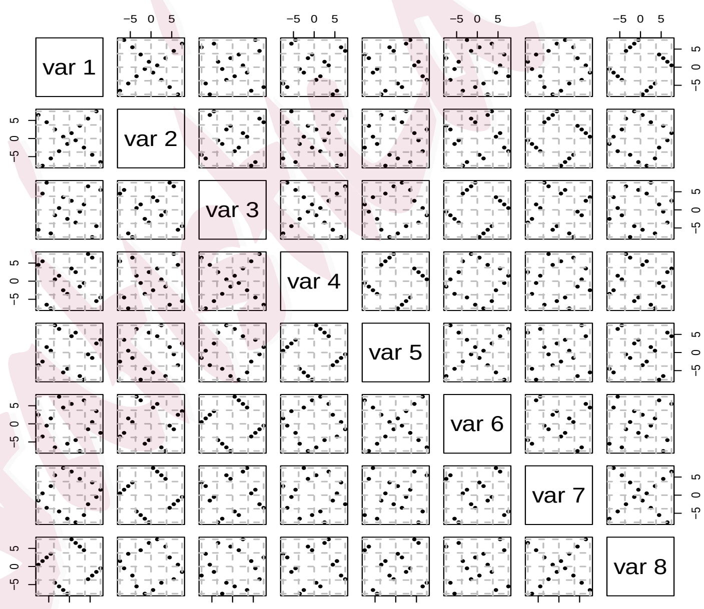
Theorem 3. *Given an even $q \geq 4$, the resulting design $(L, \widetilde{L})$ of Construction 2 is an OSLHD$(2^q, 2^{q-1})$ possessing the following stratification properties:*

(i) *this design achieves a stratification on a 2×2 grid when projected onto any two dimensions;*

(ii) *there are $2^{q-1}(2^{q-2} - 1)$ column pairs achieving stratifications on 2×4 and 4×2 grids;*



(a) The OSLHD$(16, 8)$ obtained by Construction 2.



(b) The OSLHD$(16, 8)$ obtained by Sun, Liu and Lin (2009).

Figure 1: The two-dimensional projections of two OSLHD$(16, 8)$'s obtained by Construction 2 and Sun, Liu and Lin (2009) respectively.

(iii) *there are* 4^{q-2} *column pairs achieving stratifications on* $2^{q/2} \times 2^{q/2}$ *grids.*

Theorem 3 shows that for any resulting design of Construction 2, almost all column pairs can achieve stratifications on 2×4 and 4×2 grids, and more than half of column pairs can achieve a stratification on an $n^{1/2} \times n^{1/2}$ grid, where n is the run size. From Proposition 1 and Theorem 3, it can be seen that for any even $q \geq 4$, the stratification property of the OSLHD$(2^q, 2^{q-1})$ obtained by Construction 2 is better than that of the design constructed by Sun, Liu and Lin (2009).

4. Case study

In this section, we evaluate the performance of the OSLHD and NOSLHD generated by the proposed method against that obtained by existing methods using the flow rate of water data. This study involves eight variables: the radius of the borehole $r_w \in [0.05, 0.15]$, the radius of influence $r \in [100, 50000]$, the transmissivity of upper aquifer $T_u \in [63070, 115600]$, the transmissivity of lower aquifer $T_l \in [63.1, 116]$, the potentiometric head of upper aquifer $H_u \in [990, 1110]$, the potentiometric head of lower aquifer $H_l \in [700, 820]$, the potentiometric head of lower aquifer $L \in [1120, 1680]$, and the hydraulic conductivity of borehole $K_w \in [9855, 12045]$. The response is the flow rate through the borehole, which is determined by

$$\mathbf{y} = \frac{2\pi T_u [H_u - H_l]}{\ln\left(\frac{r}{r_w}\right)\left[1 + \frac{2LT_u}{\ln(r/r_w) r_w^2 K_w} + \frac{T_u}{T_l}\right]}.$$

We employ an OSLHD$(16, 8)$ obtained by Construction 2 and an NOSLHD$(45, 8)$ obtained by Construction 1 to collect data for fitting models. For comparison, we include the OSLHD$(16, 8)$ constructed by Sun, Liu and Lin (2009) and some randomly generated LHD$(45, 8)$'s. Three modeling approaches are examined, which are a first-order linear model (FOLM), a Gaussian process model (GP) and a second-order linear model (SOLM). Considering the significant differences in the variable domains, all variables are standardized before model fitting. Variables are centered by subtracting their mean values for linear models, and then scaled to unit variance by dividing by standard deviations for the Gaussian process model. For the second-order linear model, a stepwise selection procedure is implemented to address the high dimensionality of the parameter space.

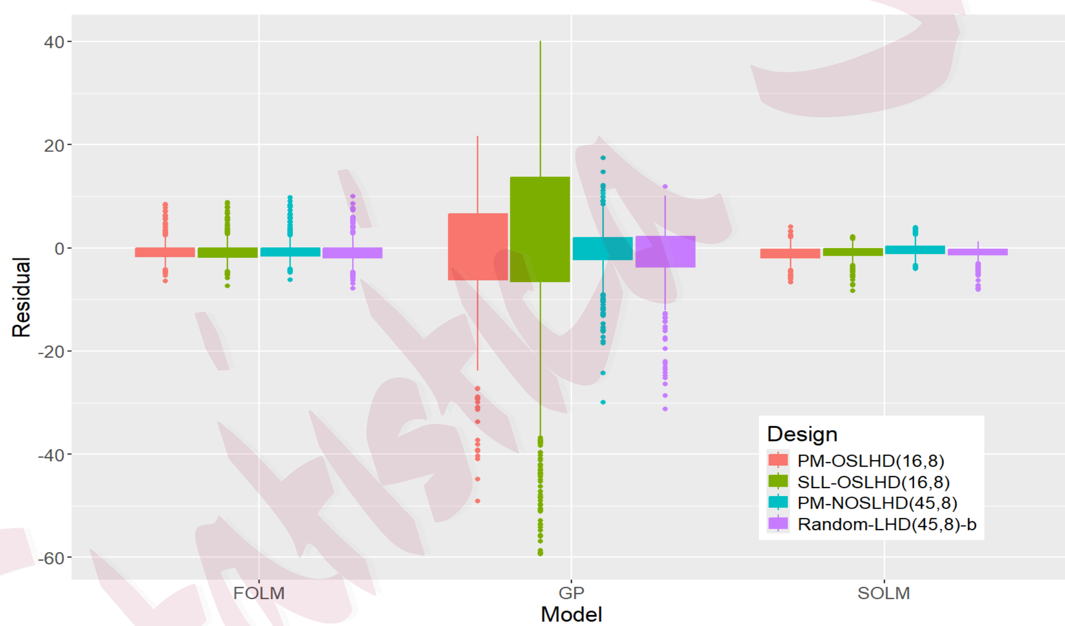


Figure 2: Prediction errors at 1000 random samples for four designs and three models.

To compare the performance of these designs, we randomly generate 1000 new samples, and calculate the prediction errors for each sample under each fitted model, Figure 2 presents the prediction results, where "PM-OSLHD$(16, 8)$" and "PM-NOSLHD$(45, 8)$" represent the OSLHD$(16, 8)$ obtained by Construction 2 and the

NOSLHD$(45, 8)$ obtained by Construction 1 respectively, "SLL-OSLHD$(16, 8)$" denotes the OSLHD$(16, 8)$ constructed by Sun, Liu and Lin (2009), and "Random-LHD$(45, 8)$-b" means the best one among 100 randomly generated LHD$(45, 8)$'s. It can be seen that, the constructed OSLHD and NOSLHD consistently outperform the existing designs across all models, which highlights the effectiveness of these constructed designs in data collection. Besides, the constructed NOSLHD$(45, 8)$ and the random generated LHD$(45, 8)$ perform better than two OSLHD$(16, 8)$'s when the Gaussian process model is fitted, which likely due to the increase in run sizes.

5. Concluding Remarks

SLHDs can not only achieve the maximum stratification when projected onto any one dimension, but also guarantee that the estimates of second-order effects and main effects are uncorrelated when a second-order linear model is fitted. In exiting literatures, Wang et al. (2018) summarized existing construction methods for OSLHDs and NOSLHDs. It can be seen that, the run sizes of the constructed OSLHDs are either powers of a specific integer, multiples of several values, or sums of these values plus one. Besides, these methods have not considered the stratification properties of the constructed SLHDs, which can reflect the space-filling properties of designs in low-dimensional projections.

In this paper, we propose two methods to construct SLHDs. Compared with existing construction methods for OSLHDs and NOSLHDs, the run sizes of the resulting designs of Construction 1 are more flexible, especially for the resulting NOSLHDs. Theorem 2 proves that the resulting NOSLHDs have good nearly orthogonality. Tables 2 and 4 summarizes the sizes of the constructed OSLHDs and NOSLHDs respectively, and Table 5 collects some new NOSLHDs obtained by Construction 1. Besides, Construction 2 can be used to obtain space-filling OSLHDs, and Theorem 3 guarantees that the stratification properties of the constructed OSLHDs are better than that of existing designs. A case study is provided to highlight the effectiveness of constructed designs in data collection.

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Appendix: Proofs

A.1. Proof of Lemma 1

First, we can prove that L is an LHD(n_1n_2, m_1m_2). For $u = 1, 2$ and $i = 1, \ldots, n_u$, let $\mathbf{l}_{u,i}$ and $\mathbf{a}_{u,i}$ be the i-th rows of L_u and A_u, respectively. Note that, the following properties hold for L_1, L_2, A_1 and A_2:

(a) both L_1 and L_2 are LHDs;

(b) for $u = 1, 2$, if i and i' satisfy that $\mathbf{l}_{u,i} = -\mathbf{l}_{u,i'}$, then $\mathbf{a}_{u,i} = \mathbf{a}_{u,i'}$.

Thus, Lemma 1 in Lin et al. (2010) implies that L is an LHD(n_1n_2, m_1m_2).

Next, let us show the symmetric property of L when n_1 is odd and n_2 is even. Denote $L = (B_1^T, B_2^T, B_3^T)^T$, where B_1 and B_3 are $(n_1-1)n_2 \times m_1m_2$ matrices, and B_2 is an $n_2 \times m_1m_2$ matrix. From Construction 1, we have

$$B_1 = A_0^{(1)} \otimes \begin{pmatrix} L_0^{(2)} \\ -L_0^{(2)} \end{pmatrix} + n_2 L_0^{(1)} \otimes \begin{pmatrix} A_0^{(2)} \\ A_0^{(2)} \end{pmatrix} = \begin{pmatrix} A_0^{(1)} \otimes L_0^{(2)} + n_2 L_0^{(1)} \otimes A_0^{(2)} \\ -A_0^{(1)} \otimes L_0^{(2)} + n_2 L_0^{(1)} \otimes A_0^{(2)} \end{pmatrix},$$
$$B_2 = \mathbf{e}^T \otimes \begin{pmatrix} L_0^{(2)} \\ -L_0^{(2)} \end{pmatrix} + n_2 \cdot \mathbf{0}_{m_1}^T \otimes \begin{pmatrix} A_0^{(2)} \\ A_0^{(2)} \end{pmatrix} = \begin{pmatrix} \mathbf{e}^T \otimes L_0^{(2)} \\ -\mathbf{e}^T \otimes L_0^{(2)} \end{pmatrix}, \quad \text{and}$$
$$B_3 = A_0^{(1)} \otimes \begin{pmatrix} L_0^{(2)} \\ -L_0^{(2)} \end{pmatrix} - n_2 L_0^{(1)} \otimes \begin{pmatrix} A_0^{(2)} \\ A_0^{(2)} \end{pmatrix} = \begin{pmatrix} A_0^{(1)} \otimes L_0^{(2)} - n_2 L_0^{(1)} \otimes A_0^{(2)} \\ -A_0^{(1)} \otimes L_0^{(2)} - n_2 L_0^{(1)} \otimes A_0^{(2)} \end{pmatrix}.$$

It can be seen that B_2 is a symmetric design, and B_1 is equivalent to $-B_3$ in the sense of reordering the rows, which implies that L is symmetric. Other cases of n_1 and n_2 can be proved similarly.

A.2. Proof of Theorem 1

We only need to prove that when conditions (i)–(iii) hold, L is column-orthogonal. From Step 3 of Construction 1, we have

$$\begin{aligned} L^T L =& [A_1 \otimes L_2 + n_2(L_1 \otimes A_2)]^T [A_1 \otimes L_2 + n_2(L_1 \otimes A_2)] \\ =& (A_1^T A_1) \otimes (L_2^T L_2) + n_2(A_1^T L_1) \otimes (L_2^T A_2) + \\ & n_2(L_1^T A_1) \otimes (A_2^T L_2) + n_2^2(L_1^T L_1) \otimes (A_2^T A_2). \end{aligned}$$

Conditions (i)–(iii) imply that $L_i^T L_i = n_i(n_i^2-1)I_{m_i}/12$ and $A_i^T A_i = n_i I_{m_i}$ for $i = 1, 2$. Besides, condition (i) and Steps 1–2 guarantee that

$$A_i^T L_i = \begin{pmatrix} (A_0^{(i)})^T & (A_0^{(i)})^T \end{pmatrix} \begin{pmatrix} L_0^{(i)} \\ -L_0^{(i)} \end{pmatrix} = (A_0^{(i)})^T L_0^{(i)} - (A_0^{(i)})^T L_0^{(i)} = \mathbf{0}_{m_i} \mathbf{0}_{m_i}^T,$$

for $i = 1, 2$. Thus, we have $L^T L = n_1 n_2[(n_1 n_2)^2 - 1] I_{m_1 m_2}/12$, which means that L is column-orthogonal. Combining with Lemma 1, it can be seen that L is an OSLHD$(n_1 n_2, m_1 m_2)$.

A.3. Proof of Corollary 1

To prove this corollary, we first demonstrate that $\widetilde{L}$ is an LHD$(n_0^2, m_1 m_2)$. Without loss of generality, we only need to prove that the first column of $\widetilde{L}$ is a permutation of $-(n_0^2-1)/2, -(n_0^2-3)/2, \ldots, (n_0^2-1)/2$. For $i = 1, 2$ and $j = 1, \ldots, n_0$, let a_{ij} and l_{ij} denote the j-th elements of the first column in A_i and L_i, respectively. Then, the elements of the first column in $\widetilde{L}$ can be represented as $l_{1k}a_{2j} - n_0 a_{1k} l_{2j}$, where $j, k = 1, \ldots, n_0$. Since L_2 is an LHD(n_0, m_2), for any given $u \in \{1, 3, \ldots, n_0 - 1\}$, there must exist $j, j' \in \{1, \ldots, n_0\}$ such that $l_{2j} = (n_0 - u)/2$ and $l_{2j'} = -(n_0 - u)/2$. Note that A_1 is a two-level design, which implies that $\{-n_0 a_{1k} l_{2j}, -n_0 a_{1k} l_{2j'}\} = \{-n_0(n_0 - u)/2, n_0(n_0 - u)/2\}$ for any $k = 1, \ldots, n_0$. Because A_2 is a two-level design and L_1 is an LHD(n_0, m_2), it easy to verify that the points of the set $\{l_{1k}a_{2j} - n_0 a_{1k} l_{2j}, l_{1k} a_{2j'} - n_0 a_{1k} l_{2j'} \mid k = 1, \ldots, n_0\}$ form an LHD$(n_0, 1)$ centered at $-n_0(n_0 - u)/2$ and an LHD$(n_0, 1)$ centered at $n_0(n_0 - u)/2$. Let u take all values from $1, 3, \ldots, n_0 - 1$, it can be seen that $l_{1k}a_{2j} - n_0 a_{1k} l_{2j}$ for $j, k = 1, \ldots, n_0$ form an LHD$(n_0^2, 1)$. Therefore, $\widetilde{L}$ is an LHD$(n_0^2, m_1 m_2)$.

Secondly, the orthogonality of $\widetilde{L}$ can be obtained in a similar way to the proof of Theorem 1. Thus, $\widetilde{L}$ is an OLHD$(n_0^2, m_1 m_2)$.

Thirdly, we can prove that each column of L is orthogonal to any column of $\widetilde{L}$ as follows. According to the expressions of L and $\widetilde{L}$, we have

$$\begin{aligned} L^T\widetilde{L} &= (A_1^T \otimes L_2^T + n_0 L_1^T \otimes A_2^T)(L_1 \otimes A_2 - n_0 A_1 \otimes L_2) \\ &= A_1^T L_1 \otimes L_2^T A_2 + n_0 L_1^T L_1 \otimes A_2^T A_2 - n_0 A_1^T A_1 \otimes L_2^T L_2 - n_0^2 L_1^T A_1 \otimes A_2^T L_2. \end{aligned}$$

For $i = 1, 2$, there are $A_i^T A_i = n_0 I_{m_i}$, $L_i^T L_i = n_0^2(n_0^2 - 1)I_{m_i}/12$ and $A_i^T L_i = \mathbf{0}_{m_i}\mathbf{0}_{mi}^T$. Thus, it can be inferred that $L^T\widetilde{L} = \mathbf{0}_{m_1m_2}\mathbf{0}_{m_1m_2}^T$, which implies that each column of L is orthogonal to any column of $\widetilde{L}$.

Finally, the symmetric property of $(L, \widetilde{L})$ can be proved as follows. Since n_0 is even, we have $A_i = \begin{pmatrix}(A_0^{(i)})^T & (A_0^{(i)})^T\end{pmatrix}^T$ and $L_i = \begin{pmatrix}(L_0^{(i)})^T & -(L_0^{(i)})^T\end{pmatrix}^T$ for $i = 1$ and 2. Thus, it can be calculated that

$$\begin{aligned} (L, \widetilde{L}) &= \begin{pmatrix} A_0^{(1)} \otimes L_2 + n_0 L_0^{(1)} \otimes A_2 & L_0^{(1)} \otimes A_2 - n_0 A_0^{(1)} \otimes L_2 \\ A_0^{(1)} \otimes L_2 - n_0 L_0^{(1)} \otimes A_2 & -L_0^{(1)} \otimes A_2 - n_0 A_0^{(1)} \otimes L_2 \end{pmatrix} \\ &= \begin{pmatrix} A_0^{(1)} \otimes L_0^{(2)} + n_0 L_0^{(1)} \otimes A_0^{(2)} & L_0^{(1)} \otimes A_0^{(2)} - n_0 A_0^{(1)} \otimes L_0^{(2)} \\ -A_0^{(1)} \otimes L_0^{(2)} + n_0 L_0^{(1)} \otimes A_0^{(2)} & L_0^{(1)} \otimes A_0^{(2)} + n_0 A_0^{(1)} \otimes L_0^{(2)} \\ A_0^{(1)} \otimes L_0^{(2)} - n_0 L_0^{(1)} \otimes A_0^{(2)} & -L_0^{(1)} \otimes A_0^{(2)} - n_0 A_0^{(1)} \otimes L_0^{(2)} \\ -A_0^{(1)} \otimes L_0^{(2)} - n_0 L_0^{(1)} \otimes A_0^{(2)} & -L_0^{(1)} \otimes A_0^{(2)} + n_0 A_0^{(1)} \otimes L_0^{(2)} \end{pmatrix}, \end{aligned}$$

which implies that $(L, \widetilde{L})$ is symmetric. The proof is completed.

A.4. Proof of Theorem 2

To obtain values of $\rho_M(L)$ and $\rho^2(L)$, we need to calculate $L^T L$. Following Step 3 of Construction 1, we have

$$L^T L = (A_1^T A_1) \otimes (L_2^T L_2) + n_2(A_1^T L_1) \otimes (L_2^T A_2) + n_2(L_1^T A_1) \otimes (A_2^T L_2) + n_2^2(L_1^T L_1) \otimes (A_2^T A_2).$$

Condition (i) implies that $L_i^T L_i = n_i(n_i^2 - 1)I_{m_i}/12$ for $i = 1, 2$. Next, we can calculate $A_i^T A_i$ and $A_i^T L_i$ with $i = 1, 2$ under three cases, and then obtain $\rho_M(L)$ and $\rho^2(L)$.

(a) When n_1 is odd and n_2 is even, A_1 and A_2 can be written as

$$A_1 = \left((A_0^{(1)})^T \quad \mathbf{e}_1 \quad (A_0^{(1)})^T\right)^T \quad \text{and} \quad A_2 = \left((A_0^{(2)})^T \quad (A_0^{(2)})^T\right)^T.$$

From Steps 1 and 2, we have

$$A_1^T L_1 = \left((A_0^{(1)})^T \quad \mathbf{e}_1 \quad (A_0^{(1)})^T\right) \begin{pmatrix} L_0^{(1)} \\ \mathbf{0}_{m_1}^T \\ -L_0^{(1)} \end{pmatrix} = \mathbf{0}_{m_1}\mathbf{0}_{m_1}^T, \quad \text{and}$$

$$A_2^T L_2 = \left((A_0^{(2)})^T \quad (A_0^{(2)})^T\right) \begin{pmatrix} L_0^{(2)} \\ -L_0^{(2)} \end{pmatrix} = \mathbf{0}_{m_2}\mathbf{0}_{m_2}^T.$$

Besides, Condition (ii) implies that $A_1^T A_1 = 2(A_0^{(1)})^T A_0^{(1)} + \mathbf{e}_1\mathbf{e}_1^T = (n_1-1)I_{m_1} + \mathbf{e}_1\mathbf{e}_1^T$ and $A_2^T A_2 = n_2 I_{m_2}$. Thus, it is easy to obtain that

$$L^T L = \frac{n_1 n_2(n_1^2 n_2^2 - 1)}{12} I_{m_1 m_2} + \frac{n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1 m_2}\right].$$

Note that, $(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1 m_2}$ is an $m_1 m_2 \times m_1 m_2$ matrix whose diagonal elements are all 0, and there are $m_1 - 1$ non-zero elements being equal to 1 or -1 with all other $m_1 m_2 - m_1 + 1$ elements being 0 in each column of it. Thus, we have

$$\begin{aligned}
\rho_M(L) =& \frac{n_2(n_2^2-1)/12}{n_1 n_2(n_1^2 n_2^2 - 1)/12} = \frac{n_2^2 - 1}{n_1(n_1^2 n_2^2 - 1)}, \quad \text{and} \\
\rho^2(L) =& \frac{[n_2(n_2^2-1)/12]^2 \times (m_1 - 1) \times m_1 m_2/[m_1 m_2(m_1 m_2 - 1)]}{[n_1 n_2(n_1^2 n_2^2 - 1)/12]^2} \\
=& \frac{(m_1-1)(n_2^2-1)^2}{(m_1 m_2 - 1) n_1^2 (n_1^2 n_2^2 - 1)^2}.
\end{aligned}$$

(b) When n_1 is even and n_2 is odd, we have

$$A_1 = \left((A_0^{(1)})^T \quad (A_0^{(1)})^T\right)^T \quad \text{and} \quad A_2 = \left((A_0^{(2)})^T \quad \mathbf{e}_2 \quad (A_0^{(2)})^T\right)^T.$$

Similarly, following Construction 1 and condition (ii), it can be obtained that $A_i^T L_i = \mathbf{0}_{m_i}\mathbf{0}_{m_i}^T$ for $i = 1, 2$, $A_1^T A_1 = n_1 I_{m_1}$ and $A_2^T A_2 = (n_2 - 1)I_{m_2} + \mathbf{e}_2\mathbf{e}_2^T$, which implies that

$$L^T L = \frac{n_1 n_2(n_1^2 n_2^2 - 1)}{12} I_{m_1 m_2} + \frac{n_1 n_2^2(n_1^2-1)}{12}\left[I_{m_1}\otimes(\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1 m_2}\right].$$

Note that, $I_{m_1} \otimes (\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}$ is an $m_1m_2 \times m_1m_2$ block diagonal matrix, and there are $m_2 - 1$ non-zero elements being equal to 1 or -1 with all other $m_1m_2 - m_2 + 1$ elements being 0 in each column of it. Thus, we have

$$\begin{aligned}
\rho_M(L) =& \frac{n_1(n_1^2-1)n_2^2/12}{n_1n_2(n_1^2n_2^2-1)/12} = \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}, \quad \text{and} \\
\rho^2(L) =& \frac{[n_1(n_1^2-1)n_2^2/12]^2 \times (m_2-1) \times m_1m_2/[m_1m_2(m_1m_2-1)]}{[n_1n_2(n_1^2n_2^2-1)/12]^2} \\
=& \frac{(m_2-1)(n_1^2-1)^2n_2^2}{(m_1m_2-1)(n_1^2n_2^2-1)^2}.
\end{aligned}$$

(c) When both n_1 and n_2 are odd, we have

$$A_1 = \begin{pmatrix} (A_0^{(1)})^T & \mathbf{e}_1 & (A_0^{(1)})^T \end{pmatrix}^T \quad \text{and} \quad A_2 = \begin{pmatrix} (A_0^{(2)})^T & \mathbf{e}_2 & (A_0^{(2)})^T \end{pmatrix}^T.$$

Similarly, when conditions (i) and (ii) hold, it can be calculated that

$$\begin{aligned}
L^TL =& \frac{n_1n_2(n_1^2n_2^2-1)}{12} I_{m_1m_2} + \frac{n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T) \otimes I_{m_2} - I_{m_1m_2}\right] \\
&+ \frac{n_1n_2^2(n_1^2-1)}{12}\left[I_{m_1} \otimes (\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right].
\end{aligned}$$

Note that,

$$\frac{n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T) \otimes I_{m_2} - I_{m_1m_2}\right] + \frac{n_1n_2^2(n_1^2-1)}{12}\left[I_{m_1} \otimes (\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right]$$

is an $m_1m_2 \times m_1m_2$ matrix whose diagonal elements are all 0, and each column of it has m_1+m_2-2 non-zero elements, where m_1-1 non-zero elements equal $n_2(n_2^2-1)/12$ or $-n_2(n_2^2-1)/12$ and the remaining m_2-1 non-zero elements equal $n_1n_2^2(n_1^2-1)/12$ or $-n_1n_2^2(n_1^2-1)/12$. As a consequence, we have

$$\begin{aligned}
\rho_M(L) =& \max\left\{\frac{n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1(n_1^2-1)n_2^2/12}{n_1n_2(n_1^2n_2^2-1)/12}\right\} \\
=& \max\left\{\frac{n_2^2-1}{n_1(n_1^2n_2^2-1)}, \frac{(n_1^2-1)n_2}{n_1^2n_2^2-1}\right\}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}\rho^2(L) =& \frac{1}{[n_1n_2(n_1^2n_2^2-1)/12]^2} \times \Big\{[n_2(n_2^2-1)/12]^2 \times (m_1-1) \\ &+ [n_1(n_1^2-1)n_2^2/12]^2 \times (m_2-1)\Big\} \times \frac{m_1m_2}{[m_1m_2(m_1m_2-1)]} \\ =& \frac{(m_1-1)(n_2^2-1)^2 + (m_2-1)n_1^2(n_1^2-1)^2n_2^2}{(m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2}.\end{aligned}$$

Obviously, in either case, L^TL can not be a diagonal matrix. Therefore, if at least one of n_1 and n_2 is odd, the design L obtained in Construction 1 must not be orthogonal. The proof is completed.

A.5. Proof of Corollary 2

Similar to the proof of Corollary 1, it can be verified that $D = (L, \widetilde{L})$ is an SLHD$(n_1n_2, 2m_1m_2)$. Then, to obtain values of $\rho_M(D)$ and $\rho^2(D)$, we need to calculate D^TD. Following Step 3 of Construction 1 and Corollary 2, we have

$$D^TD = (L,\widetilde{L})^T(L,\widetilde{L}) = \begin{pmatrix} L^TL & L^T\widetilde{L} \\ \widetilde{L}^TL & \widetilde{L}^T\widetilde{L}\end{pmatrix},$$

$$L^TL = (A_1^TA_1)\otimes(L_2^TL_2) + n_2(A_1^TL_1)\otimes(L_2^TA_2) + n_2(L_1^TA_1)\otimes(A_2^TL_2) + n_2^2(L_1^TL_1)\otimes(A_2^TA_2),$$

$$L^T\widetilde{L} = (A_1^TL_1)\otimes(L_2^TA_2) - n_1(A_1^TA_1)\otimes(L_2^TL_2) + n_2(L_1^TL_1)\otimes(A_2^TA_2) - n_1n_2(L_1^TA_1)\otimes(A_2^TL_2),$$

$$\widetilde{L}^T\widetilde{L} = (L_1^TL_1)\otimes(A_2^TA_2) - n_1(A_1^TL_1)\otimes(L_2^TA_2) - n_1(L_1^TA_1)\otimes(A_2^TL_2) + n_1^2(A_1^TA_1)\otimes(L_2^TL_2),$$

and $A_i^TL_i = \mathbf{0}_{m_i}\mathbf{0}_{m_i}^T$ for $i = 1, 2$. Condition (i) implies that $L_i^TL_i = n_i(n_i^2-1)I_{m_i}/12$ for $i = 1, 2$. Next, we can calculate $\rho_M(D)$ and $\rho^2(D)$ under three cases.

(a) When n_1 is odd and n_2 is even, we have $A_1^TA_1 = (n_1-1)I_{m_1} + \mathbf{e}_1\mathbf{e}_1^T$ and $A_2^TA_2 = n_2I_{m_2}$. Thus, it is easy to obtain that

$$\begin{aligned}L^T\widetilde{L} =& \frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}I_{m_1m_2} - \frac{n_1n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1m_2}\right], \text{ and} \\ \widetilde{L}^T\widetilde{L} =& \frac{n_1n_2(n_1^2n_2^2-1)}{12}I_{m_1m_2} + \frac{n_1^2n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1m_2}\right].\end{aligned}$$

Combining with Theorem 2, we have

$$\begin{aligned}
\rho_M(D) =& \max\Big\{\frac{n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1^2n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \\
& \frac{n_1n_2(n_1n_2+1)(n_1-n_2)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}\Big\} \\
=& \max\Big\{\frac{n_1(n_2^2-1)}{n_1^2n_2^2-1}, \frac{n_1-n_2}{n_1n_2-1}\Big\}, \quad \text{and} \\
\rho^2(D) =& \Big\{\Big[\Big(\frac{n_2(n_2^2-1)}{12}\Big)^2 + \Big(\frac{n_1^2n_2(n_2^2-1)}{12}\Big)^2 + 2\times\Big(\frac{n_1n_2(n_2^2-1)}{12}\Big)^2\Big] \\
& \times(m_1-1)m_1m_2 + \Big(\frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}\Big)^2\times 2m_1m_2\Big\} \\
& \times\Big(\frac{12}{n_1n_2(n_1^2n_2^2-1)}\Big)^2\times\frac{1}{2m_1m_2(2m_1m_2-1)} \\
=& \frac{(m_1-1)(n_1^2+1)^2(n_2^2-1)^2+2n_1^2(n_1n_2+1)^2(n_1-n_2)^2}{2(2m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2}.
\end{aligned}$$

(b) When n_1 is even and n_2 is odd, we have $A_1^TA_1 = n_1I_{m_1}$ and $A_2^TA_2 = (n_2 - 1)I_{m_2} + \mathbf{e}_2\mathbf{e}_2^T$, which implies that

$$\begin{aligned}
L^T\widetilde{L} =& \frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}I_{m_1m_2} - \frac{n_1n_2(n_1^2-1)}{12}\left[I_{m_1}\otimes(\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right], \text{ and} \\
\widetilde{L}^T\widetilde{L} =& \frac{n_1n_2(n_1^2n_2^2-1)}{12}I_{m_1m_2} + \frac{n_1(n_1^2-1)}{12}\left[I_{m_1}\otimes(\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right].
\end{aligned}$$

Combining with Theorem 2, we have

$$\begin{aligned}
\rho_M(D) =& \max\Big\{\frac{n_1n_2^2(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \\
& \frac{n_1n_2(n_1n_2+1)(n_1-n_2)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1n_2(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}\Big\} \\
=& \max\Big\{\frac{n_2(n_1^2-1)}{n_1^2n_2^2-1}, \frac{n_1-n_2}{n_1n_2-1}\Big\} = \frac{n_2(n_1^2-1)}{n_1^2n_2^2-1}, \quad \text{and} \\
\rho^2(D) =& \Big\{\Big[\Big(\frac{n_1n_2^2(n_1^2-1)}{12}\Big)^2 + \Big(\frac{n_1(n_1^2-1)}{12}\Big)^2 + 2\times\Big(\frac{n_1n_2(n_1^2-1)}{12}\Big)^2\Big] \\
& \times(m_2-1)m_1m_2 + \Big(\frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}\Big)^2\times 2m_1m_2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{12}{n_1n_2(n_1^2n_2^2-1)}\right)^2 \times \frac{1}{2m_1m_2(2m_1m_2-1)}\\
=&\frac{(m_2-1)(n_1^2-1)^2(n_2^2+1)^2+2n_2^2(n_1n_2+1)^2(n_1-n_2)^2}{2(2m_1m_2-1)n_2^2(n_1^2n_2^2-1)^2}.
\end{aligned}
$$

(c) When both n_1 and n_2 are odd, we have $A_1^TA_1 = (n_1-1)I_{m_1} + \mathbf{e}_1\mathbf{e}_1^T$ and $A_2^TA_2 = (n_2-1)I_{m_2} + \mathbf{e}_2\mathbf{e}_2^T$, which implies that

$$
\begin{aligned}
L^T\widetilde{L} =&\frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}I_{m_1m_2} + \frac{n_1n_2(n_1^2-1)}{12}\left[I_{m_1}\otimes(\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right]\\
&-\frac{n_1n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1m_2}\right], \text{ and}\\
\widetilde{L}^T\widetilde{L} =&\frac{n_1n_2(n_1^2n_2^2-1)}{12}I_{m_1m_2} + \frac{n_1(n_1^2-1)}{12}\left[I_{m_1}\otimes(\mathbf{e}_2\mathbf{e}_2^T) - I_{m_1m_2}\right]\\
&+\frac{n_1^2n_2(n_2^2-1)}{12}\left[(\mathbf{e}_1\mathbf{e}_1^T)\otimes I_{m_2} - I_{m_1m_2}\right].
\end{aligned}
$$

Combining with Theorem 2, we have

$$
\begin{aligned}
\rho_M(D) =&\max\Big\{\frac{n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1n_2^2(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12},\\
&\frac{n_1^2n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1n_2(n_1n_2+1)(n_1-n_2)/12}{n_1n_2(n_1^2n_2^2-1)/12},\\
&\frac{n_1n_2(n_1^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}, \frac{n_1n_2(n_2^2-1)/12}{n_1n_2(n_1^2n_2^2-1)/12}\Big\}\\
=&\max\left\{\frac{n_1(n_2^2-1)}{n_1^2n_2^2-1}\frac{n_2(n_1^2-1)}{n_1^2n_2^2-1}, \frac{n_1-n_2}{n_1n_2-1}\right\} = \max\left\{\frac{n_1(n_2^2-1)}{n_1^2n_2^2-1}\frac{n_2(n_1^2-1)}{n_1^2n_2^2-1}\right\}, \text{ and}\\
\rho^2(D) =&\Big\{\Big[\left(\frac{n_2(n_2^2-1)}{12}\right)^2 + \left(\frac{n_1^2n_2(n_2^2-1)}{12}\right)^2 + 2\times\left(\frac{n_1n_2(n_2^2-1)}{12}\right)^2\Big]\\
&\times(m_1-1)m_1m_2 + \Big[\left(\frac{n_1n_2^2(n_1^2-1)}{12}\right)^2 + \left(\frac{n_1(n_1^2-1)}{12}\right)^2 +\\
&2\times\left(\frac{n_1n_2(n_1^2-1)}{12}\right)^2\Big]\times(m_2-1)m_1m_2 + \left(\frac{n_1n_2(n_1n_2+1)(n_1-n_2)}{12}\right)^2\\
&\times 2m_1m_2\Big\}\times\left(\frac{12}{n_1n_2(n_1^2n_2^2-1)}\right)^2\times\frac{1}{2m_1m_2(2m_1m_2-1)}\\
=&\frac{(m_1-1)(n_1^2+1)^2(n_2^2-1)^2}{2(2m_1m_2-1)n_1^2(n_1^2n_2^2-1)^2} + \frac{(m_2-1)(n_1^2-1)^2(n_2^2+1)^2}{2(2m_1m_2-1)n_2^2(n_1^2n_2^2-1)^2}
\end{aligned}
$$

$$+\frac{(n_1-n_2)^2}{(2m_1m_2-1)(n_1^2n_2^2-1)^2}.$$

The proof is completed.

A.6. Proof of Proposition 1

The following lemmas are used to prove Proposition 1.

Lemma A.1. *For any $k \geq 1$, the signs of elements in T_k correspond one-to-one with that in S_k, where T_k and S_k are defined in Step 2 of Construction 2.*

Proof. This result obviously holds as can be seen from the definitions of T_k and S_k.

Lemma A.2. *For any $k \geq 1$, let $s_{ij}^{(k)}$ be the (i,j)-element of S_k, where S_k is defined in Step 2 of Construction 2. Then, for any $j = 1, \ldots, 2^{k-1}$, we have*

$$s_{ij}^{(k)} = \begin{cases} s_{i(2^k+1-j)}^{(k)}, & \text{for } 1 \leq i \leq 2^{k-1}; \\ -s_{i(2^k+1-j)}^{(k)}, & \text{for } 2^{k-1}+1 \leq i \leq 2^k. \end{cases}$$

Proof. When $k = 1$, it is easy to verify that the result holds. Suppose it holds for a particular integer $k-1$ and consider the next integer k. Let $s_{ij}^{(k)*}$ be the (i,j)-element of S_k^*. Given any $j = 1, \ldots, 2^{k-1}$, consider the following four cases:

(a) When $1 \leq i \leq 2^{k-2}$, we have

$$s_{ij}^{(k)} = s_{ij}^{(k-1)} = s_{i(2^{k-1}+1-j)}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)*} = s_{i(2^k+1-j)}^{(k)},$$

where the first and the fourth equalities are guaranteed by the definition of S_k, the third equality is guaranteed by the definition of S_k^*, and the second equality holds since the result is supposed to be true for $k-1$.

(b) When $2^{k-2}+1 \le i \le 2^{k-1}$, we have

$$s_{ij}^{(k)} = s_{ij}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)*} = s_{i(2^k+1-j)}^{(k)}.$$

(c) When $2^{k-1}+1 \le i \le 2^{k-1}+2^{k-2}$, we have

$$s_{ij}^{(k)} = s_{ij}^{(k-1)} = s_{i(2^{k-1}+1-j)}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)*} = -s_{i(2^k+1-j)}^{(k)}.$$

(d) When $2^{k-1}+2^{k-2}+1 \le i \le 2^k$, we have

$$s_{ij}^{(k)} = s_{ij}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)} = -s_{i(2^{k-1}+1-j)}^{(k-1)*} = -s_{i(2^k+1-j)}^{(k)}.$$

Thus, the result follows by induction.

Lemma A.3. *For any $k \ge 1$, the j_1-th and j_2-th columns of the matrix (S_k, S_k^*) with $j_1 \ne j_2$ are orthogonal if and only if $j_1 + j_2 \ne 2^{k+1}+1$, where S_k and S_k^* are defined in Step 2 of Construction 2.*

Proof. Without of generality, we assume that $1 \le j_1 \le 2^k < j_2 \le 2^{k+1}$ for any j_1 and j_2 satisfying $j_1 + j_2 = 2^{k+1}+1$. For $i, j = 1, \ldots, 2^k$, let $s_{ij}^{(k)}$ and $s_{ij}^{(k)*}$ be the (i,j)-elements of S_k and S_k^*, respectively. According to Lemma A.2, the definition of S_k and the assumption $j_1 + j_2 - 2^k = 2^k + 1$, we have

$$s_{ij_1}^{(k)} = \begin{cases} s_{i(2^k+1-j_1)}^{(k)} = s_{i(j_2-2^k)}^{(k)} = -s_{i(j_2-2^k)}^{(k)*}, & \text{for } 1 \le i \le 2^{k-1}; \\ -s_{i(2^k+1-j_1)}^{(k)} = -s_{i(j_2-2^k)}^{(k)} = -s_{i(j_2-2^k)}^{(k)*}, & \text{for } 2^{k-1}+1 \le i \le 2^k. \end{cases}$$

Thus, $s_{ij_1}^{(k)}$ equals $-s_{i(j_2-2^k)}^{(k)*}$ for any j_1 and j_2 satisfying $j_1 + j_2 = 2^{k+1}+1$, which implies that the j_1-th and j_2-th columns of (S_k, S_k^*) with $j_1 + j_2 = 2^{k+1}+1$ are not orthogonal.

Obviously, the first column of S_1 is orthogonal to the second column of S_1^*, and the second column of S_1 is orthogonal to the first column of S_1^*. Suppose this lemma holds for a particular integer $k-1$ and consider the next integer k. Note that,

$$\begin{pmatrix} S_k & S_k^* \end{pmatrix} = \begin{pmatrix} S_{k-1} & -S_{k-1}^* & -S_{k-1} & S_{k-1}^* \\ S_{k-1} & S_{k-1}^* & S_{k-1} & S_{k-1}^* \end{pmatrix}.$$

Thus, the j_1-th and j_2-th columns of (S_k, S_k^*) with $j_1 + j_2 \neq 2^{k+1}+1$ are always orthogonal, and the result follows by induction.

Proof of Proposition 1. We first prove property (i). With Lemma A.1, we know that the signs of elements in D_q correspond one-to-one with that in $(S_{q-1}^T, -S_{q-1}^T)^T$. Thus, when collapsed into two levels, D_q becomes $(S_{q-1}^T, -S_{q-1}^T)^T/2$. It is easy to verify that

$$\begin{pmatrix} S_{q-1}^T & -S_{q-1}^T \end{pmatrix} \begin{pmatrix} S_{q-1} \\ -S_{q-1} \end{pmatrix} = 2S_{q-1}^T S_{q-1} = 2^q I_{2^{q-1}}.$$

So, when collapsed into 2 levels, D_q becomes an $\mathrm{OA}(2^q, 2^{q-1}, 2, 2)$, which implies that D_q achieves a stratification on a 2×2 grid when projected onto any two dimensions.

Then, we show that if $j_1 + j_2 = 2^{q-1}+1$, the column pair consisting of the j_1-th and j_2-th columns of D_q can not achieve a stratification on a 2×4 or 4×2 grid. Without loss of generality, we assume that $j_1 < j_2$ and $j_1 + j_2 = 2^{q-1}+1$. For $i = 1,\ldots,2^k$, $j = 1,\ldots,2^{k-1}$ and $k = 1,\ldots,q$, let $\mathbf{d}_j^{(k)}$ be the j-th column of D_k and $d_{ij}^{(k)}$ be the i-th element of $\mathbf{d}_j^{(k)}$. With the definition of D_q, it can be seen that for $j = 1,\ldots,2^{q-1}$, there must be $|d_{ij}^{(q)}| \in \{1/2, 3/2, \ldots, (2^{q-1}-1)/2\}$ when $i = 1,\ldots,2^{q-2}, 2^{q-1}+1,\ldots,2^{q-1}+2^{q-2}$, and $|d_{ij}^{(q)}| \in \{(2^{q-1}+1)/2, (2^{q-1}+3)/2, \ldots, (2^q-1)/2\}$ when $i = 2^{q-2}+1,\ldots,2^{q-1}, 2^{q-1}+2^{q-2}+1,\ldots,2^q$. From Lemmas A.1 and A.2, we know that the signs of $d_{ij_1}^{(q-1)}$ and $d_{ij_2}^{(q)}$ are the same when $i = 1,\ldots,2^{q-2}, 2^{q-1}+1,\ldots,2^{q-1}+2^{q-2}$, and the signs of them are opposite when $i = 2^{q-2}+1,\ldots,2^{q-1}, 2^{q-1}+2^{q-2}+1,\ldots,2^q$. Thus, when $d_{ij_1}^{(q)} \in \{-(2^q-1)/2, -(2^q -$

$3)/2, \ldots, -(2^{q-1}+1)/2, 1/2, 3/2, \ldots, (2^{q-1}-1)/2\}$, there must be $d_{ij_1}^{(q)} > 0$, and when $d_{ij_1}^{(q)} \in \{-(2^{q-1}-1)/2, -(2^{q-1}-3)/2, \ldots, -1/2, (2^{q-1}+1)/2, (2^{q-1}+3)/2, \ldots, (2^q-1)/2\}$, there must be $d_{ij_1}^{(q)} < 0$. As a result, when collapsing $\mathbf{d}_{j_1}^{(q)}$ and $\mathbf{d}_{j_2}^{(q)}$ into four and two levels respectively, there are only four different 2-tuples appearing as rows in the resulting column pair, which are $(1/2, 1/2), (-1/2, -1/2), (3/2, -1/2)$ and $(-3/2, 1/2)$. In other words, $(\mathbf{d}_{j_1}^{(q)}, \mathbf{d}_{j_2}^{(q)})$ can not achieve a stratification on a 4×2 grid. Similarly, it can also be proved that this column pair can not achieve a stratification on a 2×4 grid.

Next, we prove that if $j_1 + j_2 \neq 2^{q-1}+1$ and $j_1 \neq j_2$, the column pair consisting of the j_1-th and j_2-th columns of D_q can achieve stratifications on 2×4 and 4×2 grids. Consider the following two column pairs

$$\begin{pmatrix} d_{1j_1}^{(q)} & d_{2j_1}^{(q)} & \cdots & d_{2^{q-2}j_1}^{(q)} & d_{(2^{q-1}+1)j_1}^{(q)} & d_{(2^{q-1}+2)j_1}^{(q)} & \cdots & d_{(2^{q-1}+2^{q-2})j_1}^{(q)} \\ d_{1j_2}^{(q)} & d_{2j_2}^{(q)} & \cdots & d_{2^{q-2}j_2}^{(q)} & d_{(2^{q-1}+1)j_2}^{(q)} & d_{(2^{q-1}+2)j_2}^{(q)} & \cdots & d_{(2^{q-1}+2^{q-2})j_2}^{(q)} \end{pmatrix}^T \quad \text{and}$$

$$\begin{pmatrix} d_{(2^{q-2}+1)j_1}^{(q)} & d_{(2^{q-2}+1)j_1}^{(q)} & \cdots & d_{2^{q-1}j_1}^{(q)} & d_{(2^{q-1}+2^{q-2}+1)j_1}^{(q)} & d_{(2^{q-1}+2^{q-2}+2)j_1}^{(q)} & \cdots & d_{2^qj_1}^{(q)} \\ d_{(2^{q-2}+1)j_2}^{(q)} & d_{(2^{q-2}+2)j_1}^{(q)} & \cdots & d_{2^{q-1}j_2}^{(q)} & d_{(2^{q-1}+2^{q-2}+1)j_2}^{(q)} & d_{(2^{q-1}+2^{q-2}+2)j_2}^{(q)} & \cdots & d_{2^qj_2}^{(q)} \end{pmatrix}^T,$$

where $j_1 + j_2 \neq 2^{q-1}+1$ and $j_1 \neq j_2$. Obviously, the row juxtaposition of the above two column pairs is equivalent to $(\mathbf{d}_{j_1}^{(q)}, \mathbf{d}_{j_2}^{(q)})$ in the sense of reordering the rows. Lemma A.3 shows that the j_1-th and j_2-th columns of (S_{q-2}, S_{q-2}^*) form an OA$(2^{q-2}, 2, 2, 2)$ when $j_1 + j_2 \neq 2^{q-1}+1$ and $j_1 \neq j_2$. Combining with Lemma A.1 and the definition of D_q, it can be inferred that each of the 2-tuples $(1/2, 1/2), (-1/2, 1/2), (1/2, -1/2)$ and $(-1/2, -1/2)$ appears 2^{q-3} times as rows in the above first column pairs after collapsing the two columns into two levels and four levels respectively. And each of the 2-tuples $(1/2, 3/2), (-1/2, 3/2), (1/2, -3/2)$ and $(-1/2, -3/2)$ appears 2^{q-3} times as rows in the above second column pairs after collapsing the two columns into two levels and four levels respectively. Thus, when $j_1 + j_2 \neq 2^{q-1}+1$ and $j_1 \neq j_2$, $(\mathbf{d}_{j_1}^{(q)}, \mathbf{d}_{j_2}^{(q)})$ can achieve a stratification on a 2×4 grid. Similarly, it can be proved that it also achieves a stratification on a 4×2 grid. So, the number of column pairs of D_q that can achieve stratifications on 2×4 and 4×2 grids is

$$\frac{2^{q-1}(2^{q-1}-1)}{2} - 2^{q-2} = 2^{q-1}(2^{q-2}-1).$$

Property (ii) has been proved.

Finally, we verify property (iii). For $i = 1, \ldots, 2^{q-1}$ and $j = 1, \ldots, 2^{q-2}$, let $t_{ij}^{(q-1)}$ be the (i, j)-th element of T_{q-1}. With the definition of T_{q-1}, we know that there must be $1 \leq |t_{ij}^{(q-1)}| \leq 2^{q-2}$ when $i, j \in \{1, \ldots, 2^{q-2}\}$ or $i, j \in \{2^{q-1}+1, \ldots, 2^{q-1}+2^{q-2}\}$, and $2^{q-2}+1 \leq |t_{ij}^{(q-1)}| \leq 2^{q-1}$ otherwise. Thus, there only exist two cases of $(d_{ij_1}^{(q)}, d_{ij_2}^{(q)})$: (a) when $1 \leq j_1 \leq 2^{q-2}$ and $2^{q-2}+1 \leq j_2 \leq 2^{q-1}$, we have $\big||d_{ij_1}^{(q)}| - |d_{ij_2}^{(q)}|\big| \geq 2^{q-2}$; (b) when $1 \leq j_1, j_2 \leq 2^{q-2}$ or $2^{q-2}+1 \leq j_1, j_2 \leq 2^{q-1}$, we have $\big||d_{ij_1}^{(q)}| - |d_{ij_2}^{(q)}|\big| \leq 2^{q-2}-1$. So, after collapsing D_q into four levels, any column pairs of the resulting matrix must meet one of the following cases: (a) for each 2-tuple appearing as rows, two elements of it have the same absolute value; (b) for each 2-tuple appearing as rows, the difference between absolute values of two elements of it equals 1. It implies that D_q can not be collapsed into an OA$(2^q, 2^{q-1}, 4, 2)$. As a result, none of column pairs in D_q can achieve a stratification on a 4×4 grid.

A.7. Proof of Theorem 3

With Corollary 1 and the fact that $D_{q/2}$ is an OSLHD$(2^{q/2}, 2^{q/2-1})$, it can be seen that $(L, \widetilde{L})$ is an OSLHD$(2^q, 2^{q-1})$. To prove the stratification property, we first verify property (iii). Collapse L and $\widetilde{L}$ into $2^{q/2}$ levels, and denote the resulting matrices as M and $\widetilde{M}$ respectively. Obviously, there are $M = D_{q/2} \otimes A_1$ and $\widetilde{M} = -A_1 \otimes D_{q/2}$. For $i, j = 1, \ldots, 2^{q/2-1}$, let $\mathbf{a}_i$ and $\mathbf{b}_j$ be the i-th and j-th columns of A_0 and $T_{q/2-1} - S_{q/2-1}/2$, respectively. Then, the column pair consisting of any one column in M and any one column in $\widetilde{M}$ can be represented as

$$\begin{pmatrix} \mathbf{b}_{j_1} \otimes \mathbf{a}_{i_1} & -\mathbf{a}_{i_2} \otimes \mathbf{b}_{j_2} \\ \mathbf{b}_{j_1} \otimes \mathbf{a}_{i_1} & \mathbf{a}_{i_2} \otimes \mathbf{b}_{j_2} \\ -\mathbf{b}_{j_1} \otimes \mathbf{a}_{i_1} & -\mathbf{a}_{i_2} \otimes \mathbf{b}_{j_2} \\ -\mathbf{b}_{j_1} \otimes \mathbf{a}_{i_1} & \mathbf{a}_{i_2} \otimes \mathbf{b}_{j_2} \end{pmatrix}. \tag{A.1}$$

Note that, A_0 is a Hadamard matrix, $D_{q/2}$ is an LHD and M is an OA$(2^q, 2^{q-2}, 2^{q/2}, 1)$. So, the matrix in (A.1) is an OA$(2^q, 2, 2^{q/2}, 2)$, which implies that the column pair consisting of any one column in L and any one column in $\widetilde{L}$ can achieve a stratification on a $2^{q/2} \times 2^{q/2}$ grid. As a result, for the resulting design of Construction 2, the number of column pairs that can achieve a stratification on a $2^{q/2} \times 2^{q/2}$ grid is $2^{q-2} \times 2^{q-2} = 4^{q-2}$.

Because we have demonstrated that each column pair consisting of any one column in L and any one column in $\widetilde{L}$ can achieve a stratification on a $2^{q/2} \times 2^{q/2}$ grid, such column pairs can also achieve stratifications on 2×4 and 4×2 grids. So, we only need to prove the stratification properties of L and $\widetilde{L}$ respectively. Note that $L = A_1 \otimes D_{q/2} + 2^{q/2}(D_{q/2} \otimes A_1)$, and elements in $A_1 \otimes D_{q/2}$ are $\{\pm 1/2, \pm 3/2, \ldots, \pm(2^{q/2} - 1)/1\}$. The stratification properties on 2×4 and 4×2 grids of column pairs of L and $D_{q/2} \otimes A_1$ are equivalent. With Proportion 1, we know that $D_{q/2}$ can achieve a stratification on a 2×2 grid when projected onto any two dimensions. So, $D_{q/2} \otimes A_1$ can also achieve a stratification on a 2×2 grid when projected onto any two dimensions since A_1 is an OA. Besides, the proof of Proportion 1 shows that the column pair consisting of the

j_1-th and j_2-th columns of $D_{q/2}$ achieve stratifications on 2×4 and 4×2 grids if and only if $j_1 + j_2 \neq 2^{q/2-1} + 1$ and $j_1 \neq j_2$, where $j_1, j_2 = 1, \ldots, 2^{q/2-1}$. Thus, for $D_{q/2} \otimes A_1$, the number of column pairs that can achieve stratifications on 2×4 and 4×2 grids is $2^{q-2} \times (2^{q-2} - 1)/2 - 2^{q/2-1} \times 2^{q/2-2} = 2^{q-2}(2^{q-3} - 1)$. As a result, L can achieve a stratification on a 2×2 grid when projected onto any two dimensions, and there are $2^{q-2}(2^{q-3} - 1)$ column pairs of L that can achieve stratifications on 2×4 and 4×2 grids. Similarly, it can be proved that $\widetilde{L}$ can achieve a stratification on a 2×2 grid when projected onto any two dimensions, and there are $2^{q-2}(2^{q-3} - 1)$ column pairs of $\widetilde{L}$ that can achieve stratifications on 2×4 and 4×2 grids. Thus, $(L, \widetilde{L})$ can achieve a stratification on a 2×2 grid when projected onto any two dimensions, and the number of column pairs of $(L, \widetilde{L})$ that can achieve stratifications on 2×4 and 4×2 grids is $2^{q-2}(2^{q-3} - 1) \times 2 + 4^{q-2} = 2^{q-1}(2^{q-2} - 1)$. The proof is completed.

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