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Title	Multiple Quantile Modelling via Reduced Rank Regression
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apparent relation between the parameters corresponding to different quantile levels. Because the ϵ_τ's at different τ values are correlated with each other, a generalized estimating equation (GEE) principle can be used to combine the individual quantile level estimating functions in a more efficient way. Specifically, let $\boldsymbol{\psi}(Y_i - \mathbf{a} - \mathbf{B}^{\mathrm{T}}\mathbf{X}_i) \equiv \{\psi_{\tau_1}(Y_i - a_1 - \mathbf{X}_i^{\mathrm{T}}\boldsymbol{\beta}_1), \ldots, \psi_{\tau_K}(Y_i - a_K - \mathbf{X}_i^{\mathrm{T}}\boldsymbol{\beta}_K)\}^{\mathrm{T}}$ and let $\mathbf{V}$ be the variance-covariance matrix of $\boldsymbol{\psi}$, i.e. let $\mathbf{V}$ be a $K \times K$ matrix with the (k, k') entry $\mathbf{V}_{k,k'} = \min(\tau_k, \tau_{k'}) - \tau_k\tau_{k'}$. Then we can solve

$$\sum_{i=1}^{n} (\mathbf{f}_i \otimes \mathbf{Z}_i)\mathbf{V}^{-1}\boldsymbol{\psi}(Y_i - \mathbf{a} - \mathbf{B}^{\mathrm{T}}\mathbf{X}_i) = \mathbf{0} \tag{4}$$

to obtain a more efficient estimator of $\mathbf{a}$ and $\mathbf{B}$ under multiple linear quantile regression without any constraints, where $\otimes$ is the Kronecker product. Here $\mathbf{f}_i \equiv \mathrm{diag}\{f_{\epsilon_{\tau_k}|\mathbf{X}}(0, \mathbf{X}_i), k = 1, \ldots, K\}$. Although (4) yields a more efficient estimator, it is not a very popular choice even in the single linear quantile regression literature. This is because the estimation involves the conditional pdf $f_{\epsilon_\tau|\mathbf{X}}(0, \mathbf{X})$, the estimation of which usually involves nonparametric methods and is nearly impossible when the dimension of $\mathbf{X}$ is large.

One compromise, following the general idea of using a "working model", is to replace the hard to estimate $\mathbf{f}_i$ quantity with a guessed model $\mathbf{f}_i^*$, and calculate an estimator from the estimating equation

$$\sum_{i=1}^{n} (\mathbf{f}_i^* \otimes \mathbf{Z}_i)\mathbf{V}^{-1}\boldsymbol{\psi}(Y_i - \mathbf{a} - \mathbf{B}^{\mathrm{T}}\mathbf{X}_i) = \mathbf{0}. \tag{5}$$

If the guessed model $\mathbf{f}_i^*$ happens to be correct, we obtain an estimator as efficient as from (4). But even if we guessed the model wrong, we still can have a consistent estimator.

In the special case when we used a uniform model for $\mathbf{f}_i^*$, we actually obtain a GEE improved version. To see this, we first recognize that minimizing the check function at a single quantile level can be reexpressed as solving an estimating equation

$$\sum_{i=1}^{n} \psi_\tau(Y_i - a_k - \mathbf{X}_i^{\mathrm{T}}\boldsymbol{\beta}_k)\mathbf{Z}_i = \mathbf{0}.$$

Thus, we can follow the same "GEE principle" that we used to obtain (4) to take into account the correlation between the K sets of such estimating functions and form the estimating equation

$$\sum_{i=1}^{n} (\mathbf{I}_K \otimes \mathbf{Z}_i)\mathbf{V}^{-1}\boldsymbol{\psi}(Y_i - \mathbf{a} - \mathbf{B}^{\mathrm{T}}\mathbf{X}_i) = \mathbf{0}, \tag{6}$$