Locally most powerful naive test for the identity of the covariance matrix

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Supplementary Material

The supplementary material consists of the following parts:

- Section S1: Proofs of key lemmas.
- Section S2: Proof of Theorem 1.
- Section S3: Additional simulation studies.
- Section S4: Additional details for the real-data analysis.

S1 Proofs of key lemmas

Throughout the proofs, unless otherwise specified, we use the same notation as in the main text. In addition, $\mathbf{B}_{ij}$ denotes the (i, j)-th element of a matrix $\mathbf{B}$, and K denotes a generic positive constant independent of (p, n), whose value may vary from line to line. Without loss of generality, we assume that

$\boldsymbol{\mu} = \mathbf{0}$, since $U_n(1)$ and $U_n(2)$ remain unchanged if each observation $\mathbf{x}_i$ is replaced by $\mathbf{x}_i + \boldsymbol{\mu}_0$ for an arbitrary fixed vector $\boldsymbol{\mu}_0$.

In the following lemmas, we suppose that $\mathbf{z}_1$ and $\mathbf{z}_2$ are two i.i.d. m-dimensional random vectors satisfying Assumption 1.

Lemma 1. (Bai and Silverstein (2010), (9.8.6)) *For any $m \times m$ non-random matrices $\mathbf{B}_1$ and $\mathbf{B}_2$, we have*

$$\mathrm{E}\{(\mathbf{z}_1'\mathbf{B}_1\mathbf{z}_1)(\mathbf{z}_1'\mathbf{B}_2\mathbf{z}_1)\} = \mathrm{tr}\,(\mathbf{B}_1)\,\mathrm{tr}\,(\mathbf{B}_2) + \mathrm{tr}\,(\mathbf{B}_1\mathbf{B}_2) + \mathrm{tr}\,(\mathbf{B}_1\mathbf{B}_2') + \Delta\mathrm{tr}\,(\mathbf{B}_1 \circ \mathbf{B}_2)\,.$$

Lemma 2. *For any $m \times m$ non-random matrices $\mathbf{B}$ and $\mathbf{C}$, we have*

$$\begin{aligned}\mathrm{E}\left\{(\mathbf{z}_1'\mathbf{B}\mathbf{z}_2)^2(\mathbf{z}_1'\mathbf{C}\mathbf{z}_2)^2\right\} = {} & 2\left\{\mathrm{tr}\,(\mathbf{B}\mathbf{C})\right\}^2 + \mathrm{tr}\,(\mathbf{B}\mathbf{B}')\,\mathrm{tr}\,(\mathbf{C}\mathbf{C}') + 2\mathrm{tr}\,(\mathbf{B}\mathbf{C}\mathbf{B}\mathbf{C}) \\ & + 2\mathrm{tr}\,(\mathbf{B}\mathbf{C}\mathbf{C}'\mathbf{B}') + 2\mathrm{tr}\,(\mathbf{B}\mathbf{B}'\mathbf{C}\mathbf{C}') + 2\Delta\mathrm{tr}\,(\mathbf{B}\mathbf{C} \circ \mathbf{B}\mathbf{C}) \\ & + 2\Delta\mathrm{tr}\,(\mathbf{B}\mathbf{B}' \circ \mathbf{C}\mathbf{C}') + 2\Delta\mathrm{tr}\,(\mathbf{C}\mathbf{B} \circ \mathbf{C}\mathbf{B}) + \Delta^2\sum_{i,j}\mathbf{B}_{ij}^2\mathbf{C}_{ij}^2.\end{aligned}$$

Specifically, we have

$$\begin{aligned}\mathrm{E}\left\{(\mathbf{z}_1'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{z}_2)^4\right\} = {} & 3\left\{\mathrm{tr}\big((\mathbf{W}\boldsymbol{\Sigma})^2\big)\right\}^2 + 6\mathrm{tr}\big((\mathbf{W}\boldsymbol{\Sigma})^4\big) + \Delta^2\sum_{i,k=1}^{m}(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma})_{ik}^4 \\ & + 6\Delta\mathrm{tr}\,(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma} \circ \boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma})\,.\end{aligned}$$

Proof. The first equality can be verified directly by using Lemma 1. The details are omitted. The second equality is a special case of the first one by taking $\mathbf{B} = \mathbf{C} = \boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}$. $\square$

Lemma 3. *For any $m \times m$ non-random matrix $\mathbf{B}$, there exists a constant K such that*

$$\mathrm{E}\left[\mathbf{z}_1'\mathbf{B}\mathbf{z}_1 - \mathrm{tr}\,(\mathbf{B})\right]^4 \le K\left\{\mathrm{tr}\,(\mathbf{B}\mathbf{B}')\right\}^2.$$

Proof. This lemma is a direct consequence of Lemma B.26 in Bai and Silverstein (2010). □

Lemma 4. *We have*

$$\sum_{i,k}\left(\mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{\Gamma}\right)_{ik}^4 \le \mathrm{tr}\left(\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)^2\right).$$

Proof. This lemma follows from the inequality

$$\sum_{i,k}\mathbf{B}_{ik}^4 \le \sum_i\left(\sum_k \mathbf{B}_{ik}^2\right)^2 = \sum_i\left(\mathbf{B}\mathbf{B}'\right)_{ii}^2 = \mathrm{tr}\left(\mathbf{B}\mathbf{B}' \circ \mathbf{B}\mathbf{B}'\right) \le \mathrm{tr}\left(\left(\mathbf{B}\mathbf{B}'\right)^2\right).$$

□

Lemma 5. *For any $p \times p$ nonnegative definite matrices $\mathbf{B}$ and $\mathbf{C}$, if $\lambda_{max}(\mathbf{B}) = o\left(\mathrm{tr}\,(\mathbf{B})\right)$, then*

$$\mathrm{tr}\,(\mathbf{B}\mathbf{C}) = o\left(\mathrm{tr}\,(\mathbf{B})\,\mathrm{tr}\,(\mathbf{C})\right). \tag{S1.1}$$

Proof. Denote the spectral decomposition of $\mathbf{B}$ by $\mathbf{B} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$, where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1(\mathbf{B}), \ldots, \lambda_p(\mathbf{B}))$. Then we have

$$\mathrm{tr}\,(\mathbf{B}\mathbf{C}) = \sum_{k=1}^p \lambda_k(\mathbf{B})[\mathbf{U}^{-1}\mathbf{C}\mathbf{U}]_{kk} \le \lambda_{max}(\mathbf{B})\sum_{k=1}^p[\mathbf{U}^{-1}\mathbf{C}\mathbf{U}]_{kk}$$

$$= \lambda_{max}(\mathbf{B})\mathrm{tr}[\mathbf{U}^{-1}\mathbf{C}\mathbf{U}] = \lambda_{max}(\mathbf{B})\mathrm{tr}[\mathbf{C}] = o\left(\mathrm{tr}\left(\mathbf{B}\right)\mathrm{tr}\left(\mathbf{C}\right)\right),$$

where the first inequality follows from the non-negativity of $\mathbf{C}$. The proof is complete. □

Lemma 6. *We have*

$$\mathrm{E}\left(U_n\right) = \frac{1}{p}\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right) - \frac{2}{p}\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\right) + \frac{\mathrm{tr}\left(\mathbf{W}\right)}{p}, \quad \mathrm{Var}\left(pU_n\right) = \sigma^2_{1,n}\left(1 + o(1)\right).$$

Proof. Recall that

$$U_n = \frac{1}{p}U_n(2) - \frac{2}{p}U_n(1) + \frac{\mathrm{tr}\left(\mathbf{W}\right)}{p},$$

where $U_n(1) = U_n^1 - U_n^3$ and $U_n(2) = U_n^2 - 2U_n^4 + U_n^5$ with

$$U_n^1 = \frac{1}{A_{n,1}}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{W}\mathbf{x}_i, \quad U_n^2 = \frac{1}{A_{n,2}}\sum_{i\neq j}\mathbf{x}_i'\mathbf{x}_j\,\mathbf{x}_i'\mathbf{W}\mathbf{x}_j, \quad U_n^3 = \frac{1}{A_{n,2}}\sum_{i\neq j}\mathbf{x}_i'\mathbf{W}\mathbf{x}_j,$$

$$U_n^4 = \frac{1}{A_{n,3}}\sum_{i,j,k}^{*}\mathbf{x}_i'\mathbf{x}_j\,\mathbf{x}_j'\mathbf{W}\mathbf{x}_k, \quad U_n^5 = \frac{1}{A_{n,4}}\sum_{i,j,k,l}^{*}\mathbf{x}_i'\mathbf{x}_j\,\mathbf{x}_k'\mathbf{W}\mathbf{x}_l,$$

The first conclusion can be verified by taking the expectation of U_n^i, $i = 1, \ldots, 5$. For the variance of pU_n, we get

$$\begin{aligned}
\mathrm{Var}\left(pU_n\right) = {} & 4\mathrm{Var}\left(U_n^1\right) + \mathrm{Var}\left(U_n^2\right) + 4\mathrm{Var}\left(U_n^3\right) + 4\mathrm{Var}\left(U_n^4\right) + \mathrm{Var}\left(U_n^5\right) \\
& - 4\mathrm{Cov}\left(U_n^1, U_n^2\right) - 8\mathrm{Cov}\left(U_n^1, U_n^3\right) + 8\mathrm{Cov}\left(U_n^1, U_n^4\right) - 4\mathrm{Cov}\left(U_n^1, U_n^5\right) \\
& + 4\mathrm{Cov}\left(U_n^2, U_n^3\right) - 4\mathrm{Cov}\left(U_n^2, U_n^4\right) + 2\mathrm{Cov}\left(U_n^2, U_n^5\right) - 8\mathrm{Cov}\left(U_n^3, U_n^4\right) \\
& + 4\mathrm{Cov}\left(U_n^3, U_n^5\right) - 4\mathrm{Cov}\left(U_n^4, U_n^5\right).
\end{aligned}$$

Using Lemmas 1–4, we have

$$
\begin{aligned}
\mathrm{Var}\left(U_n^2\right) &= \mathrm{E}\left(U_n^2 - \mathrm{E}U_n^2\right)^2 \\
&= \mathrm{E}\left[\frac{1}{n(n-1)}\sum_{i\neq j}\left(\mathbf{x}_i'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{x}_i - \mathrm{tr}(\mathbf{W}\boldsymbol{\Sigma}^2)\right)\right]^2 \\
&= \frac{2}{n^2(n-1)^2}\sum_{i\neq j}\mathrm{E}\left[\mathbf{x}_i'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{x}_i - \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right]^2 \\
&\quad + \frac{4}{n^2(n-1)^2}\sum_{i,j,l}^{*}\mathrm{E}\left[\mathbf{x}_i'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{x}_i - \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right] \\
&\qquad\qquad\qquad\qquad \left[\mathbf{x}_i'\mathbf{x}_l\mathbf{x}_l'\mathbf{W}\mathbf{x}_i - \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right] \\
&= \frac{2}{n(n-1)}\mathrm{E}\Big[\mathbf{x}_1'\mathbf{x}_2\mathbf{x}_2'\mathbf{W}\mathbf{x}_1\mathbf{x}_1'\mathbf{W}\mathbf{x}_2\mathbf{x}_2'\mathbf{x}_1 - 2\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\mathbf{x}_1'\mathbf{x}_2\mathbf{x}_2'\mathbf{W}\mathbf{x}_1 \\
&\quad + \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2\Big] + \frac{4(n-2)}{n(n-1)}\mathrm{E}\Big[\mathbf{x}_1'\mathbf{x}_2\mathbf{x}_2'\mathbf{W}\mathbf{x}_1\mathbf{x}_1'\mathbf{x}_3\mathbf{x}_3'\mathbf{W}\mathbf{x}_1 \\
&\quad + \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2 - \mathbf{x}_1'\mathbf{x}_2\mathbf{x}_2'\mathbf{W}\mathbf{x}_1\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right) - \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\mathbf{x}_1'\mathbf{x}_3\mathbf{x}_3'\mathbf{W}\mathbf{x}_1\Big] \\
&= \frac{2}{n^2}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right) + \frac{2}{n^2}\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2 + \frac{4}{n}\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right) \\
&\quad + \frac{4}{n}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma}^2)^2\right) + \frac{4\Delta}{n}\mathrm{tr}\left(\mathbf{A}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\circ\mathbf{A}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\right) \\
&\quad + O\Bigg[\frac{1}{n^3}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right) + \frac{1}{n^3}\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2 \\
&\qquad + \frac{1}{n^2}\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right) + \frac{1}{n^2}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma}^2)^2\right)\Bigg],
\end{aligned}
\tag{S1.2}
$$

where $\mathbf{A} = \boldsymbol{\Gamma}'\boldsymbol{\Gamma}$. Similarly, it can be shown that

$$
\mathrm{Var}\left(U_n^1\right) = \frac{2}{n}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right) + \frac{\Delta}{n}\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\circ\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\right), \tag{S1.3}
$$

$$
\mathrm{Var}\left(U_n^3\right) = \frac{2}{n(n-1)}\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right) = o\left(\mathrm{Var}\left(U_n^1\right)\right), \tag{S1.4}
$$

$$\begin{aligned}\mathrm{Var}\left(U_n^4\right) =& \frac{4}{n^2}\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right) + \frac{2}{n^3}\left\{\mathrm{tr}\left(\mathbf{W\Sigma}^2\right)\right\}^2 + \frac{4}{n^3}\mathrm{tr}\left((\mathbf{W\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right) \\ &+ O\left(\frac{\left\{\mathrm{tr}\left(\mathbf{W\Sigma}^2\right)\right\}^2}{n^4} + \frac{\mathrm{tr}\left((\mathbf{W\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)}{n^4}\right) = o\left(\mathrm{Var}\left(U_n^2\right)\right),\end{aligned} \tag{S1.5}$$

$$\begin{aligned}\mathrm{Var}\left(U_n^5\right) =& \frac{8}{n^4}\mathrm{tr}\left(\mathbf{\Sigma}^2\right)\mathrm{tr}\left((\mathbf{W\Sigma})^2\right) + \frac{16}{n^4}\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right) \\ &+ O\left(\frac{\mathrm{tr}\left(\mathbf{\Sigma}^2\right)\mathrm{tr}\left((\mathbf{W\Sigma})^2\right)}{n^5} + \frac{\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right)}{n^5}\right) = o\left(\mathrm{Var}\left(U_n^2\right)\right),\end{aligned} \tag{S1.6}$$

$$\mathrm{Cov}\left(U_n^1, U_n^3\right) = \mathrm{E}\left(U_n^1, U_n^3\right) - \mathrm{E}\left(U_n^1\right)\mathrm{E}\left(U_n^3\right) = 0, \tag{S1.7}$$

and

$$\begin{aligned}\mathrm{Cov}\left(U_n^1, U_n^2\right) =& \mathrm{E}\left(U_n^1, U_n^2\right) - \mathrm{E}\left(U_n^1\right)\mathrm{E}\left(U_n^2\right) \\ =& \frac{4}{n}\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^2\right) + \frac{2\Delta}{n}\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{W\Gamma} \circ \mathbf{\Gamma}'\mathbf{\Sigma W\Gamma}\right).\end{aligned} \tag{S1.8}$$

Collecting the results above, we obtain that $\mathrm{Var}\left(pU_n\right) = \sigma_{1,n}^2\left(1 + o(1)\right)$. The proof is complete. $\square$

S2 Proof of theorem 1

Rewrite pU_n as follows:

$$pU_n = Z_n - 2U_n^4 + U_n^5 - 2U_n^3 + \mathrm{tr}(\mathbf{W}),$$

where $Z_n = U_n^2 - 2U_n^1$. From the expectations and variances of U_n^i, $i = 1, \ldots, 5$ as demonstrated in the proof of Lemma 6, it can be concluded that

$$\mathrm{E}\left[Z_n + \mathrm{tr}(\mathbf{W})\right] = \mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma} - \mathbf{I}_p\right)^2\right], \quad \mathrm{Var}(Z_n) = \sigma_{1,n}^2(1 + o(1)),$$
$$\frac{U_n^3}{\sigma_{1,n}} \xrightarrow{p} 0, \quad \frac{U_n^4}{\sigma_{1,n}} \xrightarrow{p} 0 \quad \text{and} \quad \frac{U_n^5}{\sigma_{1,n}} \xrightarrow{p} 0.$$

Therefore, it suffices to prove

$$\frac{Z_n - \mathrm{E}Z_n}{\sqrt{\mathrm{Var}\left(Z_n\right)}} \xrightarrow{d} \mathrm{N}(0,1). \tag{S2.9}$$

as $p, n \to \infty$.

Let $\mathrm{E}_0(\cdot)$ denote expectation and $\mathrm{E}_k(\cdot)$ denote conditional expectation with respect to the σ-field generated by $\mathbf{x}_1, \ldots, \mathbf{x}_k$, $k = 1, 2, \ldots, n$. Then we have the martingale decomposition

$$Z_n - \mathrm{EZ}_n = \sum_{k=1}^{n} D_{n,k} \quad \text{with} \quad D_{n,k} = (\mathrm{E}_k - \mathrm{E}_{k-1})Z_n.$$

By the martingale central limit theorem (Billingsley (1995)), it is sufficient to show that

$$\frac{\sum_{k=1}^n \mathbb{E}_{k-1}\left(D_{n,k}^2\right)}{\mathrm{Var}(Z_n)} \xrightarrow{p} 1 \quad \text{and} \quad \frac{\sum_{k=1}^n \mathbb{E}(D_{n,k}^4)}{\mathrm{Var}^2(Z_n)} \xrightarrow{p} 0. \tag{S2.10}$$

S2.1 Proof of the first part of (S2.10)

Define

$$u_{n,k} = (\mathrm{E}_k - \mathrm{E}_{k-1})U_n^2 \quad \nu_{n,k} = (\mathrm{E}_k - \mathrm{E}_{k-1})U_n^1, \quad \mathbf{Q}_{k-1} = \sum_{i=1}^{k-1}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W} - \boldsymbol{\Sigma}\mathbf{W}),$$

$$\mathbf{M}_{k-1} = \mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}, \quad \boldsymbol{\xi}_{2,k} = \sum_{i=1}^{k-1} \left\{\mathbf{x}_i'\mathbf{W\Sigma W\Sigma x}_i - \operatorname{tr}\left(\mathbf{W\Sigma W\Sigma}^2\right)\right\},$$

$$\boldsymbol{\xi}_{3,k} = \sum_{i=1}^{k-1} \left\{\mathbf{x}_i'\mathbf{W\Sigma W\Sigma}^2\mathbf{x}_i - \operatorname{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right)\right\},$$

$$\boldsymbol{\xi}_{3,k}' = \sum_{i=1}^{k-1} \left\{\mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i - \operatorname{tr}\left(\mathbf{W\Sigma}^2\mathbf{W\Sigma}^2\right)\right\}.$$

Then, we have

$$\begin{aligned}
\nu_{n,k} =& \left(\mathrm{E}_k - \mathrm{E}_{k-1}\right) U_n^1 = \left(\mathrm{E}_k - \mathrm{E}_{k-1}\right) \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i'\mathbf{W}\mathbf{x}_i \\
=& \frac{\mathrm{E}_k - \mathrm{E}_{k-1}}{n}\mathbf{x}_k'\mathbf{W}\mathbf{x}_k + \frac{\mathrm{E}_k - \mathrm{E}_{k-1}}{n} \sum_{i\neq k}^{n} \mathbf{x}_i'\mathbf{W}\mathbf{x}_i \\
=& \frac{\mathbf{x}_k'\mathbf{W}\mathbf{x}_k}{n} - \frac{\mathrm{E}\mathbf{x}_k'\mathbf{W}\mathbf{x}_k}{n} = \frac{1}{n}\left(\mathbf{x}_k'\mathbf{W}\mathbf{x}_k - \operatorname{tr}\left(\mathbf{W\Sigma}\right)\right), \\
u_{n,k} =& (\mathrm{E}_k - \mathrm{E}_{k-1})U_n^2 = (\mathrm{E}_k - \mathrm{E}_{k-1})\frac{1}{A_{n,2}} \sum_{i\neq j} \mathbf{x}_i'\mathbf{x}_j\mathbf{x}_i'\mathbf{W}\mathbf{x}_j \\
=& \frac{2}{n(n-1)}(\mathrm{E}_k - \mathrm{E}_{k-1}) \sum_{i\neq k} \mathbf{x}_i'\mathbf{x}_k\mathbf{x}_i'\mathbf{W}\mathbf{x}_k \\
=& \frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k - \operatorname{tr}\left(\mathbf{\Sigma Q}_{k-1}\right)\right] + \frac{2}{n}\left[\mathbf{x}_k'\mathbf{W\Sigma x}_k - \operatorname{tr}\left(\mathbf{W\Sigma}^2\right)\right].
\end{aligned}$$

Using the expansions above, we get

$$\begin{aligned}
&\mathrm{E}_{k-1}\left(D_{n,k}^2\right) = \mathrm{E}_{k-1}\left[u_{n,k} - 2\nu_{n,k}\right]^2 \\
=&\mathrm{E}_{k-1}\Bigg[\frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k - \operatorname{tr}\left(\mathbf{\Sigma Q}_{k-1}\right)\right] \\
&\qquad + \frac{2}{n}\left[\mathbf{x}_k'\mathbf{W\Sigma x}_k - \operatorname{tr}\left(\mathbf{W\Sigma}^2\right)\right] - 2\frac{1}{n}\left(\mathbf{x}_k'\mathbf{W}\mathbf{x}_k - \operatorname{tr}\left(\mathbf{W\Sigma}\right)\right)\Bigg]^2 \\
=&\mathrm{E}_{k-1}\Bigg[\frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k - \operatorname{tr}\left(\mathbf{\Sigma Q}_{k-1}\right)\right]
\end{aligned}$$

$$+\frac{2}{n}\left[\mathbf{x}_k'\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\mathbf{x}_k-\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2-\mathbf{W}\boldsymbol{\Sigma}\right)\right]\Bigg]^2$$

$$\triangleq \mathrm{E}_{k-1}\left[A_1+A_2\right]^2=\mathrm{E}_{k-1}\left(A_1^2\right)+\mathrm{E}_{k-1}\left(A_2^2\right)+2\mathrm{E}_{k-1}\left(A_1A_2\right),$$

where

$$A_1=\frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k-\operatorname{tr}\left(\boldsymbol{\Sigma}\mathbf{Q}_{k-1}\right)\right],$$

and

$$A_2=\frac{2}{n}\left[\mathbf{x}_k'\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\mathbf{x}_k-\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2-\mathbf{W}\boldsymbol{\Sigma}\right)\right].$$

Therefore, to prove the first part of (S2.10), we only need to verify

$$\frac{\sum_{k=1}^n \mathrm{E}_{k-1}A_1A_2}{\operatorname{Var}\left(Z_n\right)}\xrightarrow{p}0, \tag{S2.11}$$

and

$$\frac{\sum_{k=1}^n \mathrm{E}_{k-1}A_1^2+\sum_{k=1}^n \mathrm{E}_{k-1}A_2^2}{\operatorname{Var}\left(Z_n\right)}\xrightarrow{p}1. \tag{S2.12}$$

Proof of (S2.11).

This can be shown by demonstrating that

$$\frac{\mathrm{E}\left(\sum_{k=1}^n \mathrm{E}_{k-1}A_1A_2\right)^2}{\operatorname{Var}^2\left(Z_n\right)}\xrightarrow{P}0. \tag{S2.13}$$

For the numerator part, we have

$$\mathrm{E}_{k-1}\left(A_1A_2\right)$$

$$
\begin{aligned}
=&\frac{4}{n^2(n-1)}\mathrm{E}_{k-1}\Big[\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k-\mathrm{tr}\left(\mathbf{\Sigma}\mathbf{Q}_{k-1}\right)\right]\\
&\qquad\qquad\left[\mathbf{x}_k'\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{x}_k-\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2-\mathbf{W}\mathbf{\Sigma}\right)\right]\Big]\\
=&\frac{4}{n^2(n-1)}\Bigg[\mathrm{tr}\left[\mathbf{Q}_{k-1}\mathbf{\Sigma}\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Sigma}\right]+\mathrm{tr}\left[\mathbf{Q}_{k-1}'\mathbf{\Sigma}\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Sigma}\right]\\
&\qquad\qquad+\Delta\mathrm{tr}\left[\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Gamma}\right]\Bigg]\\
=&\frac{4}{n^2(n-1)}\left[\boldsymbol{\xi}_{3,k}+\boldsymbol{\xi}_{3,k}'-2\boldsymbol{\xi}_{2,k}+\Delta\mathrm{tr}\left[\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Gamma}\right]\right].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\mathrm{E}\left(\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1A_2\right)^2\\
=&\mathrm{E}\left[\frac{4}{n^2(n-1)}\sum_{k=1}^{n}\left[\boldsymbol{\xi}_{3,k}+\boldsymbol{\xi}_{3,k}'-2\boldsymbol{\xi}_{2,k}+\Delta\mathrm{tr}\left[\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Gamma}\right]\right]\right]^2\\
\le& n\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\mathrm{E}\left[\left(\boldsymbol{\xi}_{3,k}-\boldsymbol{\xi}_{2,k}\right)+\left(\boldsymbol{\xi}_{3,k}'-\boldsymbol{\xi}_{2,k}\right)\right]^2\\
&+n\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\mathrm{E}\Big[\Delta\mathrm{tr}\left[\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma}-\mathbf{W}\right)\mathbf{\Gamma}\right]\Big]^2.
\end{aligned}
\qquad \text{(S2.14)}
$$

By direct calculations, one can show that

$$
\begin{aligned}
&\boldsymbol{\xi}_{3,k}-\boldsymbol{\xi}_{2,k}\\
=&\sum_{i=1}^{k-1}\left[\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i-\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]-\sum_{i=1}^{k-1}\left[\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}\mathbf{x}_i-\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\right)\right]\\
=&\sum_{i=1}^{k-1}\left[\mathbf{x}_i'\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2-\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}\right)\mathbf{x}_i-\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3-\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\right)\right]\\
=&\sum_{i=1}^{k-1}\left[\mathbf{z}_i'\left(\mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Gamma}\right)\mathbf{z}_i-\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}\left(\boldsymbol{\xi}_{3,k}-\boldsymbol{\xi}_{2,k}\right)^2\\
=&\mathrm{E}\left[\sum_{i=1}^{k-1}\left[\mathbf{z}_i'\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Gamma}\right)\mathbf{z}_i-\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\right)\right]\right]^2\\
=&\sum_{i=1}^{k-1}\mathrm{E}\left[\mathbf{z}_i'\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Gamma}\right)\mathbf{z}_i-\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\right)\right]^2\\
\leq&K\sum_{i=1}^{k-1}\mathrm{tr}\left[\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\right]\\
=&K\sum_{i=1}^{k-1}o\Big(\mathrm{tr}\big((\mathbf{W}\boldsymbol{\Sigma})^2\big)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big),
\end{aligned}
$$

where the last equality follows from Lemma 5. Therefore, we obtain

$$
\begin{aligned}
&n\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\mathrm{E}\left(\boldsymbol{\xi}_{3,k}-\boldsymbol{\xi}_{2,k}\right)^2\\
\leq&Kn\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\sum_{i=1}^{k-1}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big)\\
\leq&Kn^{-3}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big). \qquad \text{(S2.15)}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&n\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\mathrm{E}\left(\boldsymbol{\xi}_{3,k}'-\boldsymbol{\xi}_{2,k}\right)^2\\
\leq&Kn\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\sum_{i=1}^{k-1}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big)\\
\leq&Kn^{-3}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big). \qquad \text{(S2.16)}
\end{aligned}
$$

On the other hand, one can show that

$$
\mathrm{tr}\left[\boldsymbol{\Gamma}'\mathbf{Q}_{k-1}\boldsymbol{\Gamma}\circ\boldsymbol{\Gamma}'\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Gamma}\right]
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k-1}\sum_{j=1}^{m} \mathbf{e}_j'\mathbf{\Gamma}'(\mathbf{x}_i\mathbf{x}_i'\mathbf{W} - \mathbf{\Sigma}\mathbf{W})\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{m} \Big[\mathbf{z}_i'\mathbf{\Gamma}'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\mathbf{\Gamma}\mathbf{z}_i \\
&\qquad\qquad - \mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_j\Big] \\
&\triangleq \sum_{i=1}^{k-1}\left[\mathbf{z}_i'\sum_{j=1}^{m}\mathbf{M}_j\mathbf{z}_i - \mathrm{tr}\left(\sum_{j=1}^{m}\mathbf{M}_j\right)\right] \triangleq \sum_{i=1}^{k-1}\left[\mathbf{z}_i'\mathbf{M}\mathbf{z}_i - \mathrm{tr}\left(\mathbf{M}\right)\right],
\end{aligned}
$$

where

$$
\mathbf{M}_j \triangleq \mathbf{\Gamma}'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\mathbf{\Gamma} \quad \text{and} \quad \mathbf{M} \triangleq \sum_{j=1}^{m}\mathbf{M}_j.
$$

Then, by using Lemma 5 and the notations

$$
\mathbf{D} = \mathbf{\Gamma}'\mathbf{W}\left(\mathbf{\Sigma} - \mathbf{I}\right)\mathbf{\Gamma}, \quad \mathbf{F} = \mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{\Gamma} \quad \text{and} \quad \mathbf{G} = \mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma},
$$

we have

$$
\begin{aligned}
&\mathrm{E}\left[\Delta\mathrm{tr}\left[\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\right]\right]^2 \\
=&\Delta^2\mathrm{E}\left[\sum_{i=1}^{k-1}\left[\mathbf{z}_i'\mathbf{M}\mathbf{z}_i - \mathrm{tr}\left(\mathbf{M}\right)\right]\right]^2 \le \Delta^2\sum_{i=1}^{k-1}\mathrm{tr}\left(\mathbf{M}\mathbf{M}'\right) = K\sum_{i=1}^{k-1}\mathrm{tr}\left[\sum_{j=1}^{m}\sum_{l=1}^{m}\mathbf{M}_j\mathbf{M}_l'\right] \\
=&K\sum_{i=1}^{k-1}\sum_{j=1}^{m}\sum_{l=1}^{m}\mathrm{tr}\Big[\mathbf{\Gamma}'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j\mathbf{e}_j'\mathbf{\Gamma}'\mathbf{\Gamma}\mathbf{\Gamma}'\mathbf{\Gamma}\mathbf{e}_l \\
&\qquad\qquad \mathbf{e}_l'\mathbf{\Gamma}'\left(\mathbf{\Sigma}\mathbf{W} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{W}\mathbf{\Gamma}\Big] \\
=&K\sum_{i=1}^{k-1}\sum_{j=1}^{m}\sum_{l=1}^{m}\Big[\mathbf{e}_j'\mathbf{\Gamma}'\left(\mathbf{W}\mathbf{\Sigma} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_j \times \mathbf{e}_j'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{\Gamma}\mathbf{e}_l \\
&\qquad\qquad \times \mathbf{e}_l'\mathbf{\Gamma}\left(\mathbf{\Sigma}\mathbf{W} - \mathbf{W}\right)\mathbf{\Gamma}\mathbf{e}_l \times \mathbf{e}_l'\mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_j\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&K\sum_{i=1}^{k-1}\sum_{j=1}^{m}\sum_{l=1}^{m}\mathbf{D}_{jj}\mathbf{D}_{ll}\mathbf{F}_{jl}\mathbf{G}_{jl} \le K\sum_{i=1}^{k-1}\sqrt{\left[\sum_{j,l}^{m}\mathbf{D}_{jj}^2\mathbf{F}_{jl}^2\right]\left[\sum_{j,l}^{m}\mathbf{D}_{ll}^2\mathbf{G}_{jl}^2\right]}\\
\le&K\sum_{i=1}^{k-1}\sqrt{\left[\sum_{j=1}^{m}\mathbf{D}_{jj}^2\sum_{l=1}^{m}\mathbf{F}_{jl}\mathbf{F}_{lj}\right]\left[\sum_{l=1}^{m}\mathbf{D}_{ll}^2\sum_{j=1}^{m}\mathbf{G}_{lj}\mathbf{G}_{jl}\right]}\\
\le&K\sum_{i=1}^{k-1}\sqrt{\left[\sum_{j=1}^{m}\mathbf{D}_{jj}^2\mathbf{F}_{jj}^2\right]\left[\sum_{l=1}^{m}\mathbf{D}_{ll}^2\mathbf{G}_{ll}^2\right]} \le K\sum_{i=1}^{k-1}\sqrt{\mathrm{tr}\left(\mathbf{D}\mathbf{D}'\right)\mathrm{tr}\left(\mathbf{F}^2\right)\mathrm{tr}\left(\mathbf{D}\mathbf{D}'\right)\mathrm{tr}\left(\mathbf{G}^2\right)}\\
=&K\sum_{i=1}^{k-1}\mathrm{tr}\left(\mathbf{D}\mathbf{D}'\right)\sqrt{\mathrm{tr}\left(\mathbf{F}^2\right)\mathrm{tr}\left(\mathbf{G}^2\right)}\\
=&K\sum_{i=1}^{k-1}\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\sqrt{\mathrm{tr}\left(\boldsymbol{\Sigma}^4\right)\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^4\right)}\\
=&K\sum_{i=1}^{k-1}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big).
\end{aligned}
$$

This gives that

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\mathrm{E}\left[\Delta\mathrm{tr}\left[\boldsymbol{\Gamma}'\mathbf{Q}_{k-1}\boldsymbol{\Gamma}\circ\boldsymbol{\Gamma}'\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Gamma}\right]\right]^2\\
\le&Kn\sum_{k=1}^{n}\frac{16}{n^4(n-1)^2}\sum_{i=1}^{k-1}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big)\\
\le&Kn^{-3}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big). \qquad \text{(S2.17)}
\end{aligned}
$$

Combining (S2.14)-(S2.17), we get

$$
\mathrm{E}\left(\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1A_2\right)^2 = Kn^{-3}o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big). \qquad \text{(S2.18)}
$$

Next, we deal with the denominator of (S2.13), which can be rewritten as

$$\mathrm{Var}^2\left(Z_n\right)=\left[\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2+\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_2^2\right]^2.$$

We consider the lower bound of $\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2$ and $\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_2^2$ separately. For the latter, it can be demonstrate that

$$\begin{aligned}
&\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_2^2\\
=&\sum_{k=1}^{n}\mathrm{E}\frac{4}{n^2}\left[\mathbf{x}_k'\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{x}_k-\mathrm{tr}\left(\mathbf{W\Sigma}^2-\mathbf{W\Sigma}\right)\right]^2\\
\geq&\frac{4}{n^2}\sum_{k=1}^{n}\Big[\mathrm{tr}\big[\mathbf{\Gamma}'\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Gamma\Gamma}'\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Gamma}\big]\\
&\qquad+\mathrm{tr}\big[\mathbf{\Gamma}'\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Gamma\Gamma}'\left(\mathbf{\Sigma W}-\mathbf{W}\right)'\mathbf{\Gamma}\big]\Big]\\
=&\frac{4}{n^2}\sum_{k=1}^{n}\Big[\mathrm{tr}\big[\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Sigma}\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Sigma}\big]\\
&\qquad+\mathrm{tr}\big[\left(\mathbf{W\Sigma}-\mathbf{W}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma W}-\mathbf{W}\right)\mathbf{\Sigma}\big]\Big]\\
=&\frac{4}{n^2}\sum_{k=1}^{n}\Big[\mathrm{tr}\big[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\big]+\mathrm{tr}\big[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{W\Sigma}\big]\Big].
\end{aligned}\tag{S2.19}$$

For $\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2$, we have

$$\begin{aligned}
\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2=&\sum_{k=1}^{n}\mathrm{E}_{k-1}\frac{2}{n(n-1)}\frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k-\mathrm{tr}\left(\mathbf{\Sigma Q}_{k-1}\right)\right]^2\\
\geq&\sum_{k=1}^{n}\frac{4}{n^2(n-1)^2}\left[\mathrm{tr}\left(\Gamma'\mathbf{Q}_{k-1}\Gamma\Gamma'\mathbf{Q}_{k-1}\Gamma\right)+\mathrm{tr}\left(\Gamma'\mathbf{Q}_{k-1}\Gamma\Gamma'\mathbf{Q}_{k-1}'\Gamma\right)\right]
\end{aligned}$$

$$=\sum_{k=1}^{n}\frac{4}{n^2(n-1)^2}\left[\operatorname{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\right)+\operatorname{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}'_{k-1}\boldsymbol{\Sigma}\right)\right] \tag{S2.20}$$

Applying Lemma 1 and the expansion

$$\begin{aligned}
&\operatorname{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\right)\\
=&\operatorname{tr}\left[\sum_{i=1}^{k-1}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}\sum_{j=1}^{k-1}(\mathbf{x}_j\mathbf{x}_j'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}\right]\\
=&\operatorname{tr}\left[\sum_{i=j}^{k-1}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}\right]\\
&+\operatorname{tr}\left[\sum_{i\neq j}^{k-1}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}(\mathbf{x}_j\mathbf{x}_j'\mathbf{W}-\boldsymbol{\Sigma}\mathbf{W})\boldsymbol{\Sigma}\right]\\
=&\sum_{i=j}^{k-1}\left[\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i-2\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i+\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}^2\right)\right]\\
&+\sum_{i\neq j}^{k-1}\Big[\mathbf{x}_i\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\\
&\qquad-\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_j+\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}^2\right)\Big],
\end{aligned}$$

we obtain that

$$\mathrm{E}\left[\operatorname{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\right)\right]\geq\sum_{i=j}^{k-1}\left[\left\{\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2+\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right].$$

Similarly, we get

$$\mathrm{E}\left[\operatorname{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}'_{k-1}\boldsymbol{\Sigma}\right)\right]\geq\sum_{i=j}^{k-1}\left[\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)+\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right].$$

From the above two inequalities and (S2.20), we have

$$
\begin{aligned}
&\mathrm{E}\left(\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2\right)\\
\geq&\sum_{k=1}^{n}\frac{4}{n^2(n-1)^2}\left[\sum_{i=j}^{k-1}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]\right]\\
=&\sum_{k=1}^{n}\frac{4}{n(n-1)^2}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right].
\end{aligned}
\tag{S2.21}
$$

Combining (S2.19) and (S2.21), it yields that

$$
\begin{aligned}
&\mathrm{Var}^2\left(Z_n\right)\\
=&\left[\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2+\mathrm{E}\sum_{k=1}^{n}\mathrm{E}_{k-1}A_2^2\right]^2\\
\geq&\left[\sum_{k=1}^{n}\frac{4}{n(n-1)^2}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]\right.\\
&\left.+\frac{4}{n^2}\sum_{k=1}^{n}\left[\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\right]+\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{W}\mathbf{\Sigma}\right]\right]\right]^2\\
=&\frac{16}{n^2(n-1)^4}\sum_{k=1}^{n}\sum_{k=1}^{n}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]^2\\
&+\frac{16}{n^4}\sum_{k=1}^{n}\sum_{k=1}^{n}\left[\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\right]+\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{W}\mathbf{\Sigma}\right]\right]^2\\
&+\frac{32}{n^3(n-1)^2}\sum_{k=1}^{n}\sum_{k=1}^{n}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]\\
&\left[\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\right]+\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{W}\mathbf{\Sigma}\right]\right]\\
=&Kn^{-4}\left[\left\{\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}^2\right)\right\}^2+\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\mathrm{tr}\left(\mathbf{\Sigma}^2\right)+2\mathrm{tr}\left(\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^3\right)\right]^2\\
&+Kn^{-2}\left[\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\right]+\mathrm{tr}\left[\mathbf{W}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{\Sigma}\left(\mathbf{\Sigma}-\mathbf{I}\right)\mathbf{W}\mathbf{\Sigma}\right]\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ Kn^{-3}\Big[\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2 \mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\right] \\
&\qquad\qquad + \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^2 \mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right] \\
&\qquad\qquad + \mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\right] \\
&\qquad\qquad + \mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right] \\
&\qquad\qquad + 2\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\right] \\
&\qquad\qquad + 2\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big]. \qquad \text{(S2.22)}
\end{aligned}
$$

This, together with (S2.18), shows that

$$
\begin{aligned}
\frac{\mathrm{E}\left(\sum_{k=1}^n \mathrm{E}_{k-1}A_1A_2\right)^2}{\mathrm{Var}^2\left(Z_n\right)} \leq & \frac{o\Big(\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\Big)}{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]} \\
= & o\left(1\right). \qquad \text{(S2.23)}
\end{aligned}
$$

The proof of (S2.11) is thus complete.

Proof of (S2.12).

Since

$$
\frac{\mathrm{E}\left(\sum_{k=1}^n \mathrm{E}_{k-1}A_1^2 + \sum_{k=1}^n \mathrm{E}_{k-1}A_2^2\right)}{\mathrm{Var}\left(Z_n\right)} = 1,
$$

it suffices to prove

$$
\frac{\mathrm{Var}\left(\sum_{k=1}^n \mathrm{E}_{k-1}A_1^2 + \sum_{k=1}^n \mathrm{E}_{k-1}A_2^2\right)}{\mathrm{Var}^2\left(Z_n\right)} = \frac{\mathrm{Var}\left(\sum_{k=1}^n \mathrm{E}_{k-1}A_1^2\right)}{\mathrm{Var}^2\left(Z_n\right)} \xrightarrow{P} 0. \qquad \text{(S2.24)}
$$

The lower bound of the denominator is given in (S2.22). We next calculate the upper bound of the numerator:

$$
\begin{aligned}
&\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{E}_{k-1}A_1^2\right)\\
=&\mathrm{Var}\Big[\frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\big[\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)+\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\mathbf{\Gamma}'\mathbf{Q}_{k-1}'\mathbf{\Gamma}\right)\\
&\qquad\qquad\qquad\qquad+\Delta\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\big]\Big]\\
=&\mathrm{Var}\Big[\frac{4}{n^2(n-1)^2}\sum_{k=1}^{n}\big[\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}\mathbf{\Sigma}\right)+\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\\
&\qquad\qquad\qquad\qquad+\Delta\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\big]\Big]\\
\leq&\frac{16}{n^4(n-1)^4}\left[\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}\mathbf{\Sigma}\right)\right)+\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\right)\right.\\
&\qquad\qquad\left.+\mathrm{Var}\left(\sum_{k=1}^{n}\Delta\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\circ\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\right)\right]. \qquad \text{(S2.25)}
\end{aligned}
$$

For the term $\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}\mathbf{\Sigma}\right)\right)$, define

$$
L_{ij}=\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}^2\mathbf{W}\mathbf{\Sigma}\mathbf{x}_i-\mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}^2\mathbf{W}\mathbf{\Sigma}\mathbf{x}_j\right).
$$

Then we have

$$
\begin{aligned}
&\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}\mathbf{\Sigma}\right)\right)\leq n\sum_{k=1}^{n}\mathrm{Var}\left(\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}\mathbf{\Sigma}\right)\right)\\
=&n\sum_{k=1}^{n}\mathrm{Var}\left[\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}^2\mathbf{W}\mathbf{\Sigma}\mathbf{x}_i-\mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}^2\mathbf{W}\mathbf{\Sigma}\mathbf{x}_j\right)\right]\\
=&n\sum_{k=1}^{n}\mathrm{Var}\left(\sum_{i,j}^{k-1}L_{ij}\right)\leq 2n\sum_{k=1}^{n}\left[\mathrm{Var}\left(\sum_{i=j}^{k-1}L_{ii}\right)+\mathrm{Var}\left(\sum_{i\neq j}^{k-1}L_{ij}\right)\right].
\end{aligned}
\qquad \text{(S2.26)}
$$

Employing Lemmas 1-5, we obtain

$$\mathrm{Var}\left(\sum_{i=j}^{k-1} L_{ii}\right) = \sum_{i=j}^{k-1} \mathrm{Var}\left(L_{ii}\right) \leq \sum_{i=j}^{k-1} \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i - 2\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)^2$$

$$\leq K(k-1)\left[\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)^4 + \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)^2\right]$$

$$= K(k-1)\left[\mathrm{E}\left(\mathbf{z}_i'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\boldsymbol{\Gamma}\mathbf{z}_i - \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right) + \mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right)^4 + \mathrm{E}\left(\mathbf{z}_i'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\boldsymbol{\Gamma}\mathbf{z}_i\right)^2\right]$$

$$\leq K(k-1)\left[\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2 + \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^4 + \mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Gamma}\right)\right]$$

$$= K(k-1)\left[\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^4 + o\left(\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\right)\right].$$

This implies that

$$\frac{16}{n^4(n-1)^4} 2n \sum_{k=1}^{n} \mathrm{Var}\left(\sum_{i=j}^{k-1} L_{ii}\right)$$
$$\leq K n^{-5}\left[\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)\right\}^4 + o\left(\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\right)\right]. \qquad \text{(S2.27)}$$

Now consider

$$\mathrm{Var}\left(\sum_{i\neq j}^{k-1} L_{ij}\right) = \sum_{i_1\neq j_1}^{k-1} \sum_{i_2\neq j_2}^{k-1} \mathrm{Cov}\left(L_{i_1j_1}, L_{i_2j_2}\right). \qquad \text{(S2.28)}$$

We will divide these terms into three cases for discussion.

Case 1: $i_1 \neq i_2 \neq j_1 \neq j_2$. It is straightforward to verify that $\mathrm{Cov}\left(L_{i_1j_1}, L_{i_2j_2}\right) = 0$;

Case 2: $i_1 = i_2 = i \neq j_1 = j_2 = j$. We have

$$\mathrm{Cov}\left(L_{ij}, L_{ij}\right)$$
$$= \mathrm{Cov}\Big[\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i - \mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_j\right),$$

$$
\begin{aligned}
& \left(\mathbf{x}_i'\mathbf{W\Sigma x}_j\mathbf{x}_j'\mathbf{W\Sigma x}_i - \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i - \mathbf{x}_j'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_j\right)\Big] \\
\leq & \mathrm{E}\left(\mathbf{x}_i'\mathbf{W\Sigma x}_j\mathbf{x}_j'\mathbf{W\Sigma x}_i - \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i - \mathbf{x}_j'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_j\right)^2 \\
\leq & K\left[\mathrm{E}\left(\mathbf{x}_i'\mathbf{W\Sigma x}_j\mathbf{x}_j'\mathbf{W\Sigma x}_i\right)^2 + 2\mathrm{E}\left(\mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i\right)^2\right] \\
\leq & K\Big[\mathrm{E}\left(\mathbf{x}_j'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_j\right)^2 + 2\mathrm{E}\left(\mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i\right)^2 \\
& + \mathrm{E}\left(\mathbf{x}_j'\mathbf{W\Sigma}^3\mathbf{W x}_j\mathbf{x}_j'\mathbf{\Sigma W\Sigma W\Sigma x}_j\right)\Big] \\
\leq & K\Big[\left\{\mathrm{tr}\left(\left(\mathbf{W\Sigma}^2\right)^2\right)\right\}^2 + \mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{W\Sigma}^2\mathbf{W\Sigma\Gamma\Gamma}'\mathbf{\Sigma W\Sigma}^2\mathbf{W\Gamma}\right) \\
& + 2\left\{\mathrm{tr}\left(\left(\mathbf{W\Sigma}^2\right)^2\right)\right\}^2 + 2\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{W\Sigma}^2\mathbf{W\Sigma\Gamma\Gamma}'\mathbf{\Sigma W\Sigma}^2\mathbf{W\Gamma}\right) \\
& + \left\{\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right)\right\}^2\Big] \\
\leq & K\left[4\left\{\mathrm{tr}\left(\mathbf{W\Sigma W\Sigma}^3\right)\right\}^2 + 3\mathrm{tr}\left(\mathbf{W\Sigma}^2\mathbf{W\Sigma}^3\mathbf{W\Sigma}^2\mathbf{W\Sigma}\right)\right] \\
= & o\left[\left\{\mathrm{tr}\left(\mathbf{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left(\left(\mathbf{W\Sigma}\right)^2\right)\right\}^2\right].
\end{aligned}
$$

Case 3: $i_1 = i_2 = i \neq j_1 \neq j_2$. We have

$$
\begin{aligned}
& \mathrm{Cov}\left(L_{ij_1}, L_{ij_2}\right) \\
= & \mathrm{Cov}\Big[\left(\mathbf{x}_i'\mathbf{W\Sigma x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W\Sigma x}_i - \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i - \mathbf{x}_{j_1}'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_{j_1}\right), \\
& \left(\mathbf{x}_i'\mathbf{W\Sigma x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W\Sigma x}_i - \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma}^2\mathbf{x}_i - \mathbf{x}_{j_2}'\mathbf{W\Sigma W\Sigma}^2\mathbf{x}_{j_2}\right)\Big] \\
= & \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W\Sigma x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W\Sigma x}_i, \mathbf{x}_i'\mathbf{W\Sigma x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W\Sigma x}_i\right) && \text{(S2.29)} \\
& - \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W\Sigma x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W\Sigma x}_i, \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i\right) && \text{(S2.30)} \\
& - \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i, \mathbf{x}_i'\mathbf{W\Sigma x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W\Sigma x}_i\right) && \text{(S2.31)} \\
& + \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i, \mathbf{x}_i'\mathbf{W\Sigma}^2\mathbf{W\Sigma x}_i\right). && \text{(S2.32)}
\end{aligned}
$$

The terms (S2.29)-(S2.32) can be further expanded by

$$
\begin{aligned}
\text{(S2.29)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
& - \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
=& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)^2\right)\right\}^2 \\
\text{(S2.30)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
& - \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
=& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)^2\right)\right\}^2 \\
\text{(S2.31)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
& - \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
=& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)^2\right)\right\}^2 \\
\text{(S2.32)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
& - \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) \\
=& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\left(\mathbf{W}\boldsymbol{\Sigma}^2\right)^2\right)\right\}^2.
\end{aligned}
$$

Thus, $\mathrm{Cov}\left(L_{ij_1}, L_{ij_2}\right) = 0$. Collecting the three cases above, we get

$$
\frac{16}{n^4(n-1)^4} n \sum_{k=1}^{n} \mathrm{Var}\left(\sum_{i\neq j}^{k-1} L_{ij}\right) = n^{-4} \times o\left[\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2 \left\{\mathrm{tr}\left(\left(\mathbf{W}\boldsymbol{\Sigma}\right)^2\right)\right\}^2\right]. \tag{S2.33}
$$

Combining (S2.22), (S2.26), (S2.27) and (S2.33), we obtain

$$
\frac{\frac{16}{n^4(n-1)^4}\mathrm{Var}\left(\sum_{k=1}^{n}\mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\right)\right)}{\mathrm{Var}^2\left(Z_n\right)} \xrightarrow{P} 0. \tag{S2.34}
$$

For the term $\mathrm{Var}\left(\sum_{k=1}^n \mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\right)$, define

$$G_{ij} = \left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\mathbf{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i - \mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_j\right).$$

Then we have

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{k=1}^n \mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\right) \le n\sum_{k=1}^n \mathrm{Var}\left(\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\right)\\
=&n\sum_{k=1}^n \mathrm{Var}\left[\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\mathbf{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i - \mathbf{x}_j'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\Sigma^2\mathbf{x}_j\right)\right]\\
=&n\sum_{k=1}^n \mathrm{Var}\left(\sum_{i,j}^{k-1} G_{ij}\right) \le 2n\sum_{k=1}^n\left[\mathrm{Var}\left(\sum_{i=j}^{k-1} G_{ii}\right) + \mathrm{Var}\left(\sum_{i\neq j}^{k-1} G_{ij}\right)\right].
\end{aligned} \tag{S2.35}$$

On the one hand, for $\mathrm{Var}\left(\sum_{i=j}^{k-1} G_{ii}\right)$, employing Lemmas 1-5, we obtain

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{i=j}^{k-1} G_{ii}\right) = \sum_{i=j}^{k-1}\mathrm{Var}\left(G_{ii}\right)\\
\le&\sum_{i=j}^{k-1}\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{x}_i\mathbf{x}_i'\mathbf{\Sigma}\mathbf{x}_i - 2\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i\right)^2\\
\le&K(k-1)\left[\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{x}_i\mathbf{x}_i'\mathbf{\Sigma}\mathbf{x}_i\right)^2 + \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i\right)^2\right]\\
\le&K(k-1)\left[\left[\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{x}_i\right)^4\mathrm{E}\left(\mathbf{x}_i'\mathbf{\Sigma}\mathbf{x}_i\right)^4\right]^{1/2} + \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{x}_i\right)^2\right]\\
=&K(k-1)\Bigg[\left[\mathrm{E}\left(\mathbf{z}_i'\mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{z}_i - \mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right) + \mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\right)^4\right]^{1/2}\\
&\qquad\times\left[\mathrm{E}\left(\mathbf{z}_i'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{\Gamma}\mathbf{z}_i - \mathrm{tr}\left(\mathbf{\Sigma}^2\right) + \mathrm{tr}\left(\mathbf{\Sigma}^2\right)\right)^4\right]^{1/2} + \mathrm{E}\left(\mathbf{z}_i'\mathbf{\Gamma}'\mathbf{W}\mathbf{\Sigma}\mathbf{W}\mathbf{\Sigma}^2\mathbf{\Gamma}\mathbf{z}_i\right)^2\Bigg]\\
=&K(k-1)\Bigg[\left[\left\{\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^4\right)\right\}^2 + \left\{\mathrm{tr}\left((\mathbf{W}\mathbf{\Sigma})^2\right)\right\}^4\right]^{1/2}\left[\left\{\mathrm{tr}\left(\mathbf{\Sigma}^4\right)\right\}^2 + \left\{\mathrm{tr}\left(\mathbf{\Sigma}^2\right)\right\}^4\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{E}\left(\mathbf{z}_i'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\boldsymbol{\Gamma}\mathbf{z}_i\right)^2\Bigg] \\
\leq& K(k-1)\Big[\left\{\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2(1+O(1))+\left\{\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2 \\
&+\operatorname{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^5\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\right)\Big] \\
\leq& K(k-1)\left[\left\{\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2(1+O(1))+o\left(\left\{\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\right)\right] \\
\leq& K(k-1)\left[\left\{\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\right].
\end{aligned}
$$

This implies that

$$
\frac{16}{n^4(n-1)^4}2n\sum_{k=1}^{n}\operatorname{Var}\left(\sum_{i=j}^{k-1}G_{ii}\right)\leq Kn^{-5}\left[\left\{\operatorname{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\right]. \tag{S2.36}
$$

On the other hand, we consider

$$
\operatorname{Var}\left(\sum_{i\neq j}^{k-1}G_{ij}\right)=\sum_{i_1\neq j_1}^{k-1}\sum_{i_2\neq j_2}^{k-1}\operatorname{Cov}\left(G_{i_1j_1},G_{i_2j_2}\right) \tag{S2.37}
$$

We will divide these terms into three cases for discussion.

Case 1: $i_1\neq i_2\neq j_1\neq j_2$. It is straightforward to verify that $\operatorname{Cov}\left(G_{i_1j_1},G_{i_2j_2}\right)=0$;

Case 2: $i_1=i_2=i\neq j_1=j_2=j$. We have

$$
\begin{aligned}
&\operatorname{Cov}\left(G_{ij},G_{ij}\right) \\
=&\operatorname{Cov}\Big[\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i-\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_j\right), \\
&\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i-\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_j\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\leq & \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i - \mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_j\right)^2 \\
\leq & K\left[\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\mathbf{x}_i\right)^2 + 2\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)^2\right] \\
\leq & K\Big[\mathrm{E}\left\{\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\right)\right\}^2 + 2\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)^2 \\
& \quad + \mathrm{Etr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\right)\Big] \\
\leq & K\Big[\mathrm{E}\left(\mathbf{x}_j'\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\right)^2 + 2\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)^2 \\
& \quad + \mathrm{E}\left(\mathbf{x}_j'\boldsymbol{\Sigma}^3\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\right)\Big] \\
\leq & K\Big[\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2 + \mathrm{tr}\left(\boldsymbol{\Gamma}'\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\boldsymbol{\Gamma}\right) + 2\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2 \\
& \quad + 2\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\boldsymbol{\Sigma}^2\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\right) + \mathrm{tr}\left(\boldsymbol{\Sigma}^4\right)\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^4\right)\Big] \\
\leq & K\Big[3\left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2 + 3\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^5\right) + \mathrm{tr}\left(\boldsymbol{\Sigma}^4\right)\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^4\right)\Big] \\
= & Ko\left[\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right].
\end{aligned}
$$

Case 3: $i_1 = i_2 = i \neq j_1 \neq j_2$. We have

$$
\begin{aligned}
& \mathrm{Cov}\left(G_{ij_1}, G_{ij_2}\right) \\
= & \mathrm{Cov}\Big[\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\boldsymbol{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i - \mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_{j_1}\right), \\
& \quad \left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i - \mathbf{x}_{j_2}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_{j_2}\right)\Big] \\
= & \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\boldsymbol{\Sigma}\mathbf{x}_i, \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\right) \qquad \text{(S2.38)} \\
& - \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i, \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right) \qquad \text{(S2.39)} \\
& - \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i, \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\right) \qquad \text{(S2.40)} \\
& + \mathrm{Cov}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i, \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right). \qquad \text{(S2.41)}
\end{aligned}
$$

We calculate separately (S2.38)-(S2.41):

$$\begin{aligned}
\text{(S2.38)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\right)\\
&- \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\right)\\
=&\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2\\
\text{(S2.39)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
&- \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_1}\mathbf{x}_{j_1}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
=&\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2\\
\text{(S2.40)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
&- \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_{j_2}\mathbf{x}_{j_2}'\boldsymbol{\Sigma}\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
=&\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2\\
\text{(S2.41)} =& \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i, \mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
&- \mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right)\\
=&\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i\right) - \left\{\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right\}^2.
\end{aligned}$$

Thus,we have $\mathrm{Cov}\left(G_{ij_1}, G_{ij_2}\right) = 0$. Collecting the three cases above, we get

$$\frac{16}{n^4(n-1)^4} n \sum_{k=1}^{n} \mathrm{Var}\left(\sum_{i\neq j}^{k-1} G_{ij}\right) = n^{-4} \times o\left[\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2 \left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right]. \tag{S2.42}$$

Combining (S2.22), (S2.35), (S2.36) and (S2.42), we obtain

$$\frac{\frac{16}{n^4(n-1)^4}\mathrm{Var}\left(\sum_{k=1}^{n} \mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}'\boldsymbol{\Sigma}\right)\right)}{\mathrm{Var}^2\left(Z_n\right)} \xrightarrow{p} 0. \tag{S2.43}$$

For the term $\mathrm{Var}\left(\sum_{k=1}^{n} \Delta \mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma} \circ \mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\right)$, define

$$\begin{aligned} H_{ij} =& \mathbf{x}_i'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_i \\ & - \mathbf{x}_j'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_j. \end{aligned}$$

Then we have

$$\begin{aligned} & \mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma} \circ \mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right) \\ =& \mathrm{tr}\left(\mathbf{\Gamma}'\sum_{i=1}^{k-1}(\mathbf{x}_i\mathbf{x}_i'\mathbf{W} - \mathbf{\Sigma}\mathbf{W})\mathbf{\Gamma} \circ \mathbf{\Gamma}'\sum_{j=1}^{k-1}(\mathbf{x}_j\mathbf{x}_j'\mathbf{W} - \mathbf{\Sigma}\mathbf{W})\mathbf{\Gamma}\right) \\ =& \sum_{i,j}^{k-1}\sum_{l=1}^{m} \mathbf{e}_l'\mathbf{\Gamma}'(\mathbf{x}_i\mathbf{x}_i'\mathbf{W} - \mathbf{\Sigma}\mathbf{W})\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'(\mathbf{x}_j\mathbf{x}_j'\mathbf{W} - \mathbf{\Sigma}\mathbf{W})\mathbf{\Gamma}\mathbf{e}_l \\ =& \sum_{i,j}^{k-1}\sum_{l=1}^{m}\Big[\mathbf{x}_i'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_i \\ & \qquad - \mathbf{x}_j'\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{x}_j + \mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\mathbf{e}_l'\mathbf{\Gamma}'\mathbf{\Sigma}\mathbf{W}\mathbf{\Gamma}\mathbf{e}_l\Big] \\ =& \sum_{i,j}^{k-1}\sum_{l=1}^{m} H_{ij} \end{aligned}$$

and

$$\begin{aligned} & \mathrm{Var}\left[\sum_{k=1}^{n}\Delta\mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma} \circ \mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\right] = n\sum_{k=1}^{n}\Delta^2\mathrm{Var}\left(\sum_{i,j}^{k-1}\sum_{l=1}^{m}H_{ij}\right) \\ \leq & 2n\sum_{k=1}^{n}\Delta^2\left[\mathrm{Var}\left(\sum_{i=j}^{k-1}\sum_{l=1}^{p}H_{ii}\right) + \mathrm{Var}\left(\sum_{i\neq j}^{k-1}\sum_{l=1}^{m}H_{ij}\right)\right]. \end{aligned} \tag{S2.44}$$

For $\mathrm{Var}\left(\sum_{i=j}^{k-1}\sum_{l=1}^{m}H_{ii}\right)$, we have

$$\mathrm{Var}\left(\sum_{i=j}^{k-1}\sum_{l=1}^{m}H_{ii}\right) = \sum_{i=j}^{k-1}\mathrm{Var}\left(\sum_{l=1}^{m}H_{ii}\right) \leq \sum_{i=j}^{k-1}\mathrm{E}\left(\sum_{l=1}^{m}H_{ii}\right)^2$$

$$
\begin{aligned}
=&\sum_{i=j}^{k-1}\mathrm{E}\left[\sum_{l=1}^{m}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i-2\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\right)\right]^2\\
\leq& K(k-1)\left(H_{ii1}+4H_{ii2}\right). \qquad \text{(S2.45)}
\end{aligned}
$$

where

$$
\begin{aligned}
H_{ii1}&=\mathrm{E}\Big[\sum_{l=1}^{m}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\right)\Big]^2,\\
H_{ii2}&=\mathrm{E}\Big[\sum_{l=1}^{m}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\right)\Big]^2.
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
H_{ii1}=&\sum_{l_1l_2}^{m}\mathrm{E}\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\mathbf{x}_i\\
\leq&\sum_{l_1=1}^{m}\mathrm{E}^{1/2}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^4\sum_{l_2=1}^{m}\mathrm{E}^{1/2}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^4\\
\leq&\left[\sum_{l_1=1}^{m}\left[\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}'\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2+\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^4\right]^{1/2}\right]^2\\
\leq&\Big[K\sum_{l_1=1}^{m}\left[\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}'\mathbf{e}_{l_1}\right)^2\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\right]^{1/2}+K\sum_{l_1=1}^{m}\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\Big]^2\\
\leq&\left[K\left[\mathrm{tr}\left(\boldsymbol{\Sigma}^4\right)\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^4\right)\right]^{1/2}+K\mathrm{tr}\left(\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\boldsymbol{\Gamma}\right)\right]^2\\
=&o\left(\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right). \qquad \text{(S2.46)}
\end{aligned}
$$

where the second inequality follows from

$$
\begin{aligned}
&\mathrm{E}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^4\\
=&\mathrm{E}\left[\mathbf{z}_i'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\mathbf{z}_i-\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\right)+\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\right)\right]^4
\end{aligned}
$$

$$
\begin{aligned}
\le& K\left[\mathrm{E}\left(\mathbf{z}_i'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\mathbf{z}_i-\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\right)\right)^4+\left\{\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\right)\right\}^4\right]\\
\le& K\left[\left\{\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\boldsymbol{\Gamma}\boldsymbol{\Gamma}'\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\right)\right\}^2+\left\{\mathrm{tr}\left(\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Gamma}\right)\right\}^4\right]\\
=&\left[\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}'\mathbf{e}_{l_1}\right)^2\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2+\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^4\right].
\end{aligned}
$$

by employing Lemma 3. Similarly, we have

$$
\begin{aligned}
H_{ii2}=&\sum_{l_1l_2}^m \mathrm{E}\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\mathbf{x}_i\\
\le&\sum_{l_1}^m \mathrm{E}^{1/2}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^2\\
&\sum_{l_2}^m \mathrm{E}^{1/2}\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_2}\mathbf{e}_{l_2}'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^2\\
\le&\Big[\sum_{l=1}^m\left[\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)\right]^{1/2}\\
&+\sum_{l_1}^m\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\Big]^2\\
\le&\Big[\sum_{l=1}^m\left[\frac{1}{2}\left(\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^4+\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\right)\right]^{1/2}\\
&+\sum_{l_1}^m\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2\Big]^2\\
\le&K\left[\sum_{l_1}^m\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)^2+\sum_{l=1}^p\left(\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\mathbf{e}_{l_1}'\boldsymbol{\Gamma}'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_{l_1}\right)\right]^2\\
=&o\left(\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right) \qquad \text{(S2.47)}
\end{aligned}
$$

From (S2.45)-(S2.47), we get

$$\frac{16}{n^4(n-1)^4} 2n \sum_{k=1}^{n} \Delta^2 \mathrm{Var}\left(\sum_{i=j}^{k-1}\sum_{l=1}^{m} H_{ii}\right) \leq Kn^{-5} o\left(\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2 \left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right). \tag{S2.48}$$

Following the same procedures used to address (S2.28), it can be verified that

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i\neq j}^{k-1}\sum_{l=1}^{m} H_{ij}\right) &= \sum_{i_1\neq j_1}^{k-1}\sum_{i_2\neq j_2}^{k-1} \mathrm{Cov}\left(\sum_{l=1}^{m} H_{i_1 j_1}, \sum_{l=1}^{m} H_{i_2 j_2}\right) \\
&= \sum_{i\neq j}^{k-1} \mathrm{Cov}\left(\sum_{l=1}^{m} H_{ij}, \sum_{l=1}^{m} H_{ij}\right).
\end{aligned}$$

For $i \neq j$, similar to (S2.45)-(S2.47), we have

$$\begin{aligned}
&\mathrm{Cov}\left(\sum_{l=1}^{m} H_{ij}, \sum_{l=1}^{m} H_{ij}\right) \\
\leq &\mathrm{E}\Bigg[\sum_{l=1}^{m}\Big[\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i - \mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\Gamma'\mathbf{x}_i \\
&\qquad - \mathbf{x}_j'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_j\Big]\Bigg]^2 \\
\leq &K\left[\mathrm{E}\left(\sum_{l=1}^{m}\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_j\mathbf{x}_j'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^2\right. \\
&\qquad \left. + 2\mathrm{E}\left(\sum_{l=1}^{m}\mathbf{x}_i'\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Gamma}\mathbf{e}_l\mathbf{e}_l'\boldsymbol{\Gamma}'\mathbf{x}_i\right)^2\right] \\
= &o\left(\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2 \left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\right).
\end{aligned}$$

The above two results imply that

$$\frac{16\Delta^2}{n^4(n-1)^4} \times 2n \sum_{k=1}^{n} \mathrm{Var}\left(\sum_{i \neq j}^{k-1} \sum_{l=1}^{m} H_{ij}\right) = n^{-4} \times o\left[\left\{\mathrm{tr}\left(\mathbf{\Sigma}^2\right)\right\}^2 \left\{\mathrm{tr}\left((\mathbf{W\Sigma})^2\right)\right\}^2\right] \tag{S2.49}$$

This, together with (S2.22), (S2.44) and (S2.48), gives that

$$\frac{\frac{16}{n^4(n-1)^4}\mathrm{Var}\left(\sum_{k=1}^{n} \Delta \mathrm{tr}\left(\mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma} \circ \mathbf{\Gamma}'\mathbf{Q}_{k-1}\mathbf{\Gamma}\right)\right)}{\mathrm{Var}^2\left(Z_n\right)} \xrightarrow{P} 0. \tag{S2.50}$$

Combining (S2.25), (S2.34), (S2.43) and (S2.50), we arrive at

$$\frac{\mathrm{Var}\left(\sum_{k=1}^{n} \mathrm{E}_{k-1} A_1^2\right)}{\mathrm{Var}^2\left(Z_n\right)} \xrightarrow{P} 0. \tag{S2.51}$$

The proof of (S2.12) is complete.

S2.2 Proof of the second part of (S2.10)

By recalling the definition of A_1 and A_2 and applying Lemma 3, we have

$$\begin{aligned}
\sum_{k=1}^{n} \mathrm{E}\left(D_{n,k}^4\right) =& \sum_{k=1}^{n} \mathrm{E}\left[A_1 + A_2\right]^4 \\
\leq& K \sum_{k=1}^{n} \left[\mathrm{E}A_1^4 + \mathrm{E}A_2^4\right] \\
=& K \sum_{k=1}^{n} \Bigg[\mathrm{E}\left(\frac{2}{n(n-1)}\left[\mathbf{x}_k'\mathbf{Q}_{k-1}\mathbf{x}_k - \mathrm{tr}\left(\mathbf{\Sigma}\mathbf{Q}_{k-1}\right)\right]\right)^4 \\
&\quad + \mathrm{E}\left(\frac{2}{n}\left[\mathbf{x}_k'\left(\mathbf{W\Sigma} - \mathbf{W}\right)\mathbf{x}_k - \mathrm{tr}\left(\mathbf{W\Sigma}^2 - \mathbf{W\Sigma}\right)\right]\right)^4\Bigg] \\
\leq& K \sum_{k=1}^{n} \Bigg[\frac{16}{n^4(n-1)^4}\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{Q}_{k-1}\mathbf{\Sigma}\mathbf{Q}_{k-1}'\mathbf{\Sigma}\right)\right\}^2
\end{aligned}$$

$$+\frac{16}{n^4}\left\{\mathrm{tr}\left[\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}\mathbf{W}-\mathbf{W}\right)\boldsymbol{\Sigma}\right]\right\}^2\Bigg]. \tag{S2.52}$$

To establish the second part of (S2.10), we need only verify two conditions: first, that

$$\frac{\sum_{k=1}^{n}\frac{16}{n^4(n-1)^4}\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}'\boldsymbol{\Sigma}\right)\right\}^2}{\mathrm{Var}^2(Z_n)}\to 0, \tag{S2.53}$$

and second, that

$$\frac{\sum_{k=1}^{n}\frac{16}{n^4}\left\{\mathrm{tr}\left[\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}\mathbf{W}-\mathbf{W}\right)\boldsymbol{\Sigma}\right]\right\}^2}{\mathrm{Var}^2(Z_n)}\to 0. \tag{S2.54}$$

Proof of (S2.53).

For $i,j=1,\dots,n$, define

$$R_{ij}=\left(\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\mathbf{x}_j\mathbf{x}_j'\boldsymbol{\Sigma}\mathbf{x}_i-\mathbf{x}_i'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_i-\mathbf{x}_j'\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^2\mathbf{x}_j+\mathrm{tr}\left(\mathbf{W}\boldsymbol{\Sigma}\mathbf{W}\boldsymbol{\Sigma}^3\right)\right).$$

We then obtain

$$\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}'\boldsymbol{\Sigma}\right)\right\}^2=\mathrm{E}\left[\sum_{i,j}^{k-1}R_{ij}\right]^2\le 2\left[\mathrm{E}\left(\sum_{i=j}^{k-1}R_{ii}\right)^2+\mathrm{E}\left(\sum_{i\neq j}^{k-1}R_{ij}\right)^2\right].$$

Similar to the proof of (S2.35)-(S2.43), one can also verify

$$\mathrm{E}\left(\sum_{i=j}^{k-1}R_{ii}\right)^2+\mathrm{E}\left(\sum_{i\neq j}^{k-1}R_{ij}\right)^2\le K(k-1)^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2. \tag{S2.55}$$

Thus we get

$$K\sum_{k=1}^{n}\frac{16}{n^4(n-1)^4}\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}'\boldsymbol{\Sigma}\right)\right\}^2\le Kn^{-5}\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2. \tag{S2.56}$$

Combining (S2.22) and (S2.56), we have

$$\frac{K\sum_{k=1}^{n}\frac{16}{n^4(n-1)^4}\mathrm{E}\left\{\mathrm{tr}\left(\mathbf{Q}_{k-1}\boldsymbol{\Sigma}\mathbf{Q}_{k-1}'\boldsymbol{\Sigma}\right)\right\}^2}{\mathrm{Var}^2(Z_n)} \leq \frac{Kn^{-5}\left\{\mathrm{tr}\left(\boldsymbol{\Sigma}^2\right)\right\}^2\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2}{Kn^{-4}\left\{\mathrm{tr}\left((\mathbf{W}\boldsymbol{\Sigma})^2\right)\right\}^2} = O\left(\frac{1}{n}\right). \tag{S2.57}$$

The proof of (S2.53) is thus complete.

Proof of (S2.54).

Note that

$$\{\mathrm{tr}\left[\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}\mathbf{W}-\mathbf{W}\right)\boldsymbol{\Sigma}\right]\}^2 = \{\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\}^2.$$

This implies that

$$\begin{aligned}&K\sum_{k=1}^{n}\frac{16}{n^4}\{\mathrm{tr}\left[\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}\mathbf{W}-\mathbf{W}\right)\boldsymbol{\Sigma}\right]\}^2\\ \leq &Kn^{-3}\{\mathrm{tr}\left[\mathbf{W}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}-\mathbf{I}\right)\mathbf{W}\boldsymbol{\Sigma}\right]\}^2.\end{aligned} \tag{S2.58}$$

Combining (S2.22) and (S2.58),we arrive at

$$\frac{K\sum_{k=1}^{n}\frac{16}{n^4}\{\mathrm{tr}\left[\left(\mathbf{W}\boldsymbol{\Sigma}-\mathbf{W}\right)\boldsymbol{\Sigma}\left(\boldsymbol{\Sigma}\mathbf{W}-\mathbf{W}\right)\boldsymbol{\Sigma}\right]\}^2}{\mathrm{Var}^2(Z_n)} = O\left(\frac{1}{n}\right). \tag{S2.59}$$

The proof of (S2.54) is complete.

S3 Additional Simulation Studies

In this section, we collect additional numerical results for the oracle simulations in Scenarios 1–4 and for the plug-in implementation in Scenario 5 below.

S3.1 Empirical sizes for oracle implementations in Scenarios 1–4

The empirical sizes of the seven competing tests under the oracle implementations in Scenarios 1–4, for $z_{ij} \sim N(0,1)$ and $z_{ij} \sim \Gamma(4,2) - 2$, are summarized in Tables S.1–S.8.

S3.2 Empirical powers for Scenarios 1–2

Figures S.1–S.2 display the empirical powers of the seven tests under Scenarios 1–2 and reveal two salient features.

First, under local alternatives with $q_i = O(1/p)$ as in Scenario 1, the proposed T_{LMPNT} consistently achieves the highest power among all competitors across the considered (n,p) combinations. A clear gain over T_{CZZ} is observed, in agreement with the LMPNT theory in the main paper, which predicts that T_{LMPNT} is locally most powerful within the operator-indexed optional naive family under suitably structured local alternatives.

Second, in Scenario 2 the covariance structure violates Assumption 3, so the oracle operator $\mathbf{W}_{\mathbf{\Sigma}_{1p},\text{opt}}$ cannot be attained exactly (the corresponding regularized choice of $\mathbf{W}$ is specified in Remark 6 of the main paper). Nevertheless, the operator-based test remains noticeably more powerful than T_{CZZ} and the other competitors, especially in the moderate- and high-dimensional regimes ($p = n$ and $p = 1.5n$), indicating practical robustness when the or-

Table S.1: Empirical sizes of seven tests at 5% significance in Scenario 1 with $z_{ij} \sim N(0,1)$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0704	0.0548	0.0542	0.0594	0.0490	0.0506
	T_{CZZ}	0.0666	0.0528	0.0530	0.0568	0.0490	0.0482
	T_{Fishera}	0.0524	0.0536	0.0628	0.0578	0.0600	0.0538
	T_{Fisherb}	0.0616	0.0598	0.0636	0.0560	0.0510	0.0576
	T_{XW}	0.3028	0.1750	0.1270	0.0960	0.0818	0.0680
	T_{CLRT}	0.0610	0.0516	0.0550	0.0520	0.0428	0.0534
	T_{DW}	0.0612	0.0500	0.0516	0.0570	0.0484	0.0496
$p = n$	T_{LMPNT}	0.0594	0.0602	0.0574	0.0510	0.0556	0.0586
	T_{CZZ}	0.0578	0.0610	0.0550	0.0546	0.0552	0.0540
	T_{Fishera}	0.0524	0.0514	0.0572	0.0508	0.0586	0.0512
	T_{Fisherb}	0.0626	0.0662	0.0554	0.0526	0.0596	0.0564
	T_{XW}	0.3834	0.2002	0.1414	0.0970	0.0778	0.0682
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0562	0.0550	0.0546	0.0472	0.0510	0.0532
$p = 1.5n$	T_{LMPNT}	0.0568	0.0538	0.0574	0.0510	0.0556	0.0472
	T_{CZZ}	0.0566	0.0490	0.0508	0.0548	0.0544	0.0478
	T_{Fishera}	0.0480	0.0508	0.0548	0.0568	0.0562	0.0484
	T_{Fisherb}	0.0606	0.0574	0.0594	0.0536	0.0540	0.0460
	T_{XW}	0.4444	0.2244	0.1376	0.1088	0.0776	0.0686
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0488	0.0518	0.0516	0.0542	0.0502	0.0524

Table S.2: Empirical sizes of seven tests at 5% significance in Scenario 1 with $z_{ij} \sim \Gamma(4, 2) - 2$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0760	0.0664	0.0632	0.0588	0.0526	0.0564
	T_{CZZ}	0.0700	0.0656	0.0614	0.0566	0.0570	0.0568
	T_{Fishera}	0.0888	0.0832	0.0768	0.0678	0.0708	0.0600
	T_{Fisherb}	0.1772	0.1650	0.1634	0.1632	0.1610	0.1490
	T_{XW}	0.5900	0.4484	0.3410	0.2740	0.2224	0.2066
	T_{CLRT}	0.0534	0.0572	0.0530	0.0564	0.0544	0.0516
	T_{DW}	0.0944	0.0754	0.0624	0.0604	0.0578	0.0556
$p = n$	T_{LMPNT}	0.0712	0.0574	0.0580	0.0550	0.0520	0.0526
	T_{CZZ}	0.0732	0.0592	0.0542	0.0554	0.0532	0.0566
	T_{Fishera}	0.0830	0.0724	0.0620	0.0612	0.0632	0.0486
	T_{Fisherb}	0.1640	0.1546	0.1282	0.1338	0.1382	0.1298
	T_{XW}	0.7472	0.5586	0.4326	0.3378	0.2882	0.2308
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0888	0.0656	0.0632	0.0546	0.0528	0.0564
$p = 1.5n$	T_{LMPNT}	0.0700	0.0586	0.0564	0.0536	0.0500	0.0530
	T_{CZZ}	0.0642	0.0546	0.0560	0.0602	0.0552	0.0526
	T_{Fishera}	0.0722	0.0658	0.0582	0.0562	0.0588	0.0498
	T_{Fisherb}	0.1378	0.1250	0.1202	0.1202	0.1142	0.1084
	T_{XW}	0.8192	0.6362	0.4846	0.3862	0.3158	0.2508
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0780	0.0606	0.0572	0.0604	0.0556	0.0514

Table S.3: Empirical sizes of seven tests at 5% significance in Scenario 2 with $z_{ij} \sim N(0,1)$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0716	0.0548	0.0544	0.0572	0.0500	0.0498
	T_{CZZ}	0.0666	0.0528	0.0530	0.0568	0.0490	0.0482
	T_{Fishera}	0.0524	0.0536	0.0628	0.0578	0.0600	0.0538
	T_{Fisherb}	0.0616	0.0598	0.0636	0.0560	0.0510	0.0576
	T_{XW}	0.3028	0.1750	0.1270	0.0960	0.0818	0.0680
	T_{CLRT}	0.0610	0.0516	0.0550	0.0520	0.0428	0.0534
	T_{DW}	0.0612	0.0500	0.0516	0.0570	0.0484	0.0496
$p = n$	T_{LMPNT}	0.0596	0.0606	0.0586	0.0516	0.0556	0.0576
	T_{CZZ}	0.0578	0.0610	0.0550	0.0546	0.0552	0.0540
	T_{Fishera}	0.0524	0.0514	0.0572	0.0508	0.0586	0.0512
	T_{Fisherb}	0.0626	0.0662	0.0554	0.0526	0.0596	0.0564
	T_{XW}	0.3834	0.2002	0.1414	0.0970	0.0778	0.0682
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0562	0.0550	0.0546	0.0472	0.0510	0.0532
$p = 1.5n$	T_{LMPNT}	0.0566	0.0550	0.0548	0.0508	0.0498	0.0494
	T_{CZZ}	0.0566	0.0490	0.0508	0.0548	0.0544	0.0478
	T_{Fishera}	0.0480	0.0508	0.0548	0.0568	0.0562	0.0484
	T_{Fisherb}	0.0606	0.0574	0.0594	0.0536	0.0540	0.0460
	T_{XW}	0.4444	0.2244	0.1376	0.1088	0.0776	0.0686
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0488	0.0518	0.0516	0.0542	0.0502	0.0524

Table S.4: Empirical sizes of seven tests at 5% significance in Scenario 2 with $z_{ij} \sim \Gamma(4,2) - 2$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0776	0.0660	0.0630	0.0584	0.0520	0.0574
	T_{CZZ}	0.0700	0.0656	0.0614	0.0566	0.0570	0.0568
	T_{Fishera}	0.0888	0.0832	0.0768	0.0678	0.0708	0.0600
	T_{Fisherb}	0.1772	0.1650	0.1634	0.1632	0.1610	0.1490
	T_{XW}	0.5900	0.4484	0.3410	0.2740	0.2224	0.2066
	T_{CLRT}	0.0534	0.0572	0.0530	0.0564	0.0544	0.0516
	T_{DW}	0.0944	0.0754	0.0624	0.0604	0.0578	0.0556
$p = n$	T_{LMPNT}	0.0706	0.0578	0.0584	0.0542	0.0532	0.0524
	T_{CZZ}	0.0732	0.0592	0.0542	0.0554	0.0532	0.0566
	T_{Fishera}	0.0830	0.0724	0.0620	0.0612	0.0632	0.0486
	T_{Fisherb}	0.1640	0.1546	0.1282	0.1338	0.1382	0.1298
	T_{XW}	0.7472	0.5586	0.4326	0.3378	0.2882	0.2308
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0888	0.0656	0.0632	0.0546	0.0528	0.0564
$p = 1.5n$	T_{LMPNT}	0.0702	0.0594	0.0554	0.0532	0.0504	0.0530
	T_{CZZ}	0.0642	0.0546	0.0560	0.0602	0.0552	0.0526
	T_{Fishera}	0.0722	0.0658	0.0582	0.0562	0.0588	0.0498
	T_{Fisherb}	0.1378	0.1250	0.1202	0.1202	0.1142	0.1084
	T_{XW}	0.8192	0.6362	0.4846	0.3862	0.3158	0.2508
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0780	0.0606	0.0572	0.0604	0.0556	0.0514

Table S.5: Empirical sizes of seven tests at 5% significance in Scenario 3 with $z_{ij} \sim N(0,1)$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0694	0.0608	0.0588	0.0562	0.0564	0.0576
	T_{CZZ}	0.0666	0.0528	0.0530	0.0568	0.0490	0.0482
	T_{Fishera}	0.0524	0.0536	0.0628	0.0578	0.0600	0.0538
	T_{Fisherb}	0.0616	0.0598	0.0636	0.0560	0.0510	0.0576
	T_{XW}	0.3028	0.1750	0.1270	0.0960	0.0818	0.0680
	T_{CLRT}	0.0610	0.0516	0.0550	0.0520	0.0428	0.0534
	T_{DW}	0.0612	0.0500	0.0516	0.0570	0.0484	0.0496
$p = n$	T_{LMPNT}	0.0672	0.0646	0.0602	0.0568	0.0576	0.0530
	T_{CZZ}	0.0578	0.0610	0.0550	0.0546	0.0552	0.0540
	T_{Fishera}	0.0524	0.0514	0.0572	0.0508	0.0586	0.0512
	T_{Fisherb}	0.0626	0.0662	0.0554	0.0526	0.0596	0.0564
	T_{XW}	0.3834	0.2002	0.1414	0.0970	0.0778	0.0682
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0562	0.0550	0.0546	0.0472	0.0510	0.0532
$p = 1.5n$	T_{LMPNT}	0.0614	0.0536	0.0512	0.0492	0.0536	0.0574
	T_{CZZ}	0.0566	0.0490	0.0508	0.0548	0.0544	0.0478
	T_{Fishera}	0.0480	0.0508	0.0548	0.0568	0.0562	0.0484
	T_{Fisherb}	0.0606	0.0574	0.0594	0.0536	0.0540	0.0460
	T_{XW}	0.4444	0.2244	0.1376	0.1088	0.0776	0.0686
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0488	0.0518	0.0516	0.0542	0.0502	0.0524

Table S.6: Empirical sizes of seven tests at 5% significance in Scenario 3 with $z_{ij} \sim \Gamma(4,2)-2$.

		n					
p	Method	50	100	150	200	250	300
$p=0.5n$	T_{LMPNT}	0.0710	0.0672	0.0610	0.0612	0.0560	0.0620
	T_{CZZ}	0.0700	0.0656	0.0614	0.0566	0.0570	0.0568
	T_{Fishera}	0.0888	0.0832	0.0768	0.0678	0.0708	0.0600
	T_{Fisherb}	0.1772	0.1650	0.1634	0.1632	0.1610	0.1490
	T_{XW}	0.5900	0.4484	0.3410	0.2740	0.2224	0.2066
	T_{CLRT}	0.0534	0.0572	0.0530	0.0564	0.0544	0.0516
	T_{DW}	0.0944	0.0754	0.0624	0.0604	0.0578	0.0556
$p=n$	T_{LMPNT}	0.0696	0.0646	0.0602	0.0600	0.0530	0.0606
	T_{CZZ}	0.0732	0.0592	0.0542	0.0554	0.0532	0.0566
	T_{Fishera}	0.0830	0.0724	0.0620	0.0612	0.0632	0.0486
	T_{Fisherb}	0.1640	0.1546	0.1282	0.1338	0.1382	0.1298
	T_{XW}	0.7472	0.5586	0.4326	0.3378	0.2882	0.2308
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0888	0.0656	0.0632	0.0546	0.0528	0.0564
$p=1.5n$	T_{LMPNT}	0.0714	0.0596	0.0608	0.0604	0.0548	0.0522
	T_{CZZ}	0.0642	0.0546	0.0560	0.0602	0.0552	0.0526
	T_{Fishera}	0.0722	0.0658	0.0582	0.0562	0.0588	0.0498
	T_{Fisherb}	0.1378	0.1250	0.1202	0.1202	0.1142	0.1084
	T_{XW}	0.8192	0.6362	0.4846	0.3862	0.3158	0.2508
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0780	0.0606	0.0572	0.0604	0.0556	0.0514

Table S.7: Empirical sizes of seven tests at 5% significance in Scenario 4 with $z_{ij} \sim N(0,1)$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0648	0.0546	0.0568	0.0564	0.0532	0.0538
	T_{CZZ}	0.0666	0.0528	0.0530	0.0568	0.0490	0.0482
	T_{Fishera}	0.0524	0.0536	0.0628	0.0578	0.0600	0.0538
	T_{Fisherb}	0.0616	0.0598	0.0636	0.0560	0.0510	0.0576
	T_{XW}	0.3028	0.1750	0.1270	0.0960	0.0818	0.0680
	T_{CLRT}	0.0610	0.0516	0.0550	0.0520	0.0428	0.0534
	T_{DW}	0.0612	0.0500	0.0516	0.0570	0.0484	0.0496
$p = n$	T_{LMPNT}	0.0576	0.0636	0.0570	0.0516	0.0564	0.0552
	T_{CZZ}	0.0578	0.0610	0.0550	0.0546	0.0552	0.0540
	T_{Fishera}	0.0524	0.0514	0.0572	0.0508	0.0586	0.0512
	T_{Fisherb}	0.0626	0.0662	0.0554	0.0526	0.0596	0.0564
	T_{XW}	0.3834	0.2002	0.1414	0.0970	0.0778	0.0682
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0562	0.0550	0.0546	0.0472	0.0510	0.0532
$p = 1.5n$	T_{LMPNT}	0.0550	0.0558	0.0556	0.0506	0.0504	0.0554
	T_{CZZ}	0.0566	0.0490	0.0508	0.0548	0.0544	0.0478
	T_{Fishera}	0.0480	0.0508	0.0548	0.0568	0.0562	0.0484
	T_{Fisherb}	0.0606	0.0574	0.0594	0.0536	0.0540	0.0460
	T_{XW}	0.4444	0.2244	0.1376	0.1088	0.0776	0.0686
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0488	0.0518	0.0516	0.0542	0.0502	0.0524

Table S.8: Empirical sizes of seven tests at 5% significance in Scenario 4 with $z_{ij} \sim \Gamma(4,2)-2$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{LMPNT}	0.0766	0.0686	0.0596	0.0654	0.0630	0.0632
	T_{CZZ}	0.0700	0.0656	0.0614	0.0566	0.0570	0.0568
	T_{Fishera}	0.0888	0.0832	0.0768	0.0678	0.0708	0.0600
	T_{Fisherb}	0.1772	0.1650	0.1634	0.1632	0.1610	0.1490
	T_{XW}	0.5900	0.4484	0.3410	0.2740	0.2224	0.2066
	T_{CLRT}	0.0534	0.0572	0.0530	0.0564	0.0544	0.0516
	T_{DW}	0.0944	0.0754	0.0624	0.0604	0.0578	0.0556
$p = n$	T_{LMPNT}	0.0764	0.0672	0.0628	0.0604	0.0554	0.0612
	T_{CZZ}	0.0732	0.0592	0.0542	0.0554	0.0532	0.0566
	T_{Fishera}	0.0830	0.0724	0.0620	0.0612	0.0632	0.0486
	T_{Fisherb}	0.1640	0.1546	0.1282	0.1338	0.1382	0.1298
	T_{XW}	0.7472	0.5586	0.4326	0.3378	0.2882	0.2308
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0888	0.0656	0.0632	0.0546	0.0528	0.0564
$p = 1.5n$	T_{LMPNT}	0.0668	0.0654	0.0574	0.0572	0.0502	0.0532
	T_{CZZ}	0.0642	0.0546	0.0560	0.0602	0.0552	0.0526
	T_{Fishera}	0.0722	0.0658	0.0582	0.0562	0.0588	0.0498
	T_{Fisherb}	0.1378	0.1250	0.1202	0.1202	0.1142	0.1084
	T_{XW}	0.8192	0.6362	0.4846	0.3862	0.3158	0.2508
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0780	0.0606	0.0572	0.0604	0.0556	0.0514

acle conditions are not met.

These findings complement the main-paper results for Scenarios 3–4 reported in Section 5 and further corroborate the advantage of spectral alignment in enhancing power under heterogeneous alternatives.

S3.3 Details and numerical results for the plug-in implementation in Scenario 5

This subsection provides the detailed simulation setting and numerical results for the sample-splitting plug-in implementation discussed in Section 5.2 of the main paper. The purpose is to examine the practical implementability of the proposed operator-indexed procedure when the oracle operator $\mathbf{W}_{\boldsymbol{\Sigma}_{1p},\mathrm{opt}}$ is unavailable.

Let n_{tr} and n_{te} denote the sizes of the training and testing samples, respectively. In each Monte Carlo replication, the observations are randomly divided into two independent parts, with $n_{\mathrm{tr}} = \lfloor 0.3n \rfloor$ observations used for training and $n_{\mathrm{te}} = n - n_{\mathrm{tr}}$ observations used for testing. Based on the training sample, we compute the Ledoit–Wolf shrinkage covariance estimator $\widehat{\boldsymbol{\Sigma}}_{\mathrm{shrink}}$ (Ledoit and Wolf, 2004). We then apply Algorithm 1 of the main paper to $(\widehat{\boldsymbol{\Sigma}}_{\mathrm{shrink}} - \mathbf{I}_p)^2$ to obtain a plug-in operator $\widehat{\mathbf{W}}$. When Algorithm 1 yields nonpositive eigenvalues, the eigenvalue truncation rule in Remark 6

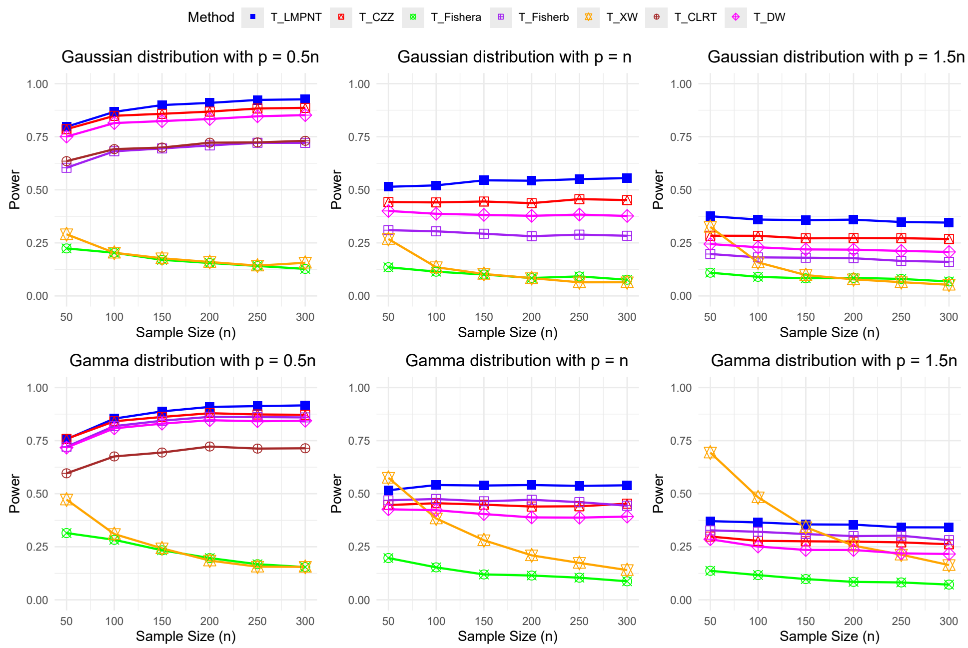

Figure S.1: Empirical powers of seven tests at 5% significance in Scenario 1.

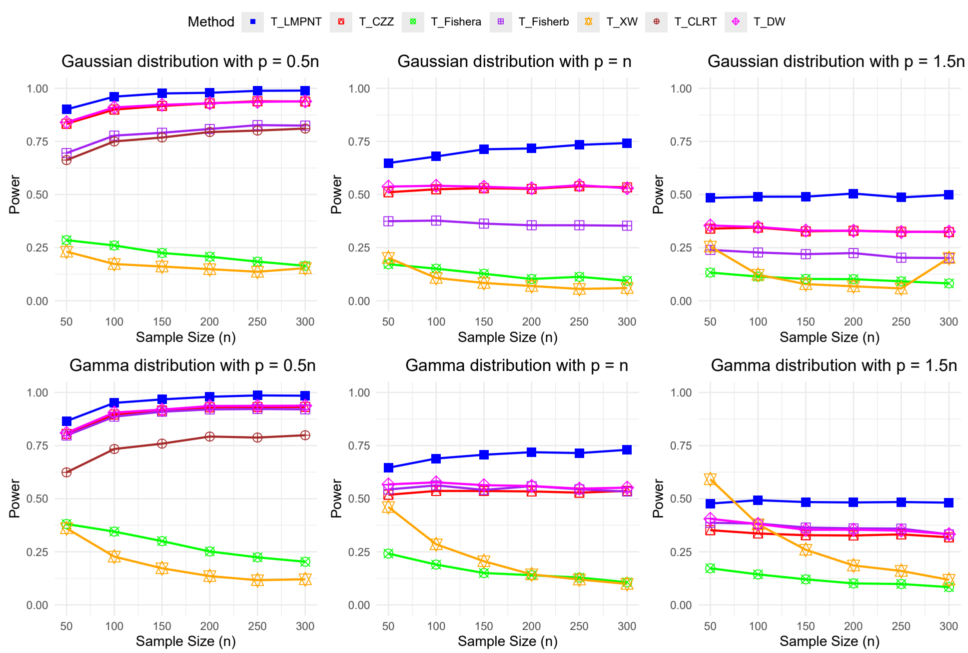

Figure S.2: Empirical powers of seven tests at 5% significance in Scenario 2.

of the main paper is applied. The testing sample is used only to evaluate $T_{n_{\text{te}}}(\widehat{\mathbf{W}})$. We denote the resulting sample-splitting plug-in test based on $T_{n_{\text{te}}}(\widehat{\mathbf{W}})$ by T_{SLMPNT}. Conditional on the training sample, $\widehat{\mathbf{W}}$ is fixed with respect to the testing sample. Hence, the fixed-$\mathbf{W}$ null calibration in Theorem 1 of the main paper applies conditionally. We next consider the following scenario for the data-driven implementation.

Scenario 5 (Diagonal local perturbation). Let $r_+ = r_- = \lceil 0.03p \rceil$. For $i = 1, \dots, r_+$, set $\tilde{q}_i = 36/p$ and $\tilde{m}_i = 1 + \sqrt{\tilde{q}_i}$; for $i = r_+ + 1, \dots, r_+ + r_-$, set $\tilde{q}_i = 24/p$ and $\tilde{m}_i = 1 - \sqrt{\tilde{q}_i}$; for $i = r_+ + r_- + 1, \dots, p$, set $\tilde{q}_i = 0$ and $\tilde{m}_i = 1$. The values $\{\tilde{m}_i\}_{i=1}^{p}$ are then arranged in decreasing order as $m_1 \geq m_2 \geq \cdots \geq m_p$, and the covariance matrix under the alternative is set to be $\mathbf{\Sigma}_2 = \text{diag}(m_1, \dots, m_p)$. This construction yields a diagonal local deviation from $\mathbf{I}_p$, in which approximately 3% of the eigenvalues are shifted upward, another 3% are shifted downward, and the remaining eigenvalues are equal to one.

The simulations are conducted under $z_{ij} \sim N(0, 1)$ and $z_{ij} \sim U(-\sqrt{3}, \sqrt{3})$, using the same (n, p) combinations as in the oracle implementation study. The significance level is 5%, and the results are based on 5,000 Monte Carlo replications.

Tables S.9–S.10 report the empirical sizes of the seven competing tests for the plug-in implementation in Scenario 5. The results show that T_{SLMPNT} maintains reasonable size control across the considered distributions.

The corresponding empirical powers are displayed in Figure S.3. Under Scenario 5, T_{SLMPNT} often improves over T_{CZZ} and several competing methods, especially when $p = 0.5n$ and $p = n$. Along the considered (n, p) regimes, the empirical power of T_{SLMPNT} tends to increase with n. A possible explanation is that, as the training sample size increases, the estimated operator $\widehat{\mathbf{W}}$ may identify the relatively informative departure directions more stably, although the shrinkage estimator $\widehat{\boldsymbol{\Sigma}}_{\mathrm{shrink}}$ itself need not become close to the true high-dimensional covariance matrix.

The maximum-type test T_{XW} exhibits high empirical power in this diagonal setting. This is expected, since the present alternative contains a small fraction of perturbed eigendirections and is therefore favorable to maximum-type procedures. However, its empirical size is noticeably liberal in several finite-sample settings, particularly for small n. Accordingly, its empirical power should be interpreted together with the corresponding size results.

Overall, these additional simulations suggest that the proposed operator-indexed framework remains practically implementable when the oracle operator is unavailable.

Table S.9: Empirical sizes of seven tests at 5% significance in Scenario 5 with $z_{ij} \sim N(0,1)$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{SLMPNT}	0.0676	0.0580	0.0566	0.0534	0.0534	0.0566
	T_{CZZ}	0.0666	0.0528	0.0530	0.0568	0.0490	0.0482
	T_{Fishera}	0.0524	0.0536	0.0628	0.0578	0.0600	0.0538
	T_{Fisherb}	0.0616	0.0598	0.0636	0.0560	0.0510	0.0576
	T_{XW}	0.3028	0.1750	0.1270	0.0960	0.0818	0.0680
	T_{CLRT}	0.0610	0.0516	0.0550	0.0520	0.0428	0.0534
	T_{DW}	0.0612	0.0500	0.0516	0.0570	0.0484	0.0496
$p = n$	T_{SLMPNT}	0.0652	0.0640	0.0616	0.0532	0.0572	0.0568
	T_{CZZ}	0.0578	0.0610	0.0550	0.0546	0.0552	0.0540
	T_{Fishera}	0.0524	0.0514	0.0572	0.0508	0.0586	0.0512
	T_{Fisherb}	0.0626	0.0662	0.0554	0.0526	0.0596	0.0564
	T_{XW}	0.3834	0.2002	0.1414	0.0970	0.0778	0.0682
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0562	0.0550	0.0546	0.0472	0.0510	0.0532
$p = 1.5n$	T_{SLMPNT}	0.0654	0.0598	0.0556	0.0522	0.0496	0.0504
	T_{CZZ}	0.0566	0.0490	0.0508	0.0548	0.0544	0.0478
	T_{Fishera}	0.0480	0.0508	0.0548	0.0568	0.0562	0.0484
	T_{Fisherb}	0.0606	0.0574	0.0594	0.0536	0.0540	0.0460
	T_{XW}	0.4444	0.2244	0.1376	0.1088	0.0776	0.0686
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0488	0.0518	0.0516	0.0542	0.0502	0.0524

Table S.10: Empirical sizes of seven tests at 5% significance in Scenario 5 with $z_{ij} \sim U(-\sqrt{3}, \sqrt{3})$.

		n					
p	Method	50	100	150	200	250	300
$p = 0.5n$	T_{SLMPNT}	0.0654	0.0606	0.0598	0.0542	0.0504	0.0558
	T_{CZZ}	0.0674	0.0584	0.0554	0.0546	0.0502	0.0474
	T_{Fishera}	0.0398	0.0516	0.0510	0.0492	0.0500	0.0534
	T_{Fisherb}	0.0290	0.0268	0.0258	0.0232	0.0202	0.0208
	T_{XW}	0.1694	0.1186	0.0880	0.0778	0.0794	0.0734
	T_{CLRT}	0.0604	0.0578	0.0522	0.0562	0.0486	0.0454
	T_{DW}	0.0480	0.0482	0.0500	0.0518	0.0482	0.0452
$p = n$	T_{SLMPNT}	0.0624	0.0540	0.0570	0.0514	0.0464	0.0498
	T_{CZZ}	0.0598	0.0578	0.0522	0.0532	0.0466	0.0512
	T_{Fishera}	0.0402	0.0482	0.0532	0.0464	0.0512	0.0540
	T_{Fisherb}	0.0306	0.0292	0.0276	0.0234	0.0198	0.0234
	T_{XW}	0.2142	0.1304	0.0994	0.1052	0.0898	0.0812
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0478	0.0494	0.0488	0.0514	0.0502	0.0514
$p = 1.5n$	T_{SLMPNT}	0.0608	0.0598	0.0492	0.0524	0.0516	0.0496
	T_{CZZ}	0.0528	0.0588	0.0538	0.0516	0.0552	0.0484
	T_{Fishera}	0.0370	0.0442	0.0516	0.0494	0.0536	0.0484
	T_{Fisherb}	0.0282	0.0308	0.0270	0.0280	0.0288	0.0274
	T_{XW}	0.2530	0.1454	0.1124	0.1008	0.0888	0.0820
	T_{CLRT}	NA	NA	NA	NA	NA	NA
	T_{DW}	0.0444	0.0452	0.0530	0.0460	0.0564	0.0470

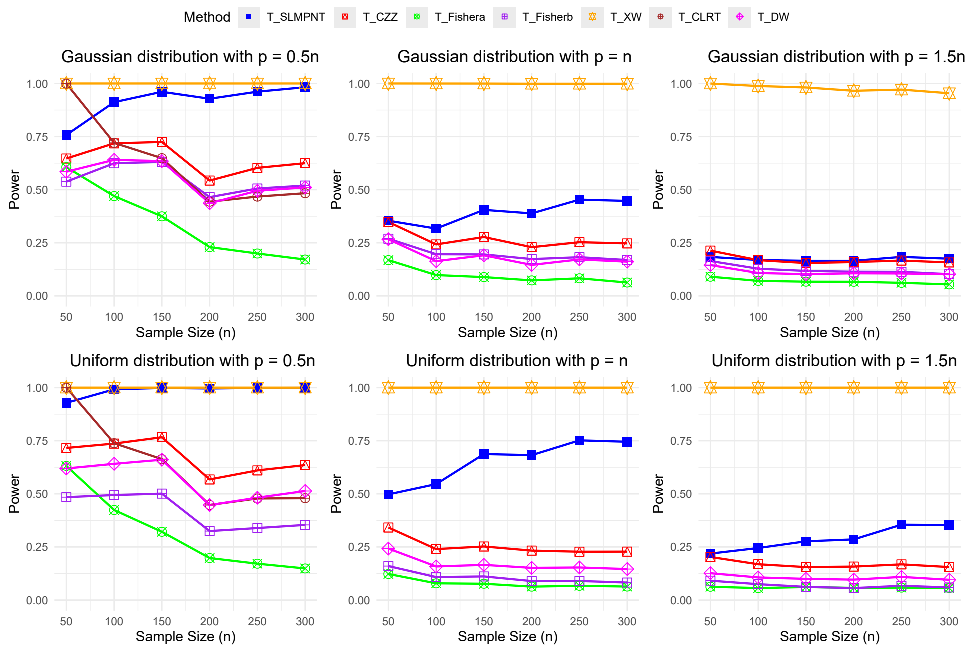

Figure S.3: Empirical powers of seven tests at 5% significance in Scenario 5 for the plug-in implementation with estimated $\mathbf{W}$.

S4 Additional details for the real-data analysis

This section provides the data-acquisition, feature-screening, preprocessing, and operator-construction details for the real-data analysis in Section 6 of the main paper. The TRANSBIG cohort (GSE7390; $n = 198$) is used as the external cohort for feature screening and construction of the operator $\mathbf{W}$, whereas the NKI cohort (GSE2034; $n = 286$) is used as the target cohort for hypothesis testing. Both cohorts were profiled on the Affymetrix

Human Genome U133A Array and contain the same 22,283 probe sets.

The expression matrices were obtained from the NCBI Gene Expression Omnibus through the Bioconductor package `GEOquery` (Davis and Meltzer, 2007). The data were processed according to the following steps.

1. **Probe-set alignment.** We first aligned the probe sets shared by the TRANSBIG and NKI cohorts and placed them in the same order in the two expression matrices.

2. **External feature screening.** Feature screening was performed using the original TRANSBIG expression matrix only. For each probe set, we computed its sample mean expression level across the TRANSBIG samples and retained the $p = 2000$ probe sets with the largest sample means. The NKI cohort was then restricted to exactly the same probe-set indices.

3. **Within-cohort centering.** After feature screening, each retained probe set was centered separately within each cohort by subtracting its cohort-specific sample mean. This produced the centered matrices $\mathbf{X}_{\text{TRANSBIG}}$ and $\mathbf{X}_{\text{NKI}}$.

4. **Covariance estimation and operator construction.** Using the centered TRANSBIG matrix, we computed the Ledoit–Wolf shrinkage

covariance estimator $\widehat{\boldsymbol{\Sigma}}_{\mathrm{LW}}$ (Ledoit and Wolf, 2004). Algorithm 1 of the main paper was then applied to $\widehat{\boldsymbol{\Sigma}}_{\mathrm{LW}}$ to construct the operator $\mathbf{W}$. The resulting operator was kept fixed when the covariance identity tests were applied to the centered NKI data.

Both the selected probe-set subset and the operator $\mathbf{W}$ are determined entirely from the external TRANSBIG cohort, whereas the NKI cohort is used only for the final hypothesis test. Accordingly, conditional on the external TRANSBIG cohort, both the selected probe-set subset and the operator $\mathbf{W}$ are fixed with respect to the NKI testing sample, and the NKI testing step falls within the fixed-$\mathbf{W}$ framework of Theorem 1 of the main paper. This design avoids target-cohort-dependent feature selection and operator construction and allows the null calibration in Theorem 1 to be applied conditionally.

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