Spatial subsample bootstraps for high-dimensional spatial data with irregular spacing

Yue Li, Shibin Zhang*

Shanghai Normal University

Supplementary Material

This supplement contains additional simulation results and proofs of the main results. The contents are summarized as below.

- Section A reports additional simulation results.
- Section B gives illustrative examples satisfying Assumptions 1 and 2.
- Section C provides the proof of Lemma 1.
- Section D presents the proof of Theorem 1.
- Section E includes proofs of Proposition 1 and Theorems 2 and 3.
- Section F contains proofs of Proposition 2 and Theorem 4.

Notation. For any set $A \subset \mathbb{R}^d$, let $|A|$, $[\![A]\!]$ denote the Lebesgue measure of A and the number of elements of A, respectively. For any vector $\mathbf{x} = (x_1, \ldots, x_p)^\top \in \mathbb{R}^p$, $\|\mathbf{x}\|_\infty = \max_{1\le j\le p} |x_j|$. For any $p \times p$ matrix $\mathbf{X} = (x_{ij})_{p\times p}$, $\|\mathbf{X}\|_{\max} = \max_{1\le i,j\le p} |x_{ij}|$. Let $\mathbb{I}_A(\cdot)$ denote the indicator function and let $\mathbf{1}$ denote the vector whose entries are

all equal to 1. For any positive sequences a_n, b_n, we write $a_n \lesssim b_n$ if $a_n \leq C\, b_n$ for some constant $C > 0$, $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. For any $a, b \in \mathbb{R}$, let $a \wedge b = \min\{a, b\}$. Let X be real-valued random variable. Let $\mathrm{P}_{\cdot|\mathbf{s}}$, $\mathrm{E}_{\cdot|\mathbf{s}}$ and $\mathrm{Var}_{\cdot|\mathbf{s}}$ denote the conditional probability, expectation and variance given the sampling locations $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$, respectively. Let $\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}$, $\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}$ and $\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}$ denote the conditional probability, expectation and variance given the sampling locations $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ and the observations $\{\mathbf{Y}(\mathbf{s}_1), \ldots, \mathbf{Y}(\mathbf{s}_n)\}$, respectively. For a random variable X, define the ψ_1-norm as $\|X\|_{\psi_1} = \inf\{c > 0 : \mathrm{E}[\exp(|X|/c)] \leq 2\}$, and the ψ_2-norm as $\|X\|_{\psi_2} = \inf\{c > 0 : \mathrm{E}[\exp(X^2/c^2)] \leq 2\}$.

General remark. Without loss of generality, we assume $\boldsymbol{\mu} = \mathrm{E}[\mathbf{Y}(\mathbf{u})] = \mathbf{0}$ throughout this supplement.

A Additional simulation results

The data generation process and simulation settings are identical to those described in Section 6.1 of the main article.

Tables A.1 – A.3 are the empirical probabilities (in %) of the event $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for the seven competing approaches (SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA) under the (UL) design. Specifically, Tables A.1, A.2, and A.3 correspond to $(n, p) = (50, 50)$, $(n, p) = (50, 400)$, and $(n, p) = (100, 400)$, respectively.

To provide a graphical assessment of the sensitivity of the competing methods

Table A.1: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (UL) design, with $n = 50$ and $p = 50$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	90.4	91.0	86.2	90.8	95.7	97.5	96.6	96.4
	SMSB-R	91.6	92.5	85.6	89.6	95.6	97.5	96.8	96.1
	SMSB-M	92.2	92.6	85.3	90.0	96.1	97.5	97.1	96.5
	SESB	91.1	88.6	69.2	89.1	95.5	97.1	94.7	96.1
	SDWB	91.6	90.4	88.5		96.7	96.5	94.7	
	GBBB	92.3	98.6	79.6		95.4	94.2	98.6	
	WIA	96.8	92.6	90.0		98.6	99.9	99.5	
$(1/2, 0, 0)$	SMSB-G	82.6	85.9	80.5	83.0	92.0	95.8	96.0	92.7
	SMSB-R	81.0	85.1	79.0	83.2	92.0	96.0	95.9	92.8
	SMSB-M	81.7	85.4	79.6	82.8	92.0	96.4	95.7	92.5
	SESB	79.0	78.3	60.3	81.3	91.8	95.3	91.9	92.4
	SDWB	81.1	81.9	79.9		94.2	95.2	93.1	
	GBBB	87.0	96.7	71.6		94.2	93.3	98.5	
	WIA	95.5	93.9	88.6		97.4	99.1	98.0	
$(1/3, 1/2, 0)$	SMSB-G	93.8	93.4	90.0	93.7	97.7	98.7	98.1	97.9
	SMSB-R	93.2	93.2	88.9	93.4	97.7	98.6	98.1	97.7
	SMSB-M	93.9	93.3	89.0	93.5	98.0	98.6	98.2	97.9
	SESB	92.7	90.1	75.6	92.4	98.0	98.4	97.5	97.7
	SDWB	94.7	92.8	91.8		98.1	97.7	97.2	
	GBBB	96.5	99.3	84.4		97.0	95.5	99.2	
	WIA	98.6	95.8	93.0		99.5	99.7	99.7	
$(1/2, 1/2, 0)$	SMSB-G	87.9	89.8	84.2	88.2	95.4	97.6	97.4	95.7
	SMSB-R	86.5	89.5	82.5	87.9	95.3	97.4	97.2	96.0
	SMSB-M	87.0	89.5	83.4	87.2	95.5	97.5	97.0	95.5
	SESB	84.7	82.1	63.7	86.1	95.4	97.4	95.2	95.5
	SDWB	86.6	86.6	84.4		96.6	96.4	95.2	
	GBBB	90.5	97.9	76.8		96.3	95.4	98.7	
	WIA	96.7	95.8	92.8		98.9	99.8	99.1	
$(1/3, 1/2, 0.05)$	SMSB-G	89.9	90.2	84.7	90.3	95.6	98.1	96.6	96.8
	SMSB-R	89.5	89.8	83.0	89.7	96.2	98.2	96.5	96.0
	SMSB-M	90.2	89.8	83.2	89.8	96.0	98.1	96.7	96.6
	SESB	88.9	84.9	66.6	88.1	95.8	97.8	93.4	96.3
	SDWB	91.1	90.0	88.1		96.2	95.5	93.5	
	GBBB	91.9	98.4	79.3		94.9	93.1	98.0	
	WIA	97.0	93.6	89.8		98.6	99.7	99.2	
$(1/2, 1/2, 0.05)$	SMSB-G	76.8	80.8	72.7	78.3	83.6	92.0	90.5	85.6
	SMSB-R	76.0	80.6	71.8	77.1	84.1	91.5	90.2	85.3
	SMSB-M	77.0	80.4	73.0	77.9	83.5	91.9	90.0	85.8
	SESB	72.9	71.1	49.6	74.7	83.6	89.5	82.9	84.3
	SDWB	76.6	75.7	71.6		87.3	87.6	83.8	
	GBBB	81.0	95.0	61.9		88.1	85.4	95.7	
	WIA	95.1	92.7	87.5		95.8	98.7	97.5	

Table A.2: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (UL), with $n = 50$ and $p = 400$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	95.4	96.6	95.4	94.9	95.9	97.3	97.8	96.3
	SMSB-R	94.9	96.6	95.3	94.5	96.0	97.5	98.0	96.2
	SMSB-M	94.8	96.7	96.2	94.4	95.9	97.5	98.2	96.6
	SESB	93.3	91.7	77.4	92.9	95.0	96.4	94.2	96.0
	SDWB	96.9	96.3	96.1		98.1	97.9	98.1	
	GBBB	95.8	99.9	83.5		98.2	97.8	99.8	
	WIA	87.5	87.5	88.6		96.1	94.2	92.1	
$(1/2, 0, 0)$	SMSB-G	86.2	91.6	90.5	86.3	91.8	96.6	97.9	93.7
	SMSB-R	84.9	90.6	90.5	85.7	92.4	96.9	97.6	93.3
	SMSB-M	85.8	90.9	90.4	86.2	92.6	97.1	97.5	93.8
	SESB	81.2	81.8	64.6	80.8	91.2	94.7	92.6	92.1
	SDWB	89.1	92.4	92.3		96.6	97.4	97.1	
	GBBB	88.9	99.1	72.9		96.9	96.9	99.8	
	WIA	76.7	78.2	83.7		92.7	91.6	90.4	
$(1/3, 1/2, 0)$	SMSB-G	97.5	98.2	96.6	97.7	98.5	99.3	99.1	98.9
	SMSB-R	96.9	97.7	96.8	97.5	98.3	99.0	98.9	98.5
	SMSB-M	97.1	97.9	97.2	97.2	98.7	99.3	99.2	98.8
	SESB	97.0	95.7	88.8	96.8	98.2	98.4	97.0	98.7
	SDWB	98.5	98.5	97.9		99.3	99.2	99.1	
	GBBB	98.2	99.8	90.7		99.5	99.1	100.0	
	WIA	96.1	94.9	93.5		98.5	98.0	96.3	
$(1/2, 1/2, 0)$	SMSB-G	93.5	95.9	95.1	93.6	96.3	97.6	97.9	96.4
	SMSB-R	92.4	95.7	95.2	93.2	96.0	97.5	98.1	96.6
	SMSB-M	92.9	95.4	94.7	93.4	95.9	97.5	98.1	96.5
	SESB	90.2	91.1	75.3	90.8	95.6	96.9	94.8	96.2
	SDWB	94.2	95.6	95.2		98.1	98.3	97.8	
	GBBB	94.6	99.2	81.9		98.3	97.8	100.0	
	WIA	89.4	87.0	88.4		96.3	95.9	94.7	
$(1/3, 1/2, 0.05)$	SMSB-G	95.4	95.9	95.4	95.6	95.6	97.3	97.3	96.1
	SMSB-R	95.2	95.8	95.1	94.6	95.6	96.8	96.9	95.9
	SMSB-M	95.3	95.5	94.5	95.0	95.4	96.9	97.3	95.9
	SESB	94.5	91.9	80.0	94.1	95.1	96.0	93.6	95.8
	SDWB	96.7	96.8	95.4		97.5	97.4	96.8	
	GBBB	96.5	99.6	84.0		98.3	97.4	99.7	
	WIA	92.7	90.7	89.3		95.9	93.2	92.5	
$(1/2, 1/2, 0.05)$	SMSB-G	79.8	85.5	84.3	80.5	84.9	91.5	93.0	87.4
	SMSB-R	79.1	85.5	83.8	79.5	84.8	91.9	92.8	85.7
	SMSB-M	79.5	85.2	84.5	81.0	84.7	91.8	92.8	87.1
	SESB	75.2	75.5	57.9	76.7	83.7	88.3	81.7	84.4
	SDWB	84.0	87.2	85.5		90.3	92.1	90.4	
	GBBB	86.1	98.1	65.5		93.7	91.8	98.6	
	WIA	76.9	77.8	79.0		86.4	84.9	81.6	

Table A.3: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (UL), with $n = 100$ and $p = 400$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	95.0	97.0	96.1	93.0	96.5	97.6	97.9	96.1
	SMSB-R	94.8	96.2	96.1	92.3	96.0	97.8	97.8	96.5
	SMSB-M	94.2	96.9	96.4	91.9	96.0	97.8	98.1	96.1
	SESB	92.2	91.0	76.8	91.1	95.2	96.7	93.7	95.7
	SDWB	97.1	97.3	97.0		97.8	98.2	98.4	
	GBBB	94.2	99.6	79.7		97.7	97.3	99.6	
	WIA	86.9	90.2	88.8		94.7	93.8	93.4	
$(1/2, 0, 0)$	SMSB-G	83.7	92.2	91.8	83.9	94.2	97.6	97.8	94.6
	SMSB-R	81.7	90.6	90.3	80.3	93.6	97.1	97.7	94.2
	SMSB-M	83.2	91.0	90.5	82.5	93.7	97.1	97.6	94.3
	SESB	78.3	80.1	62.3	76.7	92.9	95.8	92.5	93.7
	SDWB	86.9	91.9	93.5		96.8	98.1	98.3	
	GBBB	84.4	99.2	67.0		96.1	96.1	99.6	
	WIA	73.4	80.3	81.7		90.5	93.0	92.1	
$(1/3, 1/2, 0)$	SMSB-G	97.8	98.7	97.1	97.4	98.4	98.6	98.6	98.3
	SMSB-R	96.9	97.9	97.4	96.5	98.1	98.6	98.3	98.1
	SMSB-M	97.5	98.4	97.6	96.9	98.5	98.7	98.8	98.2
	SESB	96.2	95.0	87.6	96.4	98.2	97.9	96.4	98.4
	SDWB	98.6	98.6	98.5		99.2	99.2	99.3	
	GBBB	98.3	99.7	89.2		99.1	98.7	100.0	
	WIA	94.1	94.0	92.7		97.4	97.0	96.4	
$(1/2, 1/2, 0)$	SMSB-G	90.7	94.2	94.2	90.4	96.1	98.3	98.3	96.5
	SMSB-R	90.7	94.1	93.4	89.7	96.0	98.0	98.2	96.6
	SMSB-M	90.5	94.3	93.9	90.1	95.8	98.1	98.4	96.5
	SESB	88.3	88.5	73.7	87.8	95.1	96.7	94.8	95.4
	SDWB	91.8	93.9	94.7		97.3	97.9	97.9	
	GBBB	90.6	99.5	75.8		97.1	97.5	99.6	
	WIA	84.9	86.4	86.5		95.6	95.9	94.3	
$(1/3, 1/2, 0.05)$	SMSB-G	93.6	95.0	93.6	93.0	93.2	95.0	94.3	92.9
	SMSB-R	92.2	94.4	93.6	91.6	92.9	94.6	94.1	92.2
	SMSB-M	93.2	94.7	93.5	91.7	93.1	94.7	94.4	92.7
	SESB	91.3	89.6	74.9	90.7	92.3	92.6	87.6	92.7
	SDWB	95.0	95.9	94.7		94.7	95.8	95.6	
	GBBB	93.2	99.5	76.2		95.1	93.9	99.0	
	WIA	88.2	89.2	86.2		91.4	90.9	86.9	
$(1/2, 1/2, 0.05)$	SMSB-G	75.1	84.4	82.5	73.5	69.1	82.9	85.7	72.6
	SMSB-R	73.1	83.2	81.6	72.0	68.1	81.4	85.0	70.5
	SMSB-M	73.6	82.9	82.8	73.3	70.2	83.2	85.0	71.6
	SESB	68.0	69.2	46.2	67.3	67.8	75.7	68.4	68.9
	SDWB	76.6	81.1	80.9		78.3	84.2	86.7	
	GBBB	76.4	97.4	49.9		82.8	80.6	97.6	
	WIA	68.3	71.9	69.4		68.6	73.7	73.5	

to the choice of subsample size, we have added Figure A-1 to the online supplementary material. The figure reports the empirical coverage probabilities of the event $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for the seven competing approaches, namely SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA, under the (UL) design, as a function of the subsample size $b \in \{4, 6, 8, 10, 12\}$ for $(n, p) = (100, 50)$ and $\lambda = 24$.

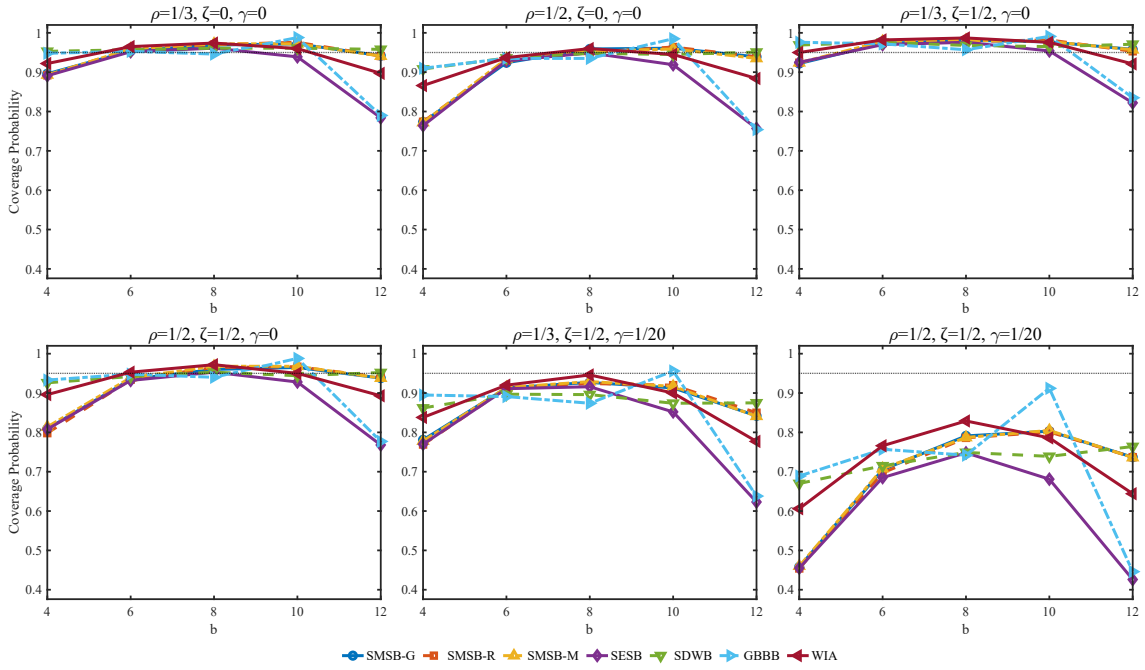

Figure A-1: Empirical coverage probabilities of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ as a function of the subsample size $b \in \{4, 6, 8, 10, 12\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (UL) design with $(n, p) = (100, 50)$ and $\lambda = 24$. The dashed horizontal line indicates the target coverage level of 95%. Each panel corresponds to a different parameter setting (ρ, ζ, γ).

The figure reveals three main patterns. First, under the settings with $\gamma = 0$, the proposed SMSB procedures generally attain coverage probabilities close to the nominal

level of 95% over a relatively broad range of subsample sizes, particularly for moderate values of b. Second, the coverage probabilities are generally non-monotone in b, and relatively large subsample sizes, particularly $b = 12$, can lead to noticeable undercoverage for several methods. Third, when $\gamma = 1/20$, the coverage probabilities deteriorate for all competing procedures, with the deterioration being more pronounced under the stronger dependence setting. Nevertheless, the proposed SMSB procedures remain competitive with the existing methods across the considered settings. Overall, the results illustrate that the choice of subsample size can have a substantial impact on finite-sample coverage and provide empirical support for the data-driven subsample-size selection procedure proposed in the paper.

Tables A.4–A.7 report the empirical probabilities (in %) of the event $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for the seven competitive approaches (SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA) under the (NUL) design. Tables A.4, A.5, A.6, and A.7 correspond to $(n, p) = (50, 50)$, $(n, p) = (100, 50)$, $(n, p) = (50, 400)$ and $(n, p) = (100, 400)$, respectively.

Table A.4: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (NUL) design, with $n = 50$ and $p = 50$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	93.4	92.2	92.7	93.6	97.3	98.7	98.3	97.8
	SMSB-R	92.7	91.8	92.4	92.9	97.3	98.4	98.6	97.5
	SMSB-M	92.6	92.3	92.5	92.7	97.1	98.6	98.3	97.5
	SESB	90.6	86.6	78.6	91.9	96.6	97.1	92.8	96.9
	SDWB	94.8	94.8	94.6		97.8	97.5	97.8	
	GBBB	91.9	98.9	78.1		96.2	95.4	99.0	
	WIA	95.2	88.1	87.0		95.3	98.2	94.8	
$(1/2, 0, 0)$	SMSB-G	86.6	90.1	91.9	86.6	94.3	97.4	98.5	96.2
	SMSB-R	85.0	90.5	92.0	85.3	94.5	97.4	98.6	95.6
	SMSB-M	85.4	90.4	91.5	85.4	94.8	97.3	98.4	96.1
	SESB	81.7	82.0	70.2	82.1	93.6	96.0	94.0	94.3
	SDWB	88.2	91.0	91.2		96.7	97.4	97.4	
	GBBB	85.2	96.6	68.4		94.1	93.7	99.2	
	WIA	88.3	80.5	82.0		92.9	96.9	91.7	
$(1/3, 1/2, 0)$	SMSB-G	96.6	95.8	96.8	96.8	98.8	99.4	99.1	98.7
	SMSB-R	96.1	95.6	95.4	96.4	98.8	99.2	99.2	98.7
	SMSB-M	96.6	95.5	95.4	96.5	98.7	99.0	99.1	98.7
	SESB	95.2	91.6	85.5	96.1	98.1	98.8	95.8	98.4
	SDWB	97.2	97.5	96.9		98.2	98.8	98.3	
	GBBB	94.9	98.4	83.8		97.1	95.8	99.0	
	WIA	98.4	92.9	91.3		97.5	99.4	96.9	
$(1/2, 1/2, 0)$	SMSB-G	89.6	92.2	93.3	89.7	96.6	98.1	97.8	97.5
	SMSB-R	88.9	91.8	92.3	89.1	96.2	98.0	98.1	97.2
	SMSB-M	89.2	90.9	92.4	89.3	96.2	97.9	98.4	97.4
	SESB	86.5	84.7	76.9	86.8	96.0	96.5	93.4	96.9
	SDWB	91.6	92.4	92.8		96.5	96.8	96.5	
	GBBB	87.9	96.7	74.1		95.7	94.8	98.7	
	WIA	92.2	86.7	85.1		95.8	97.6	95.1	
$(1/3, 1/2, 0.05)$	SMSB-G	93.0	90.0	89.1	93.7	94.9	96.5	95.7	95.3
	SMSB-R	92.6	89.7	88.9	93.0	94.7	96.4	95.4	95.6
	SMSB-M	92.6	89.6	89.6	93.1	94.8	96.6	95.7	95.5
	SESB	90.8	85.4	77.6	92.5	94.0	94.9	90.0	94.4
	SDWB	95.0	94.9	94.3		94.3	94.6	93.7	
	GBBB	91.8	98.0	76.7		94.5	92.8	97.2	
	WIA	96.4	88.7	87.0		92.7	95.7	91.3	
$(1/2, 1/2, 0.05)$	SMSB-G	80.9	83.4	85.8	80.8	84.6	91.6	93.3	87.5
	SMSB-R	78.9	84.4	85.2	80.1	84.0	91.3	92.9	87.3
	SMSB-M	79.8	84.0	85.8	79.7	84.3	91.4	93.5	87.5
	SESB	74.4	72.1	59.8	75.3	82.3	86.5	79.4	84.7
	SDWB	83.1	85.3	86.4		87.4	88.4	87.8	
	GBBB	80.5	94.5	61.5		85.2	83.4	94.7	
	WIA	85.6	74.8	72.2		81.9	90.8	79.6	

Table A.5: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (NUL) design, with $n = 100$ and $p = 50$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	95.6	97.2	97.0	94.4	94.9	97.2	97.3	95.6
	SMSB-R	95.1	96.7	96.6	94.1	95.1	97.0	97.3	95.4
	SMSB-M	95.7	96.7	96.5	93.9	95.1	96.8	97.3	95.6
	SESB	94.0	93.1	83.7	92.7	94.3	95.7	93.1	94.5
	SDWB	92.2	94.0	93.2		96.9	96.9	97.3	
	GBBB	88.2	96.7	73.0		94.2	92.9	98.9	
	WIA	93.5	93.5	90.1		95.0	96.3	96.1	
$(1/2, 0, 0)$	SMSB-G	87.6	93.8	93.8	86.2	91.5	94.9	96.4	93.0
	SMSB-R	86.8	93.2	92.9	85.4	91.9	95.2	96.5	92.6
	SMSB-M	87.7	93.0	93.1	85.6	91.9	95.2	96.0	92.7
	SESB	83.3	85.4	74.3	82.2	90.2	94.0	91.7	91.4
	SDWB	82.8	87.4	88.8		94.0	94.9	94.9	
	GBBB	78.6	94.6	62.9		91.3	90.8	98.1	
	WIA	80.3	83.8	82.0		89.9	92.0	92.0	
$(1/3, 1/2, 0)$	SMSB-G	98.3	98.4	98.3	97.5	96.4	97.7	97.8	96.9
	SMSB-R	98.0	98.8	98.0	97.5	96.4	97.7	97.3	96.7
	SMSB-M	97.8	98.5	98.3	97.4	96.4	98.0	97.7	96.9
	SESB	97.1	96.8	89.9	96.9	96.1	96.0	94.2	96.0
	SDWB	96.1	96.6	95.9		97.5	97.6	97.3	
	GBBB	91.3	98.0	78.1		97.4	95.6	99.3	
	WIA	97.1	96.7	94.7		97.0	97.6	96.6	
$(1/2, 1/2, 0)$	SMSB-G	89.4	92.9	93.2	88.9	93.1	96.3	97.2	94.5
	SMSB-R	89.3	92.8	93.3	87.2	92.8	96.3	96.9	94.2
	SMSB-M	89.5	92.7	92.9	87.8	93.7	96.7	97.3	94.0
	SESB	86.5	87.7	77.9	86.0	92.4	94.8	92.9	93.0
	SDWB	86.3	88.2	88.6		94.5	95.0	95.0	
	GBBB	82.8	95.8	65.1		93.8	93.4	98.4	
	WIA	86.2	87.5	85.0		91.7	93.5	92.7	
$(1/3, 1/2, 0.05)$	SMSB-G	94.0	94.4	93.0	92.9	88.2	92.2	92.6	88.8
	SMSB-R	93.7	93.9	93.2	92.8	87.7	92.0	93.2	88.2
	SMSB-M	93.5	94.6	93.3	92.8	87.5	92.7	92.5	88.9
	SESB	92.0	89.5	78.8	91.6	85.9	88.4	82.5	87.6
	SDWB	90.3	89.9	89.0		90.0	90.5	90.2	
	GBBB	84.4	95.9	63.5		89.2	87.6	95.5	
	WIA	92.2	92.2	87.6		87.9	91.0	88.5	
$(1/2, 1/2, 0.05)$	SMSB-G	78.3	85.9	84.7	75.7	69.7	82.4	85.2	74.2
	SMSB-R	77.5	84.4	83.9	75.4	69.9	82.5	85.7	73.1
	SMSB-M	76.9	85.2	84.9	76.1	70.6	82.4	85.1	74.6
	SESB	72.2	74.4	61.2	71.8	66.4	75.7	69.9	70.3
	SDWB	71.9	75.4	76.9		76.4	78.6	78.5	
	GBBB	68.3	88.8	48.2		73.5	72.7	91.2	
	WIA	70.7	73.8	71.8		66.5	75.4	71.1	

Table A.6: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (NUL) design, with $n = 50$ and $p = 400$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	94.8	97.1	96.6	95.0	95.7	97.7	97.9	96.1
	SMSB-R	94.5	96.6	95.8	94.2	95.6	97.3	97.8	96.1
	SMSB-M	94.7	96.2	96.0	94.5	95.9	97.5	98.1	95.9
	SESB	92.9	91.8	78.8	93.2	94.9	96.7	94.7	95.2
	SDWB	96.4	96.7	96.3		97.9	98.3	98.3	
	GBBB	95.8	99.8	78.2		98.2	98.0	99.6	
	WIA	89.7	90.2	90.6		96.0	94.3	92.3	
$(1/2, 0, 0)$	SMSB-G	85.7	92.7	92.5	86.3	93.8	97.0	97.8	94.5
	SMSB-R	84.5	92.0	90.9	84.9	93.9	96.8	97.9	94.5
	SMSB-M	84.9	92.2	91.5	85.8	93.7	96.9	97.9	94.5
	SESB	80.9	80.7	62.4	79.8	93.0	95.4	94.6	93.9
	SDWB	89.5	92.2	92.0		97.1	97.4	97.5	
	GBBB	86.8	99.3	66.0		97.1	96.9	99.6	
	WIA	79.9	82.3	84.6		94.8	93.8	92.6	
$(1/3, 1/2, 0)$	SMSB-G	98.6	98.9	98.5	98.5	98.6	99.4	99.2	98.2
	SMSB-R	98.0	98.7	98.4	98.3	98.4	99.2	99.0	98.5
	SMSB-M	98.3	98.7	98.2	98.4	98.4	98.6	99.1	98.7
	SESB	98.0	97.1	89.2	97.8	98.1	98.1	96.9	98.1
	SDWB	99.5	99.4	99.2		99.4	99.3	98.5	
	GBBB	98.9	99.9	89.1		99.8	99.0	99.9	
	WIA	96.3	96.3	94.6		98.5	97.9	95.8	
$(1/2, 1/2, 0)$	SMSB-G	94.2	96.6	95.9	94.8	96.0	98.7	98.7	97.5
	SMSB-R	93.0	96.6	95.6	93.7	96.0	99.0	98.4	96.7
	SMSB-M	93.5	96.5	95.2	94.1	96.1	98.5	98.9	96.7
	SESB	91.5	90.8	76.2	91.5	95.8	97.7	95.9	96.2
	SDWB	95.6	96.5	95.6		98.7	99.0	98.6	
	GBBB	95.4	99.6	77.9		98.9	98.6	99.8	
	WIA	91.8	90.6	91.0		97.4	96.7	95.1	
$(1/3, 1/2, 0.05)$	SMSB-G	95.0	95.8	94.6	94.7	96.8	97.7	97.2	96.6
	SMSB-R	94.6	95.5	93.7	94.0	96.1	97.7	97.5	96.3
	SMSB-M	94.9	96.0	94.3	93.9	96.4	97.6	97.3	97.3
	SESB	93.8	91.1	80.2	93.1	96.3	96.7	93.7	96.5
	SDWB	96.2	95.8	94.8		97.8	97.9	97.6	
	GBBB	95.9	99.9	80.5		98.2	97.4	99.5	
	WIA	93.1	91.7	90.4		96.3	94.3	92.8	
$(1/2, 1/2, 0.05)$	SMSB-G	81.8	88.0	85.5	82.9	85.8	92.1	93.5	88.0
	SMSB-R	79.6	87.1	85.3	81.0	85.6	91.8	93.0	87.7
	SMSB-M	81.5	88.3	85.4	81.6	86.0	91.8	93.3	87.8
	SESB	77.1	74.9	56.1	76.7	83.9	88.1	83.7	86.0
	SDWB	83.6	85.8	84.8		90.6	91.1	90.0	
	GBBB	84.0	99.0	59.5		92.5	91.6	98.6	
	WIA	80.0	79.1	77.5		89.4	86.9	84.6	

Table A.7: Empirical probabilities (in %) of $\{\|\mathbf{T}_{\lambda,n}\|_\infty \leq q_{0.95}\}$ for SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB and WIA under the (NUL) design, with $n = 100$ and $p = 400$.

		$\lambda = 12$				$\lambda = 24$			
(ρ, ζ, γ)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(1/3, 0, 0)$	SMSB-G	95.3	96.6	95.5	93.3	95.9	97.0	97.8	96.0
	SMSB-R	94.2	95.7	95.5	93.0	95.6	96.8	97.5	95.8
	SMSB-M	94.4	96.2	95.6	92.8	95.7	97.5	97.6	95.6
	SESB	93.1	92.2	78.4	92.5	95.3	96.0	94.0	95.0
	SDWB	95.8	96.7	96.3		97.4	98.4	98.6	
	GBBB	93.4	99.5	76.2		96.5	96.8	99.8	
	WIA	91.3	92.4	91.0		94.6	94.6	93.5	
$(1/2, 0, 0)$	SMSB-G	84.4	91.8	91.2	84.2	90.8	95.7	97.0	92.2
	SMSB-R	83.5	91.0	90.3	83.5	90.8	95.6	96.6	91.6
	SMSB-M	83.3	91.5	90.9	84.1	90.7	96.2	96.8	92.3
	SESB	81.2	81.4	61.9	79.3	89.6	93.4	90.4	90.5
	SDWB	86.5	90.5	91.3		94.6	97.5	97.5	
	GBBB	82.9	98.8	63.0		92.7	94.4	99.5	
	WIA	79.0	84.1	84.4		90.7	92.6	91.4	
$(1/3, 1/2, 0)$	SMSB-G	97.5	97.7	97.7	97.1	98.2	99.0	99.3	98.5
	SMSB-R	97.2	97.7	97.3	97.0	98.3	98.7	99.0	97.9
	SMSB-M	97.4	97.9	97.2	97.0	98.2	98.9	98.7	98.3
	SESB	97.1	96.3	86.8	97.0	98.0	98.0	96.5	97.9
	SDWB	98.2	98.4	97.8		98.7	99.3	99.1	
	GBBB	97.3	99.7	85.6		99.4	98.7	99.8	
	WIA	97.1	96.8	94.9		97.7	98.0	96.8	
$(1/2, 1/2, 0)$	SMSB-G	90.8	94.4	93.3	89.9	94.4	97.3	97.9	95.0
	SMSB-R	89.7	94.3	93.2	88.9	94.1	97.5	97.5	95.2
	SMSB-M	90.0	94.2	92.8	89.3	94.1	97.5	97.7	95.2
	SESB	87.3	87.6	72.7	87.9	93.3	95.6	93.3	94.2
	SDWB	91.0	92.8	93.1		96.3	98.0	98.2	
	GBBB	89.7	99.2	71.7		96.4	96.7	99.7	
	WIA	89.0	90.0	87.3		94.5	95.8	94.8	
$(1/3, 1/2, 0.05)$	SMSB-G	93.8	94.8	93.8	92.4	93.4	94.5	94.3	92.7
	SMSB-R	93.0	94.7	92.7	91.5	92.5	94.6	93.9	92.4
	SMSB-M	92.7	94.7	93.0	91.2	92.8	94.4	93.9	92.2
	SESB	90.9	88.8	72.5	91.1	91.8	91.9	87.0	92.3
	SDWB	93.7	93.6	93.1		93.6	94.9	95.1	
	GBBB	91.4	99.0	70.4		94.8	93.5	99.0	
	WIA	91.1	91.0	86.8		91.1	92.6	89.3	
$(1/2, 1/2, 0.05)$	SMSB-G	75.8	82.7	81.0	75.0	73.0	83.9	86.9	73.9
	SMSB-R	73.3	82.1	80.5	73.2	71.4	83.4	85.5	73.5
	SMSB-M	74.0	83.5	81.3	73.6	72.2	84.1	86.1	74.0
	SESB	70.4	68.3	46.2	68.6	69.1	78.0	70.2	71.6
	SDWB	75.8	81.8	80.3		78.8	84.7	86.6	
	GBBB	73.6	95.3	47.4		79.6	79.7	96.4	
	WIA	73.9	77.1	73.5		74.4	76.6	74.9	

In addition to their robust performance across diverse model settings, the proposed SSBs exhibit favorable computational performance. Table A.8 reports the average computation times (in seconds) required by each method (SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, GBBB, and WIA) to obtain the 0.95-quantile of $\|\mathbf{T}_{\lambda,n}\|_\infty$ for model (6.1) with $(\rho, \zeta, \gamma) = (1/3, 0, 0)$ under the (UL) design, for different combinations of n and λ. The simulations were conducted using MATLAB R2019b on a Linux 7.3 workstation equipped with a 2.10 GHz Intel CPU. For SMSB-G, SMSB-R, SMSB-M, SESB, SDWB, and GBBB, the reported average computation time is based on 1,000 Monte Carlo simulations, with 1,000 bootstrap replications performed in each simulation. For WIA, the average computation time is based on 1,000 Monte Carlo simulations without bootstrap replications.

Table A.8 shows that the proposed SSBs exhibit favorable computational performance across the considered settings. Their computation times are generally stable across different choices of the subsample size b and are substantially lower than those of SDWB and GBBB, particularly as n increases. For example, when $\lambda = 24$ and $n = 1000$, the computation times of the proposed SMSBs and SESB range from approximately 0.12 to 0.21 seconds, compared with 0.40–0.43 seconds for SDWB and 0.77–1.31 seconds for GBBB. The three SMSB variants have comparable computational costs, and the use of the data-driven subsample size $\hat{b}^{\text{opt}}$ does not result in a substantial increase in computation time. As expected, WIA has a lower computation time because it does

Table A.8: Average computation time (in seconds) over 1,000 Monte Carlo simulations for obtaining the 0.95-quantile of $\|\mathbf{T}_{\lambda,n}\|_\infty$ for model (6.1) with $(\rho, \zeta, \gamma) = (1/3, 0, 0)$ under the (UL) design.

		$\lambda = 12$				$\lambda = 24$			
(n, p)	Method	$b = 4$	$b = 5$	$b = 6$	$b = \hat{b}^{\text{opt}}$	$b = 6$	$b = 8$	$b = 10$	$b = \hat{b}^{\text{opt}}$
$(100, 50)$	SMSB-G	0.038	0.037	0.037	0.028	0.134	0.134	0.133	0.119
	SMSB-R	0.025	0.024	0.024	0.028	0.116	0.116	0.115	0.120
	SMSB-M	0.032	0.031	0.031	0.033	0.161	0.156	0.153	0.131
	SESB	0.040	0.040	0.039	0.032	0.186	0.187	0.184	0.146
	SDWB	0.043	0.044	0.046		0.053	0.053	0.053	
	GBBB	0.060	0.040	0.040		0.547	0.409	0.293	
	WIA	0.013	0.011	0.008		0.015	0.014	0.013	
$(500, 50)$	SMSB-G	0.030	0.035	0.036	0.033	0.118	0.117	0.120	0.125
	SMSB-R	0.024	0.025	0.026	0.032	0.113	0.113	0.114	0.126
	SMSB-M	0.032	0.032	0.033	0.037	0.152	0.146	0.140	0.138
	SESB	0.040	0.041	0.042	0.034	0.185	0.179	0.173	0.151
	SDWB	0.241	0.235	0.234		0.307	0.297	0.280	
	GBBB	0.532	0.285	0.312		1.122	0.982	0.575	
	WIA	0.063	0.052	0.045		0.070	0.063	0.053	
$(1000, 50)$	SMSB-G	0.043	0.044	0.051	0.039	0.139	0.126	0.157	0.148
	SMSB-R	0.033	0.028	0.037	0.039	0.120	0.124	0.120	0.149
	SMSB-M	0.038	0.037	0.046	0.044	0.135	0.131	0.155	0.162
	SESB	0.045	0.045	0.047	0.051	0.202	0.185	0.206	0.206
	SDWB	0.379	0.385	0.369		0.425	0.433	0.402	
	GBBB	1.010	0.733	0.822		1.311	1.054	0.773	
	WIA	0.099	0.071	0.067		0.123	0.098	0.070	

not require bootstrap replications. Overall, the reported runtimes indicate that the proposed SSBs can provide bootstrap-based inference at a moderate computational cost, with the runtime advantage over SDWB and GBBB becoming more pronounced as n increases in the considered settings.

B Illustrative examples satisfying Assumptions 1 and 2

B.1 An illustrative example satisfying Assumption 1

Let $\mathbf{L}(\cdot) = (L_1(\cdot), \ldots, L_p(\cdot))^\top$ be a centered homogeneous $\mathbb{R}^p$-valued Lévy basis on $\mathbb{R}^d$ that is independently scattered, meaning that the random vectors associated with disjoint spatial sets are independent. Consider the Lévy-driven moving-average (LMA) field defined by

$$\mathbf{X}(\mathbf{u}) = \int_{\mathbb{R}^d} G(\mathbf{u} - \mathbf{v})\, \mathbf{L}(\mathrm{d}\mathbf{v}), \qquad \mathbf{u} \in \mathbb{R}^d, \tag{B.1}$$

where $G : \mathbb{R}^d \to \mathbb{R}^{p\times p}$ is a measurable kernel.

Assumption B.1 (Kernel structure)**.** Let ξ_0, r_0, M, m_0 and ξ^* be positive constants independent of n. Let $\psi_{m_0} : \mathbb{R}^d \to [0,1]$ satisfy $\psi_{m_0}(\mathbf{v}) = 1$ if $\|\mathbf{v}\|_1 \le m_0/2$, $\psi_{m_0}(\mathbf{v}) = 0$ if $\|\mathbf{v}\|_1 > m_0$. Consider the kernel $G(\mathbf{v}) = \big(g_{j,k}(\mathbf{v})\big)_{p\times p}$ with entries $g_{j,k}(\mathbf{v}) = \xi_{j,k}\, \mathrm{e}^{-r_{j,k}\|\mathbf{v}\|_1}\psi_{m_0}(\mathbf{v})$, where $|\xi_{j,j}| \ge \xi_*$, $\max_{1\le j\le p}[\![\{k \ne j : \xi_{j,k} \ne 0\}]\!] \le M$, and $|\xi_{j,k}| \le \xi_0$, $r_0 \le r_{j,k} \le \xi_0$ for all $1 \le j, k \le p$.

Assumption B.2 (Lévy tails, moments and non-degeneracy)**.** For some even integer

$q_0 \geq 8$, $\sup_n \max_{1\leq j\leq p} \mathrm{E}\big|L_j([0,1]^d)\big|^{q_0} \leq C_L < \infty$. The marginal Lévy measures admit densities $\nu_{0,j}(x) = C_j|x|^{-1-\beta_j}\exp(-c_j|x|)\,\mathbb{I}_{\mathbb{R}\setminus\{0\}}(x)$, where $-1 < \beta_j < 1$ and $0 < c_* \leq c_j \leq c^* < \infty$, $0 < C_* \leq C_j \leq C^* < \infty$ uniformly over $1 \leq j \leq p$. Moreover, letting $\Sigma_L = \mathrm{Cov}\{\mathbf{L}([0,1]^d)\}$, there exists a constant $c_L > 0$ independent of n, such that $\lambda_{\min}(\Sigma_L) \geq c_L$.

Proposition B.1. Suppose that Assumptions B.1 and B.2 hold. Then, the field $\{\mathbf{X}(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ defined in (B.1) is strictly stationary and satisfies Assumption 1 with $D_n = \mathcal{O}(1)$.

Proposition B.1 provides an example of a non-Gaussian field satisfying Assumption 1. The construction is motivated by the localization argument (Kurisu et al., 2024, Supplementary Material, Section B).

Proof of Proposition B.1. Strict stationarity follows from the homogeneity of the Lévy basis and the translation-invariant form of the kernel.

We first verify Assumption 1(a).

By Assumption B.2 and the Cauchy–Schwarz inequality, the entries of Σ_L are uniformly bounded. The covariance function of the LMA field is $\sigma_{ij}(\mathbf{h}) = \int_{\mathbb{R}^d} g_{i,\cdot}(\mathbf{v})\Sigma_L g_{j,\cdot}(\mathbf{v}+\mathbf{h})^\top \mathrm{d}\mathbf{v}$. By Assumption B.1, each row of $G(\cdot)$ contains at most $M+1$ nonzero entries and $|g_{j,k}(\mathbf{v})| \leq \xi_0 \mathbb{I}\{\|\mathbf{v}\|_1 \leq m_0\}$. Hence, uniformly over i, j, $|\sigma_{ij}(\mathbf{h})| \lesssim \mathbb{I}\{\|\mathbf{h}\|_1 \leq 2m_0\}$. After enlarging the constant in the definition of ρ_2, if necessary, this implies $|\sigma_{ij}(\mathbf{h})| \leq \rho_2(\mathbf{h})$. Moreover, by the compact support of $\sigma_{ij}(\cdot)$, $\sup_{1\leq i,j\leq p}\int_{\mathbb{R}^d}\|\mathbf{h}\|_1|\sigma_{ij}(\mathbf{h})|\,\mathrm{d}\mathbf{h} < \infty$.

Define $\mathbf{a}_j = \int_{\mathbb{R}^d} g_{j,\cdot}(\mathbf{v})^\top \,\mathrm{d}\mathbf{v}$. By Fubini's theorem,

$$\int_{\mathbb{R}^d} \sigma_{jj}(\mathbf{h})\,\mathrm{d}\mathbf{h} = \mathbf{a}_j^\top \Sigma_L \mathbf{a}_j \geq \lambda_{\min}(\Sigma_L)\|\mathbf{a}_j\|_2^2 \geq c_L |a_{j,j}|^2.$$

Since $a_{j,j} = \xi_{j,j} \int_{\mathbb{R}^d} \mathrm{e}^{-r_{j,j}\|\mathbf{v}\|_1}\psi_{m_0}(\mathbf{v})\,\mathrm{d}\mathbf{v}$, Assumption B.1 yields

$$|a_{j,j}| \geq \xi_* \int_{\{\|\mathbf{v}\|_1 \leq m_0/2\}} \mathrm{e}^{-\xi_0\|\mathbf{v}\|_1}\,\mathrm{d}\mathbf{v} =: c_g.$$

Therefore, $\inf_{1\leq j\leq p}\int_{\mathbb{R}^d}\sigma_{jj}(\mathbf{h})\,\mathrm{d}\mathbf{h} \geq c_L c_g^2 > 0$. This proves the non-degeneracy of the integrated marginal covariance.

Let $\kappa^L_{a,b,c,e} = \mathrm{cum}\{L_a([0,1]^d), L_b([0,1]^d), L_c([0,1]^d), L_e([0,1]^d)\}$. Assumption B.2 and Hölder's inequality imply $\sup_{a,b,c,e}|\kappa^L_{a,b,c,e}| < \infty$. By the cumulant formula for Lévy stochastic integrals,

$$c_4^{(i,j,l,m)}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3) = \sum_{a,b,c,e=1}^{p} \kappa^L_{a,b,c,e}\int_{\mathbb{R}^d} g_{i,a}(\mathbf{v})g_{j,b}(\mathbf{v}+\mathbf{h}_1)g_{l,c}(\mathbf{v}+\mathbf{h}_2)g_{m,e}(\mathbf{v}+\mathbf{h}_3)\,\mathrm{d}\mathbf{v}.$$

For fixed (i,j,l,m), at most $(M+1)^4$ terms in the sum are nonzero. By the compact support of $G(\cdot)$, $|c_4^{(i,j,l,m)}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)| \lesssim \prod_{r=1}^3 \mathbb{I}\{\|\mathbf{h}_r\|_1 \leq 2m_0\}$. Thus, after enlarging the constant in ρ_2, if necessary, $|c_4^{(i,j,l,m)}(\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3)| \lesssim \prod_{\ell=1}^3 \rho_2(\mathbf{h}_\ell)$ uniformly over i,j,l,m. In particular, setting $\mathbf{h}_1 = \mathbf{h}_2 = \mathbf{h}_3 = \mathbf{0}$ gives a uniform bound on the fourth cumulants of the marginal field components. Together with the uniform bound on $\sigma_{jj}(\mathbf{0})$ and the identity $\mathrm{E}X_j(\mathbf{u})^4 = \mathrm{cum}_4\{X_j(\mathbf{u})\} + 3\sigma_{jj}(\mathbf{0})^2$ for centered random variables, this yields $\sup_{\mathbf{u}\in\mathbb{R}^d}\max_{1\leq j\leq p}\mathrm{E}|X_j(\mathbf{u})|^4 < \infty$. This verifies the fourth-moment and fourth-order cumulant conditions and completes the verification of Assumption 1(a).

We next verify Assumption 1(b).

Since $g_{j,k}(\mathbf{v}) = 0$ whenever $\|\mathbf{v}\|_1 > m_0$, the field $\{\mathbf{X}(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ is $2m_0$-dependent. Hence, $\beta(a;b) = 0$ for $a > 2m_0$, $b > 0$. Let $m_* = \max\{1, 2m_0\}$. For any fixed $\eta > 1$, define

$$\beta_1(a) = \begin{cases} 1, & 0 < a \le m_*, \\ (m_*/a)^\eta, & a > m_*, \end{cases} \qquad \beta_2(b) = 1.$$

Then $\beta(a;b) \le \beta_1(a)\beta_2(b)$ for $a, b > 0$, and for all $a \ge 1$, $\beta_1(a) \le m_*^\eta a^{-\eta}$. Since $\lambda_2 \to \infty$, we have $\lambda_2 > 2m_0$ eventually, and hence $\beta(\lambda_2; \lambda^d) = 0$ for all sufficiently large n. Consequently, $(\lambda/\lambda_1)^d \beta(\lambda_2; \lambda^d) = 0$ eventually. Thus, Assumption 1(b) holds.

Finally, we verify Assumption 1(c).

For each k, let $K_k(t) = \log \mathrm{E} \exp\{t\, L_k([0,1]^d)\}$. Assumption B.2 implies that there exists a constant $t_* > 0$ such that $\sup_{1 \le k \le p} \sup_{|t| \le t_*} |K_k(t)| < \infty$. For $\xi_{j,k} \ne 0$, define $Z_{j,k}(\mathbf{u}) = \int_{\mathbb{R}^d} g_{j,k}(\mathbf{u} - \mathbf{v}) L_k(\mathrm{d}\mathbf{v})$, and choose $t_0 > 0$ such that $t_0 \xi_0 \le t_*$. By the cumulant formula for Lévy stochastic integrals (Sato, 1999),

$$\log \mathrm{E} \exp\{\pm t_0 Z_{j,k}(\mathbf{u})\} = \int_{\mathbb{R}^d} K_k(\pm t_0 g_{j,k}(\mathbf{u} - \mathbf{v}))\, \mathrm{d}\mathbf{v}.$$

Since $g_{j,k}$ is supported on $\|\mathbf{v}\|_1 \le m_0$ and $|t_0 g_{j,k}| \le t_*$, the right-hand side is uniformly bounded in $\mathbf{u}, j, k, n$ and p. It follows that $\sup_{\mathbf{u}, j, k: \xi_{j,k} \ne 0} \mathrm{E} \exp\{t_0 |Z_{j,k}(\mathbf{u})|\} < \infty$, and hence $\sup_{\mathbf{u} \in \mathbb{R}^d} \max_{j,k: \xi_{j,k} \ne 0} \|Z_{j,k}(\mathbf{u})\|_{\psi_1} < \infty$. Since $X_j(\mathbf{u}) = \sum_{k: \xi_{j,k} \ne 0} Z_{j,k}(\mathbf{u})$ contains at most $M + 1$ terms, the triangle inequality for the ψ_1-norm gives

$$\sup_{\mathbf{u} \in \mathbb{R}^d} \max_{1 \le j \le p} \|X_j(\mathbf{u})\|_{\psi_1} = \mathcal{O}(1).$$

Thus, Assumption 1(c) holds with $D_n = \mathcal{O}(1)$. □

B.2 An illustrative example satisfying Assumption 2

Let $\{\mathbf{Y}_\lambda(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ be a p-dimensional random field with the decomposition

$$\mathbf{Y}_\lambda(\mathbf{u}) = \mathbf{X}(\mathbf{u}) + \boldsymbol{\Upsilon}_\lambda(\mathbf{u}),$$

where $\{\mathbf{X}(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ is a strictly stationary random field and $\{\boldsymbol{\Upsilon}_\lambda(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ is a mean-zero perturbation field. Specifically, we consider

$$\boldsymbol{\Upsilon}_\lambda(\mathbf{u}) = a_\lambda \mathbf{W}(\mathbf{u}) h(\mathbf{u}/\lambda), \qquad a_\lambda = \mathcal{O}\big(\frac{\log \lambda}{\lambda^{d+1}}\big), \tag{B.2}$$

where $\{\mathbf{W}(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ is a second-order stationary, mean-zero random field taking values in $\mathbb{R}^p$, independent of $\{\mathbf{X}(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ and of the sampling locations. Suppose further that $\sup_{\mathbf{u}\in\mathbb{R}^d} \max_{1\le j\le p} |W_j(\mathbf{u})| \le C_W$ almost surely, for some constant C_W independent of n and p, and that $h(\cdot)$ is bounded and Lipschitz with Lipschitz constant L_h.

Then, we have the following proposition.

Proposition B.2. Suppose that $\lambda \asymp n^{\alpha_0}$ and $p \asymp n^{\iota}$, where $0 < \alpha_0 \le 1/d$ and $\iota > 0$. Then, the following results hold.

(i) The perturbation field $\{\boldsymbol{\Upsilon}_\lambda(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ satisfies Assumption 2 in Section 4.1 of the article.

(ii) For fixed $\delta > 0$,

$$\sup_{\mathbf{u}\in U(\mathbf{u}_0,\delta)} \|\mathbf{Y}_\lambda(\mathbf{u}) - \widetilde{\mathbf{Y}}_\lambda(\mathbf{u})\|_\infty = \mathcal{O}\left(\frac{a_\lambda}{\lambda}\right),$$

where $\{\widetilde{\mathbf{Y}}_\lambda(\mathbf{u}),\, \mathbf{u} \in \mathbb{R}^d\}$ is a stationary field.

Proof. (i) Since both $\mathbf{W}(\mathbf{u})$ and $h(\cdot)$ are uniformly bounded,

$$\mathrm{E}[\sup_{\mathbf{u}\in R_\lambda} \max_{1\le j\le p} |\Upsilon_{\lambda,j}(\mathbf{u})|^2] = \mathcal{O}(a_\lambda^2) = \mathcal{O}(\log^2\lambda/\lambda^{2d+2}),$$

which verifies Assumption 2(a). Moreover, almost surely, $\lambda^{d/2}\max_{1\le j\le p}\left|n^{-1}\sum_{i=1}^n \Upsilon_{\lambda,j}(\mathbf{s}_i)\right| = \mathcal{O}\big(\frac{\log\lambda}{\lambda^{d/2+1}}\big)$. Let $0 < \varsigma < \alpha_0(d/2+1)$. Then

$$\frac{a_\lambda\lambda^{d/2}}{n^{-\varsigma}\log^{-1/2}p} = \mathcal{O}\big(n^{\varsigma-\alpha_0(d/2+1)}(\log n)^{3/2}\big) = o(1),$$

where we use $\lambda \asymp n^{\alpha_0}$ and $p \asymp n^\iota$. Consequently, for all sufficiently large n,

$$\max_{1\le j\le p}\left|\lambda^{d/2}n^{-1}\sum_{i=1}^n \Upsilon_{\lambda,j}(\mathbf{s}_i)\right| \le n^{-\varsigma}\log^{-1/2}p$$

almost surely. Hence,

$$\mathrm{P}_{\cdot|\mathbf{s}}\Big\{\max_{1\le j\le p}\left|\lambda^{d/2}n^{-1}\sum_{i=1}^n \Upsilon_{\lambda,j}(\mathbf{s}_i)\right| > n^{-\varsigma}\log^{-1/2}p\Big\} = 0$$

almost surely for all sufficiently large n. Thus, Assumption 2(b) holds, proving (i).

(ii) For any $\mathbf{u}_0 \in \mathbb{R}^d$, define the stationary field

$$\widetilde{\mathbf{Y}}_\lambda(\mathbf{u}) = \mathbf{X}(\mathbf{u}) + a_\lambda\mathbf{W}(\mathbf{u})h(\mathbf{u}_0/\lambda).$$

For any fixed $\delta > 0$, the Lipschitz continuity of $h(\cdot)$ gives

$$\sup_{\mathbf{u}\in U(\mathbf{u}_0,\delta)}\max_{1\le j\le p}\left|Y_{\lambda,j}(\mathbf{u}) - \widetilde{Y}_{\lambda,j}(\mathbf{u})\right| \le a_\lambda C_W L_h\frac{\delta}{\lambda},$$

almost surely, which proves part (ii). □

The construction in (B.2) allows for spatially varying variance and covariance contributions through the spatially varying factor $h(\mathbf{u}/\lambda)$, while the magnitude of the perturbation vanishes asymptotically at the rate $a_\lambda = \mathcal{O}(\log\lambda/\lambda^{d+1})$. Proposition B.2(ii) shows, within any fixed local neighborhood, the resulting nonstationary field can be uniformly approximated by a stationary field, thereby providing a concrete example of a mildly nonstationary or locally stationary spatial field covered by Assumption 2. In contrast, spatial nonstationarity of nonvanishing magnitude is not accommodated by the present construction.

C Proof of Lemma 1

C.1 Proof of part (i) of Lemma 1

Proof of part (i) of Lemma 1. Recall that $\mathbf{T}_{\lambda,n} = \lambda^{d/2}\,\bar{\mathbf{Y}}_{\lambda,n}$ with $\bar{\mathbf{Y}}_{\lambda,n} = \bar{\mathbf{X}}_{\lambda,n} + \bar{\boldsymbol{\Upsilon}}_{\lambda,n}$, where $\bar{\mathbf{X}}_{\lambda,n} = n^{-1}\sum_{i=1}^{n}\mathbf{X}(\mathbf{s}_i)$ and $\bar{\boldsymbol{\Upsilon}}_{\lambda,n} = n^{-1}\sum_{i=1}^{n}\boldsymbol{\Upsilon}(\mathbf{s}_i)$. Since $\mathrm{E}(\bar{\mathbf{Y}}_{\lambda,n}) = \mathbf{0}$, we decompose $\Xi_{\lambda,n}^{(i,j)}$ as

$$\begin{aligned}\Xi_{\lambda,n}^{(i,j)} =& \lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{Y}_{\lambda,n}^{(i)}\bar{Y}_{\lambda,n}^{(j)}\big]\\ =& \lambda^d\left\{\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}_{\lambda,n}^{(i)}\bar{X}_{\lambda,n}^{(j)}\big] + \mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}_{\lambda,n}^{(i)}\bar{\Upsilon}_{\lambda,n}^{(j)}\big] + \mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{\Upsilon}_{\lambda,n}^{(i)}\bar{X}_{\lambda,n}^{(j)}\big] + \mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{\Upsilon}_{\lambda,n}^{(i)}\bar{\Upsilon}_{\lambda,n}^{(j)}\big]\right\}, \quad \text{(C.1)}\end{aligned}$$

for each pair (i,j).

Observe that

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}^{(i)}_{\lambda,n}\bar{\Upsilon}^{(j)}_{\lambda,n}\big|\big] \le \mathrm{E}_{\cdot|\mathbf{s}}\big[\max_{1\le i_1,i_2\le n}\max_{1\le i,j\le p}\big|\Upsilon_i(\mathbf{s}_{i_1})\big|\,\big|\Upsilon_j(\mathbf{s}_{i_2})\big|\big]. \tag{C.2}$$

Using $|ab| \le (a^2+b^2)/2$ and taking expectations on both sides of (C.2) gives

$$\mathrm{E}\big\{\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}^{(i)}_{\lambda,n}\bar{\Upsilon}^{(j)}_{\lambda,n}\big|\big]\big\} \le \mathrm{E}\Big[\max_{1\le i\le n}\max_{1\le j\le p}\big|\Upsilon_j(\mathbf{s}_i)\big|^2\Big] \le \mathrm{E}\big[\sup_{\mathbf{u}\in R_\lambda}\max_{1\le j\le p}|\Upsilon_j(\mathbf{u})|^2\big] = \mathcal{O}\Big(\frac{\log^2\lambda}{\lambda^{2d+2}}\Big),$$

where the last inequality follows from Assumption 2(a). Applying Markov's inequality, we have

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}^{(i)}_{\lambda,n}\bar{\Upsilon}^{(j)}_{\lambda,n}\big|\big] = \mathcal{O}_p\Big(\frac{\log^2\lambda}{\lambda^{2d+2}}\Big). \tag{C.3}$$

By the Cauchy–Schwarz inequality,

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{X}^{(i)}_{\lambda,n}\bar{\Upsilon}^{(j)}_{\lambda,n}\big|\big] \le \sqrt{\mathrm{E}_{\cdot|\mathbf{s}}[\bar{X}^{(i)2}_{\lambda,n}]\,\mathrm{E}_{\cdot|\mathbf{s}}[\bar{\Upsilon}^{(j)2}_{\lambda,n}]} \text{ and } \mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big|\big] \le \sqrt{\mathrm{E}_{\cdot|\mathbf{s}}[\bar{\Upsilon}^{(i)2}_{\lambda,n}]\,\mathrm{E}_{\cdot|\mathbf{s}}[\bar{X}^{(j)2}_{\lambda,n}]}. \tag{C.4}$$

From (C.1), (C.3) and (C.4), to establish (3.4), it suffices to show

$$\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big] = \int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\,\sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\Big(\frac{\log\lambda}{\lambda}\Big). \tag{C.5}$$

We verify (C.5) by showing

$$\mathrm{E}\big\{\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big]\big\} = \int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\,\sigma_{ij}(\mathbf{0}) + \mathcal{O}\Big(\frac{\log\lambda}{\lambda}\Big), \tag{C.6}$$

$$\mathrm{Var}\big\{\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big]\big\} = \mathcal{O}\Big(\frac{1}{n}\Big) + \mathcal{O}\Big(\frac{\lambda^d}{n^2}\Big). \tag{C.7}$$

Note that

$$\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big(\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big) = \frac{\lambda^d}{n^2}\sum_{i_1\ne i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1}) + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}). \tag{C.8}$$

Taking expectations on both sides yields

$$\mathrm{E}\big\{\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big]\big\} = \frac{(n-1)\lambda^d}{n}\mathrm{E}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)] + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}). \tag{C.9}$$

By Assumption 3, we have

$$\mathrm{E}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)] = \frac{1}{\lambda^{2d}}\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2 + \mathcal{O}\Big(\frac{\log\lambda}{\lambda^{2d+1}}\Big)\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2. \tag{C.10}$$

Applying the change of variables $\mathbf{h}=\mathbf{u}_2-\mathbf{u}_1$ and $\mathbf{u}=\mathbf{u}_1$, we obtain

$$\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2 = \int_{R_\lambda}\int_{R_\lambda-\mathbf{u}}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h}\,\mathrm{d}\mathbf{u} = \int_{R_\lambda}\int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\mathbb{I}_{R_\lambda}(\mathbf{u}+\mathbf{h})\,\mathrm{d}\mathbf{h}\,\mathrm{d}\mathbf{u}.$$

By Fubini's theorem and $\int_{R_\lambda}\mathbb{I}_{R_\lambda}(\mathbf{u}_1+\mathbf{h})\,\mathrm{d}\mathbf{u}_1 = |R_\lambda\cap(R_\lambda-\mathbf{h})|$, we have

$$\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2 = \int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})|R_\lambda\cap(R_\lambda-\mathbf{h})|\,\mathrm{d}\mathbf{h}. \tag{C.11}$$

Since $R_\lambda=\lambda R_0$ and $|R_0|=1$, the boundary regularity condition on R_0 implies that,

$$\Big|\frac{|R_\lambda\cap(R_\lambda-\mathbf{h})|}{\lambda^d}-1\Big| \lesssim \min\big\{1,\frac{\|\mathbf{h}\|_1}{\lambda}\big\}, \qquad \mathbf{h}\in\mathbb{R}^d. \tag{C.12}$$

Therefore, by Assumption 1(a),

$$\Big|\frac{1}{\lambda^d}\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2 - \int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h}\Big| \lesssim \int_{\mathbb{R}^d}|\sigma_{ij}(\mathbf{h})|\min\{1,\frac{\|\mathbf{h}\|_1}{\lambda}\}\,\mathrm{d}\mathbf{h} = \mathcal{O}(\lambda^{-1}), \tag{C.13}$$

uniformly over $1\le i,j\le p$. Combining (C.11)–(C.13) gives

$$\frac{1}{\lambda^d}\iint_{R_\lambda^2}\sigma_{ij}(\mathbf{u}_2-\mathbf{u}_1)\,\mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2 = \int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \mathcal{O}\Big(\frac{1}{\lambda}\Big).$$

Substituting this result into (C.10) yields

$$\mathrm{E}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)] = \frac{1}{\lambda^d}\int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \mathcal{O}\Big(\frac{\log\lambda}{\lambda^{d+1}}\Big). \tag{C.14}$$

Combining (C.14) with (C.9) then yields (C.6).

Since (C.8) have been obtained, we have

$$\begin{aligned}\mathrm{Var}\big\{\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big]\big\} &= \frac{\lambda^{2d}}{n^4}\sum_{i_1\neq i_2, i_3\neq i_4}\mathrm{Cov}\big(\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1}),\sigma_{ij}(\mathbf{s}_{i_4}-\mathbf{s}_{i_3})\big)\\ &= 4\,\frac{\lambda^{2d}\,(n-1)(n-2)}{n^3}\mathrm{Cov}\big(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1),\sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\big) + 2\,\frac{\lambda^{2d}\,(n-1)}{n^3}\mathrm{Var}\big(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\big).\end{aligned}$$

Using $\mathrm{Cov}(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y]$, $\mathrm{Var}(X) = \mathrm{E}[X^2] - \big(\mathrm{E}[X]\big)^2$, and the result (C.14), after some arrangements, we obtain

$$\begin{aligned}\mathrm{Var}\big\{\lambda^d\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\bar{X}^{(i)}_{\lambda,n}\bar{X}^{(j)}_{\lambda,n}\big]\big\} =& \mathcal{O}\Big(\frac{\lambda^{2d}}{n}\Big)\,\mathrm{E}\big[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\big]\\ &+ \mathcal{O}\Big(\frac{\lambda^{2d}}{n^2}\Big)\,\mathrm{E}\big[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)^2\big] + \mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned} \tag{C.15}$$

With arguments analogous to those employed in deriving (C.14), we have

$$\mathrm{E}\big[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\big] = \mathcal{O}\Big(\frac{1}{\lambda^{2d}}\Big) \quad\text{and}\quad \mathrm{E}\big[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)^2\big] = \mathcal{O}\Big(\frac{1}{\lambda^{d}}\Big). \tag{C.16}$$

Combining (C.15) with (C.16) produces (C.7). This completes the proof of part (i). □

C.2 Proof of part (ii) of Lemma 1

Proof of part (ii) of Lemma 1. Define $\bar{\mathbf{X}}_{b,n}(k) = n_k^{-1}\sum_{i=1}^n \mathbf{X}(\mathbf{s}_i)\mathbb{I}_{R_b(k)}(\mathbf{s}_i)$, $\bar{\boldsymbol{\Upsilon}}_{b,n}(k) = n_k^{-1}\sum_{i=1}^n \boldsymbol{\Upsilon}(\mathbf{s}_i)\mathbb{I}_{R_b(k)}(\mathbf{s}_i)$, and set $\bar{\mathbf{Y}}_{b,n}(k) = \bar{\mathbf{X}}_{b,n}(k) + \bar{\boldsymbol{\Upsilon}}_{b,n}(k)$ for $1 \le k \le N$. Let

$\mathbf{T}_{b,n}(k) = (\lambda^d/n)^{1/2} n_k^{1/2}\, \bar{\mathbf{Y}}_{b,n}(k)$ and $\hat{\mathbf{T}}_{\lambda,b,n}(k) = \mathbf{T}_{b,n}(k) - (n_k/n)^{1/2}\, \mathbf{T}_{\lambda,n}$. Let $\mathrm{Cov}_{\cdot|\mathbf{s}}(\cdot,\cdot)$ denote the conditional covariance given the sampling locations $\{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$. We decompose $\Xi^{(i,j)}_{\lambda,b,n}(k)$ as

$$\begin{aligned}\Xi^{(i,j)}_{\lambda,b,n}(k) &= \mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{b,n}(k), T^{(j)}_{b,n}(k)\big) - (n_k/n)^{1/2}\, \mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{b,n}(k), T^{(j)}_{\lambda,n}\big) \\ &\quad - (n_k/n)^{1/2}\, \mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{\lambda,n}, T^{(j)}_{b,n}(k)\big) + (n_k/n)\, \mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{\lambda,n}, T^{(j)}_{\lambda,n}\big), \qquad \text{(C.17)}\end{aligned}$$

for each pair (i,j).

Recall that $n_k = \sum_{i=1}^n \mathbb{I}_{R_b(k)}(\mathbf{s}_i)$. Under Assumption 3,

$$\mathrm{E}[n_k] = n \int_{R_b(k)} f(\mathbf{s})\, \mathrm{d}\mathbf{s} = \frac{nb^d}{\lambda^d} + \mathcal{O}\Big(\frac{nb^d \log\lambda}{\lambda^{d+1}}\Big) \quad \text{and} \quad \mathrm{Var}[n_k] = \mathcal{O}\Big(\frac{nb^d}{\lambda^d}\Big).$$

By Chebyshev's inequality,

$$\frac{n_k}{n} = \frac{b^d}{\lambda^d}\Big(1 + \mathcal{O}_p\big(\frac{\log\lambda}{\lambda}\big) + \mathcal{O}_p\big(\frac{\lambda^{d/2}}{n^{1/2} b^{d/2}}\big)\Big). \qquad \text{(C.18)}$$

It then follows from (3.4) that

$$(n_k/n)\, \mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{\lambda,n}, T^{(j)}_{\lambda,n}\big) = \mathcal{O}_p(b^d/\lambda^d). \qquad \text{(C.19)}$$

Applying the Cauchy–Schwarz inequality yields

$$\mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(i)}_{b,n}(k), T^{(j)}_{\lambda,n}\big) \le \sqrt{\mathrm{Var}_{\cdot|\mathbf{s}}[T^{(i)}_{b,n}(k)]\, \mathrm{Var}_{\cdot|\mathbf{s}}[T^{(j)}_{\lambda,n}]}, \qquad \text{(C.20)}$$

$$\mathrm{Cov}_{\cdot|\mathbf{s}}\big(T^{(j)}_{b,n}(k), T^{(i)}_{\lambda,n}\big) \le \sqrt{\mathrm{Var}_{\cdot|\mathbf{s}}[T^{(j)}_{b,n}(k)]\, \mathrm{Var}_{\cdot|\mathbf{s}}[T^{(i)}_{\lambda,n}]}. \qquad \text{(C.21)}$$

Combining (C.17)–(C.21), to establish (3.5), it suffices to show

$$\mathrm{Cov}_{\cdot|\mathbf{s}}(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(k)) = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\Big(\frac{\log\lambda}{\lambda}\Big) + \mathcal{O}_p\Big(\frac{1}{b^{\min(1,d/2)}}\Big). \tag{C.22}$$

Note that

$$\begin{aligned}\mathrm{Cov}_{\cdot|\mathbf{s}}(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(k)) = \frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\Big[&\bar{X}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k) + \bar{X}_{b,n}^{(i)}(k)\bar{\Upsilon}_{b,n}^{(j)}(k) \\ &+ \bar{\Upsilon}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k) + \bar{\Upsilon}_{b,n}^{(i)}(k)\bar{\Upsilon}_{b,n}^{(j)}(k)\Big].\end{aligned}$$

Using arguments analogous to those in deriving (C.3) and (C.4), we have

$$\begin{aligned}
&\frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}_{b,n}^{(i)}(k)\bar{\Upsilon}_{b,n}^{(j)}(k)\big|\big] \le \frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\max_{1\le i\le n_k}\max_{1\le j\le p}\big|\Upsilon_j(\mathbf{s}_i)\big|^2\big] = \mathcal{O}_p\Big(\frac{b^d\,\log^2\lambda}{\lambda^{2d+2}}\Big),\\
&\frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{X}_{b,n}^{(i)}(k)\bar{\Upsilon}_{b,n}^{(j)}(k)\big|\big] \le \frac{\lambda^d}{n} n_k \sqrt{\mathrm{E}_{\cdot|\mathbf{s}}[\bar{X}_{b,n}^{(i)2}(k)]\,\mathrm{E}_{\cdot|\mathbf{s}}[\bar{\Upsilon}_{b,n}^{(j)2}(k)]},\\
&\frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\big|\bar{\Upsilon}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k)\big|\big] \le \frac{\lambda^d}{n} n_k \sqrt{\mathrm{E}_{\cdot|\mathbf{s}}[\bar{\Upsilon}_{b,n}^{(i)2}(k)]\,\mathrm{E}_{\cdot|\mathbf{s}}[\bar{X}_{b,n}^{(j)2}(k)]}.
\end{aligned}$$

Hence, to prove (C.22), it suffices to establish

$$\begin{aligned}\frac{\lambda^d}{n} n_k\,\mathrm{E}_{\cdot|\mathbf{s}}\big(\bar{X}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k)\big) = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\big(\frac{\lambda^{d/2}}{n^{1/2}b^{d/2}}\big)\\ + \mathcal{O}_p\big(\frac{\log\lambda}{\lambda}\big) + \mathcal{O}_p\big(\frac{1}{b}\big) + \mathcal{O}_p\big(\frac{\lambda^d}{nb^{d/2}}\big).\end{aligned} \tag{C.23}$$

Observe that

$$\frac{\lambda^d}{n} n_k \mathrm{E}_{\cdot|\mathbf{s}}\big(\bar{X}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k)\big) = \frac{\lambda^d}{n} n_k^{-1} \sum_{i_1\ne i_2} \sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2}) + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}). \tag{C.24}$$

From (C.18), $n_k^{-1} = \frac{\lambda^d}{nb^d}\big(1 + \mathcal{O}_p(\frac{\log\lambda}{\lambda}) + \mathcal{O}_p(\frac{\lambda^{d/2}}{n^{1/2}b^{d/2}})\big)$. Substituting into (C.24) gives

$$\frac{\lambda^d}{n} n_k \mathrm{E}_{\cdot|\mathbf{s}}\big(\bar{X}_{b,n}^{(i)}(k)\bar{X}_{b,n}^{(j)}(k)\big) = \frac{\lambda^{2d}}{n^2 b^d}\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2}) + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0})$$
$$+ \mathcal{O}_p\big(\frac{\log\lambda}{\lambda} + \mathcal{O}_p(\frac{\lambda^{d/2}}{n^{1/2}b^{d/2}})\big)\frac{\lambda^{2d}}{n^2 b^d}\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2}).$$

Therefore, to verify (C.23), it suffices to show

$$\mathrm{E}\Big\{\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2})\Big\} = \frac{n^2 b^d}{\lambda^{2d}}\int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \mathcal{O}\Big(\frac{n^2 b^d\log\lambda}{\lambda^{2d+1}}\Big) + \mathcal{O}\Big(\frac{n^2 b^{d-1}}{\lambda^{2d}}\Big), \tag{C.25}$$

and

$$\mathrm{Var}\Big\{\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2})\Big\} = \mathcal{O}\Big(\frac{n^3 b^d}{\lambda^{3d}}\Big) + \mathcal{O}\Big(\frac{n^2 b^d}{\lambda^{2d}}\Big). \tag{C.26}$$

Under Assumption 3,

$$\mathrm{E}\Big\{\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2})\Big\} = n(n-1)\mathrm{E}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b}(\mathbf{s}_1,\mathbf{s}_2)]. \tag{C.27}$$

Analogous to the derivation of (C.14), we obtain

$$\mathrm{E}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b}(\mathbf{s}_1,\mathbf{s}_2)] = \frac{b^d}{\lambda^{2d}}\int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \mathcal{O}\Big(\frac{b^d\log\lambda}{\lambda^{2d+1}}\Big) + \mathcal{O}\Big(\frac{b^{d-1}}{\lambda^{2d}}\Big). \tag{C.28}$$

Combining (C.27) with (C.28) produces (C.25).

Since $\mathbf{s}_1,\ldots,\mathbf{s}_n$ are i.i.d. sampling locations,

$$\mathrm{Var}\Big\{\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2})\Big\} = 2n(n-1)\mathrm{Var}[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2)]$$
$$+ 4n(n-1)(n-2)\mathrm{Cov}\big(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2), \sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_3)\big).$$

Following the same reasoning as in deriving (C.15), we have

$$\mathrm{Var}\Big\{\sum_{i_1\neq i_2}\sigma_{ij}(\mathbf{s}_{i_2}-\mathbf{s}_{i_1})\mathbb{I}_{R_b(k)}(\mathbf{s}_{i_1},\mathbf{s}_{i_2})\Big\} = \mathcal{O}(n^3)\,\mathrm{E}\big(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)\big)$$
$$+\,\mathcal{O}(n^2)\,\mathrm{E}\big[(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2))^2\big]+\mathcal{O}\big(\frac{n^3b^{2d}}{\lambda^{4d}}\big). \tag{C.29}$$

As in (C.28), $\mathrm{E}\big[\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\sigma_{ij}(\mathbf{s}_3-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)\big] = \mathcal{O}(b^d/\lambda^{3d})$ and $\mathrm{E}\big[(\sigma_{ij}(\mathbf{s}_2-\mathbf{s}_1)\mathbb{I}_{R_b(k)}(\mathbf{s}_1,\mathbf{s}_2))^2\big] = \mathcal{O}(b^d/\lambda^{2d})$. Substituting these bounds into (C.29) yields (C.26), which in turn establishes (C.23). This completes the proof of part (ii). □

D Proof of Theorem 1

D.1 Proof of part (i) of Theorem 1

Proof of part (i) of Theorem 1. Recall that $\mathbf{T}^*_{\lambda,n} = N^{-1/2}\sum_{k=1}^N \hat{\mathbf{T}}_{\lambda,b,n}(k)\,e_k$ with $\hat{\mathbf{T}}_{\lambda,b,n}(k) = \big(\lambda^d/n\big)^{1/2} n_k^{1/2}\big(\bar{\mathbf{Y}}_{b,n}(k)-\bar{\mathbf{Y}}_{\lambda,n}\big)$ for $k=1,\ldots,N$. It can be equivalently written as

$$\mathbf{T}^*_{\lambda,n} = N^{-1/2}\sum_{k=1}^N\big(\mathbf{T}_{b,n}(k)-(n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\big)e_k,$$

where $\mathbf{T}_{b,n}(k) = (\lambda^d/n)^{1/2}n_k^{1/2}\,\bar{\mathbf{Y}}_{b,n}(k)$. Taking conditional variances on both sides yields

$$\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^*_{\lambda,n}) = N^{-1}\sum_{k=1}^N\Big(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top-(n_k/n)^{1/2}\mathbf{T}_{b,n}(k)\mathbf{T}_{\lambda,n}^\top$$
$$-(n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\mathbf{T}_{b,n}(k)^\top+(n_k/n)\,\mathbf{T}_{\lambda,n}\mathbf{T}_{\lambda,n}^\top\Big). \tag{D.1}$$

From (C.18), we have $n_k/n = O_p(b^d/\lambda^d)$ uniformly in k. Applying the triangle inequality, we obtain

$$\begin{aligned}\big\|\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^*_{\lambda,n}) - \Xi_{n/\lambda^d}\big\|_{\max} \le \Big\|N^{-1}\sum_{k=1}^{N}\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \Xi_{n/\lambda^d}\Big\|_{\max} \\ + \mathcal{O}_p(b^{d/2}/\lambda^{d/2})\,\big\|\bar{\mathbf{T}}_{b,n}\mathbf{T}_{\lambda,n}^\top\big\|_{\max} + \mathcal{O}_p(b^d/\lambda^d)\,\big\|\mathbf{T}_{\lambda,n}\mathbf{T}_{\lambda,n}^\top\big\|_{\max}, \qquad \text{(D.2)}\end{aligned}$$

where $\bar{\mathbf{T}}_{b,n} = N^{-1}\sum_{k=1}^{N}\mathbf{T}_{b,n}(k)$ and $\mathbf{T}_{\lambda,n} = \lambda^{d/2}\bar{\mathbf{Y}}_{\lambda,n}$.

From (E.4) in the supplementary material of Kurisu et al. (2024), $\mathrm{E}_{\cdot|\mathbf{s}}\big[\|\bar{\mathbf{Y}}_{\lambda,n}\|_\infty\big] = \mathcal{O}_p(\lambda^{-d/2}\sqrt{\log p})$. Thus, by Markov's inequality,

$$\|\mathbf{T}_{\lambda,n}\|_\infty = \mathcal{O}_p\Big(\sqrt{\log p}\Big) \quad \text{and} \quad \|\bar{\mathbf{T}}_{b,n}\|_\infty = \mathcal{O}_p\Big(\sqrt{\log p}\Big). \qquad \text{(D.3)}$$

Since $\|\mathbf{X}\mathbf{Y}^\top\|_{\max} = \|\mathbf{X}\|_\infty\|\mathbf{Y}\|_\infty$, we further obtain

$$\big\|\bar{\mathbf{T}}_{b,n}\mathbf{T}_{\lambda,n}^\top\big\|_{\max} = \mathcal{O}_p\big(\log p\big) \quad \text{and} \quad \big\|\mathbf{T}_{\lambda,n}\mathbf{T}_{\lambda,n}^\top\big\|_{\max} = \mathcal{O}_p(\log p). \qquad \text{(D.4)}$$

Combining (D.2) and (D.4), to complete the proof of part (i), it suffices to show that

$$\Big\|N^{-1}\sum_{k=1}^{N}\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \Xi_{n/\lambda^d}\Big\|_{\max} = \mathcal{O}_p\big(\frac{\log\lambda}{\lambda}\big) + \mathcal{O}_p\big(\frac{1}{b^{\min(1,d/2)}}\big) + \mathcal{O}_p\big(\frac{b^d\sqrt{\log p}}{\lambda^{d/2}}\big). \qquad \text{(D.5)}$$

By the triangle inequality,

$$\begin{aligned}\Big\|N^{-1}\sum_{k=1}^{N}\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \Xi_{n/\lambda^d}\Big\|_{\max} \le \Big\|N^{-1}\sum_{k=1}^{N}\mathrm{E}(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top) - \Xi_{n/\lambda^d}\Big\|_{\max} \\ + \Big\|N^{-1}\sum_{k=1}^{N}\{\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \mathrm{E}(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top)\}\Big\|_{\max}. \qquad \text{(D.6)}\end{aligned}$$

From (C.22), for all i, j, k,

$$\mathrm{E}(T_{b,n}^{(i)}(k)T_{b,n}^{(j)}(k)) = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) + \mathcal{O}\Big(\frac{\log\lambda}{\lambda}\Big) + \mathcal{O}\Big(\frac{1}{b^{\min(1,d/2)}}\Big).$$

Thus,

$$\big\|N^{-1}\sum_{k=1}^{N}\mathrm{E}(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top) - \Xi_{n/\lambda^d}\big\|_{\max} = \mathcal{O}\Big(\frac{\log\lambda}{\lambda}\Big) + \mathcal{O}\Big(\frac{1}{b^{\min(1,d/2)}}\Big). \tag{D.7}$$

Therefore, using (D.6) and (D.7), in order to prove (D.5), it remains to verify

$$\begin{aligned}&\big\|N^{-1}\sum_{k=1}^{N}\{\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \mathrm{E}(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top)\}\big\|_{\max}\\ &\quad= \mathcal{O}_p\Big(\frac{\sqrt{\log p}(\log b)^{d/2}}{b^{d/2}}\Big) + \mathcal{O}_p\Big(\frac{b^{d/2}\sqrt{\log p}}{\lambda^{d/2}}\Big).\end{aligned} \tag{D.8}$$

Let $\mathbf{S}_k = \mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \mathrm{E}(\mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top)$. By the union bound, for any $\tau > 0$,

$$\mathrm{P}\Big(\Big\|N^{-1}\sum_{k=1}^{N}\mathbf{S}_k\Big\|_{\max} \geq \tau\Big) \leq p^2 \max_{i,j}\mathrm{P}\Big(\Big|N^{-1}\sum_{k=1}^{N}S_k^{(i,j)}\Big| \geq \tau\Big), \tag{D.9}$$

where $S_k^{(i,j)}$ is the (i,j)th entry of $\mathbf{S}_k$. Applying Theorem 6.2 of Rio (2017) to the centered β-mixing sequence $\{S_k^{(i,j)},\, k \geq 1\}$ (cf. Assumption 1), we obtain, for any $\tau > 0$ and $r \geq 1$,

$$\begin{aligned}\mathrm{P}\Big(\Big|N^{-1}\sum_{k=1}^{N}S_k^{(i,j)}\Big| \geq 4\tau\Big) &\leq \mathrm{P}\Big(\sup_{1\leq m\leq N}\Big|\sum_{k=1}^{m}S_k^{(i,j)}\Big| \geq 4N\tau\Big)\\ &\leq 4\Big(1 + \frac{N^2\tau^2}{r s_N^2}\Big)^{-r/2} + 4\tau^{-1}\int_0^{H(N\tau/r)} Q(u)\,\mathrm{d}u,\end{aligned} \tag{D.10}$$

where $s_N^2 = \sum_{k=1}^{N}\sum_{l=1}^{N}\big|\mathrm{Cov}(S_k^{(i,j)}, S_l^{(i,j)})\big|$, $Q(u) = \sup_k Q_k(u)$ with $Q_k(u)$ being the u-upper quantile of $|S_k^{(i,j)}|$, and $H(u) = R^{-1}(u)$ is the generalized inverse of $R(u) =$

$\beta_1^{-1}(u)Q(u)$ with $\alpha_1^{-1}(\cdot)$ being the generalized inverse of the β-mixing coefficient $\beta_1(\cdot)$ defined in Assumption 1(b).

If we can show that

$$\tau^{-1}\int_0^{H(N\tau/r)} Q(u)\,\mathrm{d}u = o\Big(\big(1+\frac{N^2\tau^2}{r s_N^2}\big)^{-r/2}\Big) \quad \text{and} \tag{D.11}$$

$$s_N^2 = \sum_{k=1}^{N}\sum_{l=1}^{N}|\mathrm{Cov}(S_k^{(i,j)}, S_l^{(i,j)})| = \mathcal{O}\big(\frac{\lambda^{2d}(\log b)^d}{b^d}\big) + \mathcal{O}\big(\lambda^d b^d\big), \tag{D.12}$$

then substituting (D.11) and (D.12) into (D.10), and setting $r = 5\log p$ together with $\tau^2 = 5\log p((\log b)^d/b^d + b^d/\lambda^d)$, yields

$$\mathrm{P}\Big(\Big|\, N^{-1}\sum_{k=1}^{N} S_k^{(i,j)}\Big| \geq 4\tau\Big) \leq 2^{2-5\log p/2}(1+o(1)) = \mathcal{O}(p^{-5/2}). \tag{D.13}$$

Finally, combining (D.9) with (D.13) establishes (D.8), which in turn yields (D.5) and completes the proof of part (i).

Proof of (D.11)**.** By Assumption 1(b,c), together with the standard Orlicz–norm product inequality $\|XY\|_{\psi_1} \leq \|X\|_{\psi_2}\|Y\|_{\psi_2}$ (Vershynin, 2018, Lemma 2.8.6), we obtain the uniform bounds

$$\|\bar{Y}_{b,n}^{(j)}(k)\|_{\psi_2} = \mathcal{O}\big(\frac{D_n\lambda^{d/2}}{n^{1/2}b^{d/2}}\big) \quad \text{and} \quad \|\bar{Y}_{b,n}^{(i)}(k)\bar{Y}_{b,n}^{(j)}(k)\|_{\psi_1} = \mathcal{O}\big(\frac{D_n^2\lambda^d}{nb^d}\big)$$

for all $i, j = 1, \ldots, p$ and $k = 1, \ldots, N$, where D_n is given in Assumption 1(c). Recalling that $\mathbf{T}_{b,n}(k) = (\lambda^d/n)^{1/2} n_k^{1/2}\,\bar{\mathbf{Y}}_{b,n}(k)$, it follows that

$$\|S_k^{(i,j)}\|_{\psi_1} = \big\|T_{b,n}^{(i)}(k)T_{b,n}^{(j)}(k) - \mathrm{E}[T_{b,n}^{(i)}(k)T_{b,n}^{(j)}(k)]\big\|_{\psi_1} = \mathcal{O}\big(\frac{D_n^2\lambda^d}{n}\big).$$

Applying the sub-exponential tail inequality (Vershynin, 2018, Proposition 2.8.1), we have

$$Q(u) = \mathcal{O}\Big(\frac{D_n^2\lambda^d}{n}\log(1/u)\Big) \qquad \text{for } u \in (0,1). \tag{D.14}$$

Moreover, Assumption 1(b) implies $\beta_1^{-1}(u) = \mathcal{O}(u^{-1/\eta})$, and hence

$$R(u) = \beta_1^{-1}(u)Q(u) = \mathcal{O}\Big(\frac{D_n^2\lambda^d}{n}u^{-1/\eta}\log(1/u)\Big). \tag{D.15}$$

Taking u_t such that $u_t \lesssim (\frac{nt}{D_n^2\lambda^d})^{-\eta}[\log(\frac{D_n^2\lambda^d}{nt})]^{-\eta}$, then we have $u_t^{-1/\eta} = \mathcal{O}(\frac{nt}{D_n^2\lambda^d}(\log\frac{nt}{D_n^2\lambda^d})^{-1})$ and $\log(1/u_t) = \mathcal{O}(\log(\frac{nt}{D_n^2\lambda^d}))$. Substituting u_t into (D.14), we obtain $R(u_t) = \mathcal{O}(t)$. Therefore,

$$H(t) = R^{-1}(t) = \mathcal{O}\Big((\frac{D_n^2\lambda^d}{nt})^{\eta}\big[\log(\frac{nt}{D_n^2\lambda^d})\big]^{\eta}\Big), \quad \frac{nt}{D_n^2\lambda^d} \to \infty. \tag{D.16}$$

Combining (D.14), (D.16) and the identity $\int_0^y \log(1/u)\,\mathrm{d}u = y(1-\log y)$ for $y \in (0,1)$, we obtain

$$\begin{aligned}\tau^{-1}\int_0^{H(N\tau/r)} Q(u)\,\mathrm{d}u &= \mathcal{O}\Big(\frac{D_n^2\lambda^d}{n\tau}H(N\tau/r)\big(1-\log H(N\tau/r)\big)\Big)\\ &= \mathcal{O}\Big((\frac{D_n^2\lambda^d}{n\tau})(\frac{D_n^2\log p}{n\tau})^{\eta}\big[\log\frac{n\tau}{D_n^2\log p}\big]^{\eta+1}\Big).\end{aligned}$$

Let $E_{\lambda,b,n} := \frac{D_n^2\lambda^{3d/2}}{nb^{d/2}}$. Since $\tau^2 = 5\log p((\log b)^d/b^d + b^d/\lambda^d)$, we have

$$\frac{D_n^2\lambda^d}{n\tau} = \mathcal{O}(E_{\lambda,b,n}), \quad \frac{D_n^2\log p}{n\tau} = \mathcal{O}(E_{\lambda,b,n}\frac{\sqrt{\log p}}{\lambda^d}).$$

By conditions $p = \mathcal{O}(n^{\iota})$ and $D_n\log n/b^{d/4} \to 0$, we have $\frac{D_n^2\log p}{n\tau} = \mathcal{O}(\frac{D_n^2\lambda^{d/2}\sqrt{\log p}}{nb^{d/2}}) = o(1)$. Combining this result with $\tau = \mathcal{O}(\sqrt{\log p})$ gives $\log\frac{n\tau}{D_n^2\log p} = \mathcal{O}(\log n)$. Conse-

quently,

$$\tau^{-1}\int_0^{H(N\tau/r)} Q(u)\,\mathrm{d}u = \mathcal{O}\big[(E_{\lambda,b,n})^{\eta+1}\big(\frac{\sqrt{\log p}}{\lambda^d}\big)^{\eta}(\log n)^{\eta+1}\big].$$

Now, to establish (D.11), we need only to verify

$$(E_{\lambda,b,n})^{\eta+1}\big(\frac{\log p}{\lambda^d}\big)^{\eta}(\log n)^{\eta+1} = o(n^{-5\iota/2}). \tag{D.17}$$

Substituting $E_{\lambda,b,n}$ with its expression $\frac{D_n^2\lambda^{3d/2}}{nb^{d/2}}$, we find the condition (D.17) is equivalent to

$$\frac{D_n^4\lambda^{d(3-\frac{2\eta}{1+\eta})}(\log n)^{4-\frac{2}{\eta+1}}}{n^{2-\frac{5\iota}{\eta+1}}b^d} = o(1).$$

This equality follows from the conditions $\eta > 1$, $0 < \iota \le (\eta-1)/5$, and $D_n\log n/b^{d/4} \to 0$. This proves (D.11).

Proof of (D.12)**.** Observe that $\mathrm{Cov}\left(S_k^{(i,j)},\, S_l^{(i,j)}\right) = \mathrm{cum}\left(T_{b,n}^{(i)}(k)\,T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l)\,T_{b,n}^{(j)}(l)\right)$. By Theorem 2.3.2 of Brillinger (2001), the fourth-order cumulant expands as

$$\begin{aligned}\mathrm{cum}\left(T_{b,n}^{(i)}(k)\,T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l)\,T_{b,n}^{(j)}(l)\right) &= \mathrm{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(l)\big)\mathrm{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l)\big)\\ &+ \mathrm{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(i)}(l)\big)\mathrm{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(j)}(l)\big) + \mathrm{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l), T_{b,n}^{(j)}(l)\big).\end{aligned} \tag{D.18}$$

Let $\mathrm{Cum}_{\cdot|\mathbf{s}}$ denote the conditional cumulant given the sampling locations $\{\mathbf{s}_1,\ldots,\mathbf{s}_n\}$. With arguments similar to deriving Proposition B.2(iv) in the online supplementary material of Zhang (2025), we obtain

$$\begin{aligned}\mathrm{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l), T_{b,n}^{(j)}(l)\big) &= \mathrm{E}\big\{\mathrm{cum}_{\cdot|\mathbf{s}}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l), T_{b,n}^{(j)}(l)\big)\big\}\\ &= \mathcal{O}\big(\frac{1}{b^d}\big) + \mathcal{O}\big(\frac{\lambda^d}{nb^d}\big) + \mathcal{O}\big(\frac{\lambda^{2d}}{n^2b^d}\big) + \mathcal{O}\big(\frac{\lambda^{3d}}{n^3b^d}\big)\end{aligned} \tag{D.19}$$

uniformly over $k, l = 1, \ldots, N$. According to (D.18) and (D.19), to prove (D.12), it remains to verify

$$\sum_{k=1}^{N}\sum_{l=1}^{N}\operatorname{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(i)}(l)\big)\operatorname{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(j)}(l)\big) = \mathcal{O}\big(\frac{\lambda^{2d}(\log b)^d}{b^d}\big) + \mathcal{O}\big(\lambda^d b^d\big) \quad \text{(D.20)}$$

and

$$\sum_{k=1}^{N}\sum_{l=1}^{N}\operatorname{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(l)\big)\operatorname{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l)\big) = \mathcal{O}\big(\frac{\lambda^{2d}(\log b)^d}{b^d}\big) + \mathcal{O}\big(\lambda^d b^d\big). \quad \text{(D.21)}$$

From (C.22) and the Cauchy–Schwarz inequality, we obtain

$$\operatorname{cum}_{\cdot|\mathbf{s}}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(i)}(l)\big) \le \sqrt{\operatorname{Var}_{\cdot|\mathbf{s}}\big(T_{b,n}^{(i)}(k)\big)\operatorname{Var}_{\cdot|\mathbf{s}}\big(T_{b,n}^{(i)}(l)\big)} = \mathcal{O}_p(1)$$

uniformly in k, l. Using the same arguments as in the proof of Proposition B.2(iii) in the online supplementary material of Zhang (2025), we obtain

$$\begin{aligned}
&\sum_{k=1}^{N}\sum_{l=1}^{N}\operatorname{cum}\big(T_{b,n}^{(i)}(k), T_{b,n}^{(i)}(l)\big)\operatorname{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(j)}(l)\big) \\
&= \mathcal{O}(1)\sum_{k=1}^{N}\sum_{l=1}^{N}\operatorname{cum}\big(T_{b,n}^{(j)}(k), T_{b,n}^{(j)}(l)\big) = \mathcal{O}\big(\frac{\lambda^{2d}(\log b)^d}{b^d}\big) + \mathcal{O}\big(\lambda^d b^d\big),
\end{aligned}$$

which proves (D.20). The same reasoning yields (D.21). This, in turn, completes the proof of (D.12). □

D.2 Proof of part (ii) of Theorem 1

Proof of part (ii) of Theorem 1. Using the triangle inequality,

$$\big\|\operatorname{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}^{\circ}\big) - \Xi_{n/\lambda^d}\big\|_{\max} \le \big\|\operatorname{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}^{\circ}\big) - \operatorname{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}^{*}\big)\big\|_{\max} + \big\|\operatorname{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}^{*}\big) - \Xi_{n/\lambda^d}\big\|_{\max}.$$

From Theorem 1(i), $\big\|\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^*_{\lambda,n}) - \Xi_{n/\lambda^d}\big\|_{\max} = \mathcal{O}_p(\mathcal{U}_{\lambda,b}) + \mathcal{O}_p(\mathcal{V}_{\lambda,b,p})$. Thus, to prove part (ii), it suffices to show that

$$\big\|\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}^\circ_{\lambda,n}\big) - \mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}^*_{\lambda,n}\big)\big\|_{\max} = \mathcal{O}_p(b^d \log p/\lambda^d). \tag{D.22}$$

Observe that $\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^\circ_{\lambda,n}) = N^{-1}\sum_{k=1}^N (\hat{\mathbf{T}}_{\lambda,b,n}(k) - \tilde{\mathbf{T}}_{\lambda,b,n})(\hat{\mathbf{T}}_{\lambda,b,n}(k) - \tilde{\mathbf{T}}_{\lambda,b,n})^\top$, where $\tilde{\mathbf{T}}_{\lambda,b,n} = N^{-1}\sum_{k=1}^N \hat{\mathbf{T}}_{\lambda,b,n}(k)$, $\hat{\mathbf{T}}_{\lambda,b,n}(k) = \mathbf{T}_{b,n}(k) - (n_k/n)^{1/2}\,\mathbf{T}_{\lambda,n}$, $k = 1,\ldots,N$. Rewriting it yields

$$\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^\circ_{\lambda,n}) = N^{-1}\sum_{k=1}^N \mathbf{T}_{b,n}(k)\mathbf{T}_{b,n}(k)^\top - \bar{\mathbf{T}}_{b,n}\bar{\mathbf{T}}_{b,n}^\top + \mathcal{R}, \tag{D.23}$$

where $\bar{\mathbf{T}}_{b,n} = N^{-1}\sum_{k=1}^N \mathbf{T}_{b,n}(k)$ and

$$\mathcal{R} = -\frac{1}{N}\sum_{k=1}^N \big\{(\mathbf{T}_{b,n}(k) - \bar{\mathbf{T}}_{b,n})\,\delta_k\mathbf{T}_{\lambda,n}^\top + \delta_k\mathbf{T}_{\lambda,n}(\mathbf{T}_{b,n}(k) - \bar{\mathbf{T}}_{b,n})^\top\big\} + \frac{1}{N}\sum_{k=1}^N \delta_k^2\,\mathbf{T}_{\lambda,n}\mathbf{T}_{\lambda,n}^\top,$$

with $\delta_k := (n_k/n)^{1/2} - N^{-1}\sum_{k=1}^N (n_k/n)^{1/2}$. Comparing (D.1) with (D.23) produces

$$\begin{aligned}
&\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}^*_{\lambda,n}\big) - \mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\mathbf{T}^\circ_{\lambda,n}) \\
&= N^{-1}\sum_{k=1}^N \big(\bar{\mathbf{T}}_{b,n}\bar{\mathbf{T}}_{b,n}^\top - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\mathbf{T}_{b,n}(k)^\top - (n_k/n)^{1/2}\mathbf{T}_{b,n}(k)\mathbf{T}_{\lambda,n}^\top + (n_k/n)\mathbf{T}_{\lambda,n}\mathbf{T}_{\lambda,n}^\top\big) - \mathcal{R} \\
&= N^{-1}\sum_{k=1}^N \big[\bar{\mathbf{T}}_{b,n} - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\big]\big[\bar{\mathbf{T}}_{b,n} - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\big]^\top.
\end{aligned}$$

Thus, to establish (D.22), it suffices to show

$$\sup_{1\le k\le N}\big\|\bar{\mathbf{T}}_{b,n} - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\big\|_\infty = \mathcal{O}_p\big(\frac{b^{d/2}\sqrt{\log p}}{\lambda^{d/2}}\big). \tag{D.24}$$

Note that

$$\bar{\mathbf{T}}_{b,n} - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n} = \frac{\lambda^{d/2}}{n^{1/2}N}\sum_{i=1}^n \mathbf{Y}(\mathbf{s}_i)\sum_{k=1}^N n_k^{-1/2}\mathbb{I}_{R_b(k)}(\mathbf{s}_i) - (n_k/n)^{1/2}\lambda^{d/2}\bar{\mathbf{Y}}_{\lambda,n}.$$

Combining this with (D.3) and using $n_k/n = \mathcal{O}_p(b^d/\lambda^d)$, we obtain

$$\sup_{1\le k\le N} \big\|\bar{\mathbf{T}}_{b,n} - (n_k/n)^{1/2}\mathbf{T}_{\lambda,n}\big\|_\infty = \mathcal{O}_p(b^{d/2})\|\bar{\mathbf{Y}}_{\lambda,n}\|_\infty = \mathcal{O}_p\Big(\frac{b^{d/2}\sqrt{\log p}}{\lambda^{d/2}}\Big),$$

which establishes (D.24), and therefore completes the proof of part (ii). □

E Proofs of Proposition 1 and Theorems 2 and 3

E.1 Proof of part (i) of Proposition 1

Proof of part (i) of Proposition 1. Let $\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}$ denote the conditional probability given the sampling locations $\{\mathbf{s}_1,\dots,\mathbf{s}_n\}$ and the observations $\{\mathbf{X}(\mathbf{s}_1),\dots,\mathbf{X}(\mathbf{s}_n)\}$, and let $\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}$ denote the corresponding expectation.

Recall that $\mathbf{T}^*_{\lambda,n} = N^{-1/2}\sum_{k=1}^N (\lambda^d/n)^{1/2} n_k^{1/2}(\bar{\mathbf{Y}}_{b,n}(k) - \bar{\mathbf{Y}}_{\lambda,n})e_k$. Using (C.18), this becomes $\mathbf{T}^*_{\lambda,n} = \big(1+o_p(1)\big)\, N^{-1/2} b^{d/2}\sum_{k=1}^N (\bar{\mathbf{Y}}_{b,n}(k) - \bar{\mathbf{Y}}_{\lambda,n})e_k$. Define

$$\begin{aligned}\gamma_{\lambda,n,b} := \sum_{k=1}^N (\bar{\mathbf{Y}}_{b,n}(k) - \bar{\mathbf{Y}}_{\lambda,n})e_k &= \sum_{k=1}^N \bar{\mathbf{X}}_{b,n}(k)e_k - \sum_{k=1}^N \bar{\mathbf{Y}}_{\lambda,n}e_k + \sum_{k=1}^N \bar{\mathbf{\Upsilon}}_{b,n}(k)e_k \\ &=: \gamma^{(1)}_{\lambda,n,b} - \gamma^{(2)}_{\lambda,n,b} + \gamma^{(3)}_{\lambda,n,b}.\end{aligned}$$

Suppose that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})$,

$$\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\Big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty > n^{-c_1/2}\log^2 p\Big) \le n^{-c_1/3}, \tag{E.1}$$

$$\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\Big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty > n^{-c_1/2}\log^2 p\Big) \le n^{-c_1/3}. \tag{E.2}$$

Then, by arguments analogous to those leading to (E.3) in the supplementary materials of Kurisu et al. (2024), we obtain that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})$, for

any rectangle $A = \prod_{j=1}^{p}[a_j, b_j]$ with $\mathbf{a} = (a_1, \ldots, a_p)^\top$ and $\mathbf{b} = (b_1, \ldots, b_p)^\top$,

$$\begin{aligned}
&\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\boldsymbol{\gamma}_{\lambda,n,b} \in A\big) \leq \mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\{-N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq -\mathbf{a} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\} \\
&\qquad \cap \{N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq \mathbf{b} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\}\big) + \mathcal{O}(n^{-c_1/3}), \qquad \text{(E.3)} \\
&\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\boldsymbol{\gamma}_{\lambda,n,b} \in A\big) \geq \mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\{-N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq -\mathbf{a} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\} \\
&\qquad \cap \{N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq \mathbf{b} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\}\big) - \mathcal{O}(n^{-c_1/3}). \qquad \text{(E.4)}
\end{aligned}$$

Next, suppose we can show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\begin{aligned}
&\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\{-N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq -\mathbf{a} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\} \cap \{N^{-1/2}b^{d/2}\boldsymbol{\gamma}^{(1)}_{\lambda,n,b} \leq \mathbf{b} + 2n^{-c_1/2}\log^2 p \cdot \mathbf{1}\}\big) \\
&\quad \leq \mathrm{P}_{\cdot|\mathbf{s}}\big(\{-\mathbf{Z}_{\lambda,n} \leq -\mathbf{a} + 2^{d+1}n^{-c_1/2}\log^2 p \cdot \mathbf{1}\} \cap \{\mathbf{Z}_{\lambda,n} \leq \mathbf{b} + 2^{d+1}n^{-c_1/2}\log^2 p \cdot \mathbf{1}\}\big) \\
&\qquad + \mathcal{O}(n^{-c_1/4}). \qquad \text{(E.5)}
\end{aligned}$$

Applying Nazarov's inequality (cf. Chernozhukov et al., 2017, Lemma A.1), it then follows from (E.3) and (E.4) that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\begin{aligned}
\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\boldsymbol{\gamma}_{\lambda,n,b} \in A\big) &\leq \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n} \in A\big) + \mathcal{O}(n^{-c_1/4}), \\
\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\boldsymbol{\gamma}_{\lambda,n,b} \in A\big) &\geq \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n} \in A\big) - \mathcal{O}(n^{-c_1/4}),
\end{aligned}$$

hold uniformly over $A \in \mathcal{A}$. This completes the proof. It therefore remains to establish (E.1), (E.2), and (E.5).

Proof of (E.1)**.** For any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, applying Markov's inequality to the outer

probability $\mathrm{P}_{\cdot|\mathbf{s}}$ yields

$$\mathrm{P}_{\cdot|\mathbf{s}}\Big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty > \varepsilon_1\big) > \varepsilon_2\Big) \leq \frac{1}{\varepsilon_2}\mathrm{E}_{\cdot|\mathbf{s}}\Big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty > \varepsilon_1\big)\Big)$$
$$\leq \frac{1}{\varepsilon_1\varepsilon_2}\mathrm{E}_{\cdot|\mathbf{s}}\Big(\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty\big)\Big). \tag{E.6}$$

Observe that

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty\big)\big] = N^{-1/2}b^{d/2}\,\mathrm{E}_{\cdot|\mathbf{s}}\Big[\|\bar{\mathbf{Y}}_{\lambda,n}\|_\infty\,\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\big|\sum_{k=1}^{N} e_k\big|\big)\Big].$$

Utilizing the Cauchy-Schwarz inequality gives

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[\big|\sum_{k=1}^{N} e_k\big|\big] \leq \big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\big|\sum_{k=1}^{N} e_k\big|\big)^2\big]^{1/2} = \sqrt{N}. \tag{E.7}$$

Combining (E.7) with the result $\mathrm{E}_{\cdot|\mathbf{s}}\big[\|\bar{\mathbf{Y}}_{\lambda,n}\|_\infty\big] = \mathcal{O}_p\big(\lambda^{-d/2}\sqrt{\log p}\,\big)$ (see Kurisu et al., 2024, (E.4) in the supplementary material) yields

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(2)}_{\lambda,n,b}\|_\infty\big)\Big] = \mathcal{O}_p\Big(b^{d/2}\lambda^{-d/2}\sqrt{\log p}\,\Big). \tag{E.8}$$

Substituting (E.8) into (E.6) and taking $\varepsilon_1 = n^{-c_1/2}\log^2 p$ and $\varepsilon_2 = n^{-c_1/3}$ together establish (E.1).

Proof of (E.2)**.** Again applying Markov's inequality, we obtain

$$\mathrm{P}_{\cdot|\mathbf{s}}\Big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty > n^{-c_1/2}\log^2 p\big) > n^{-c_1/3}\Big)$$
$$\leq \frac{1}{n^{-5c_1/6}\log^{1/2} p}\mathrm{E}_{\cdot|\mathbf{s}}\Big(\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty\big)\Big). \tag{E.9}$$

Next,

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big(\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty\big)\Big) \leq N^{-1/2}b^{d/2}\mathrm{E}_{\cdot|\mathbf{s}}\Big(\max_{1\leq k\leq N}\|\bar{\boldsymbol{\Upsilon}}_{b,n}(k)\|_\infty\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\sum_{k=1}^{N}\big|e_k\big|\big)\Big).$$

Using Assumption 2, we have the uniform bound

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty\big)\Big] \le N^{1/2}b^{d/2}\,\mathrm{E}_{\cdot|\mathbf{s}}\Big[\max_{1\le k\le N}\|\bar{\boldsymbol{\Upsilon}}_{b,n}(k)\|_\infty\Big] = \mathcal{O}_p\Big(\frac{b^{d/2}\log\lambda}{\lambda^{d/2+1}}\Big).$$

Consequently,

$$\frac{1}{n^{-5c_1/6}\log^{1/2}p}\,\mathrm{E}_{\cdot|\mathbf{s}}\Big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}b^{d/2}\|\gamma^{(3)}_{\lambda,n,b}\|_\infty\big)\Big] = \mathcal{O}_p(n^{-c_1/6}). \tag{E.10}$$

Combining (E.9) and (E.10) establishes (E.2).

Proof of (E.5)**.** We establish (E.5) in three steps.

Step 1. We show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$, the following inequality holds:

$$\begin{aligned}
&\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\{-N^{-1/2}b^{d/2}\gamma^{(1)}_{\lambda,n,b}\le -\mathbf{a}+2n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\cap\{N^{-1/2}b^{d/2}\gamma^{(1)}_{\lambda,n,b}\le \mathbf{b}+2n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\big)\\
&\le \mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\{-\omega_{\lambda,n,b}\tilde{\gamma}^{(11)}_{n,b}\le -\mathbf{a}+2^{d+1}n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\\
&\qquad\cap\{\omega_{\lambda,n,b}\tilde{\gamma}^{(11)}_{n,b}\le \mathbf{b}+2^{d+1}n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\big)+\mathcal{O}(n^{-c_1/3})
\end{aligned} \tag{E.11}$$

using the large-block–small-block construction scheme (cf. Kurisu et al., 2024, the proof of Theorem 4.1 in the supplementary materials), where $\omega_{\lambda,n,b}=N^{-1/2}\lambda^d/(nb^{d/2})$, $\tilde{\gamma}^{(11)}_{n,b}$ denotes a sum of independent random variables, detailed in (E.12).

We set $\lambda_3=\lambda_1+\lambda_2$, where λ_1 and λ_2 are the sequences from Assumption 1(b). Partition $\mathbb{R}^d$ into hypercubes of side length λ_3,

$$\Gamma_n(\boldsymbol{\ell};\mathbf{0})=(\boldsymbol{\ell}+(0,1]^d)\lambda_3,\quad \boldsymbol{\ell}=(\ell_1,\dots,\ell_d)^\top\in\mathbb{Z}^d.$$

Each cube $\Gamma_n(\boldsymbol{\ell};\mathbf{0})$ is further subdivided into 2^d disjoint subcubes $\Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon})=\prod_{j=1}^d I_j(\epsilon_j)$,

$\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_d)^\top \in \{1,2\}^d$, where

$$I_j(\epsilon_j) = \begin{cases} (\ell_j\lambda_3, \ell_j\lambda_3 + \lambda_1), & \epsilon_j = 1, \\ (\ell_j\lambda_3 + \lambda_1, \ell_j\lambda_3 + \lambda_3), & \epsilon_j = 2, \end{cases} \qquad j = 1, \ldots, d.$$

Let $q(\boldsymbol{\epsilon}) = [\![\{1 \le j \le d : \epsilon_j = 1\}]\!]$. Then $|\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon})| = \lambda_1^{q(\boldsymbol{\epsilon})}\lambda_2^{d-q(\boldsymbol{\epsilon})}$. Let $\boldsymbol{\epsilon}_0 = (1, \ldots, 1)^\top$. Since $\lambda_2/\lambda_1 \to 0$, $|\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon})| = o(|\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon}_0)|)$ for all $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0$. Thus the cubes $\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon}_0)$ are "big blocks", while the remaining $2^d - 1$ subcubes $\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon})$ with $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0$ are "small blocks".

Define the index sets $L_{1n} = \left\{\boldsymbol{\ell} \in \mathbb{Z}^d : \Gamma_n(\boldsymbol{\ell}; \mathbf{0}) \subset R_\lambda\right\}$ and $L_{2n} = \left\{\boldsymbol{\ell} \in \mathbb{Z}^d : \Gamma_n(\boldsymbol{\ell}; \mathbf{0}) \cap R_\lambda \neq \emptyset,\ \Gamma_n(\boldsymbol{\ell}; \mathbf{0}) \cap R_\lambda^c \neq \emptyset\right\}$. Thus L_{1n} indexes cubes fully contained in R_λ, whereas L_{2n} indexes boundary cubes intersecting both R_λ and R_λ^c. Let $L_n = L_{1n} \cup L_{2n}$. Elementary geometric arguments imply

$$[\![L_{1n}]\!] \asymp (\lambda/\lambda_3)^d \asymp (\lambda/\lambda_1)^d, \quad [\![L_{2n}]\!] \asymp (\lambda/\lambda_3)^{d-1} \asymp (\lambda/\lambda_1)^{d-1}.$$

For $\boldsymbol{\ell} \in L_n$ and $\boldsymbol{\epsilon} \in \{1,2\}^d$, define

$$\tilde{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon}) = \left(\tilde{\gamma}_{n,b}^{(1)}(\boldsymbol{\ell}; \boldsymbol{\epsilon}), \ldots, \tilde{\gamma}_{n,b}^{(p)}(\boldsymbol{\ell}; \boldsymbol{\epsilon})\right)^\top = \sum_{k=1}^{N} \sum_{i:\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon}) \cap R_b(k)} \mathbf{X}(\mathbf{s}_i)\, e_k.$$

Since each $\mathbf{s}_i \in R_b(k)$ lies in exactly one subcube $\Gamma_n(\boldsymbol{\ell}; \boldsymbol{\epsilon})$, $N^{-1/2} b^{d/2} \boldsymbol{\gamma}_{\lambda,n,b}^{(1)} = \omega_{\lambda,n,b}\, \tilde{\gamma}_{n,b}(1+$

$o_p(1)$), where

$$\tilde{\gamma}_{n,b} = \sum_{k=1}^{N} \sum_{\mathbf{s}_i \in R_b(k)} \mathbf{X}(\mathbf{s}_i) e_k = \underbrace{\sum_{\boldsymbol{\ell} \in L_n} \tilde{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon}_0)}_{=:\tilde{\gamma}_{n,b}^{(11)}} + \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0} \underbrace{\sum_{\boldsymbol{\ell} \in L_{1n}} \tilde{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon})}_{=:\tilde{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon})} + \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0} \underbrace{\sum_{\boldsymbol{\ell} \in L_{2n}} \tilde{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon})}_{=:\tilde{\gamma}_{n,b}^{(13)}(\boldsymbol{\epsilon})}$$

$$=: \tilde{\gamma}_{n,b}^{(11)} + \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0} \tilde{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon}) + \sum_{\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0} \tilde{\gamma}_{n,b}^{(13)}(\boldsymbol{\epsilon}). \tag{E.12}$$

If we can show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\big\|\omega_{\lambda,n,b}\tilde{\gamma}_{n,b}^{(1\kappa)}(\boldsymbol{\epsilon})\big\|_\infty > n^{-c_1/2}\log^2 p\big) \leq n^{-c_1/3}, \quad \text{for } \kappa = 2,3,\ \boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0\ , \tag{E.13}$$

then, using arguments parallel to those in (E.8) of the supplementary materials of Kurisu et al. (2024), we obtain that the inequality (E.11) holds with the stated probability.

In what follows, we prove (E.13).

For any $\boldsymbol{\ell}_1, \boldsymbol{\ell}_2 \in L_n$ with $\boldsymbol{\ell}_1 \neq \boldsymbol{\ell}_2$, applying the same arguments as those used in the proof of (E.7) in the supplementary materials of Kurisu et al. (2024), we obtain, for any $\varepsilon_3, \varepsilon_4 > 0$,

$$\sup_{\varepsilon_4 \geq 0} \big|\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}(\|\tilde{\gamma}_{n,b}^{(1\kappa)}(\boldsymbol{\epsilon})\|_\infty > \varepsilon_3) > \varepsilon_4\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}(\|\breve{\gamma}_{n,b}^{(1\kappa)}(\boldsymbol{\epsilon})\|_\infty > \varepsilon_3) > \varepsilon_4\big)\big| \lesssim \mathcal{W}_{\lambda,\lambda_1,\lambda_2},$$

which holds for $\kappa = 2,3$, where $\breve{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{\ell} \in L_{1n}} \breve{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon})$, $\breve{\gamma}_{n,b}^{(13)}(\boldsymbol{\epsilon}) = \sum_{\boldsymbol{\ell} \in L_{2n}} \breve{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon})$, and $\{\breve{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon}) : \boldsymbol{\ell} \in L_n, \boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0\}$ is a sequence of independent random vectors satisfying $\breve{\gamma}_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon}) \overset{d}{=} \gamma_{n,b}(\boldsymbol{\ell}; \boldsymbol{\epsilon})$ under $\mathrm{P}_{\cdot|\mathbf{s}}$. Hence, to establish (E.13), it suffices to show that, for $\kappa = 2,3$ and $\boldsymbol{\epsilon} \neq \boldsymbol{\epsilon}_0$,

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}(\big\|\omega_{\lambda,n,b}\breve{\gamma}_{n,b}^{(1\kappa)}(\boldsymbol{\epsilon})\big\|_\infty > n^{-c_1/2}\log^2 p) > n^{-c_1/3}\big) = \mathcal{O}(n^{-c_1/6}). \tag{E.14}$$

We detail the case $\kappa = 2$. By Markov's inequality,

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}(\big\|\omega_{\lambda,n,b}\breve{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon})\big\|_\infty > \varepsilon_3) > \varepsilon_4\big) \leq \frac{1}{\varepsilon_3\varepsilon_4}\mathrm{E}_{\cdot|\mathbf{s}}\big(\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}(\|\omega_{\lambda,n,b}\breve{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon})\|_\infty)\big). \quad \text{(E.15)}$$

Applying Lemma H.1 of the supplementary materials in Kurisu et al. (2024), there exists a constant $K_1 > 0$ such that

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}(\|\omega_{\lambda,n,b}\breve{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon})\|_\infty)\big] \leq K_1\omega_{\lambda,n,b}\Big\{\mathrm{E}_{\cdot|\mathbf{s}}[\tilde{\sigma}]\sqrt{\log p} + \mathrm{E}_{\cdot|\mathbf{s}}\big[\sqrt{\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}(M^2)}\big]\log p\Big\}, \quad \text{(E.16)}$$

where $M = \max_{\boldsymbol{\ell}\in L_{1n}}\|\breve{\gamma}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon})\|_\infty$ and $\tilde{\sigma}^2 = \max_{1\leq j\leq p}\sum_{\boldsymbol{\ell}\in L_{1n}}\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}[\breve{\gamma}_{n,b}^{(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon})^2]$. Therefore, to prove (E.14) for $\kappa = 2$, it remains to verify that

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\sqrt{\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}(M^2)}\big] = \mathcal{O}_p\big(b^d\lambda_1^{d-1}\lambda_2 n\lambda^{-d}D_n\log n\log p\big), \quad \text{(E.17)}$$

$$\mathrm{E}_{\cdot|\mathbf{s}}[\tilde{\sigma}] = \mathcal{O}_p\big(n\lambda^{-d/2}b^{d/2}\lambda_1^{-1/2}\lambda_2^{1/2}(\log n)(\log p)D_n\big). \quad \text{(E.18)}$$

Let $\mathcal{D}_{\boldsymbol{\ell},k} = \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon})\cap R_b(k)$ and $\mathcal{S}_{\boldsymbol{\ell},k} = \sum_{\mathbf{s}_i\in\mathcal{D}_{\boldsymbol{\ell},k}}\mathbf{X}(\mathbf{s}_i)$. Then we have $\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}[M^2] = \mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[\max_{\boldsymbol{\ell}\in L_{1n}}\big\|\sum_{k=1}^N e_k\mathcal{S}_{\boldsymbol{\ell},k}\big\|_\infty^2\big]$. By the triangle inequality and the Cauchy-Schwarz inequality,

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[M^2\big] \leq \mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\Big[\Big(\sum_{k=1}^N|e_k|\max_{\boldsymbol{\ell}\in L_{1n}}\|\mathcal{S}_{\boldsymbol{\ell},k}\|_\infty\Big)^2\Big] \leq \Big(\sum_{k=1}^N\max_{\boldsymbol{\ell}\in L_{1n}}\|\mathcal{S}_{\boldsymbol{\ell},k}\|_\infty\Big)^2.$$

Lemma D.1 of the supplementary materials of Kurisu et al. (2024) implies that $\sum_{i=1}^n\mathbb{I}_{\mathcal{D}_{\boldsymbol{\ell},k}}(\mathbf{s}_i) \lesssim |\mathcal{D}_{\boldsymbol{\ell},k}|n\lambda^{-d}$ hold uniformly in $\boldsymbol{\ell}\in L_{1n}$ and $k = 1,\ldots,N$. Since $\sum_{k=1}^N|\mathcal{D}_{\boldsymbol{\ell},k}| \leq$

$b^d|\Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon})| = b^d\lambda_1^{d-1}\lambda_2$, we obtain

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big(\sum_{k=1}^{N}\max_{\boldsymbol{\ell}\in L_{1n}}\|\mathcal{S}_{\boldsymbol{\ell},k}\|_\infty\Big) \lesssim b^d\lambda_1^{d-1}\lambda_2 n\lambda^{-d}\mathrm{E}_{\cdot|\mathbf{s}}\big(\max_{1\le i\le n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty\big)$$

$$\lesssim b^d\lambda_1^{d-1}\lambda_2 n\lambda^{-d}(\log n)\big\|\|\mathbf{X}(\mathbf{u})\|_\infty\big\|_{\psi_1} \lesssim b^d\lambda_1^{d-1}\lambda_2 n\lambda^{-d}(\log n)(\log p)D_n, \tag{E.19}$$

where the second inequality follows from a standard bound on the Orlicz norm (cf. van der Vaart and Wellner, 1996, pp. 146), and the last inequality follows from Assumption 1(c). This proves (E.17).

Next, observe that

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big\{\max_{1\le j\le p}\sum_{\boldsymbol{\ell}\in L_{1n}}\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\Big[(\sum_{k=1}^{N}e_k\mathcal{S}_{\boldsymbol{\ell},k}^{(j)})^2\Big]\Big\} = \mathrm{E}_{\cdot|\mathbf{s}}\Big\{\max_{1\le j\le p}\sum_{\boldsymbol{\ell}\in L_{1n}}\sum_{k=1}^{N}(\mathcal{S}_{\boldsymbol{\ell},k}^{(j)})^2\Big\}. \tag{E.20}$$

Using again the bound $\sum_{i=1}^n \mathbb{I}_{\mathcal{D}_{\boldsymbol{\ell},k}}(\mathbf{s}_i) \lesssim |\mathcal{D}_{\boldsymbol{\ell},k}|n\lambda^{-d}$, we obtain

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big\{\max_{1\le j\le p}\sum_{\boldsymbol{\ell}\in L_{1n}}\sum_{k=1}^{N}(\mathcal{S}_{\boldsymbol{\ell},k}^{(j)})^2\Big\} \lesssim \sum_{\boldsymbol{\ell}\in L_{1n}}\sum_{k=1}^{N}\mathrm{E}_{\cdot|\mathbf{s}}\big[(n\lambda^{-d}|\mathcal{D}_{\boldsymbol{\ell},k}|\max_{1\le i\le n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty\big)^2\big]. \tag{E.21}$$

Since $\sum_{k=1}^N|\mathcal{D}_{\boldsymbol{\ell},k}|^2 \le b^d\lambda_1^{2d-2}\lambda_2^2$, arguments analogous to those used in (E.19) yield

$$\sum_{k=1}^{N}\mathrm{E}_{\cdot|\mathbf{s}}\Big[\big(n\lambda^{-d}|\mathcal{D}_{\boldsymbol{\ell},k}|\max_{1\le i\le n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty\big)^2\Big] \le n^2\lambda^{-2d}b^d\lambda_1^{d-1}\lambda_2\,\mathrm{E}_{\cdot|\mathbf{s}}\Big[\max_{1\le i\le n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty^2\Big]$$

$$\lesssim n^2\lambda^{-2d}b^d\lambda_1^{d-1}\lambda_2(\log n)^2(\log p)^2D_n^2. \tag{E.22}$$

Combining (E.20)–(E.22) yields

$$\mathrm{E}_{\cdot|\mathbf{s}}\big\{\max_{1\le j\le p}\sum_{\boldsymbol{\ell}\in L_{1n}}\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[(\sum_{k=1}^{N}e_k\mathcal{S}_{\boldsymbol{\ell},k}^{(j)})^2\big]\big\} \lesssim n^2\lambda^{-d}b^d\lambda_1^{-1}\lambda_2(\log n)^2(\log p)^2D_n^2. \tag{E.23}$$

Then, an application of Jensen's inequality leads to (E.18).

Substituting (E.17) and (E.18) into (E.16), we conclude that

$$\begin{aligned}\mathrm{E}_{\cdot|\mathbf{s}}\big[\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}(\|\omega_{\lambda,n,b}\breve{\gamma}_{n,b}^{(12)}(\boldsymbol{\epsilon})\|_\infty)\big] \lesssim & \frac{\lambda_2^{1/2}(\log p)^{3/2}(\log n)D_n}{\lambda_1^{1/2}} + \frac{b^{d/2}\lambda_1^{d-1}\lambda_2 D_n \log n \log^2 p}{\lambda^{d/2}} \\ \lesssim & n^{-c_1}\log^2 p,\end{aligned}$$

where the last equality follows from Assumption 4(a). Taking $\varepsilon_3 = n^{-c_1/2}\log^2 p$ and $\varepsilon_4 = n^{-c_1/3}$ in (E.15) then establishes (E.14) for $\kappa = 2$. The proof for $\kappa = 3$ is analogous and is therefore omitted. This completes the proof of (E.11).

Step 2. We show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6})$,

$$\sup_{A\in\mathcal{A}}\big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\omega_{\lambda,n,b}\tilde{\gamma}_{n,b}^{(11)} \in A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{G}_{\lambda,n,b} \in A\big)\big| = \mathcal{O}(n^{-c_1/4}), \tag{E.24}$$

where $\mathbf{G}_{\lambda,n,b}$ is a centered Gaussian random vector under $\mathrm{P}_{|\mathbf{s}}$ with variance matrix given as $\omega_{\lambda,n,b}^2 \sum_{\boldsymbol{\ell}\in L_n} \mathrm{E}_{\cdot|\mathbf{s}}\big[\tilde{\gamma}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\tilde{\gamma}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)^\top\big]$.

To establish (E.24), it suffices to show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6})$,

$$\sup_{A\in\mathcal{A}}\big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\omega_{\lambda,n,b}\tilde{\gamma}_{n,b}^{(11)} \in A\big) - \mathrm{P}_{|\mathbf{s},\mathbf{X}}\big(\omega_{\lambda,n,b}\sum_{\boldsymbol{\ell}\in L_n}\tilde{\gamma}_{n,b}^{*}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \in A\big)\big| = \mathcal{O}\big(n^{-c_1/4}\big), \tag{E.25}$$

$$\sup_{A\in\mathcal{A}}\big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}\big(\omega_{\lambda,n,b}\sum_{\boldsymbol{\ell}\in L_n}\tilde{\gamma}_{n,b}^{*}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \in A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{G}_{\lambda,n,b} \in A\big)\big| = \mathcal{O}\big(n^{-c_1/4}\big), \tag{E.26}$$

where $\tilde{\gamma}_{n,b}^{*}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)$ is the version of $\tilde{\gamma}_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)$ constructed using an i.i.d. Gaussian sequence $e_1^*, \ldots, e_N^*$.

Using arguments similar to those in Section B.1.2 of the supplementary materials of Kurisu et al. (2024), we have that $\{\tilde{\gamma}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) : \boldsymbol{\ell} \in L_n\}$ is a collection of independent

random vectors under $\mathrm{P}_{\cdot|\mathbf{s},\mathbf{X}}$. Therefore, we apply Corollary 3.2 and equality (47) in Chernozhukov et al. (2022) to establish (E.25). We now verify that the present setting satisfies Conditions E and M in Chernozhukov et al. (2022).

For convenience, define $\Psi_{\boldsymbol{\ell},j} = \sum_{k=1}^{N} \sum_{\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \cap R_b(k)} X_j(\mathbf{s}_i)$. We will verify that, for all $\boldsymbol{\ell} \in L_n$ and $1 \le j \le p$,

$$\frac{1}{[\![L_n]\!]} \sum_{\boldsymbol{\ell} \in L_n} \mathrm{E}_{\cdot|\mathbf{s}}\big[(\sqrt{[\![L_n]\!]}\omega_{\lambda,n,b}\Psi_{\boldsymbol{\ell},j})^2\big] \ge c_3, \tag{E.27}$$

$$\frac{1}{[\![L_n]\!]} \sum_{\boldsymbol{\ell} \in L_n} \mathrm{E}_{\cdot|\mathbf{s}}\big[(\sqrt{[\![L_n]\!]}\omega_{\lambda,n,b}\Psi_{\boldsymbol{\ell},j})^4\big] \le c_4 B_n^2, \tag{E.28}$$

$$\mathrm{E}_{\cdot|\mathbf{s}}\big[\exp\big(|\sqrt{[\![L_n]\!]}\omega_{\lambda,n,b}\Psi_{\boldsymbol{\ell},j}|/B_n\big)\big] \le 2, \tag{E.29}$$

where $c_4 \ge c_3 > 0$ are constants and $\{B_n\}_{n\ge1}$ is a sequence of constants with $B_n \ge 1$ for all $n \ge 1$.

Observe that $\Psi_{\boldsymbol{\ell},j} \le b^d \sum_{\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)} X_j(\mathbf{s}_i)$ and $[\![L_n]\!] \sim (\lambda/\lambda_1)^d$. Using arguments analogous to those employed in the derivation of (C.23), we obtain

$$[\![L_n]\!]^{-1} \sum_{\boldsymbol{\ell} \in L_n} \mathrm{E}_{\cdot|\mathbf{s}}\big[(\sqrt{[\![L_n]\!]}\omega_{\lambda,n,b}\Psi_{\boldsymbol{\ell},j})^2\big] \lesssim \lambda_1^{-d}\lambda^d\omega_{\lambda,n,b}^2 b^{2d}\,\frac{n^2\lambda_1^d}{\lambda^{2d}} \ge c_3$$

uniformly over $1 \le j \le p$, establishing (E.27). In addition,

$$[\![L_n]\!]^2\omega_{\lambda,n,b}^4\,\mathrm{E}_{\cdot|\mathbf{s}}\big[\|\Psi_{\boldsymbol{\ell}}\|_\infty^4\big] \lesssim \frac{\lambda^{4d}b^{2d}}{n^4\lambda_1^{2d}}(n\lambda^{-d}\lambda_1^d)^4\mathrm{E}_{\cdot|\mathbf{s}}\big[\max_{1\le i\le n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty^4\big]$$

$$\lesssim \lambda_1^{2d}b^{2d}\big((\log n)(\log p)\max_{1\le i\le n}\max_{1\le j\le p}\|X_j(\mathbf{s}_i)\|_{\psi_1}\big)^4 \lesssim \lambda_1^{2d}b^{2d}(\log n)^4(\log p)^4 D_n^4.$$

Thus, (E.28) holds with $B_n = \lambda_1^d b^d (\log n)^2(\log p)^2 D_n^2$.

Next, applying the triangle inequality for the Orlicz-norm yields

$$\Big\| \sum_{\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)} |X_j(\mathbf{s}_i)| \Big\|_{\psi_1} \leq \sum_{\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)} \Big\| X_j(\mathbf{s}_i) \Big\|_{\psi_1} \lesssim n\lambda^{-d}\lambda_1^d D_n. \tag{E.30}$$

Moreover,

$$\mathrm{E}_{\cdot|\mathbf{s}}\big(\exp\big([\![L_n]\!]^{1/2} \omega_{\lambda,n,b} \big| \Psi_{\boldsymbol{\ell},j} \big| / B_n \big) \big) \leq \mathrm{E}_{\cdot|\mathbf{s}}\big[\exp\big((n\lambda^{-d}\lambda_1^d D_n)^{-1} \sum_{\mathbf{s}_i \in \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)} \big| X_j(\mathbf{s}_i) \big| \big) \big]. \tag{E.31}$$

Combining (E.30) and (E.31), and using the definition of the sub-exponential norm, we obtain (E.29).

Then, we establish (E.26). Since both $\omega_{\lambda,n,b} \sum_{\boldsymbol{\ell} \in L_{1n}} \tilde{\gamma}^*_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)$ and $\mathbf{G}_{\lambda,n,b}$ are conditionally Gaussian, invoking Lemma H.3 in the supplementary materials of Kurisu et al. (2024) implies that the left-hand side of (E.26) is bounded by $C\hat{\Delta}^{1/3}\log^{2/3} p$, where

$$\hat{\Delta} := \frac{\lambda^{2d}}{n^2 b^d N} \max_{1 \leq j_1, j_2 \leq p} \Big| \sum_{\boldsymbol{\ell} \in L_n} \big(\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[\tilde{\gamma}^{*(j_1)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \tilde{\gamma}^{*(j_2)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \big] - \mathrm{E}_{\cdot|\mathbf{s}}\big[\tilde{\gamma}^{*(j_1)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \tilde{\gamma}^{*(j_2)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \big] \big) \Big|.$$

Thus, to prove (E.26), it suffices to show that

$$\mathrm{P}_{\cdot|\mathbf{s}}(\hat{\Delta} > n^{-5c_1/6} \log^{1/2} p) = \mathcal{O}_p(n^{-c_1/6}). \tag{E.32}$$

Let $\mathcal{D}'_{\boldsymbol{\ell},k} = \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \cap R_b(k)$. Using the Cauchy-Schwarz inequality,

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[\tilde{\gamma}^{*(j_1)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \tilde{\gamma}^{*(j_2)}_{n,b}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \big] = \sum_{k=1}^{N} \big(\sum_{\mathbf{s}_i, \mathbf{s}'_i \in \mathcal{D}'_{\boldsymbol{\ell},k}} |X_{j_1}(\mathbf{s}_i)| |X_{j_2}(\mathbf{s}_{i'})| \big) \leq \sum_{k=1}^{N} \big(\sum_{\mathbf{s}_i \in \mathcal{D}'_{\boldsymbol{\ell},k}} |X_j(\mathbf{s}_i)| \big)^2. \tag{E.33}$$

Since $\sum_{i=1}^{n} \mathbb{I}_{\mathcal{D}'_{\boldsymbol{\ell},k}}(\mathbf{s}_i) \lesssim n\lambda^{-d}|\mathcal{D}'_{\boldsymbol{\ell},k}|$ for all $\boldsymbol{\ell} \in L_n$ and $\sum_{k=1}^{N} |\mathcal{D}'_{\boldsymbol{\ell},k}| \leq b^d\lambda_1^d$, an argument parallel to that used when deriving (E.23) gives

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big[\max_{\boldsymbol{\ell}\in L_n}\max_{1\leq j\leq p}\Big(\sum_{k=1}^{N}\Big(\sum_{\mathbf{s}_i\in\mathcal{D}'_{\boldsymbol{\ell},k}}|X_j(\mathbf{s}_i)|\Big)^2\Big)^2\Big] \lesssim \mathrm{E}_{\cdot|\mathbf{s}}\Big[\Big(\sum_{k=1}^{N}(n\lambda^{-d}|\mathcal{D}'_{\boldsymbol{\ell},k}|\max_{1\leq i\leq n}\|\mathbf{X}(\mathbf{s}_i)\|_\infty\Big)^2\Big)^2\Big]$$

$$\lesssim n^4\lambda^{-4d}b^{2d}\lambda_1^{4d}(\log n)^4(\log p)^4D_n^4. \tag{E.34}$$

Combining (E.33) and (E.34) yields

$$\mathrm{E}_{\cdot|\mathbf{s}}\Big(\max_{\boldsymbol{\ell}\in L_n}\max_{1\leq j_1,j_2\leq p}\big|\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[\tilde{\gamma}_{n,b}^{*(j_1)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\tilde{\gamma}_{n,b}^{*(j_2)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\big]\big|^2\Big) \lesssim n^4\lambda^{-4d}b^{2d}\lambda_1^{4d}(\log n)^4(\log p)^4D_n^4. \tag{E.35}$$

Similarly,

$$\max_{1\leq j_1,j_2\leq p}\sum_{\boldsymbol{\ell}\in L_n}\mathrm{E}_{\cdot|\mathbf{s}}\Big(\big|\mathrm{E}_{\cdot|\mathbf{s},\mathbf{X}}\big[\tilde{\gamma}_{n,b}^{*(j_1)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\tilde{\gamma}_{n,b}^{*(j_2)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\big]\big|^2\Big) \lesssim n^4\lambda^{-3d}b^{2d}\lambda_1^{3d}(\log n)^4D_n^4. \tag{E.36}$$

Combining (E.35) with (E.36) and applying Lemma H.1 in the supplementary materials of Kurisu et al. (2024), we obtain

$$\begin{aligned}\mathrm{E}_{\cdot|\mathbf{s}}[\hat{\Delta}] &\lesssim \frac{\lambda^d}{n^2b^d}\Big(n^2\lambda^{-2d}b^d\lambda_1^{2d}(\log n)^2(\log p)^3D_n^2 + n^2\lambda^{-3d/2}b^d\lambda_1^{3d/2}(\log n)^2(\log p)^{1/2}D_n^2\Big)\\ &\lesssim \lambda^{-d}\lambda_1^{2d}(\log n)^2(\log p)^3D_n^2 + \lambda^{-d/2}\lambda_1^{3d/2}(\log n)^2(\log p)^{1/2}D_n^2 \lesssim n^{-c_1}\log^{1/2}p.\end{aligned}$$

Consequently, applying Markov's inequality establishes (E.32). This completes the proof of (E.24).

Step 3. We show that

$$\sup_{A\in\mathcal{A}}\big|\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{G}_{\lambda,n,b}\in A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)\big| = \mathcal{O}_p(n^{-c_1/3}). \tag{E.37}$$

It follows from Theorem 1(i) that

$$\tilde{\Delta} := \max_{1\le j_1,j_2\le p} \left|\left(\omega_{\lambda,n,b}^2 \sum_{\boldsymbol{\ell}\in L_n} \mathrm{E}_{\cdot|\mathbf{s}}\left[\tilde{\gamma}_{n,b}^{*(j_1)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\tilde{\gamma}_{n,b}^{*(j_2)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\right] - \Xi_{n/\lambda^d}^{(j_1,j_2)}\right)\right| = \mathcal{O}_p(\mathcal{U}_{\lambda,b}).$$

Given that $\mathcal{U}_{\lambda,b}\log^2 p = \mathcal{O}(n^{-c_1})$, we obtain (E.37) by applying Lemma H.3 in the supplementary materials of Kurisu et al. (2024).

Finally, combining (E.11), (E.24) and (E.37) yields (E.5). This completes the proof of (E.5). □

E.2 Proof of part (ii) of Proposition 1

Proof of part (ii) of Proposition 1. Recall that $\mathbf{T}_{\lambda,n}^{\circ} = N^{-1/2}\sum_{k=1}^{N}\left(\mathbf{T}_{\lambda,b,n}^{\circ}(k) - \tilde{\mathbf{T}}_{\lambda,b,n}\right)$, where $\mathbf{T}_{\lambda,b,n}^{\circ}(k)$ are i.i.d. draws from the empirical distribution of $\{\hat{\mathbf{T}}_{\lambda,b,n}(k),\ k = 1,\ldots,N\}$, and $\tilde{\mathbf{T}}_{\lambda,b,n} = N^{-1}\sum_{k=1}^{N}\hat{\mathbf{T}}_{\lambda,b,n}(k)$. As in the proof of part (i), for each $\boldsymbol{\ell}\in L_n$ and $\boldsymbol{\epsilon}\in\{1,2\}^d$, define

$$\boldsymbol{\gamma}_{n,b}^{\circ}(\boldsymbol{\ell};\boldsymbol{\epsilon}) = \left(\gamma_{n,b}^{\circ(1)}(\boldsymbol{\ell};\boldsymbol{\epsilon}),\ldots,\gamma_{n,b}^{\circ(p)}(\boldsymbol{\ell};\boldsymbol{\epsilon})\right)^{\top} = \sum_{k:\Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon})\cap R_b(k)\neq\emptyset}\left(\mathbf{T}_{\lambda,b,n}^{\circ}(k) - \tilde{\mathbf{T}}_{\lambda,b,n}\right),$$

and note that $\mathbf{T}_{\lambda,n}^{\circ} = N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ}$, where

$$\boldsymbol{\gamma}_{n,b}^{\circ} := \sum_{k=1}^{N}\left(\mathbf{T}_{\lambda,b,n}^{\circ}(k) - \tilde{\mathbf{T}}_{\lambda,b,n}\right) = \underbrace{\sum_{\boldsymbol{\ell}\in L_n}\boldsymbol{\gamma}_{n,b}^{\circ}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)}_{=:\boldsymbol{\gamma}_{n,b}^{\circ(11)}} + \sum_{\boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}_0}\underbrace{\sum_{\boldsymbol{\ell}\in L_{1n}}\boldsymbol{\gamma}_{n,b}^{\circ}(\boldsymbol{\ell};\boldsymbol{\epsilon})}_{=:\boldsymbol{\gamma}_{n,b}^{\circ(12)}(\boldsymbol{\epsilon})} + \sum_{\boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}_0}\underbrace{\sum_{\boldsymbol{\ell}\in L_{2n}}\boldsymbol{\gamma}_{n,b}^{\circ}(\boldsymbol{\ell};\boldsymbol{\epsilon})}_{=:\boldsymbol{\gamma}_{n,b}^{\circ(13)}(\boldsymbol{\epsilon})}$$

$$=: \boldsymbol{\gamma}_{n,b}^{\circ(11)} + \sum_{\boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}_0}\boldsymbol{\gamma}_{n,b}^{\circ(12)}(\boldsymbol{\epsilon}) + \sum_{\boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}_0}\boldsymbol{\gamma}_{n,b}^{\circ(13)}(\boldsymbol{\epsilon}).$$

Using arguments identical to those leading to (E.13), we obtain that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\big\|N^{1/2}\boldsymbol{\gamma}_{n,b}^{\circ(1\kappa)}(\boldsymbol{\epsilon})\big\|_\infty > n^{-c_1/2}\log^2 p\big)\le n^{-c_1/3},\quad \text{for } \kappa=2,3,\ \boldsymbol{\epsilon}\neq\boldsymbol{\epsilon}_0\ .$$

Consequently, for any hyper-rectangle $A=\prod_{j=1}^p[a_j,b_j]$ with $\mathbf{a}=(a_1,\ldots,a_p)^\top$ and $\mathbf{b}=(b_1,\ldots,b_p)^\top$,

$$\begin{aligned}
&\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\{-N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ}\le-\mathbf{a}\}\cap\{N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ}\le\mathbf{b}\}\big)\\
&\quad\le \mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\{-N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(11)}\le-\mathbf{a}+2(2^d-1)n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\\
&\qquad\cap\{N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(11)}\le\mathbf{b}+2(2^d-1)n^{-c_1/2}\log^2 p\cdot\mathbf{1}\}\big)+\mathcal{O}(n^{-c_1/3})
\end{aligned}\tag{E.38}$$

holds with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$.

Next, suppose that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})$,

$$\sup_{A\in\mathcal{A}}\big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(11)}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)\big|=\mathcal{O}(n^{-c_1/4}).\tag{E.39}$$

Combining (E.38) with (E.39), and applying Nazarov's inequality (cf. Chernozhukov et al., 2017, Lemma A.1), we conclude that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\begin{aligned}
\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ}\in A\big)&\le\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)+\mathcal{O}(n^{-c_1/4}),\\
\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ}\in A\big)&\ge\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)-\mathcal{O}(n^{-c_1/4}).
\end{aligned}$$

This completes the proof of part (ii). It therefore remains to establish (E.39).

Let $\mathbf{U}_{\lambda,n,b}$ be a random vector, independent of $\mathbf{s}$ and $\mathbf{Y}$, and distributed as $\mathbf{U}_{\lambda,n,b} \sim N\big(\mathbf{0}, \mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(11)})\big)$. To establish (E.39), it suffices to verify that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6})$,

$$\sup_{A\in\mathcal{A}} \big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(11)} \in A\big) - \mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{U}_{\lambda,n,b} \in A\big)\big| = \mathcal{O}(n^{-c_1/4}), \tag{E.40}$$

$$\sup_{A\in\mathcal{A}} \big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{U}_{\lambda,n,b} \in A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{G}_{\lambda,n,b} \in A\big)\big| = \mathcal{O}(n^{-c_1/4}). \tag{E.41}$$

Together with (E.37), these bounds imply (E.39).

Proof of (E.40)**.** Since the collection $\{\boldsymbol{\gamma}_{n,b}^{\circ}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) : \boldsymbol{\ell} \in L_n\}$ consists of independent random vectors under $\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}$, we apply Theorem 2.1 of Chernozhukov et al. (2022). As in the proof of (E.26), it suffices to verify that the following conditions hold uniformly over all $\boldsymbol{\ell} \in L_n$ and $1 \le j \le p$:

$$\frac{1}{[\![L_n]\!]} \sum_{\boldsymbol{\ell}\in L_n} \mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[(\sqrt{[\![L_n]\!]}N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0))^2\big] \ge \frac{k_1}{2}, \tag{E.42}$$

$$\frac{1}{[\![L_n]\!]} \sum_{\boldsymbol{\ell}\in L_n} \mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[\big(\sqrt{[\![L_n]\!]}N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)\big)^4\big] \le k_2 C_n^2, \tag{E.43}$$

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[\exp\big(|\sqrt{[\![L_n]\!]}N^{-1/2}\boldsymbol{\gamma}_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)|/C_n\big)\big] \le 2, \tag{E.44}$$

for some constants $k_2 \ge k_1 > 0$ and a sequence $\{C_n\}_{n\ge1}$ satisfying $C_n \ge 1$.

Note that $\|\hat{T}_{\lambda,b,n}^{(j)}(k)\|_{\psi_1} = \mathcal{O}(\lambda^{d/2}n^{-1/2}D_n)$, and the same bound holds for $T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)}$. Let $\mathcal{K}_n = \{k : \Gamma_n(\boldsymbol{\ell};\boldsymbol{\epsilon}_0) \cap R_b(k) \neq \emptyset\}$. Then $[\![\mathcal{K}_n]\!] \sim \lambda_1^d$. By the triangle inequality for the Orlicz-norm,

$$\Big\| \sum_{k\in\mathcal{K}_n} |T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)}| \Big\|_{\psi_1} \le \Big\| \sum_{k\in\mathcal{K}_n} \big(T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)}\big) \Big\|_{\psi_1} \lesssim \lambda_1^d \lambda^{d/2} n^{-1/2} D_n$$

uniformly over $\boldsymbol{\ell}$ and j. Therefore, the inequality (E.44) holds with $C_n = \lambda_1^{d/2}\lambda^{d/2}n^{-1/2}D_n$.

Moreover,

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[(\sqrt{[\![L_n]\!]}N^{-1/2}\gamma_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0))^2\big] = [\![L_n]\!]N^{-1}\sum_{k\in\mathcal{K}_n}\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)}\big). \quad \text{(E.45)}$$

From Theorem 1(ii), for all k and j,

$$\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(T_{\lambda,b,n}^{\circ(j)}(k)\big) = \frac{1}{N}\sum_{k'=1}^{N}\big(\hat{T}_{\lambda,b,n}^{(j)}(k') - \bar{T}_{\lambda,b,n}^{(j)}\big)^2 = \Xi_{n/\lambda^d} + \mathcal{O}_p(\mathcal{U}_{\lambda,b} + \mathcal{V}_{\lambda,b,p}) \geq k_1, \quad \text{(E.46)}$$

where $\bar{T}_{\lambda,b,n}^{(j)} = N^{-1}\sum_{k'=1}^{N}\hat{T}_{\lambda,b,n}(k')$. Since $\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)}\big) = (1-\frac{1}{N})\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(T_{\lambda,b,n}^{\circ(j)}(k)\big)$, combining (E.45) with (E.46) yields (E.42).

Finally, by Rosenthal's inequality (Rosenthal, 1970),

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[\gamma_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)^4\big] \lesssim [\![\mathcal{K}_n]\!]\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[(T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)})^4\big] + [\![\mathcal{K}_n]\!]^2\Big(\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(T_{\lambda,b,n}^{\circ(j)}(k))\Big)^2.$$

Using moment bounds analogous to those in the proof of (E.28) and Markov's inequality, we obtain

$$\mathrm{E}_{\cdot|\mathbf{s},\mathbf{Y}}\big[(T_{\lambda,b,n}^{\circ(j)}(k) - \tilde{T}_{\lambda,b,n}^{(j)})^4\big] = \mathcal{O}_p(\lambda^{2d}n^{-2}(\log p)^4 D_n^4),$$

which, together with (E.46), implies (E.43).

Proof of (E.41)**.** It follows from the proof of (E.42) that

$$\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}\big(U_{\lambda,n,b}^{(j)}\big) = N^{-1}\sum_{\boldsymbol{\ell}\in L_n}\mathrm{Var}_{\cdot|\mathbf{s},\mathbf{Y}}(\gamma_{n,b}^{\circ(j)}(\boldsymbol{\ell};\boldsymbol{\epsilon}_0)) = \Xi_{n/\lambda^d}^{(j,j)} + \mathcal{O}_p(\mathcal{U}_{\lambda,b} + \mathcal{V}_{\lambda,b,p})$$

uniformly over $1 \leq j \leq p$. Since $\mathrm{Var}_{\cdot|\mathbf{s}}(\mathbf{G}_{\lambda,n,b}) = \Xi_{n/\lambda^d} + \mathcal{O}_p(\mathcal{U}_{\lambda,b})$, and given that $\mathcal{U}_{\lambda,b}\log^2 p = \mathcal{O}(n^{-c_1})$ and Assumption 4(a) hold, applying Lemma H.3 in the supplementary materials of Kurisu et al. (2024) yields (E.41). □

E.3 Proof of Theorem 2

Proof of Theorem 2. By the triangle inequality,

$$\sup_{A\in\mathcal{A}}\left|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}^*_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{T}_{\lambda,n}\in A\big)\right|$$

$$\leq \sup_{A\in\mathcal{A}}\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}^*_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)\big|+\sup_{A\in\mathcal{A}}\left|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)\right|.$$

The first term on the right-hand side is controlled in Proposition 1(i). Moreover, by Theorem 4.1 of Kurisu et al. (2024),

$$\sup_{A\in\mathcal{A}}\left|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(\mathbf{T}_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n}\in A\big)\right|=\mathcal{O}_p(n^{-(\frac{c_2}{6}\wedge\varsigma)})+\mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}),$$

where c_2 is the constant specified in Assumption 4(b) and $\varsigma>0$ is defined in Assumption 2(b). Combining these bounds completes the proof of part (i). Part (ii) follows by arguments analogous to those used for part (i). □

E.4 Proof of Theorem 3

Proof of Theorem 3. (i) Let $D_{\lambda,n}=\mathrm{diag}(\Xi_{n/\lambda^d})^{1/2}$. To prove part (i), it suffices to show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1-\mathcal{O}(n^{-c_1/6})-\mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$:

$$\sup_{A\in\mathcal{A}}\left|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(D^{*-1}_{\lambda,n}\mathbf{T}^*_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(D^{-1}_{\lambda,n}\mathbf{Z}_{\lambda,n}\in A\big)\right|=\mathcal{O}(n^{-c_1/4}),\tag{E.47}$$

$$\sup_{A\in\mathcal{A}}\left|\mathrm{P}_{\cdot|\mathbf{s}}\big(D^{-1}_{\lambda,n}\mathbf{Z}_{\lambda,n}\in A\big)-\mathrm{P}_{\cdot|\mathbf{s}}\big(D^{*-1}_{\lambda,n}\mathbf{T}_{\lambda,n}\in A\big)\right|=\mathcal{O}(n^{-\frac{c_2}{6}\wedge\varsigma})+\mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2})+o(n^{-c_1/4}).\tag{E.48}$$

Proof of (E.47)**.** By Assumption 1, $\Xi^{(j,j)}_{n/\lambda^d}\geq c>0$ uniformly over $1\leq j\leq p$, which

implies $\Xi^{*(j,j)}_{\lambda,n} \geq c/2$ uniformly for all j. Proposition 1(i) then yields that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\sup_{A\in\mathcal{A}} \big|\mathrm{P}_{\cdot|\mathbf{s},\mathbf{Y}}\big(D^{*-1}_{\lambda,n}\mathbf{T}^{*}_{\lambda,n} \in A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(D^{*-1}_{\lambda,n}\mathbf{Z}_{\lambda,n} \in A\big)\big| = \mathcal{O}(n^{-c_1/4}), \tag{E.49}$$

where we use the fact that $D^{*}_{\lambda,n}A \in \mathcal{A}$ for any $A \in \mathcal{A}$. Recall that $\mathbf{Z}_{\lambda,n}$ is a Gaussian vector under $\mathrm{P}_{\cdot|\mathbf{s}}$ with covariance matrix Ξ_{n/λ^d}. To complete the proof of (E.47), it remains to control the effect of replacing $D^{*}_{\lambda,n}$ with $D_{\lambda,n}$ by Nazarov's inequality (cf. Chernozhukov et al., 2017, Lemma A.1).

From the proofs of (E.1), (E.2), and (E.5) in Section E.1, we obtain that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\begin{aligned}\max_{1\leq j_1,j_2\leq p} \big|\Xi^{*(j_1,j_2)}_{\lambda,n} - \Xi^{(j_1,j_2)}_{n/\lambda^d}\big| &\leq \hat{\Delta} + \tilde{\Delta} + \mathcal{O}(n^{-c_1/2}) \\ &= \mathcal{O}(n^{-5c_1/6}\log^{1/2} p) + \mathcal{O}(n^{-c_1/2}) = \mathcal{O}(n^{-c_1/2}).\end{aligned}$$

Consequently,

$$\begin{aligned}\max_{1\leq j\leq p}\left|\sqrt{\frac{\Xi^{(j,j)}_{n/\lambda^d}}{\Xi^{*(j,j)}_{\lambda,n}}} - 1\right| &\leq \max_{1\leq j\leq p}\left|\frac{\Xi^{(j,j)}_{n/\lambda^d}}{\Xi^{*(j,j)}_{\lambda,n}} - 1\right| \leq \max_{1\leq j\leq p}\left|\frac{1}{\Xi^{*(j,j)}_{\lambda,n}}\right| \max_{1\leq j\leq p}\left|\Xi^{*(j,j)}_{\lambda,n} - \Xi^{(j,j)}_{n/\lambda^d}\right| \\ &\leq \frac{\mathcal{O}(n^{-c_1/2})}{\min_{1\leq j\leq p}\Xi^{(j,j)}_{n/\lambda^d} - \max_{1\leq j\leq p}\left|\Xi^{*(j,j)}_{\lambda,n} - \Xi^{(j,j)}_{n/\lambda^d}\right|} = \mathcal{O}(n^{-c_1/2}) =: \delta_n \end{aligned} \tag{E.50}$$

holds with the same probability. This implies that $D^{*-1}_{\lambda,n}D_{\lambda,n} = I + E_n$, where E_n is diagonal and $\|E_n\|_{\max} \leq \delta_n$.

Let Ω_n denote the event defined by the inequality $\max_{1\leq j\leq p}|D^{*(j,j)}_{\lambda,n}/D^{(j,j)}_{\lambda,n} - 1| \leq \delta_n$.

From (E.50), we have $\mathrm{P}_{\cdot|\mathbf{s}}(\Omega_n) \geq 1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2})$. Note that

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{-1}\mathbf{Z}_{\lambda,n} \in A\big) = \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{Z}_{\lambda,n} \in D_{\lambda,n}^{*-1}D_{\lambda,n}A\big) = \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{Z}_{\lambda,n} \in (I+E_n)A\big).$$

On Ω_n, Nazarov's inequality (cf. Chernozhukov et al., 2017, Lemma A.1) yields

$$\sup_{A\in\mathcal{A}} \big|\mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{Z}_{\lambda,n} \in (I+E_n)A\big) - \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{Z}_{\lambda,n} \in A\big)\big| = \mathcal{O}(\delta_n\sqrt{\log p}) = o(n^{-c_1/4}).$$

Combining this bound with (E.49) establishes (E.47).

Proof of (E.48)**.** To verify (E.48), it suffices to show that, with $\mathrm{P}_{\cdot|\mathbf{s}}$-probability at least $1 - \mathcal{O}(n^{-c_1/6}) - \mathcal{O}\big(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}\big)$,

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{T}_{\lambda,n} \in A\big) \leq \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{-1}\mathbf{Z}_{\lambda,n} \in A\big) + \mathcal{O}(n^{-\frac{c_2}{6}\wedge\varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}) + o(n^{-c_1/4}), \tag{E.51}$$

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{*-1}\mathbf{T}_{\lambda,n} \in A\big) \geq \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{-1}\mathbf{Z}_{\lambda,n} \in A\big) + \mathcal{O}(n^{-\frac{c_2}{6}\wedge\varsigma}) - \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}) + o(n^{-c_1/4}). \tag{E.52}$$

On Ω_n, we have $D_{\lambda,n}^{(j,j)}(1-\delta_n) \leq D_{\lambda,n}^{*(j,j)} \leq D_{\lambda,n}^{(j,j)}(1+\delta_n)$ for all $j = 1,\ldots,p$. For any rectangle $A = \prod_{j=1}^p [a_j, b_j] \in \mathcal{A}$, define the outer and inner rectangles

$$A_{\text{outer}} = \prod_{j=1}^p [\min(D_{\lambda,n}^{(j,j)}(1-\delta_n)a_j, D_{\lambda,n}^{(j,j)}(1+\delta_n)a_j), \max(D_{\lambda,n}^{(j,j)}(1-\delta_n)b_j, D_{\lambda,n}^{(j,j)}(1+\delta_n)b_j)],$$

$$A_{\text{inner}} = \prod_{j=1}^p [\max(D_{\lambda,n}^{(j,j)}(1-\delta_n)a_j, D_{\lambda,n}^{(j,j)}(1+\delta_n)a_j), \min(D_{\lambda,n}^{(j,j)}(1-\delta_n)b_j, D_{\lambda,n}^{(j,j)}(1+\delta_n)b_j)].$$

Then $A_{\text{outer}}, A_{\text{inner}} \in \mathcal{A}$. Therefore, for any $A \in \mathcal{A}$, $\{\mathbf{T}_{\lambda,n} \in A_{\text{inner}}\} \subseteq \{D_{\lambda,n}^{*-1}\mathbf{T}_{\lambda,n} \in A\} \subseteq \{\mathbf{T}_{\lambda,n} \in A_{\text{outer}}\}$. Invoking Theorem 4.1 of Kurisu et al. (2024),

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{T}_{\lambda,n} \in A_{\text{outer}}\big) \leq \mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n} \in A_{\text{outer}}\big) + \mathcal{O}(n^{-\frac{c_2}{6}\wedge\varsigma}) + \mathcal{O}(\mathcal{W}_{\lambda,\lambda_1,\lambda_2}). \tag{E.53}$$

Again applying Nazarov's inequality (cf. Chernozhukov et al., 2017, Lemma A.1) yields

$$\mathrm{P}_{\cdot|\mathbf{s}}\big(\mathbf{Z}_{\lambda,n} \in A_{\text{outer}}\big) = \mathrm{P}_{\cdot|\mathbf{s}}\big(D_{\lambda,n}^{-1}\mathbf{Z}_{\lambda,n} \in A\big) + \mathcal{O}(\delta_n\sqrt{\log p}). \tag{E.54}$$

Since $\delta_n\sqrt{\log p} = o(n^{-c_1/4})$, combining (E.53) and (E.54) yields (E.51). A symmetric argument based on A_{inner} establishes (E.52). This completes the proof of (E.48).

(ii) Part (ii) follows by replacing $\mathbf{T}^*_{\lambda,n}, D^*_{\lambda,n}$ with $\mathbf{T}^\circ_{\lambda,n}, D^\circ_{\lambda,n}$ throughout and applying arguments analogous to those in the proof of part (i). □

F Proofs of Proposition 2 and Theorem 4

Proof of Proposition 2. For illustration, we prove only parts (i) and (iii).

(i) It follows from (D.1) that

$$\begin{aligned}\mathrm{E}(\Xi^{*(i,j)}_{\lambda,n}) = N^{-1}\sum_{k=1}^{N}\mathrm{E}\Big(&T^{(i)}_{b,n}(k)T^{(j)}_{b,n}(k) - (n_k/n)^{1/2}T^{(i)}_{b,n}(k)T^{(j)}_{\lambda,n}\\ &- (n_k/n)^{1/2}T^{(i)}_{\lambda,n}T^{(j)}_{b,n}(k) + (n_k/n)\,T^{(i)}_{\lambda,n}T^{(j)}_{\lambda,n}\Big).\end{aligned} \tag{F.1}$$

From the proof of part (i) of Theorem 1, together with arguments analogous to those used in deriving (C.22), to complete the proof, it suffices to establish

$$N^{-1}\sum_{k=1}^{N}\mathrm{E}\big(\frac{\lambda^d}{n}n_k\bar{X}^{(i)}_{b,n}(k)\bar{X}^{(j)}_{b,n}(k) - \Xi^{(i,j)}_{\lambda,n}\big) = -\frac{\int_{\mathbb{R}^d}V(\mathbf{h})\sigma_{ij}(\mathbf{h})\,\mathrm{d}\mathbf{h}}{b} + o(b^{-1}). \tag{F.2}$$

Let $\pi_k = \mathrm{P}\{\mathbf{s}_1 \in R_b(k)\} = \int_{R_b(k)} f(\mathbf{u})\,\mathrm{d}\mathbf{u}$. Conditional on $n_k = m \geq 1$, the m sampling

locations falling in $R_b(k)$ are i.i.d. random variables. Hence, from (C.24),

$$\mathrm{E}\Big[\frac{\lambda^d}{n} n_k \bar{X}^{(i)}_{b,n}(k)\bar{X}^{(j)}_{b,n}(k) | n_k = m\Big] = \frac{\lambda^d}{n}(m-1)\pi_k^{-2} \iint_{R_b^2(k)} \sigma_{ij}(\mathbf{v}-\mathbf{u}) f(\mathbf{u}) f(\mathbf{v})\, \mathrm{d}\mathbf{u}\, \mathrm{d}\mathbf{v} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}).$$

Since $\pi_k = \frac{b^d}{\lambda^d}(1 + \mathcal{O}(\frac{\log\lambda}{\lambda}))$ and $\mathrm{E}(n_k) = n\pi_k$, averaging over n_k, we obtain

$$\mathrm{E}\big[\frac{\lambda^d}{n} n_k \bar{X}^{(i)}_{b,n}(k)\bar{X}^{(j)}_{b,n}(k)\big] = \frac{1}{b^d}(1 + \mathcal{O}(\frac{\log\lambda}{\lambda}) + \mathcal{O}(\frac{\lambda^d}{nb^d})) \iint_{R_b^2(k)} \sigma_{ij}(\mathbf{v}-\mathbf{u})\, \mathrm{d}\mathbf{u}\, \mathrm{d}\mathbf{v} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) \tag{F.3}$$

uniformly over i, j and k. With arguments similar to proving (C.11), we have

$$\iint_{R_b^2} \sigma_{ij}(\mathbf{u}_2 - \mathbf{u}_1)\, \mathrm{d}\mathbf{u}_1\, \mathrm{d}\mathbf{u}_2 = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) |R_b \cap (R_b + \mathbf{h})|\, \mathrm{d}\mathbf{h}. \tag{F.4}$$

By the boundary regularity condition on R_0, $|(R_0 + \mathbf{h}) \triangle R_0| \lesssim \min\{1, \|\mathbf{h}\|_1\}$. Since $R_b(k) = c_k + bR_0$, this implies $0 \le b(1 - \frac{|R_b \cap (R_b+\mathbf{h})|}{b^d}) \lesssim b \min\{1, \frac{\|\mathbf{h}\|_1}{b}\} \lesssim \|\mathbf{h}\|_1$. Therefore, by Assumption 1(a), dominated convergence gives, uniformly over $1 \le i, j \le p$,

$$\int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) \frac{|R_b \cap (R_b + \mathbf{h})|}{b^d}\, \mathrm{d}\mathbf{h} = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\, \mathrm{d}\mathbf{h} - \frac{1}{b}\int_{\mathbb{R}^d} V(\mathbf{h})\sigma_{ij}(\mathbf{h})\, \mathrm{d}\mathbf{h} + o(b^{-1}). \tag{F.5}$$

Combining (F.3)–(F.5) and $\log\lambda/\lambda = o(b^{-1})$, $\lambda^d/(nb^d) = o(b^{-1})$, we obtain

$$\mathrm{E}\Big(\frac{\lambda^d}{n} n_k \bar{X}^{(i)}_{b,n}(k)\bar{X}^{(j)}_{b,n}(k)\Big) = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\, \mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) - \frac{\Phi_{ij}}{b} + o(\frac{1}{b}).$$

Recalling that $\Xi^{(i,j)}_{\lambda,n} = \int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h})\, \mathrm{d}\mathbf{h} + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0}) + \mathcal{O}_p\Big(\frac{\log\lambda}{\lambda}\Big)$, we conclude that (F.2) holds.

(iii) We expand the variance of $\Xi_{\lambda,n}^{*(i,j)}$ as

$$\mathrm{Var}(\Xi_{\lambda,n}^{*(i,j)}) = N^{-2}\mathrm{Var}\Big\{ \sum_{k=1}^{N} \big(T_{b,n}^{(i)}(k)T_{b,n}^{(j)}(k) - (n_k/n)^{1/2}T_{b,n}^{(i)}(k)T_{\lambda,n}^{(j)} \\ - (n_k/n)^{1/2}T_{\lambda,n}^{(i)}T_{b,n}^{(j)}(k) + (n_k/n)\, T_{\lambda,n}^{(i)}T_{\lambda,n}^{(j)}\big)\Big\}. \tag{F.6}$$

As in (F.1), the leading contribution to (F.6) is

$$N^{-2}\mathrm{Var}\big(\sum_{k=1}^{N} T_{b,n}^{(i)}(k)T_{b,n}^{(j)}(k)\big) = N^{-2}\sum_{k=1}^{N}\sum_{l=1}^{N}\mathrm{Cov}\big(T_{b,n}^{(i)}(k)\, T_{b,n}^{(j)}(k), T_{b,n}^{(i)}(l)\, T_{b,n}^{(j)}(l)\big).$$

Using arguments similar to those in the proof of (D.12), to complete the proof, it suffices to establish

$$\frac{1}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N}\mathrm{cum}(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(l))\mathrm{cum}(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(l)) = K_0 b^d\lambda^{-d}(\Xi_{n/\lambda^d}^{(i,j)})^2(1+o(1)), \tag{F.7}$$

$$\frac{1}{N^2}\sum_{k=1}^{N}\sum_{l=1}^{N}\mathrm{cum}(T_{b,n}^{(i)}(k), T_{b,n}^{(i)}(l))\mathrm{cum}(T_{b,n}^{(j)}(k), T_{b,n}^{(j)}(l)) = K_0 b^d\lambda^{-d}\Xi_{n/\lambda^d}^{(i,i)}\Xi_{n/\lambda^d}^{(j,j)}(1+o(1)). \tag{F.8}$$

It follows from (F.3) and (F.4) that, for any $k, l = 1, \ldots, N$,

$$\mathrm{cum}(T_{b,n}^{(i)}(k), T_{b,n}^{(j)}(l)) = \Big\{\frac{1}{b^d}\int_{\mathbb{R}^d}\sigma_{ij}(\mathbf{h})\big|R_b(k)\cap(R_b(l)+\mathbf{h})\big|\mathrm{d}\mathbf{h}\Big\}(1+o(1)) \\ + \frac{\lambda^d}{n}\sigma_{ij}(\mathbf{0})\frac{\big|R_b(k)\cap R_b(l)\big|}{b^d}. \tag{F.9}$$

Recall that $R_b(k) = \mathbf{c}_k + bR_0$, where $\mathbf{c}_k$ denotes the center of the subregion $R_b(k)$. Define $\mathcal{I}_n := \big\{(k,l) : \|\mathbf{c}_k - \mathbf{c}_l\|_\infty < C_R\, b\big\}$, where $C_R > 2\sup_{\mathbf{u},\mathbf{v}\in R_0}\|\mathbf{u}-\mathbf{v}\|_\infty$ is a fixed constant. Since the block centers lie on the lattice $\mathbb{Z}_\delta^d$ with $\delta \to 1$, $[\![\mathcal{I}_n]\!] = \mathcal{O}(Nb^d)$. For

$(k,l) \notin \mathcal{I}_n$, Assumption 1(a) implies $\sum_{(k,l)\notin\mathcal{I}_n}\{\mathrm{cum}(T^{(i)}_{b,n}(k), T^{(j)}_{b,n}(l))\} = o(Nb^d)$. For any $\mathbf{h} \in \mathbb{R}^d$ and each pair (k,l), we have

$$|R_b(k) \cap (R_b(l) + \mathbf{h})| = b^d \big| R_0 \cap \big(R_0 + \frac{\mathbf{c}_k - \mathbf{c}_l - \mathbf{h}}{b}\big)\big| = b^d\, \psi\big(\frac{\mathbf{c}_k - \mathbf{c}_l - \mathbf{h}}{b}\big), \tag{F.10}$$

where $\psi(\mathbf{v}) := |R_0 \cap (R_0 + \mathbf{v})|$. The boundary regularity condition on the sampling region implies $|(R_0 + \mathbf{h}) \triangle R_0| = \mathcal{O}(\|\mathbf{h}\|_1)$. Then, letting $\mathbf{v}_0 = (\mathbf{c}_k - \mathbf{c}_l)/b$, we obtain

$$|\psi(\mathbf{v}_0 - \mathbf{h}/b) - \psi(\mathbf{v}_0)| \leq |(R_0 + \mathbf{v}_0 - \mathbf{h}/b) \triangle (R_0 + \mathbf{v}_0)| = \mathcal{O}(\|\mathbf{h}\|_1/b). \tag{F.11}$$

Combining Assumption 1(a) with (F.9)–(F.11), we obtain

$$\mathrm{cum}(T^{(i)}_{b,n}(k), T^{(j)}_{b,n}(l)) = \psi(\mathbf{v}_0)\Big(\int_{\mathbb{R}^d} \sigma_{ij}(\mathbf{h}) \mathrm{d}\mathbf{h} + \frac{\lambda^d}{n} \sigma_{ij}(\mathbf{0}) \Big) + o(1)$$

uniformly for $(k,l) \in \mathcal{I}_n$. Therefore,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \{\mathrm{cum}(T^{(i)}_{b,n}(k), T^{(j)}_{b,n}(l))\}^2 = \sum_{(k,l)\in\mathcal{I}_n} \psi^2\big(\mathbf{v}_0\big) \big(\Xi^{(i,j)}_{n/\lambda^d}\big)^2 (1 + o(1)).$$

Applying a Riemann sum argument yields

$$\sum_{(k,l)\in\mathcal{I}_n} \psi^2\big(\mathbf{v}_0\big) = Nb^d(1 + o(1)) \int_{\mathbb{R}^d} \psi^2(\mathbf{v})\, \mathrm{d}\mathbf{v}, \quad n \to \infty.$$

Since $N \asymp \lambda^d$, we obtain (F.7) immediately. By similar arguments, (F.8) follows. This completes the proof. □

Proof of Theorem 4. The result follows by minimizing the leading-order asymptotic MSE derived in Proposition 2 with respect to the subsample size b. □

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