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Power laws distributions in objective priors

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Abstract: Using objective priors in Bayesian applications has become a common practice to analyze data without subjective information. Formal rules usually obtain these prior distributions, and the data provide the dominant information in the posterior distribution. However, these priors are typically improper and may lead to improper posterior. Here, for a general family of distributions, we show that the obtained objective priors for the parameters either follow a power-law distribution or have an asymptotic power-law behavior. As a result, we observed that the exponents of the model are between 0.5 and 1. Understanding these behaviors allows us to easily verify if such priors lead to proper or improper posteriors directly from the exponent of the power-law. The general family considered in our study includes essential models such as Exponential, Gamma, Weibull, Nakagami-m, Half-Normal, Rayleigh, Erlang, and Maxwell Boltzmann distributions, to list a few. In summary, we show that comprehending the mechanisms describing the shapes of the priors provides essential information that can be used to understand the properties of the posterior distributions.

Key words and phrases: Bayesian inference; objective prior; power-law.

1 Introduction

Bayesian methods have become ubiquitous among statistical procedures and have provided essential results in areas from medicine to engineering (Lloyd-Jones et al., 2019; Wang and Matthies, 2019). In the Bayesian approach, the parameters in a statistical model are assumed to be random variables (Bernardo, 2005), differently from the frequentist approach, which considers these parameters as constant. Moreover, a subjective ingredient can be included in the model to reproduce the knowledge of a specialist (see O'Hagan et al. (2006)). On the other hand, in many situations, we are interested in obtaining a prior distribution, which guarantees that the information provided by the data will not be overshadowed by subjective information. In this case, an objective analysis is recommended by considering non-informative priors that are derived by formal rules (Consonni et al., 2018; Kass and Wasserman, 1996). Although several studies have found weakly informative priors (flat priors) as presumed non-informative priors, Bernardo (2005) argued that using simple proper priors, supposed to be non-informative, often hides significant unwarranted assumptions, which may easily dominate, or even invalidate the statistical analysis.

Objective priors are constructed by formal rules (Kass and Wasserman, 1996) and are usually improper, i.e., they do not correspond to a proper probability distribution and could lead to improper posteriors, which is undesirable. A recent discussion about their limitations has been considered by Leisen et al. (2019). According to Northrop and Attalides (2016), there are no simple conditions that can be used to prove that improper prior yields a proper posterior for a particular distribution. Therefore, a case-by-case investigation is needed to check the propriety of the posterior distribution. For the Stacy (1962) general family of distributions, we solve this problem by proving that if objective priors asymptotically follow a power-law model with the exponent in some particular regions, then the posterior distributions can be proper or improper. As a result, one can easily check if the obtained posterior is proper or improper, directly looking at the behavior of the improper prior as a power-law model.

Understanding the situations when the data follow a power-law distribution can indicate the mechanisms that describe the natural phenomenon in question. Power-law distribution appears in many physical, biological, and man-made phenomena, for instance, they can be used to describe biological networks (Pržulj, 2007), infectious diseases (Geilhufe et al., 2014), the sizes of craters on the moon (Newman, 2005), intensity function in repairable systems (Louzada et al., 2019) and energy dissipation in cyclones (Corral et al., 2010) (see also Goldstein et al. (2004); Barrat et al. (2008); Newman (2018)). The probability density function of a power-law distribution can be represented as

$$\pi(\theta) = c\,\theta^{-\lambda}, \tag{1.1}$$

where c is a normalized constant and λ is the exponent parameter. During the application of Bayesian methods, the normalizing constant is usually omitted and the prior can be represented by $\pi(\theta) \propto \theta^{-\lambda}$.

In this paper, we analyze the behaviour of different objective priors related to the parameters of many distributions. We show that its asymptotic behaviour follows power-law models with exponents between 0.5 and 1. Under these cases, they may lead to a proper or improper posterior depending on the exponent values of the priors. Situations, where a power-law distribution is observed with an exponent smaller than one were observed by Goldstein et al. (2004), Deluca and Corral (2013) and Hanel et al. (2017). The objective priors are obtained from the Jeffreys' rule (Kass and Wasserman, 1996), Jeffreys' prior (Jeffreys, 1946) and reference priors (Bernardo, 1979, 2005; Berger et al., 2015). Although the posterior distribution may be proper, the posterior moments can be infinite. Therefore, we also provided sufficient conditions to verify if the posterior moments are finite. These results play an important role in which the acknowledgment of the power-law behavior of the prior distribution related to a particular distribution can provide an understanding of the shape of the prior that can be used in situations where additional complexity (e.g. random censoring, long-term survival, among others) is presented or priors obtained from formal rules are more difficult or cannot be obtained.

The remainder of this paper is organized as follows. Section 2 presents the theorems that provide necessary and sufficient conditions for the posterior distributions to be proper depending on the asymptotic behavior of the prior as a power-law model. Additionally, we also discuss sufficient conditions to check if the posterior moments are finite. Section 3 presents the study of the behavior of the objective priors. Section 4 provides an application in a real dataset. Finally, Section 5 summarizes the study with concluding remarks.

2 A general model

The Stacy family of distributions plays an important role in statistics and has proven to be very flexible in practice for modeling data from several areas, such as climatology, meteorology, medicine, reliability, and image processing data, among others (Stacy, 1962). A random variable X follows a Stacy's distribution if its probability density function (PDF) is given by

$$f(x|\boldsymbol{\theta}) = \alpha\mu^{\alpha\phi}x^{\alpha\phi-1}\exp\left(-(\mu x)^{\alpha}\right)/\Gamma(\phi), \quad x > 0 \tag{2.2}$$

where $\Gamma(\phi) = \int_0^{\infty} e^{-x}x^{\phi-1}dx$ is the gamma function, $\boldsymbol{\theta} = (\phi, \mu, \alpha)$, $\alpha > 0$ and $\phi > 0$ are the shape parameters and $\mu > 0$ is a scale parameter. The Stacy's distribution unifies many important distributions, as shown in Table 1, and may be referred to as generalized gamma (GG) distribution.

The inference procedures related to the parameters are conducted using the joint posterior distribution for $\boldsymbol{\theta}$ that is given by the product of the likelihood function and the prior distribution $\pi(\boldsymbol{\theta})$ divided by a normalizing constant $d(\boldsymbol{x})$, resulting in

$$p(\boldsymbol{\theta}|\boldsymbol{x}) = \frac{\pi(\boldsymbol{\theta})}{d(\boldsymbol{x})}\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}, \tag{2.3}$$

where

$$d(\boldsymbol{x}) = \int\limits_{\mathcal{A}} \pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\}d\boldsymbol{\theta} \tag{2.4}$$

and $\mathcal{A} = \{(0,\infty)\times(0,\infty)\times(0,\infty)\}$ is the parameter space of $\boldsymbol{\theta}$. Considering any prior in the form $\pi\left(\boldsymbol{\theta}\right) \propto \pi(\mu)\pi(\alpha)\pi(\phi)$, our main aim is to analyze the asymptotic behavior

Table 1: Distributions included in the Stacy family of distributions (see equation 2.2).

Distribution	μ	ϕ	α
Exponential	$\cdot$	1	1
Rayleigh	$\cdot$	1	2
Haf-Normal	$\cdot$	0.5	2
Maxwell Boltzmann	$\cdot$	$\frac{3}{2}$	2
scaled chi-square	$\cdot$	0.5n	1
chi-square	2	0.5n	1
Weibull	$\cdot$	1	$\cdot$
Generalized Haf-Normal	$\cdot$	2	$\cdot$
Gamma	$\cdot$	$\cdot$	1
Erlang	$\cdot$	n	$\cdot$
Nakagami	$\cdot$	$\cdot$	2
Wilson-Hilferty	$\cdot$	$\cdot$	3
Lognormal	$\cdot$	$\phi \to \infty$	$\cdot$

$n \in \mathbb{N}$

of the priors that leads to power-law distributions, thus finding necessary and sufficient conditions for the posterior to be proper, i.e., $d(\boldsymbol{x}) < \infty$.

To study such asymptotic behavior, the following definitions and propositions will be useful to prove the results related to the posterior distribution. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

denote the *extended real number line* with the usual order ($\geq$), let $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{R}_0^+$ denote the positive real numbers including 0, and denote $\overline{\mathbb{R}}^+$ and $\overline{\mathbb{R}}_0^+$ analogously. Moreover, if $M \in \mathbb{R}^+$ and $a \in \overline{\mathbb{R}}^+$, we define $M \cdot a$ as the usual product if $a \in \mathbb{R}$, and $M \cdot a = \infty$ if $a = \infty$.

Definition 1. Let $a \in \overline{\mathbb{R}}_0^+$ and $b \in \overline{\mathbb{R}}_0^+$. We say that $a \lesssim b$ if there exist $M \in \mathbb{R}^+$ such that $a \leq M \cdot b$. If $a \lesssim b$ and $b \lesssim a$, then we say that $a \propto b$.

In other words, by Definition 1 we have $a \lesssim b$ if either $a < \infty$ or $b = \infty$, and we have $a \propto b$ if either $a < \infty$ and $b < \infty$, or $a = b = \infty$.

Definition 2. Let $\mathrm{g} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$ and $\mathrm{h} : \mathcal{U} \to \overline{\mathbb{R}}_0^+$, where $\mathcal{U} \subset \mathbb{R}$. We say that $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ if there exist $M \in \mathbb{R}^+$ such that $\mathrm{g}(x) \leq M\,\mathrm{h}(x)$ for every $x \in \mathcal{U}$. If $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ and $\mathrm{h}(x) \lesssim \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \propto \mathrm{h}(x)$.

Definition 3. Let $\mathcal{U} \subset \mathbb{R}$, $a \in \mathcal{U}' \cup \{\infty\}$, where $\mathcal{U}'$ is the closure of $\mathcal{U}$ in $\mathbb{R}$, and let $\mathrm{g} : \mathcal{U} \to \mathbb{R}^+$ and $\mathrm{h} : \mathcal{U} \to \mathbb{R}^+$. We say that $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$ if $\limsup_{x \to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} < \infty$. If $\mathrm{g}(x) \underset{x \to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{h}(x) \underset{x \to a}{\lesssim} \mathrm{g}(x)$ then we say that $\mathrm{g}(x) \underset{x \to a}{\propto} \mathrm{h}(x)$.

The meaning of the relations $\mathrm{g}(x) \underset{x \to a^+}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x \to a^-}{\lesssim} \mathrm{h}(x)$ for $a \in \mathbb{R}$ are defined analogously. Note that, if for some $d \in \mathbb{R}^+$ we have $\lim_{x \to c} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = d$, then it follows directly that $\mathrm{g}(x) \underset{x \to c}{\propto} \mathrm{h}(x)$. The following proposition is a direct consequence of the above definition.

Proposition 1. *Let $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $c \in [a, b]$, $r \in \mathbb{R}^+$, and let $f_1(x)$, $f_2(x)$, $g_1(x)$ and $g_2(x)$ be non-negative continuous functions with domain (a, b) such that $f_1(x) \underset{x \to c}{\lesssim} f_2(x)$*

and $g_1(x) \underset{x\to c}{\lesssim} g_2(x)$. *Then the following hold*

$$f_1(x)g_1(x) \underset{x\to c}{\lesssim} f_2(x)g_2(x) \quad \textit{and} \quad f_1(x)^r \underset{x\to c}{\lesssim} f_2(x)^r.$$

The following proposition relates Definition 2 and Definition 3.

Proposition 2. *Let* $\mathrm{g} : (a,b) \to \mathbb{R}_0^+$ *and* $\mathrm{h} : (a,b) \to \mathbb{R}^+$ *be continuous functions on* $(a,b) \subset \mathbb{R}$, *where* $a \in \mathbb{R}$ *and* $b \in \overline{\mathbb{R}}$. *Then* $\mathrm{g}(x) \lesssim \mathrm{h}(x)$ *if and only if* $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ *and* $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$.

Proof. See Appendix A.2. □

Note that if $\mathrm{g} : (a,b) \to \mathbb{R}^+$ and $\mathrm{h} : (a,b) \to \mathbb{R}^+$ are continuous functions on $(a,b) \subset \mathbb{R}$, then by continuity it follows directly that $\lim_{x\to c} \frac{\mathrm{g}(x)}{\mathrm{h}(x)} = \frac{\mathrm{g}(c)}{\mathrm{h}(c)} > 0$ and therefore $\mathrm{g}(x) \underset{x\to c}{\propto} \mathrm{h}(x)$ for every $c \in (a,b)$. This fact and Proposition 2 imply directly the following.

Proposition 3. *Let* $\mathrm{g} : (a,b) \to \mathbb{R}^+$ *and* $\mathrm{h} : (a,b) \to \mathbb{R}^+$ *be continuous functions in* $(a,b) \subset \mathbb{R}$, *where* $a \in \mathbb{R}$ *and* $b \in \overline{\mathbb{R}}$, *and let* $c \in (a,b)$. *Then if* $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ *(or* $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$*) we have* $\int_a^c g(t)\, dt \lesssim \int_a^c h(t)\, dt$ *(respectively* $\int_c^b g(t)\, dt \lesssim \int_c^b h(t)\, dt$ *).*

2.1 Case when α is known

Let $p(\boldsymbol{\theta}|\boldsymbol{x}, \alpha)$ be of the form (2.3) but considering α fixed and $\boldsymbol{\theta} = (\phi, \mu)$, the normalizing constant is given by

$$d(\boldsymbol{x}; \alpha) \propto \int_{\mathcal{A}} \frac{\pi(\boldsymbol{\theta})}{\Gamma(\phi)^n} \left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\} \mu^{n\alpha\phi} \exp\left\{-\mu^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right\} d\boldsymbol{\theta}, \tag{2.5}$$

where $\mathcal{A} = \{(0,\infty) \times (0,\infty)\}$ is the parameter space. Here our objective is reduced to analyzing $\pi(\boldsymbol{\theta}) \propto \pi(\mu)\pi(\phi)$ and finding sufficient and necessary conditions for $d(\boldsymbol{x};\alpha) < \infty$.

Theorem 1. *Suppose that $\pi(\mu,\phi) < \infty$ for all $(\mu,\phi) \in \mathbb{R}_+^2$, that $n \in \mathbb{N}^+$, and suppose that $\pi(\mu,\phi) = \pi(\mu)\pi(\phi)$ and the priors have asymptotic power-law behaviors with*

$$\pi(\mu) \lesssim \mu^k, \quad \pi(\phi) \underset{\phi\to 0^+}{\lesssim} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\lesssim} \phi^{r_\infty},$$

such that $k = -1$ with $n > -r_0$, or $k > -1$ with $n > -r_0 - 1$, then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.

Proof. See Appendix A.3. □

Theorem 2. *Suppose that $\pi(\mu,\phi) > 0$ $\forall(\mu,\phi) \in \mathbb{R}_+^2$, $n \in \mathbb{N}^+$, $\pi(\mu,\phi) \gtrsim \pi(\mu)\pi(\phi)$ and the priors have asymptotic power-law behaviors where $\pi(\mu) \gtrsim \mu^k$ and one of the following holds:*

i) $k < -1$; *or*

ii) $k > -1$ *where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n \leq -r_0 - 1$; *or*

iii) $k = -1$ *where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n \leq -r_0$,

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.

Proof. See Appendix A.4 □

Theorem 3. *Let $\pi(\mu,\phi) = \pi(\mu)\pi(\phi)$ and the behavior of $\pi(\mu)$, $\pi(\phi)$ following the asymptotic power-law distributions given by*

$$\pi(\mu) \propto \mu^k, \quad \pi(\phi) \underset{\mu\to 0^+}{\propto} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty},$$

for $k \in \mathbb{R}$, $r_0 \in \mathbb{R}$ and $r_\infty \in \mathbb{R}$. The posterior related to $\pi(\mu,\phi)$ is proper if and only if $k=-1$ with $n > -r_0$, or $k > -1$ with $n > -r_0 - 1$, and in this case the posterior mean of μ and ϕ are finite, as well as all moments.

Proof. Since the posterior is proper, by Theorem 1 we have $k=-1$ with $n > -r_0$ or $k > -1$ with $n > -r_0 - 1$.

Let $\pi^*(\mu,\phi) = \phi\pi(\mu,\phi)$. Then $\pi^*(\mu,\phi) = \pi^*(\mu)\pi^*(\phi)$, where $\pi^*(\mu) = \pi(\mu)$ and $\pi^*(\phi) = \phi\pi(\phi)$, and we have

$$\pi^*(\mu) \propto \mu^k, \quad \pi^*(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0+1} \quad \text{and} \quad \pi^*(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty+1}.$$

Since $k=-1$ with $n > -r_0 > -(r_0+1)$ or $k > -1$ with $n > -(r_0+1)-1$, it follows from Theorem 1 that the posterior

$$\pi^*(\mu,\phi)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

related to the prior $\pi^*(\mu,\phi)$ is proper. Therefore

$$E[\phi|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \phi\pi(\mu,\phi)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi < \infty.$$

Analogously, one can prove that

$$E[\mu|\boldsymbol{x}] = \int_0^\infty\int_0^\infty \mu\pi(\mu,\phi)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\phi < \infty.$$

Therefore, we have proved that if a prior $\pi(\mu,\phi)$ satisfying the assumptions of the theorem leads to a proper posterior, then the priors $\phi\pi(\mu,\phi)$ and $\mu\pi(\mu,\phi)$ also lead to proper posteriors. It follows by induction that $\mu^s\phi^r\pi(\mu,\phi)$ also leads to proper posteriors for any r and $s \in \mathbb{N}$, which concludes the proof. $\square$

2.2 Case when ϕ is known

Let $p(\boldsymbol{\theta}|\boldsymbol{x},\phi)$ be of the form (2.3) but considering fixed ϕ and $\boldsymbol{\theta}=(\mu,\alpha)$, the normalizing constant is given by

$$d(\boldsymbol{x};\phi)=\int\limits_{\mathcal{A}}\pi\left(\boldsymbol{\theta}\right)\alpha^{n}\left\{\prod_{i=1}^{n}x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^{\alpha}\sum_{i=1}^{n}x_i^{\alpha}\right\}d\boldsymbol{\theta}, \tag{2.6}$$

where $\mathcal{A}=\{(0,\infty)\times(0,\infty)\}$ is the parameter space. Let $\pi\left(\boldsymbol{\theta}\right)\propto\pi(\mu)\pi(\alpha)$, our purpose is to find necessary and sufficient conditions where $d(\boldsymbol{x};\phi)<\infty$.

Theorem 4. *Suppose that $\pi(\mu,\alpha)<\infty$ for all $(\mu,\alpha)\in\mathbb{R}_+^2$, that $n\in\mathbb{N}^+$, and suppose that $\pi(\mu,\alpha)=\pi(\alpha)\pi(\mu)$ and the priors have asymptotic power-law behaviors with*

$$\pi(\mu)\lesssim\mu^{k},\quad \pi(\alpha)\underset{\alpha\to0^+}{\lesssim}\alpha^{q_0},\quad \pi(\alpha)\underset{\alpha\to\infty}{\lesssim}\alpha^{q_\infty},$$

such that $k=-1$, $n>-q_0$ and $q_\infty\in\mathbb{R}$. then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.

Proof. See Appendix A.5. □

Theorem 5. *Suppose that $\pi(\mu,\alpha)>0\ \forall(\mu,\alpha)\in\mathbb{R}_+^2$ and that $n\in\mathbb{N}^+$, and suppose that $\pi(\mu,\alpha)\gtrsim\pi(\mu)\pi(\alpha)$ and the priors have asymptotic power-law behaviors where $\pi(\mu)\gtrsim\mu^{k}$ and one of the following holds*

i) $k<-1$;

ii) $k>-1$ *such that* $\pi(\alpha)\underset{\alpha\to0^+}{\gtrsim}\alpha^{q_0}$ *with* $q_0\in\mathbb{R}$; *or*

iii) $k=-1$ *such that* $\pi(\alpha)\underset{\alpha\to0^+}{\gtrsim}\alpha^{q_0}$ *with* $n\leq-q_0$

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is improper.

Proof. See Appendix A.6. □

Theorem 6. *Let $\pi(\mu,\alpha)=\pi(\mu)\pi(\alpha)$ and suppose the behavior of $\pi(\mu)$, $\pi(\alpha)$ follows an asymptotic power-law distribution given by*

$$\pi(\mu)\propto\mu^k,\quad \pi(\alpha)\underset{\mu\to 0^+}{\propto}\alpha^{q_0}\quad \textit{and}\quad \pi(\alpha)\underset{\alpha\to\infty}{\propto}\alpha^{q_\infty},$$

for $k\in\mathbb{R}$, $q_0\in\mathbb{R}$ and $q_\infty\in\mathbb{R}$. The posterior related to $\pi(\mu,\alpha)$ is proper if and only if $k=-1$ with $n>-q_0$, and in this case the posterior mean of α is finite for this prior, as well as all moments relative to α, and the posterior mean of μ is not finite.

Proof. Since the posterior is proper, by Theorem 5 we have $k=-1$ and $n>-q_0$.

Let $\pi^*(\mu,\alpha)=\alpha\pi(\mu,\alpha)$. Then, $\pi^*(\mu,\alpha)=\pi^*(\mu)\pi^*(\alpha)$, where $\pi^*(\alpha)=\alpha\pi(\alpha)$ and $\pi^*(\mu)=\pi(\mu)$, and we have

$$\pi^*(\mu)\propto\mu^{-1},\quad \pi^*(\alpha)\underset{\mu\to 0^+}{\propto}\alpha^{q_0+1}\quad \text{and}\quad \pi^*(\alpha)\underset{\alpha\to\infty}{\propto}\alpha^{q_\infty+1}.$$

However, since $n>-q_0>-(q_0+1)$ it follows from Theorem 4 that the posterior

$$\pi^*(\mu,\alpha)\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}$$

relative to the prior $\pi^*(\mu,\alpha)$ is proper. Therefore,

$$E[\alpha|\boldsymbol{x}]=\int_0^\infty\int_0^\infty \alpha\pi(\mu,\alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha<\infty.$$

Analogously, one can prove this using item ii) of the Theorem 5 that

$$E[\mu|\boldsymbol{x}]=\int_0^\infty\int_0^\infty \mu\pi(\mu,\alpha)\pi(\boldsymbol{\theta})\frac{\alpha^n}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu d\alpha=\infty$$

since in this case $\mu\pi(\mu)\propto\mu^0$.

Therefore, we have proved that if a prior $\pi(\mu,\alpha)$ satisfying the assumptions of the theorem leads to a proper posterior, then the prior $\alpha\pi(\mu,\alpha)$ also leads to a proper posterior. It follows by induction that $\alpha^r\pi(\mu,\alpha)$ also leads to proper posteriors for any r in $\mathbb{N}$, which concludes the proof. $\square$

2.3 General case when ϕ, α and μ are unknown

Theorem 7. *Suppose that $\pi(\mu,\alpha,\phi)<\infty$ for all $(\mu,\alpha,\phi)\in\mathbb{R}^3_+$, that $n\in\mathbb{N}^+$, and suppose that $\pi(\mu,\alpha,\phi)=\pi(\mu)\pi(\alpha)\pi(\mu)$ and the priors have asymptotic power-law behaviors with*

$$\pi(\mu)\lesssim\mu^k,\quad \pi(\alpha)\underset{\alpha\to 0^+}{\lesssim}\alpha^{q_0},\quad \pi(\alpha)\underset{\alpha\to\infty}{\lesssim}\alpha^{q_\infty},$$

$$\pi(\phi)\underset{\phi\to 0^+}{\lesssim}\phi^{r_0}\quad \textit{and}\quad \pi(\phi)\underset{\phi\to\infty}{\lesssim}\phi^{r_\infty},$$

such that $k=-1$, $q_\infty<r_0$, $2r_\infty+1<q_0$, $n>-q_0$ and $n>-r_0$, then $p(\boldsymbol{\theta}|\boldsymbol{x})$ is proper.

Proof. See Appendix A.7 $\square$

Theorem 8. *Suppose that $\pi(\mu,\alpha,\phi)>0\ \forall(\mu,\alpha,\phi)\in\mathbb{R}^3_+$ and that $n\in\mathbb{N}^+$, then the following items are valid*

i) If $\pi(\mu,\alpha,\phi)\gtrsim\pi(\mu)\pi(\alpha)\pi(\phi)$ for all $\phi\in[b_0,b_1]$ where $0\leq b_0<b_1$, such that $\pi(\mu)\gtrsim\mu^k$ and one of the following holds

- *$k<-1$;*

- *$k>-1$; where $\pi(\alpha)\underset{\alpha\to 0^+}{\gtrsim}\alpha^{q_0}$ with $q_0\in\mathbb{R}$; or*

- $k > -1$; *where* $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *with* $n < -r_0 - 1$ *and* $b_0 = 0$

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

ii) If $\pi(\mu,\alpha,\beta) \gtrsim \pi(\mu)\pi(\alpha)\pi(\beta)$ *such that* $\pi(\mu) \gtrsim \mu^{-1}$ *and one of the following occurs*

- $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ *and* $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ *where either* $q_\infty \geq r_0$ *or* $n \leq -r_0$;
- $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$ *and* $\pi(\phi) \underset{\phi\to\infty}{\gtrsim} \phi^{r_\infty}$ *where either* $2r_\infty + 1 \geq q_0$ *or* $n \leq -q_0$;

then $p(\boldsymbol{\theta}|\boldsymbol{x})$ *is improper.*

Proof. See Appendix A.8 □

Theorem 9. *Suppose that* $0 < \pi(\mu,\alpha,\phi) < \infty$ *for all* $(\mu,\alpha,\phi) \in \mathbb{R}^3_+$, *and suppose that* $\pi(\mu,\alpha,\phi) = \pi(\mu)\pi(\alpha)\pi(\phi)$ *where the priors have asymptotic power-law behaviors with*

$$\pi(\mu) \propto \mu^k, \quad \pi(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^{q_0}, \quad \pi(\alpha) \underset{\alpha\to\infty}{\propto} \alpha^{q_\infty},$$

$$\pi(\phi) \underset{\phi\to 0^+}{\propto} \phi^{r_0} \quad \text{and} \quad \pi(\phi) \underset{\phi\to\infty}{\propto} \phi^{r_\infty},$$

then the posterior is proper if and only if $k = -1$, $q_\infty < r_0$, $2r_\infty + 1 < q_0$, $n > -q_0$ *and* $n > -r_0$. *Moreover, if the posterior is proper, then* $\mu^j\alpha^q\phi^r\pi(\mu,\alpha,\phi)$ *leads to a proper posterior if and only if* $j = 0$, *and* $2(r + r_\infty) + 1 - q_0 < q < r + r_0 - q_\infty$.

Proof. Notice that under our hypothesis, Theorems 7 and 8 are complementary, and thus the first part of the theorem is proved. Analogously, by Theorems 7 and 8 the prior $\mu^j\alpha^q\phi^r\pi(\mu,\alpha,\phi)$ leads to a proper posterior if and only if $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) + 1 < q + q_0$, $n > -q_0 - q$ and $n > -r_0 - r$. The last two proportionals are already satisfied since $n > -q_0$ and $n > -r_0$. Combining the other inequalities, the proof is completed. $\square$

3 Objective priors with power-law asymptotic behavior

3.1 Some common priors

A common approach was suggested by Jeffreys that considered different procedures for constructing objective priors. For $\theta \in (0, \infty)$ (see, Kass and Wasserman (1996)), Jeffreys suggested using the prior $\pi(\theta) = \theta^{-1}$, i.e., a power-law distribution with exponent 1. The main justification for this choice is its invariance under power transformations of the parameters. As the parameters of the Stacy family of distributions are contained in the interval $(0, \infty)$, the prior using Jeffreys' first rule is $\pi_1(\mu, \alpha, \phi) \propto (\mu\alpha\phi)^{-1}$.

Let us consider the case when α is known. Hence, the result is valid for the Gamma, Nakagami, Wilson-Hilferty distributions, among others. The Jeffreys' first rule when α is known follows power-law distributions with $\pi(\phi) \propto \phi^{-1}$ and $\pi(\mu) \propto \mu^{-1}$. Hence, the posterior distribution obtained is proper for all $n > 1$, as well as its higher moments. This can be easily proved by noticing that as $\pi_1(\mu, \phi) \propto \mu^{-1}\phi^{-1}$, we can apply Theorem 6 with $k = r_0 = r_\infty = -1$ and it follows that the posterior is proper for $n > -r_0 = 1$, as well as its moments.

On the other hand, under the general model where all parameters are unknown, we have the posterior distribution (2.3) obtained using Jeffreys' first rule is improper for all $n \in \mathbb{N}^+$. Since $\pi(\phi) \propto \phi^{-1}$, $\pi(\alpha) \propto \alpha^{-1}$ and $\pi(\mu) \propto \mu^{-1}$, i.e., power-laws with exponent 1, we can apply Theorem 8 ii) with $k = q_\infty = r_0 = -1$, where $q_\infty \geq r_0$, and therefore we have $\pi_2(\mu, \alpha, \phi) \propto \phi^{-1}\alpha^{-1}\mu^{-1}$ that leads to an improper posterior for all $n \in \mathbb{N}^+$.

3.2 Priors based on the Fisher information matrix

Let us consider the cases where $\pi(\mu) \propto \mu^{-1}$ and the $\pi(\phi)$ have different forms which can be written as

$$\pi_j\left(\boldsymbol{\theta}\right) \propto \frac{\pi_j(\phi)}{\mu}, \tag{3.7}$$

where j is the index related to a particular prior. Therefore, our main focus will be to study the behavior of the priors $\pi_j(\phi)$.

One crucial objective prior is based on Jeffreys' general rule (Jeffreys, 1946) and known as Jeffreys' prior. This prior is obtained through the square root of the determinant of the Fisher information matrix. It has been widely used due to its invariance property under one-to-one transformations. The Fisher information matrix for the Stacy family of distributions was derived by Hager and Bain (1970) and its elements are given by

$$I_{\alpha,\alpha}(\boldsymbol{\theta}) = \frac{1 + 2\psi(\phi) + \phi\psi'(\phi) + \phi\psi(\phi)^2}{\alpha^2},\ I_{\alpha,\mu}(\boldsymbol{\theta}) = -\frac{\psi(\phi)}{\alpha},\ I_{\mu,\phi}(\boldsymbol{\theta}) = \frac{\alpha}{\mu},$$

$$I_{\alpha,\phi}(\boldsymbol{\theta}) = -\frac{1+\phi\psi(\phi)}{\mu},\quad I_{\mu,\mu}(\boldsymbol{\theta}) = \frac{\phi\alpha^2}{\mu^2}\quad \text{and}\quad I_{\phi,\phi}(\boldsymbol{\theta}) = \psi'(\phi),$$

where $\psi'(k) = \frac{\partial}{\partial k}\psi(k)$ is the trigamma function.

Van Noortwijk (2001) provided the Jeffreys' prior for the general model, which can be expressed by (3.7) with

$$\pi_3(\phi) \propto \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}. \tag{3.8}$$

Corollary 1. *The prior $\pi_3(\phi)$ has the asymptotic behavior given by*

$$\pi_3(\phi) \underset{\phi\to 0^+}{\propto} \phi^0 \quad \textit{and} \quad \pi_3(\phi) \underset{\phi\to\infty}{\propto} \phi^{-1}.$$

Then, the obtained posterior distribution is improper for all $n \in \mathbb{N}^+$.

Proof. Ramos et al. (2017) proved that

$$\sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to 0^+}{\propto} 1 \ \text{ and } \ \sqrt{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1} \underset{\phi\to\infty}{\propto} \frac{1}{\phi}. \tag{3.9}$$

Since $\pi_3(\phi) \underset{\phi\to 0^+}{\propto} 1$, the hypotheses of Theorem 8, ii) hold with $k = -1$ and $r_0 = q_\infty = 0$, where $q_\infty \geq r_0$, and therefore $\pi_3(\boldsymbol{\theta})$ leads to an improper posterior for all $n \in \mathbb{N}^+$. $\square$

Let α be known, then the Jeffreys' prior has the form (3.7) where $\pi(\phi)$ is given by

$$\pi_4(\phi) \propto \sqrt{\phi\psi'(\phi) - 1}. \tag{3.10}$$

Corollary 2. *The prior $\pi_4(\phi)$ has the asymptotic power-law behavior given by*

$$\pi_4(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}} \quad \textit{and} \quad \pi_4(\phi) \underset{\phi\to\infty}{\propto} \phi^{-\frac{1}{2}},$$

then the obtained posterior is proper for $n \geq 1$, as well as its higher moments.

Proof. Here, we have $\pi(\beta) = \beta^{-1}$, i.e, power-law distribution. Following Abramowitz and Stegun (1972) we have $\lim_{z\to 0^+} \dfrac{\psi'(z)}{z^{-2}} = 1$, then $\lim_{\phi\to 0^+} \dfrac{\phi\psi'(\phi) - 1}{\phi^{-1}} = \lim_{\phi\to 0^+} \dfrac{\psi'(\phi)}{\phi^{-2}} - \phi = 1$, and thus

$$\phi\psi'(\phi) - 1 \underset{\phi\to 0^+}{\propto} \phi^{-1}, \tag{3.11}$$

which implies $\sqrt{\phi\psi'(\phi)-1} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$. Moreover, from Abramowitz and Stegun (1972), we have $\psi'(z) = \dfrac{1}{z} + \dfrac{1}{2z^2} + o\left(\dfrac{1}{z^3}\right)$ and thus

$$\frac{\phi\psi'(\phi)-1}{\phi^{-1}} = \frac{1}{2} + o\left(\frac{1}{\phi}\right) \Rightarrow \lim_{\phi\to\infty} \frac{\sqrt{\phi\psi'(\phi)-1}}{\phi^{-\frac{1}{2}}} = \frac{1}{\sqrt{2}},$$

which implies $\sqrt{\phi\psi'(\phi)-1} \underset{\phi\to\infty}{\propto} \phi^{-\frac{1}{2}}$.

Therefore, we can apply Theorem 3 with $k=-1$ and $r_0 = r_\infty = -\frac{1}{2}$ and therefore the posterior is proper and the posterior moments are finite for all $n > -r_0 = \frac{1}{2}$. □

Fonseca et al. (2008) derived objective priors for the student-t distribution and showed that the standard Jeffreys prior returned an improper posterior. On the other hand, assuming that one of the parameters were independent, the obtained independent Jeffreys prior returned a proper posterior. The proposed Jeffreys' prior with an independent structure has the form $\pi_{J2}(\boldsymbol{\theta}) \propto \sqrt{|\operatorname{diag} I(\boldsymbol{\theta})|}$, where $\operatorname{diag} I(\cdot)$ is the diagonal matrix of $I(\cdot)$. For the general distribution, the prior is given by (3.7) with

$$\pi_5(\phi) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)}. \tag{3.12}$$

Notice that for (3.12), it is only necessary to know the behavior $\pi_5(\phi)$ when $\phi\to 0^+$ that provides enough information to verify that the posterior is improper.

Corollary 3. *The prior (3.12) has the asymptotic power-law behavior given by $\pi_5(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$ and the obtained posterior is improper for all $n \in \mathbb{N}^+$.*

Proof. By Abramowitz and Stegun (1972), we have the recurrence relations

$$\psi(\phi) = -\frac{1}{\phi} + \psi(\phi+1) \quad \text{and} \quad \psi'(\phi) = \frac{1}{\phi^2} + \psi'(\phi+1). \tag{3.13}$$

It follows that

$$
\begin{aligned}
&2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1=\\
&2\left(-\frac{1}{\phi}+\psi(\phi+1)\right)+\phi\left(\frac{1}{\phi^2}+\psi'(\phi+1)\right)+\phi\left(\frac{1}{\phi^2}-\frac{2}{\phi}\psi(\phi+1)+\psi(\phi+1)^2\right)+1=\\
&1+\phi\left(\psi(\phi+1)^2+\psi'(\phi+1)\right).
\end{aligned}
$$

Hence, $2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1 \underset{\phi\to 0^+}{\propto} 1$, which implies that

$$
\pi_5\left(\phi\right) \propto \sqrt{\phi\psi'(\phi)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}, \tag{3.14}
$$

i.e., power-law distribution with exponent $\frac{1}{2}$, then, Theorem 8 ii) can be applied with $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$ and therefore $\pi_5(\boldsymbol{\theta})$ leads to an improper posterior. □

This approach can be further extended considering that only one parameter is independent. For instance, let (θ_1,θ_2) be dependent parameters and θ_3 be independent then under the partition the $((\theta_1,\theta_2),\theta_3)$-Jeffreys' prior is given by

$$
\pi\left(\boldsymbol{\theta}\right) \propto \sqrt{\left(I_{11}(\boldsymbol{\theta})I_{22}(\boldsymbol{\theta})-I_{12}^2(\boldsymbol{\theta})\right)I_{33}(\boldsymbol{\theta})}. \tag{3.15}
$$

For the general model, the partition $((\phi,\mu),\alpha)$-Jeffreys' prior is of the form (3.7) with

$$
\pi_6\left(\phi\right) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)}. \tag{3.16}
$$

Corollary 4. *The prior (3.16) has the asymptotic power-law behavior given by* $\pi_6\left(\phi\right) \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}$ *and the obtained posterior is improper for all* $n\in\mathbb{N}^+$.

Proof. From equation (3.11), we have $\phi\psi'(\phi)-1 \underset{\phi\to 0^+}{\propto} \frac{1}{\phi}$ which combined with the relation (3.14) implies that

$$
\pi_6\left(\phi\right) \propto \sqrt{\left(\phi\psi'(\phi)-1\right)\left(1+2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2\right)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}}. \tag{3.17}
$$

i.e., power-law distribution with exponent $\frac{1}{2}$, then Theorem 8, ii) can be applied with $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$ and therefore $\pi_6(\boldsymbol{\theta})$ leads to an improper posterior. $\square$

Considering the partition $((\alpha,\mu),\phi)$-Jeffreys' prior is given by (3.7) where

$$\pi_7(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)}\,. \tag{3.18}$$

This is similar to the two cases above. From the recurrence relations (3.13), we have

$$\phi^2\psi^{'}(\phi)+\phi-1=\phi\left(1+\phi\psi^{'}(\phi+1)\right) \Rightarrow \phi^2\psi^{'}(\phi)+\phi-1 \underset{\phi\to 0^+}{\propto} \phi \tag{3.19}$$

as $\psi'(\phi) \propto \frac{1}{\phi^2}$ it follows that

$$\pi_7(\phi) \propto \sqrt{\psi'(\phi)(\phi^2\psi'(\phi)+\phi-1)} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}},$$

with the same values $k=-1$, $r_0=-\frac{1}{2}$ and $q_\infty=0$ where $q_\infty \geq r_0$, the prior $\pi_7(\boldsymbol{\theta})$ leads to an improper posterior.

3.3 Reference priors

Another important class of objective priors was introduced by Bernardo (1979) with further developments (Berger and Bernardo, 1989, 1992; Berger et al., 1992) reference priors play an important role in objective Bayesian analysis. The reference priors have desirable properties, such as invariance, consistent marginalization, and consistent sampling properties. Bernardo (2005) reviewed different procedures to derive reference priors considering the ordered parameters of interest. The following proposition will be applied to obtain the reference priors for the Generalized Gamma distribution.

Proposition 4. *[Bernardo (1979), pg 40, Theorem 14] Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ be a vector with the ordered parameters of interest and $p(\boldsymbol{\theta}|\boldsymbol{x})$ be the posterior distribution that has an asymptotically normal distribution with dispersion matrix $V(\hat{\boldsymbol{\theta}}_n)/n$, where $\hat{\boldsymbol{\theta}}_n$ is a consistent estimator of $\boldsymbol{\theta}$ and $H(\boldsymbol{\theta}) = V^{-1}(\boldsymbol{\theta})$. In addition, V_j is the upper $j \times j$ sub-matrix of V, $H_j = V_j$ and $h_{j,j}(\boldsymbol{\theta})$ is the lower right element of H_j. If the parameter space of θ_j is independent of $\boldsymbol{\theta}_{-j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_m)$, for $j = 1, \ldots, m$, and $h_{j,j}(\boldsymbol{\theta})$ are factorized in the form $h_{j,j}^{\frac{1}{2}}(\boldsymbol{\theta}) = f_j(\theta_j) g_j(\boldsymbol{\theta}_{-j}), \quad j = 1, \ldots, m$, then the reference prior for the ordered parameters $\boldsymbol{\theta}$ is given by*

$$\pi(\boldsymbol{\theta}) = \pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) \times \cdots \times \pi(\theta_2|\theta_1)\pi(\theta_1),$$

where $\pi(\theta_j|\theta_1, \ldots, \theta_{j-1}) = f_j(\theta_j)$, for $j = 1, \ldots, m$, and there is no need for compact approximations, even if the conditional priors are not proper.

The reference priors obtained from Proposition 4 belong to the class of improper priors given by

$$\pi\left(\boldsymbol{\theta}\right) \propto \pi(\phi)\alpha^{-1}\mu^{-1}, \tag{3.20}$$

therefore, both $\pi(\mu) \propto \mu^{-1}$, $\pi(\alpha) \propto \alpha^{-1}$ follows power-law distributions with exponent 1. Our focus will be to study the asymptotic power-law behavior of $\pi(\phi)$. Let (α, ϕ, μ) be the ordered parameters of interest, then conditional priors of the (α, ϕ, μ)-reference prior are given by

$$\pi(\alpha) \propto \alpha^{-1}, \quad \pi(\phi|\alpha) \propto \sqrt{\frac{\phi\psi'(\phi) - 1}{\phi}}, \quad \pi(\mu|\alpha, \phi) \propto \mu^{-1}.$$

Therefore, (α,ϕ,μ)-reference prior is of the form (3.20) with

$$\pi_8(\phi) \propto \sqrt{\frac{\phi\psi'(\phi)-1}{\phi}} \underset{\phi\to 0^+}{\propto} \phi^{-1}.$$

which is also a power-law distribution with exponent 1. Therefore, item ii) of Theorem 8 can be applied with $k = r_0 = q_\infty = 1$ where $q_\infty \geq r_0$ which implies that $\pi_8(\alpha,\phi,\mu)$ leads to an improper posterior for all $n \in \mathbb{N}^+$.

Assuming that (α,μ,ϕ) are the ordered parameters, then the conditional reference priors are

$$\pi(\alpha) \propto \alpha^{-1}, \quad \pi(\mu|\alpha) \propto \mu^{-1}, \quad \pi(\phi|\alpha,\mu) \propto \sqrt{\psi'(\phi)},$$

and the (α,μ,ϕ)-reference prior is of the form (3.20) with

$$\pi_9(\phi) \propto \sqrt{\psi'(\phi)}.$$

From $\psi'(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-2}$ we have $\sqrt{\psi'(\phi)} \underset{\phi\to 0^+}{\propto} \phi^{-1}$, i.e., a PL distribution with exponent 1. Similar to the case of $\pi_8(\mu,\alpha,\phi)$, we have $\pi_9(\mu,\alpha,\phi)$ that leads to an improper posterior for all $n \in \mathbb{N}^+$.

Consider the case where α is known with $\alpha = 1$ reducing to the Gamma distribution. Then $\pi(\phi,\mu) \propto \mu^{-1}\sqrt{\psi'(\phi)}$ is the (μ,ϕ)-reference prior and the joint posterior densities when $\alpha = 1$ using the (μ,ϕ)-reference is proper for $n \geq 2$ as well as its higher moments.

The results above follow from the fact that $\psi'(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-2}$ and
$\psi'(\phi) \underset{\phi\to\infty^+}{\propto} \phi^{-1}$ and thus $\pi_9(\phi)$ has an asymptotic power-law behavior given by

$$\pi_9(\phi) \underset{\phi\to 0^+}{\propto} \phi^{-1} \quad \text{and} \quad \pi_9(\phi) \underset{\phi\to\infty^+}{\propto} \phi^{-\frac{1}{2}},$$

therefore, from the power-law distributions above, as well as the distribution $\pi(\mu)$ that has a PL with exponent 1, we can apply Theorem 3 with $k=-1$, $r_0=-1$ and $r_\infty=-0.5$ and it follows that the posterior, as well as all its moments are proper for all $n>-r_0=1$.

Assuming now that ϕ is known with $\phi=1$, then the distribution reduces to the Weibull distribution. In this case, $\pi(\mu,\alpha)\propto\alpha^{-1}\mu^{-1}$ is the (α,μ)-reference prior, note that each prior follows a power-law distribution. The joint posterior density using the (α,μ)-reference is proper for $n\geq 2$ although its higher moments relative to μ are improper. This result is a direct consequence from Theorem 6 considering that $k=-1$ and $q_0=q_\infty=-1$ that leads to a proper posterior.

Returning to the general model, if (μ,ϕ,α) is the vector of ordered parameters, it follows that the conditional priors are

$$\pi(\mu)\propto\mu^{-1},\ \ \pi(\phi|\mu)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi^2)+1}},\ \ \pi(\alpha|\phi,\mu)\propto\alpha^{-1}$$

and the (μ,ϕ,α)-reference prior is of the form (3.20) with

$$\pi_{10}(\phi)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi^2)+1}}.$$

Corollary 5. *The prior $\pi_{10}(\phi)$ has the asymptotic power-law behavior given by $\pi_{10}(\phi)\underset{\phi\to0^+}{\propto}\phi^{-1}$ and the obtained posterior is improper for all $n\in\mathbb{N}^+$.*

Proof. From Abramowitz and Stegun (1972), we have

$$\psi(\phi)=\log(\phi)-\frac{1}{2\phi}-\frac{1}{12\phi^2}+o\left(\frac{1}{\phi^2}\right)\quad\text{and}\quad\psi'(\phi)=\frac{1}{\phi}+\frac{1}{2\phi^2}+o\left(\frac{1}{\phi^2}\right),\qquad(3.21)$$

where it follows directly that

$$\psi(\phi)^2=\log(\phi)^2-\frac{\log(\phi)}{\phi}+o\left(\frac{1}{\phi}\right).$$

Therefore, $2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1=\phi\log(\phi)^2+\log(\phi)+2+o(1)$ and

$$\begin{aligned}\pi_{10}(\phi)&\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\\&=\sqrt{\frac{\left(\frac{1}{\phi}+\frac{1}{2\phi^2}+o\left(\frac{1}{\phi^2}\right)\right)\left(\phi\log(\phi)^2+\log(\phi)+2+o(1)\right)-\log(\phi)^2+\frac{\log(\phi)}{\phi}+o\left(\frac{1}{\phi}\right)}{\phi\log(\phi)^2+\log(\phi)+2+o(1)}}\\&=\sqrt{\frac{\frac{1}{\phi}\left(\log(\phi)^2+o(\log(\phi)^2)\right)}{\phi\left(\log(\phi)^2+o(\log(\phi)^2)\right)}}=\frac{1}{\phi}\sqrt{\frac{1+o(1)}{1+o(1)}}.\end{aligned}$$

Thus,

$$\pi_{10}(\phi)\propto\sqrt{\psi'(\phi)-\frac{\psi(\phi)^2}{2\psi(\phi)+\phi\psi'(\phi)+\phi\psi(\phi)^2+1}}\underset{\phi\to0^+}{\propto}\phi^{-1},$$

and therefore Theorem 8 ii) can be applied with $k=q_0=r_\infty=-1$ where $2r_\infty+1\geq q_0$. Thus, $\pi_{10}(\boldsymbol{\theta})$ leads to an improper posterior.

$\square$

Finally, let (ϕ,α,μ) be the ordered parameters, then the conditional priors are

$$\pi(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}},\qquad\pi(\alpha|\phi)\propto\alpha^{-1},\qquad\pi(\mu|\alpha,\phi)\propto\mu^{-1}$$

and the (ϕ,α,μ)-reference prior is of the form (3.20) with

$$\pi_{11}(\phi)\propto\sqrt{\frac{\phi^2\psi'(\phi)^2-\psi'(\phi)-1}{\phi^2\psi'(\phi)+\phi-1}}.\tag{3.22}$$

It is worth mentioning that (ϕ,μ,α)-reference prior is the same as the (ϕ,α,μ)-reference prior, while (μ,α,ϕ)-reference prior has the same form of $\pi_9(\boldsymbol{\theta})$ which completes all possible reference priors obtained from Proposition 4.

Corollary 6. *The prior $\pi_{11}(\phi)$ has the asymptotic power-law behavior given by*

$$\pi_{11}(\phi)\underset{\phi\to0^+}{\propto}\phi^{-\frac{1}{2}}\quad\textit{and}\quad\pi_{11}(\phi)\underset{\phi\to\infty}{\propto}\phi^{-\frac{3}{2}}.$$

Then the obtained posterior distribution is proper for $n \geq 2$ and its higher moments are improper for all $n \in \mathbb{N}^+$.

Proof. From (3.9) and by the asymptotic relations (3.21) we have

$$\phi^2\psi^{'}(\phi) + \phi - 1 = 2\phi - \frac{1}{2} + o(1) \underset{\phi\to\infty}{\propto} \phi$$

which together with equation (3.19) implies that

$$\sqrt{\phi^2\psi'(\phi) + \phi - 1} \underset{\phi\to 0^+}{\propto} \sqrt{\phi} \quad \text{and} \quad \sqrt{\phi^2\psi'(\phi) + \phi - 1} \underset{\phi\to\infty}{\propto} \sqrt{\phi}.$$

Hence, from the above proportionalities, we have

$$\sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}} \underset{\phi\to 0^+}{\propto} \phi^{-\frac{1}{2}} \quad \text{and} \quad \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}} \underset{\phi\to\infty}{\propto} \phi^{-\frac{3}{2}}.$$

Therefore, Theorem 7 can be applied with $k = q_0 = q_\infty = -1$, $r_0 = -\frac{1}{2}$ and $r_\infty = -\frac{3}{2}$ where $k = -1$, $q_\infty < r_0$ and $2r_\infty + 1 < q_0$, and therefore $\pi_{11}(\mu, \alpha, \phi)$ leads to a proper posterior for every $n > -q_0 = 1$.

To prove that the higher moments are improper, suppose $\alpha^q\phi^r\mu^j\pi(\boldsymbol{\theta})$ leads to a proper posterior for $r \in \mathbb{N}$, $q \in \mathbb{N}$ and $k \in \mathbb{N}$. By Theorem 9, we have $j = 0$, $q + q_\infty < r + r_0$, $2(r + r_\infty) \leq q + q_0$ and $n \geq -q_0$, i.e., $k = 0$ and $2r - 1 < q < r + \frac{1}{2}$. The inequality $2r - 1 < r + \frac{1}{2}$ leads to $r < \frac{3}{2}$, i.e., $r = 0$ or $r = 1$. By the previous inequality, the case where $r = 0$ leads to $-1 < q < \frac{1}{2}$, that is, $q = 0$. Now, for $r = 1$ we have the inequality $1 < q < \frac{3}{2}$ which do not have an integer solution. Therefore, the only possible values for which $\alpha^q\phi^r\mu^j\pi(\boldsymbol{\theta})$ is proper is $q = r = j = 0$, that is, the higher moments are improper. $\square$

Zellner (1977, 1984) discussed another procedure to obtain an objective prior that is based on the information measure known as Shannon entropy. Such a prior is known as MDI prior and can be obtained by solving

$$\pi_Z\left(\boldsymbol{\theta}\right) \propto \exp\left(\int f(t\,|\,\phi,\mu,\alpha)\log f(t\,|\,\phi,\mu,\alpha)dt\right). \tag{3.23}$$

Ramos et al. (2017) showed that the MDI prior (3.23) for the GG distribution is given by

$$\pi_Z\left(\boldsymbol{\theta}\right) \propto \frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\}. \tag{3.24}$$

Notice this distribution is not a power law distribution since for any ϕ close enough to 0 such that $\psi(\phi)<0$, letting $\beta=\frac{-\psi(\phi)}{\alpha}$ we have

$$\begin{aligned}\lim_{\alpha\to 0^+}\frac{1}{\alpha^k}\frac{\alpha\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\left(\phi-\frac{1}{\alpha}\right)-\phi\right\} &= \lim_{\beta\to\infty}\frac{\beta^{k-1}}{\left(-\psi(\phi)\right)^{k-1}}\frac{\mu}{\Gamma(\phi)}\exp\left\{\psi(\phi)\phi+\beta-\phi\right\}\\ &= \frac{\mu}{\Gamma(\phi)}\frac{\exp\left\{\psi(\phi)\phi-\phi\right\}}{\left(-\psi(\phi)\right)^{k-1}}\lim_{\beta\to\infty}\beta^{k-1}\exp\left\{\beta\right\}=\infty\end{aligned} \tag{3.25}$$

and thus $\lim_{\alpha\to 0^+}\frac{\pi_Z(\boldsymbol{\theta})}{\alpha^k}=\infty$ for any $k\in\mathbb{R}$, that is, $\pi_Z\left(\boldsymbol{\theta}\right)$ is not a power-law distribution. This distribution also leads to an improper posterior as discussed in Ramos et al. (2017).

3.4 Hierarchical models

The power-law behaviour occurs in other models such as hierarchical models. For instance, Fonseca et al. (2019) derived different objective priors for such a hierarchical structure. For a student-t model with unknown degrees of freedom, where $y|\theta\ student(\theta)$ a standard Student-t model with fixed mean and precision, an unknown degree of freedom v. The model may be rewritten in an hierarchical setting as

$$y|w \sim N(0, 1/w)$$

$$w|\theta \sim Gamma(\theta/2, \theta/2)$$

The model has two levels of hierarchy where θ appears in the second level, the Jeffreys' prior may be written as

$$\pi_{h_1}(\theta) \propto \sqrt{I_y(\theta)}$$

where $I_y(\theta) = I_w(\theta) - E_y[I_w(\theta \mid y)]$, and where $I_w(\theta)$ and $E_y[I_w(\theta \mid y)]$ are given by

$$I_w(\theta) = \frac{1}{4}\psi^{(2)}\left(\frac{\theta}{2}\right) - \frac{1}{2\theta}$$

and

$$E_y[I_w(\theta \mid y)] = \frac{1}{4}\psi^{(1)}\left(\frac{\theta+1}{2}\right) + \frac{\theta+2}{2\theta(\theta+3)} - \frac{1}{\theta+1}$$

Corollary 7. *The prior $\pi_{h_1}(\phi)$ has the asymptotic power-law behavior given by*

$$\pi_{h_1}(\theta) \underset{\theta\to 0^+}{\propto} \theta^{-1} \quad \textit{and} \quad \pi_{h_1}(\theta) \underset{\theta\to\infty}{\propto} \theta^{-2},$$

Proof. First notice that

$$I_y(\theta) = \frac{1}{4}\psi^{(1)}\left(\frac{\theta}{2}\right) - \frac{1}{4}\psi^{(1)}\left(\frac{\theta+1}{2}\right) - \frac{\theta+5}{2\theta(\theta+1)(\theta+3)}$$

Now, following Abramowitz and Stegun (1972), we know that $\lim_{x\to 0^+} x^2\psi^{(1)}(x) = 1$, and $\lim_{x\to\infty} x\psi^{(1)}(x) = 1$ and thus in particular it follows that

$$\lim_{\theta\to\infty} \theta\psi^{(1)}\left(\frac{\theta+1}{2}\right) = \lim_{\theta\to\infty} 2\left(\frac{\theta}{\theta+1}\right)\left[\left(\frac{\theta+1}{2}\right)\psi^{(1)}\left(\frac{\theta+1}{2}\right)\right] = 2\times 1\times 1 = 2.$$

and similarly we have $\lim_{\theta\to 0^+} \theta^2\psi^{(1)}\left(\frac{\theta}{2}\right) = 4$, $\lim_{\theta\to 0^+} \theta^2\psi^{(1)}\left(\frac{\theta+1}{2}\right) = 0$ and $\lim_{\theta\to\infty} \theta\psi^{(1)}\left(\frac{\theta}{2}\right) = 2$. Thus, combining all these items we have

$$\lim_{\theta\to 0^+} \theta^2 I_y(\theta) = \frac{4}{4} - 0 - 0 = 1 \Rightarrow I_y(\theta) \underset{\theta\to 0^+}{\propto} \frac{1}{\theta^2}.$$

Moreover, from Abramowitz and Stegun (1972), we have the asymptotic relation

$$\psi^{(1)}(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + o(x^4), \text{ as } x \to \infty,$$

from which, letting $w(\theta) = \frac{1}{\theta} + \frac{1}{2\theta^2} + \frac{1}{6\theta^3}$ we conclude that

$$\begin{aligned}\lim_{\theta\to\infty} \theta^4 I_y(\theta) &= \lim_{\theta\to\infty} \theta^4 \left[\frac{1}{4}w\left(\frac{\theta}{2}\right) - \frac{1}{4}w\left(\frac{\theta+1}{2}\right) - \frac{\theta+5}{2\theta(\theta+1)(\theta+3)}\right] \\ &= \lim_{\theta\to\infty} \frac{21\,\theta^4 + 48\,\theta^3 + 29\,\theta^2 + 6\,\theta}{6(\theta+1)^3(\theta+3)} = \frac{21}{6} \Rightarrow I_y(\theta) \underset{\theta\to\infty}{\propto} \frac{1}{\theta^4}.\end{aligned}$$

Thus, we proved that $I_y(\theta) \underset{\theta\to 0^+}{\propto} \theta^{-2}$ and $I_y(\theta) \underset{\theta\to\infty}{\propto} \theta^{-4}$, which combined with $\pi_{h_1} \propto \sqrt{I_y(\theta)}$ concludes the proof. $\square$

Hence, the Jeffreys' prior under this hierarchical model also follows an asymptotic power-law distribution.

Another case may be obtained when we assume independent Laplace priors for the regression coefficients as discussed in Fonseca et al. (2019). They assume that,

$$y|w_1 \sim N(Xw_1, \sigma^2 I_n)$$

where y is the $n \times 1$ vector of responses, X is the $n \sim p$ matrix of covariates, and w_1 is the $n \times 1$ vector of regression coefficients

The lasso constraint, under the Bayesian context, is equivalent to using the independent Laplace prior

$$p(w_{1,j}) = \frac{\gamma}{2} \exp\left\{-\gamma\,|w_{1,j}|\right\}$$

The lasso prior is obtained as a uniform scale mixture by considering the conditional setting

$$w_{1,j} \mid w_{2,j} \sim \text{Unif}\left(-\sigma w_{2,j}, \sigma w_{2,j}\right)$$

$$w_{2,j} \sim \text{Gamma}(2, \theta).$$

After some algebraic manipulations, the final Jeffreys' prior has a closed-form given by $\pi(\theta) \propto \theta^{-1}$, i.e., a power-law distribution with $\lambda = 1$. The results show that power-law behavior also occurs in hierarchical models.

4 A Real Application

Van Noortwijk (2001) analyzed a data set related to the annual maximum discharge of the river Rhine at Lobith, the Netherlands, from 1901 to 1998, where the Dutch river dikes have to withstand water levels and discharges with an average return period of up to 1250 years. Maximum river discharge is usually associated with floods which cause much damage worldwide. The values of m^3/s are provided in Figure 1.

The authors considered the GG distribution to predict the exceedance probabilities of annual maximum discharge. The posterior distribution was constructed using the Jeffreys' prior (3.8). However, we proved in Corollary (1) that the obtained posterior is improper for all $n \in \mathbb{N}^+$ and should not be used to compute the posterior estimates. The estimates for the parameters ϕ, $1/\mu$ and α were respectively, 1.380, 4936.0 and 2.310, while the authors did not provide the credibility interval for $1/\mu$, the credibility intervals for both ϕ and α were (0.01,6.00). In this case, there is a good indication that the inference was

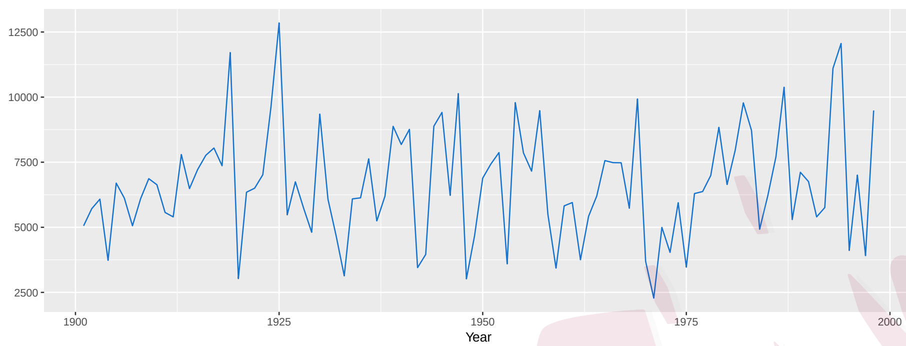

Figure 1: Time series plot for the data set related to the annual maximum discharges (m^3/s) of the river Rhine at Lobith during 1901-1998.

conducted improperly. The range of credibility intervals was probably influenced due to the use of an improper posterior distribution, and the results are not reliable.

The posterior distribution using the (ϕ, α, μ)-reference prior (3.22) is proper for $n \geq 2$ and can be used to analyze this data. Due to the consistent marginalization property of the reference prior, the reference marginal posterior distribution of ϕ and α is

$$p_{12}(\phi, \alpha|\boldsymbol{x}) \propto \alpha^{n-2} \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \sqrt{\frac{\phi^2\psi'(\phi)^2 - \psi'(\phi) - 1}{\phi^2\psi'(\phi) + \phi - 1}} \left(\frac{\sqrt[n]{\prod_{i=1}^n t_i^\alpha}}{\sum_{i=1}^n t_i^\alpha} \right)^{n\phi},$$

while the conditional posterior distributions for μ given ϕ and α are given by

$$p_{12}(\mu|\phi, \alpha, \boldsymbol{x}) \sim \mathrm{GG}\left(n\phi, \left(\sum_{i=1}^n t_i^\alpha \right)^{\frac{1}{\alpha}}, \alpha \right).$$

The distributions above are helpful to obtain posterior estimates using Markov chain Monte Carlo methods. We have conducted a simulation study available in Appendix A.9 which shows from an intensive simulation study that the obtained posterior estimates are accurate, especially when compared with simple proper flat priors. Since we proved that the posterior mean for the parameter does not return finite values, the posterior medians for ϕ, μ and α were considered as posterior estimates. Moreover, following Van Noortwijk (2001), the annual maximum river discharge (MRD) in which the probability of exceedance is 1/1250 per year is also presented. The posterior summaries are shown in Table 2.

Table 2: Posterior median, standard deviations and 95% credible intervals for ϕ, μ and α.

$\boldsymbol{\theta}$	Median	SD	$\mathrm{CI}_{95\%}(\boldsymbol{\theta})$
ϕ	3.741	1.857	(1.072; 7.609)
$1/\mu$	3,061.6	2,160.5	(999.08; 7,970)
α	1.620	0.562	(1.070; 3.194)
MRD	14,887	4,055.9	(10,535; 22,591)

The MRD presented by Van Noortwijk was 15,150, which shows that the improper analysis returned an overestimated annual maximum discharge. Therefore, based on our estimates, the Dutch River dikes will have to withstand water levels and discharges of up to $14,887$ m^3/s.

5 Discussion

Objective priors play an important role in Bayesian analysis. For several important distributions, we showed that such objective priors are improper priors and may lead to improper posterior; in these cases, the Bayesian inference cannot be conducted, which is undesirable. An exciting aspect of our findings is that such priors either follow a power-law distribution or present an asymptotic behaviour to this distribution. Our mathematical formalism is general and covers important distributions widely used in the literature. The exponent of the obtained power-law distributions is contained between 0.5 and 1. Hence, they are improper with infinite mean and variance.

We provided sufficient and necessary conditions for the posteriors to be proper, depending on the exponent of the power-law model. For instance, if ϕ is known, the (α, μ)-reference prior for the Weibull and Generalized half-normal distributions, the priors follow power-law distributions with exponent one and return proper posteriors. By considering α fixed, we showed that both the Jeffreys' first rule and the Jeffreys' prior returned proper posterior distributions, as well as finite higher moments, which are valid for the Gamma, Nakagami-m and Wilson-Hilferty distributions. Moreover, we provided many situations where the obtained posteriors are improper and should not be used, opening up new opportunities for the analysis of real data.

The observed behavior also occurs in many other classes of distributions, for instance, for the Lomax distribution, which is a modified version of the Pareto model, the reference prior for the two parameters of the model follows power-law distributions with the exponent one (Ferreira et al., 2020). This behaviour is also observed in a Gaussian distribution when μ is a known parameter, in this case, the Jeffreys prior for standard deviation σ follows a power-law distribution with exponent one and the obtained posterior is proper.

Under the Behrens-Fisher problem, the obtained Jeffreys prior for the parameters has the same behaviour with exponents two while the reference prior has exponents three (Liseo, 1993).

The proposed theoretical results were applied to show that the Bayesian approach was misused to analyze the data set related to the annual maximum discharge of the river Rhine at Lobith, Netherlands, hence, using a proper posterior distribution, the correct posterior estimates were computed. There are a large number of possible extensions of this current work. The power-law distributions could be considered as objective prior in the models when there is the presence of censored data or long-term survival. The use of our approach for other distributions, such as generalized linear models, should also be further investigated.

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A Appendix A:

A.1 Useful Proportionalities

The following proportionalities are useful to prove results related to the posterior distribution, and its proof can be seen in Ramos et al. (2017).

Proposition 5. *Let* $\mathrm{p}(\alpha) = \log\left(\frac{\frac{1}{n}\sum_{i=1}^{n} t_i^{\alpha}}{\sqrt[n]{\prod_{i=1}^{n} t_i^{\alpha}}}\right)$, $\mathrm{q}(\alpha) = \mathrm{p}(\alpha) + \log n$, *for* $t_1, t_2, \ldots, t_n$ *positive and not all equal,* $h \in \mathbb{R}^+$, $r \in \mathbb{R}^+$ *and* $t_m = \max\{t_1, \ldots, t_n\}$, *then* $\mathrm{p}(\alpha) > 0$, $\mathrm{q}(\alpha) > 0$ *and the following results hold*

$$\mathrm{p}(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^2 \quad \textit{and} \quad \mathrm{p}(\alpha) \underset{\alpha\to\infty}{\propto} \alpha;$$

$$\mathrm{q}(\alpha) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \mathrm{q}(\alpha) \underset{\alpha\to\infty}{\propto} \alpha;$$

$$\frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \underset{\phi\to 0^+}{\propto} \phi^{n-1} \quad \textit{and} \quad \frac{\Gamma(n\phi)}{\Gamma(\phi)^n} \underset{\phi\to\infty}{\propto} \phi^{\frac{n-1}{2}} n^{n\phi};$$

$$\gamma\left(h, r\,\mathrm{q}(\alpha)\right) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \gamma\left(h, r\,\mathrm{q}(\alpha)\right) \underset{\alpha\to\infty}{\propto} 1; \tag{A.26}$$

$$\Gamma\left(h, r\,\mathrm{p}(\alpha)\right) \underset{\alpha\to 0^+}{\propto} 1 \quad \textit{and} \quad \Gamma\left(h, r\,\mathrm{p}(\alpha)\right) \underset{\alpha\to\infty}{\propto} \alpha^{k-1} e^{-rk(\boldsymbol{x})\alpha}; \tag{A.27}$$

where $k(\boldsymbol{x}) = \log\left(\frac{t_m}{\sqrt[n]{\prod_{i=1}^{n} t_i}}\right) > 0$; $\gamma(y,x) = 1 - \Gamma(y,x)$ *and* $\Gamma(y,x) = \int_x^{\infty} w^{y-1} e^{-w}\, dw$ *is the upper incomplete gamma function.*

A.2 Proof of Proposition 2

Suppose that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$ and $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$. Then, by Definition 3 we have $\limsup_{x\to a} \dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} = w$ for some $w \in \mathbb{R}^+$. Therefore, from the definition of lim sup there exist some $a' \in (a, b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{3w}{2}$ for every $x \in (a, a']$. Proceeding analogously, there must exist some $v \in \mathbb{R}^+$ and $b' \in (a', b)$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{3v}{2}$ for every $x \in [b', b)$. On the other hand, since $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)}$ is continuous in $[a', b']$, the Weierstrass Extreme Value Theorem states that there exist some $x_1 \in [a', b']$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq \dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}$ for every $x \in [a', b']$. Finally, choosing $M = \max\left(\dfrac{3w}{2}, \dfrac{3v}{2}, \dfrac{\mathrm{g}(x_1)}{\mathrm{h}(x_1)}\right) < \infty$, it follows that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$, which by Definition 2 means that $g(x) \lesssim h(x)$.

Now suppose $g(x) \lesssim h(x)$. By Definition 2, there exist some $M < 0$ such that $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M$ for every $x \in (a, b)$. This implies that $\limsup_{x\to a} \frac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq M < \infty$ which by Definition 3 means that $\mathrm{g}(x) \underset{x\to a}{\lesssim} \mathrm{h}(x)$. The proof that $\mathrm{g}(x) \underset{x\to b}{\lesssim} \mathrm{h}(x)$ must also be satisfied is analogous to the previous case. Therefore, the theorem is proved.

A.3 Proof of Theorem 1

Let $\alpha \in \mathbb{R}^+$ be fixed. Since $\frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\}\pi(\mu)\mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$\begin{aligned}
d(\boldsymbol{x}; \alpha) &= \int\limits_{\mathcal{A}} \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\boldsymbol{\theta} \\
&= \int_0^\infty\int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi.
\end{aligned}$$

Since $\pi(\mu) \lesssim \mu^k$ and $k \geq -1$ by hypothesis, it follows that

$$d(\boldsymbol{x};\alpha) \lesssim \int_0^\infty \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi+k} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi$$
$$= \int_0^\infty \int_0^\infty \frac{\pi(\phi)}{\Gamma(\phi)^n} \left(\prod_{i=1}^n x_i^\alpha\right)^\phi \frac{\alpha\Gamma\left(n\phi + \frac{k+1}{\alpha}\right)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi+\frac{k+1}{\alpha}}}\, d\mu\, d\phi.$$

Now suppose that $k > -1$. Then, since $k+1>0$, $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi\to 0^+}{\propto} 1$ and $\Gamma(n\phi + \frac{k+1}{\alpha}) \underset{\phi\to\infty}{\propto} \Gamma(n\phi)(n\phi)^{\frac{k+1}{\alpha}}$ (see Abramowitz and Stegun (1972)). Therefore, from the proportionalities in Proposition 5 it follows that

$$\begin{aligned} d(\boldsymbol{x};\alpha) &\lesssim \int_0^1 \pi(\phi)\frac{1}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi} d\phi + \int_1^\infty \pi(\phi)\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}\phi^{\frac{k+1}{\alpha}} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi \\ &\propto \int_0^1 \pi(\phi)\phi^n e^{-n\,\mathrm{q}(\alpha)\phi} d\phi + \int_1^\infty \pi(\phi)\phi^{\frac{n-1}{2}+\frac{k+1}{\alpha}} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi \\ &= s_1(\boldsymbol{x};\alpha) + s_2(\boldsymbol{x};\alpha) \end{aligned} \tag{A.28}$$

where $\mathrm{q}(\alpha)$ and $\mathrm{p}(\alpha)$ are given in Proposition 5 and $s_1(\boldsymbol{x};\alpha)$ and $s_2(\boldsymbol{x};\alpha)$ denote the respective two integrals in the sum that precedes it. It follows that $d(\boldsymbol{x};\alpha) < \infty$ if $s_1(\boldsymbol{x};\alpha) < \infty$ and $s_2(\boldsymbol{x};\alpha) < \infty$. Now, using the proportionalities in Proposition 5 it follows that, since $n + r_0 > -1$, $\mathrm{q}(\alpha) > 0$ and $\mathrm{p}(\alpha) > 0$, then

$$s_1(\boldsymbol{x};\alpha) \lesssim \int_0^1 \phi^{n+r_0} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi = \frac{\gamma(n+r_0+1, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}} < \infty,$$

and

$$s_2(\boldsymbol{x};\alpha) \lesssim \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}+\frac{k+1}{\alpha}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi = \frac{\Gamma(\frac{n+1+2r_\infty}{2}+\frac{k+1}{\alpha}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}+\frac{k+1}{\alpha}}} < \infty,$$

therefore, we have $d(\boldsymbol{x};\alpha) < \infty$.

The case where $k=-1$ and $n>-r_0$ is completely analogous to the previous case, with the only difference in the proof being that $\Gamma(n\phi+\frac{k+1}{\alpha}) \underset{\phi\to 0^+}{\propto} \phi^{-1}$ in this case, instead of $\Gamma(n\phi+\frac{k+1}{\alpha}) \underset{\phi\to 0^+}{\propto} 1$.

A.4 Proof of Theorem 2

Let $\alpha\in\mathbb{R}^+$ be fixed. Suppose that hypothesis of item $i)$ hold, that is, $\pi(\mu)\gtrsim\mu^k$ with $k<-1$. Notice that, for $0<\phi\leq-\frac{(k+1)}{n\alpha}$ we have $n\alpha\phi+k\leq-1$. Moreover, for every $\alpha>0$ fixed we have $\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} \underset{\mu\to 0^+}{\propto} 1$. Hence, from Proposition 3 we have

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu \gtrsim \int_0^1 \mu^{n\alpha\phi+k}d\mu=\infty,$$

for all $\phi\in(0,-\frac{(k+1)}{n\alpha}]$. Therefore

$$\begin{aligned} d(\boldsymbol{x};\alpha) &\gtrsim \int_0^{-\frac{(k+1)}{n\alpha}} \frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \int_0^\infty \mu^{n\alpha\phi+k}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\, d\phi \\ &\gtrsim \int_0^{-\frac{(k+1)}{n\alpha}} \infty\, d\phi = \infty, \end{aligned}$$

that is, $d(\boldsymbol{x};\alpha)=\infty$.

Now suppose that the hypothesis of $ii)$ holds, first suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$, where $k>-1$ and $n<-r_0-1$. Then, following the same steps that resulted in (A.28) we have

$$d(\boldsymbol{x};\alpha)\gtrsim\int_0^1 \phi^{n+r_0}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi \propto \int_0^1 \phi^{n+r_0}\,d\phi=\infty$$

and therefore $d(\boldsymbol{x};\alpha)=\infty$.

The case where $k=-1$, and $n<-r_0$ follows analogously

A.5 Proof of Theorem 4

Let $\phi \in \mathbb{R}^+$ be fixed. Since $\pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\}\pi(\mu)\mu^{n\alpha\phi-1}\times$ $\exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$\begin{aligned} d(\boldsymbol{x};\phi) &= \int_{\mathcal{A}} \pi(\alpha)\alpha^n \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\boldsymbol{\theta} \\ &= \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n \left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\alpha. \end{aligned} \tag{A.29}$$

Now, since $\pi(\mu) \lesssim \mu^{-1}$ by hypothesis it follows that

$$\begin{aligned} d(\boldsymbol{x};\phi) &\lesssim \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n \left(\prod_{i=1}^n x_i^{\alpha}\right)^{\phi}\mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\alpha \\ &= \int_0^\infty \pi(\alpha)\alpha^{n-1}\frac{\left(\prod_{i=1}^n x_i^\alpha\right)^\phi}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}\, d\alpha = \int_0^\infty \pi(\alpha)\alpha^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha \end{aligned}$$

where $\mathrm{q}(\alpha)$ is given in Proposition 5. Therefore, from the proportionalities in Proposition 5 it follows that

$$\begin{aligned} d(\boldsymbol{x};\phi) &\lesssim \int_0^\infty \pi(\alpha)\alpha^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha \\ &\propto \int_0^1 \alpha^{q_0+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha + \int_1^\infty \alpha^{q_\infty+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha = s_1(\boldsymbol{x};\phi) + s_2(\boldsymbol{x};\phi). \end{aligned} \tag{A.30}$$

where $s_1(\boldsymbol{x};\phi)$ and $s_2(\boldsymbol{x};\phi)$ denote the respective two real numbers in the sum that precedes it. It follows that $d(\boldsymbol{x};\phi) < \infty$ if $s_1(\boldsymbol{x};\phi) < \infty$ and $s_2(\boldsymbol{x};\phi) < \infty$.

By Proposition 5, $\mathrm{q}(\alpha) > 0$, which implies that $e^{-n\,\mathrm{q}(\alpha)\phi} \leq 1$. Moreover, since $q_0+n > 0$ we have

$$s_1(\boldsymbol{x};\phi) = \int_0^1 \alpha^{q_0+n-1}e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha \quad \leq \int_0^1 \alpha^{q_0+n-1}\, d\alpha < \infty$$

Additionally, by Proposition 5, $\mathrm{q}(\alpha) \underset{\alpha\to\infty}{\propto} \alpha$ and therefore by Proposition 2 there exists $c > 0$ such that $\mathrm{q}(\alpha) \leq c\alpha$ for all $\alpha \in [1, \infty)$. Therefore,

$$s_2(\boldsymbol{x};\phi) = \int_1^\infty \alpha^{q_\infty+n-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha \leq \int_1^\infty \alpha^{q_\infty+n-1} e^{-n\phi c\alpha}\, d\alpha = \frac{\Gamma(q_\infty+n, n\phi c)}{(n\phi c)^{q_\infty+n}} < \infty,$$

hence, $d(\boldsymbol{x};\phi) < \infty$.

A.6 Proof of Theorem 5

Let $\phi \in \mathbb{R}^+$ be fixed. Suppose that $\pi(\mu) \gtrsim \mu^k$ where $k < -1$. Notice that, for $0 < \alpha \leq \frac{k+1}{n\phi}$ it follows that $n\phi + \frac{k+1}{\alpha} \leq 0$ and since $\exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \underset{\mu\to 0^+}{\propto} 1$, we have

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \gtrsim \int_0^1 \mu^{n\alpha\phi+k} d\mu = \infty,$$

for all $\alpha \in (0, \frac{k+1}{n\phi}]$. Therefore,

$$\begin{aligned} d(\boldsymbol{x};\phi) &\gtrsim \int_0^{\frac{k+1}{n\phi}} \pi(\alpha)\alpha^{n-1}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \int_0^1 \pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\alpha \\ &= \int_0^{\frac{k+1}{n\phi}} \infty\, d\alpha = \infty \end{aligned}$$

hence $d(\boldsymbol{x};\phi) = \infty$.

Now suppose that $\pi(\mu) \gtrsim \mu^k$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k > -1$ and $q_0 \in \mathbb{R}$. Then

$$\begin{aligned}
d(\boldsymbol{x};\phi) &\gtrsim \int_0^1\int_0^\infty \alpha^{n+q_0}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi+k}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\}d\mu\, d\alpha\\
&= \int_0^1\int_0^\infty \alpha^{n+q_0}\frac{\left(\prod_{i=1}^n x_i^\alpha\right)^\phi}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi+\frac{k+1}{\alpha}}}u^{n\phi+\frac{k+1}{\alpha}-1}e^{-u}\,du\,d\alpha =\\
&= \int_0^1\int_0^\infty \alpha^{n+q_0}\left(\prod_{i=1}^n x_i^\alpha\right)^{-\frac{k+1}{\alpha}} n^{-n\phi-\frac{k+1}{\alpha}}e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)}u^{n\phi+\frac{k+1}{\alpha}-1}e^{-u}\,du\,d\alpha\\
&= \int_0^\infty \left(\prod_{i=1}^n x_i\right)^{-(k+1)} n^{-n\phi}u^{n\phi-1}e^{-u}\int_0^1 \alpha^{n+q_0}e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)}e^{(\log u-\log n)\frac{k+1}{\alpha}}\,d\alpha\,du
\end{aligned}$$

where in the above we used the change of variables $u = \mu^\alpha\sum_{i=1}^n x_i^\alpha$ in the integral and $\mathrm{p}(\alpha)$ is given as in Proposition 5.

Now, since $\mathrm{p}(\alpha) \underset{\alpha\to 0^+}{\propto} \alpha^2$ from Proposition 5 it follows that $\lim_{\alpha\to 0^+} e^{-\mathrm{p}(\alpha)\left(n\phi+\frac{k+1}{\alpha}\right)} = \lim_{\alpha\to 0^+} e^{-\frac{p(\alpha)}{\alpha^2}(n\phi\alpha+k+1)\alpha} = e^0 = 1$. These two facts together applied to the above inequality lead to

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^\infty n^{-n\phi}\left(\prod_{i=1}^n x_i\right)^{-(k+1)} u^{n\phi-1}e^{-u}\int_0^1 \alpha^{n+q_0}e^{(\log u-\log n)\frac{k+1}{\alpha}}\,d\alpha\,du$$

Thus, since $n \geq 1$ and $\log u - \log n > 0$ for $u \geq 3n > e\cdot n$, and since $\int_0^1 \alpha^H e^{\frac{L}{\alpha}} = \infty$ for every $H \in \mathbb{R}$ and $L \in R^+$ (which can be easily checked via the change of variable $\beta = \frac{1}{\alpha}$ in the integral), it follows that

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^\infty n^{-n\phi}\left(\prod_{i=1}^n x_i\right)^{-(k+1)} u^{n\phi-1}e^{-u}\cdot\infty\,du = \infty, \tag{A.31}$$

and therefore $d(\boldsymbol{x};\phi) = \infty$.

Now suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k \leq -1$ and $n \leq -q_0$. Then, following the same steps that resulted in (A.30) we have

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^1 \alpha^{q_0+n-1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha.$$

but since by Proposition 5 we have $q(\alpha) \underset{\alpha\to 0^+}{\propto} 0$ it follows that $e^{-nq(\alpha)\phi} \underset{\alpha\to 0^+}{\propto} 1$ and therefore,

$$d(\boldsymbol{x};\phi) \gtrsim \int_0^1 \alpha^{q_0+n-1}\, d\alpha = \infty.$$

A.7 Proof of Theorem 7

Since $\pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi}\right\}\pi(\mu)\mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} \geq 0$ always, by Tonelli's theorem we have:

$$\begin{aligned} d(\boldsymbol{x}) &= \int_{\mathcal{A}} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\boldsymbol{\theta} \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left\{\prod_{i=1}^n x_i^{\alpha\phi-1}\right\}\pi(\mu)\mu^{n\alpha\phi}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha. \end{aligned}$$

Now, since $\pi(\mu) \lesssim \mu^{-1}$ we have

$$\begin{aligned} d(\boldsymbol{x}) &\lesssim \int_0^\infty\int_0^\infty\int_0^\infty \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \mu^{n\alpha\phi-1}\exp\left\{-\mu^\alpha\sum_{i=1}^n x_i^\alpha\right\} d\mu\, d\phi\, d\alpha \\ &= \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^{n-1}\frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^n x_i^\alpha\right)^\phi \frac{\Gamma(n\phi)}{\left(\sum_{i=1}^n x_i^\alpha\right)^{n\phi}}\, d\phi\, d\alpha \\ &= \int_0^\infty\int_0^\infty \pi(\alpha)\alpha^{n-1}\pi(\phi)\frac{\Gamma(n\phi)}{\Gamma(\phi)^n} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha \end{aligned}$$

where $\mathrm{q}(\alpha)$ is given in Proposition 5. Therefore, from the proportionalities in Proposition

5 it follows that

$$\begin{aligned}
d(\boldsymbol{x}) &\lesssim \int_0^\infty \int_0^\infty \pi(\alpha)\alpha^{n-1}\pi(\phi)\frac{\Gamma(n\phi)}{\Gamma(\phi)^n}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha \\
&\propto \int_0^1\int_0^1 f(\alpha,\phi)\,d\phi\,d\alpha + \int_1^\infty\int_0^1 f(\alpha,\phi)\,d\phi\,d\alpha + \int_0^1\int_1^\infty g(\alpha,\phi)\,d\phi\,d\alpha + \int_1^\infty\int_1^\infty g(\alpha,\phi)\,d\phi\,d\alpha \qquad \text{(A.32)} \\
&= s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x}),
\end{aligned}$$

where $f(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{n-1}e^{-n\,\mathrm{q}(\alpha)\phi}$, $g(\alpha,\phi) = \pi(\alpha)\alpha^{n-1}\pi(\phi)\phi^{\frac{n-1}{2}}e^{-n\,\mathrm{p}(\alpha)\phi}$ and $s_1(\boldsymbol{x})$, $s_2(\boldsymbol{x})$, $s_3(\boldsymbol{x})$ and $s_4(\boldsymbol{x})$ denote the respective four real numbers in the sum that precedes it. It follows that $d(\boldsymbol{x}) < \infty$, if and only if $s_1(\boldsymbol{x}) < \infty$, $s_2(\boldsymbol{x}) < \infty$, $s_3(\boldsymbol{x}) < \infty$ and $s_4(\boldsymbol{x}) < \infty$. Now, using the proportionalities in Proposition 5 it follows that

$$\begin{aligned}
s_1(\boldsymbol{x}) &\lesssim \int_0^1 \alpha^{q_0+n-1}\int_0^1 \phi^{n+r_0-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha \\
&= \int_0^1 \alpha^{q_0+n-1}\frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\,d\alpha \propto \int_0^1 \alpha^{q_0+n-1}\,d\alpha < \infty,
\end{aligned}$$

where in the last inequality the condition $n > -q_0$ was used, and in the equality that precedes it the condition $n > -r_0$ was used to ensure that $\gamma(n+r_0, n\,\mathrm{q}(\alpha))$ is well defined and that the equality holds,

$$\begin{aligned}
s_2(\boldsymbol{x}) &\lesssim \int_1^\infty \alpha^{q_\infty+n-1}\int_0^1 \phi^{n+r_0-1}e^{-n\,\mathrm{q}(\alpha)\phi}\,d\phi\,d\alpha \\
&= \int_1^\infty \alpha^{q_\infty+n-1}\frac{\gamma(n+r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n+r_0}}\,d\alpha \propto \int_1^\infty \alpha^{q_\infty-r_0-1}\,d\alpha < \infty,
\end{aligned}$$

where just as in the $s_1(\boldsymbol{x})$ case, the condition $n > -r_0$ was used in order for the above equality to hold,

$$s_3(\boldsymbol{x}) \lesssim \int_0^1 \alpha^{q_0+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_0^1 \alpha^{q_0+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{q_0-2r_\infty-2}\, d\alpha < \infty,$$

where in the last inequality the condition $q_0 > 2r_\infty + 1$ was used, and finally

$$s_4(\boldsymbol{x}) \lesssim \int_1^\infty \alpha^{q_\infty+n-1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2}-1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_1^\infty \alpha^{q_\infty+n-1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty+n-2} e^{-nk\alpha}\, d\alpha < \infty,$$

where in the above $k \in \mathbb{R}^+$ is given in Proposition 5. Therefore, from $s_i(\boldsymbol{x}) < \infty, i = 1, \ldots, 4$, we have $d = s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x}) < \infty$.

A.8 Proof of Theorem 8

Suppose that the hypothesis of item i) holds.

First suppose that $\pi(\mu) \gtrsim \mu^k$ with $k < -1$. Denoting $h = \sqrt{\frac{-k-1}{2n}} > 0$, it follows that for $0 < \alpha \le h$ and $0 < \phi \le h$ we have $n\alpha\phi + k \le nh^2 + k = \frac{(k-1)}{2} < -1$. Moreover, for every $\alpha > 0$ fixed we have $\exp\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\} \underset{\mu \to 0^+}{\propto} 1$, hence, from Proposition 3 we have

$$\int_0^\infty \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^\alpha \sum_{i=1}^n x_i^\alpha\right\} d\mu \gtrsim \int_0^\infty \mu^{n\alpha\phi+k} = \infty,$$

for all fixed $\alpha \in (0, h]$ and $\phi \in (0, h]$. Therefore,

$$d(\boldsymbol{x}) \gtrsim \int_{h/2}^{h}\int_{h/2}^{h} \pi(\alpha)\alpha^n \frac{\pi(\phi)}{\Gamma(\phi)^n}\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{\phi} \int_0^{\infty} \mu^{n\alpha\phi+k} \exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\phi\, d\alpha$$
$$\propto \int_{h/2}^{h}\int_{h/2}^{h} \infty\, d\phi\, d\alpha = \infty,$$

that is, $d(\boldsymbol{x}) = \infty$.

Now suppose that $\pi(\mu) \underset{\mu\to\infty}{\gtrsim} \mu^k$ and $\pi(\alpha) \underset{\alpha\to 0^+}{\gtrsim} \alpha^{q_0}$, where $k > -1$ and $q_0 \in \mathbb{R}$. Under these hypotheses, in equation (A.31) it was proved that

$$d(\boldsymbol{x};\phi) \propto \int_0^{\infty}\int_0^{\infty} \pi(\alpha)\alpha^n \left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\alpha = \infty$$

for every $\phi > 0$, and therefore,

$$d(\boldsymbol{x}) \propto \int_0^{\infty} \frac{\pi(\phi)}{\Gamma(\phi)^n} \int_0^{\infty}\int_0^{\infty} \pi(\alpha)\alpha^n \left\{\prod_{i=1}^{n} x_i^{\alpha\phi-1}\right\} \pi(\mu)\mu^{n\alpha\phi} \exp\left\{-\mu^{\alpha}\sum_{i=1}^{n} x_i^{\alpha}\right\} d\mu\, d\alpha\, d\phi$$
$$= \int_0^{\infty} \frac{\pi(\phi)}{\Gamma(\phi)^n} \cdot \infty\, d\phi = \infty$$

and thus $d(\boldsymbol{x}) = \infty$.

Suppose on the other hand that the hypotheses of ii) hold. Since $\pi(\mu) \gtrsim \mu^{-1}$, following the same steps that resulted in (A.32) and the same expressions for $s_i(\boldsymbol{x})$, where $i = 1, \cdots, 4$, we have $d(\boldsymbol{x}) \gtrsim s_1(\boldsymbol{x}) + s_2(\boldsymbol{x}) + s_3(\boldsymbol{x}) + s_4(\boldsymbol{x})$. We now divide the proof that $d(\boldsymbol{x}) = \infty$ in four cases:

- Suppose that $\pi(\phi) \underset{\phi\to 0^+}{\gtrsim} \phi^{r_0}$ and $\pi(\alpha) \underset{\alpha\to\infty}{\gtrsim} \alpha^{q_\infty}$ with $n \leq -r_0$. Then,

$$s_2(\boldsymbol{x}) \gtrsim \int_1^{\infty} \alpha^{q_\infty+n-1} \int_0^{1} \phi^{n+r_0-1} e^{-n\,q(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_1^{\infty} \alpha^{q_\infty+n-1} \cdot \infty\, d\alpha = \infty$$

which implies $d(\boldsymbol{x}) = \infty$.

- Suppose that $\pi(\phi) \underset{\phi \to 0^+}{\gtrsim} \phi^{r_0}$ and $\pi(\alpha) \underset{\alpha \to \infty}{\gtrsim} \alpha^{q_\infty}$ with $q_\infty \geq r_0$ and $n > -r_0$. Then,

$$s_2(\boldsymbol{x}) \gtrsim \int_1^\infty \alpha^{q_\infty + n - 1} \int_0^1 \phi^{n + r_0 - 1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_1^\infty \alpha^{q_\infty + n - 1} \frac{\gamma(n + r_0, n\,\mathrm{q}(\alpha))}{(n\,\mathrm{q}(\alpha))^{n + r_0}}\, d\alpha \propto \int_1^\infty \alpha^{q_\infty - r_0 - 1}\, d\alpha = \infty$$

which implies $d(\boldsymbol{x}) = \infty$.

- Suppose that $\pi(\alpha) \underset{\alpha \to 0^+}{\gtrsim} \alpha^{q_0}$ and $\pi(\phi) \underset{\phi \to \infty}{\gtrsim} \phi^{r_\infty}$ with $n \leq -q_0$. Then, by Proposition 5 we have $q(\alpha) \underset{\alpha \to 0^+}{\propto} 0$ from where it follows that $e^{-nq(\alpha)\phi} \underset{\alpha \to 0^+}{\propto} 1$ and therefore,

$$s_1(\boldsymbol{x}) \gtrsim \int_0^1 \pi(\phi)\phi^{n-1} \int_0^1 \alpha^{q_0 + n - 1} e^{-n\,\mathrm{q}(\alpha)\phi}\, d\alpha\, d\phi$$
$$\propto \int_0^1 \pi(\phi)\phi^{n-1} \int_0^1 \alpha^{q_0 + n - 1}\, d\alpha\, d\phi = \int_0^1 \pi(\phi)\phi^{n-1} \cdot \infty\, d\phi = \infty,$$

which implies $d(\boldsymbol{x}) = \infty$.

- Suppose that $\pi(\alpha) \underset{\alpha \to 0^+}{\gtrsim} \alpha^{q_0}$ and $\pi(\phi) \underset{\phi \to \infty}{\gtrsim} \phi^{r_\infty}$ with $2r_\infty + 1 \geq q_0$. Then,

$$s_3(\boldsymbol{x}) \gtrsim \int_0^1 \alpha^{q_0 + n - 1} \int_1^\infty \phi^{\frac{n+1+2r_\infty}{2} - 1} e^{-n\,\mathrm{p}(\alpha)\phi}\, d\phi\, d\alpha$$
$$= \int_0^1 \alpha^{q_0 + n - 1} \frac{\Gamma(\frac{n+1+2r_\infty}{2}, n\,\mathrm{p}(\alpha))}{(n\,\mathrm{p}(\alpha))^{\frac{n+1+2r_\infty}{2}}}\, d\alpha \propto \int_0^1 \alpha^{q_0 - 2r_\infty - 2}\, d\alpha = \infty$$

which implies $d(\boldsymbol{x}) = \infty$.

Therefore, the proof is completed.

A.9 Simulation Study

Here, we presented a simulation study using the Monte Carlo method to compare the posterior estimates using different priors for the parameters of Stacy's model. This is conducted by using the absolute Bias and the MSE (mean square error) given by,

$$\text{Bias}\left(\hat{\theta}_w\right) = \frac{1}{N}\sum_{j=1}^{N}\left|\hat{\theta}_{w,j} - \theta_w\right| \quad \text{and} \quad \text{MSE}\left(\hat{\theta}_w\right) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_{w,j} - \theta_w)^2,$$

for $w = 1, 2, 3$, where $N = 5,000$ corresponds to the Monte Carlo realization numbers. Additionally, the coverage probabilities (CPs) of the Bayesian 95% HPDIs, assuming 95% of credibility. Under this approach, the selected posterior estimator should provide Bias and MSE closer to one, while the relative frequencies which contain the true parameter values should be close to 0.95.

As argued by Bernardo (2005), the use of simple proper priors as non-informative often hides significant unwarranted assumptions, which may easily dominate, or even invalidate the statistical analysis. Hence, we assume simple proper priors where $\theta_i \sim$ Uniforme$(0, 40)$, $i = 1, \ldots, 3$ or $\theta_i \sim$ Gamma$(0.01, 0.01)$. The posterior estimates were obtained using OPENBUGS, and the codes are attached in the Supplemental Material. Tables 3 and 4 displays the Bias, MSE and CP of the posterior estimates using objective and flat proper priors.

We observe from the results above that flat priors with vague information can affect the posterior estimates, especially for small sample sizes. For instance, in the case of $n = 50$ and $\mu = 0.5$, we obtained a Bias of 7.001 with a MSE of 88.485, which is undesirable. On the other hand, the estimates obtained through the reference posterior returned more

Table 3: Bias and MSEs obtained from the posterior estimates assuming $\theta_i \sim \text{Uniform}(0, 40)$, $\theta_i \sim \text{Gamma}(0.01, 0.01)$ and the reference prior.

		Bias(MSE)		
Prior distribution	$\boldsymbol{\theta}$	$n = 50$	$n = 100$	$n = 200$
$\theta_i \sim \text{Uniform}(0, 40)$	ϕ	2.580(11.582)	0.899(2.594)	0.244(0.187)
	μ	7.015(88.546)	1.823(15.939)	0.176(0.398)
	α	2.409(12.950)	1.218(3.515)	0.637(0.722)
$\theta_i \sim \text{Gamma}(0.01, 0.01)$	ϕ	0.664(1.117)	0.322(0.292)	0.164(0.059)
	μ	1.301(8.102)	0.315(1.204)	0.076(0.052)
	α	4.017(102.405)	1.323(13.471)	0.619(0.739)
Reference prior	ϕ	0.257(0.222)	0.157(0.055)	0.102(0.019)
	μ	0.275(0.693)	0.131(0.059)	0.081(0.013)
	α	1.929(7.503)	1.244(2.787)	0.882(1.330)

accurate estimates in terms of Bias and MSE values for all parameters, mainly for small and moderate sample sizes. The Bayesian intervals obtained from the reference posterior also provided good CPs close to the nominal level 0.95, even for small sample sizes. We did not compare the reference prior with other priors such as Jeffreys' or MDIP priors since they returned improper posteriors. We can conclude that the Bayes estimators using the reference prior should be considered for applications.

Table 4: CPs obtained from the posterior estimates assuming $\theta_i \sim \text{Uniform}(0, 40)$, $\theta_i \sim \text{Gamma}(0.01, 0.01)$ and the reference prior.

		Coverage Probabilities		
Prior distribution	$\boldsymbol{\theta}$	$n = 50$	$n = 100$	$n = 200$
$\theta_i \sim \text{Uniform}(0, 40)$	ϕ	0.7224	0.8832	0.9210
	μ	0.7178	0.8802	0.9230
	α	0.7250	0.8824	0.9208
$\theta_i \sim \text{Gamma}(0.01, 0.01)$	ϕ	0.9366	0.9360	0.9446
	μ	0.9370	0.9374	0.9472
	α	0.9376	0.9382	0.9434
Reference prior	ϕ	0.9656	0.956	0.9522
	μ	0.9652	0.9524	0.9542
	α	0.9550	0.9564	0.9478