Supplementary Material to "Non-parametric Testing for Survival Data With Time-dependent Covariates"

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S1 Some Necessary Notations

- T: a continuous random variable representing the survival time.
- $\bar{\boldsymbol{Z}} = \{V(t), 0 \leq t < \infty\}$: a time dependent covariate process, where $V(t)$ is a scalar, either continuous or discrete.
- C: the censoring time.
- $X = \min(T, C)$: the observed survival time.
- $\delta = I(T \leq C)$: the event status.
- $\mathcal{T} = \{t_k\}_{k=1}^{K}$: the landmark time points of interest with $K < \infty$.
- $\Omega_{V(t_k)}$: the support of $V(t_k)$ for any $t_k \in \mathcal{T}$
- At each landmark time t_k, we define
  - $T^r(t_k) = T - t_k$: the residual survival time.
  - $V(t_k)$: the covariate.
  - $C^r(t_k) = C - t_k$: the residual censoring time.
  - $X^r(t_k) = \min(T^r(t_k), C^r(t_k))$: the observed residual survival time.
- $q_{1,t_k}(\tau_1) = Q_{T^r(t_k)|X>t_k}(\tau_1)$: the τ_1th quantile of $T^r(t_k)$ conditional on $X > t_k$.
- $\widehat{q}_{1,t_k}(\tau_1) = \widehat{Q}_{T^r(t_k)|X>t_k}(\tau_1)$: the τ_1th quantile of the empirical cumulative distribution function of $T^r(t_k)$ conditional on $X > t_k$
- $Q_{T^r(t_k)|V(t_k),X>t_k}(\tau_1)$: the τ_1th quantile of $T^r(t_k)$ conditional on $V(t_k)$ and $X > t_k$.
- $F_1(s; t_k) = P(T^r(t_k) \leq s \mid X > t_k)$: the cumulative distribution function of $T^r(t_k)$ conditional on $X > t_k$. The estimator of $F_1(s; t_k)$ is noted as $F_{n,1}(s; t_k)$.
- $f_1(s; t_k) = f_{T^r(t_k)|X>t_k}(s)$: the density function of $T^r(t_k)$ conditional on $X > t_k$.
- $F_2(v; t_k) = P(V(t_k) \leq v \mid X > t_k)$: the cumulative distribution function of $V(t_k)$ conditional on $X > t_k$. The estimator of $F_2(v; t_k)$ is noted as $F_{n,2}(v; t_k)$.
- $S_{12}(q_{1,t_k}(\tau_1), v) = P(T^r(t_k) > q_{1,t_k}(\tau_1), V(t_k) \leq v \mid X > t_k)$ and $S_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v)$ denotes its estimator.
- $F_{12}(q_{1,t_k}(\tau_1), v) = P(T^r(t_k) \leq q_{1,t_k}(\tau_1), V(t_k) \leq v \mid X > t_k)$ and $F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v)$ denotes its estimator.

- $c(\tau_1, v; t_k) = \text{Cov}\{I(F_1(T^r(t_k); t_k) \le \tau_1), I(V(t_k) \le v) \mid X > t_k\}$: the conditional covariance between $I(F_1(T^r(t_k); t_k) \le \tau_1)$ and $I(V(t_k) \le v)$ conditional on $X > t_k$.
- $\approx$: asymptotic equivalence uniformly with difference $\sup_{r\in\Theta} |R_n(r)| = o_p(1)$, where Θ is the prespecified support.

We state the following necessary regularity condition.

Condition S1. *π_{t_k} is bounded below by zero, i.e. $\min_{t_k\in\mathcal{T}} \pi_{t_k} > 0$; and above by some positive constant $c_{\pi_{t_k}}$, i.e. $\max_{t_k\in\mathcal{T}} \pi_{t_k} < c_{\pi_{t_k}}$.*

Condition S2. *The survival distribution $G^r_{C,t_k}\{q_{1,t_k}(\tau)\}$ is uniformly bounded below by 0 for $\tau \in \Delta$, i.e. $\inf_{\tau\in\Delta} G^r_{C,t_k}\{q_{1,t_k}(\tau)\} > 0$; and above by some positive constant $c_{G^r_{C,t_k}}$, i.e. $\sup_{\tau\in\Delta} G^r_{C,t_k}\{q_{1,t_k}(\tau)\} < c_{G^r_{C,t_k}}$.*

S2 Propositions S1 and S2

Proposition S1. *For any $\tau \in \Delta$, we have*
(a) $Q_{T^r(t_k)|V(t_k),T>t_k}(\tau) = Q_{T^r(t_k)|V(t_k),X>t_k}(\tau)$;
(b) $Q_{T^r(t_k)|T>t_k}(\tau) = Q_{T^r(t_k)|X>t_k}(\tau)$.

Proof. For $P(T^r(t_k) > s, V(t_k) = v \mid T > t_k)$, we can write

$$
\begin{aligned}
& P(T^r(t_k) > s, V(t_k) = v \mid T > t_k) = P(T^r(t_k) > s, V(t_k) = v \mid T > t_k) \\
= & \frac{P(T^r(t_k) > s, V(t_k) = v \mid T > t_k)P(C > t_k \mid T > t_k)P(T > t_k)}{P(C > t_k \mid T > t_k)P(T > t_k)} \\
= & \frac{P(T^r(t_k) > s, V(t_k) = v, X > t_k)}{P(X > t_k)} \\
= & P(T^r(t_k) > s, V(t_k) = v \mid X > t_k).
\end{aligned}
$$

The second line multiplies both the numerator and denominator by $P(C > t_k \mid T > t_k)P(T > t_k)$. The third line holds under the assumption that $(T^r(t_k), V(t_k)) \perp C \mid T > t_k$ for any $t_k \in \mathcal{T}$ and $s > 0$.

As $(T^r(t_k), V(t_k)) \perp C \mid T > t_k$ implies $T^r(t_k) \perp C \mid T > t_k$ and $V(t_k) \perp C \mid T > t_k$ for any $t_k \in \mathcal{T}$, we can similarly derive that

$$
\begin{aligned}
& P(T^r(t_k) > s \mid T > t_k) = P(T^r(t_k) > s \mid X > t_k), \\
& P(V(t_k) = v \mid T > t_k) = P(V(t_k) = v \mid X > t_k).
\end{aligned}
$$

From the above, we have

$$
\begin{aligned}
F_{T^r(t_k)|V(t_k)=v,T>t_k}(s) &= 1 - P(T^r(t_k) > s \mid V(t_k) = v, T > t_k) \\
&= 1 - \frac{P(T^r(t_k) > s, V(t_k) = v \mid T > t_k)}{P(V(t_k) = v \mid T > t_k)} \\
&= 1 - \frac{P(T^r(t_k) > s, V(t_k) = v \mid X > t_k)}{P(V(t_k) = v \mid X > t_k)} \\
&= 1 - P(T^r(t_k) > s \mid V(t_k) = v, X > t_k) \\
&= F_{T^r(t_k)|V(t_k)=v,X>t_k}(s) \\
F_{T^r(t_k)|T>t_k}(s) &= 1 - P(T^r(t_k) > s \mid T > t_k) \\
&= 1 - P(T^r(t_k) > s \mid X > t_k) = F_{T^r(t_k)|X>t_k}(s).
\end{aligned}
$$

Then from the definition of quantile function and the fact that $T^r(t_k)$ is continuous, we have

$$
\begin{aligned}
& Q_{T^r(t_k)|V(t_k),T>t_k}(\tau) = Q_{T^r(t_k)|V(t_k),X>t_k}(\tau), \\
& Q_{T^r(t_k)|T>t_k}(\tau) = Q_{T^r(t_k)|X>t_k}(\tau).
\end{aligned}
$$

This completes the proof. □

Proposition S2. *For any $\tau \in \Delta$, we have*

$$Q_{T^r(t_k)|V(t_k),X>t_k}(\tau_1) = Q_{T^r(t_k)|X>t_k}(\tau) \Leftrightarrow \int_{\Omega_{V(t_k)}} \frac{c^2(\tau, v; t_k)}{\tau(1-\tau)F_2(v;t_k)(1-F_2(v;t_k))} d\mu_2(v) = 0,$$

where $c(\tau, v; t_k) = Cov\{I(F_1(T^r(t_k); t_k) \le \tau), I(V(t_k) \le v) \mid X > t_k\}$.

Proof. From the definition of quantile function and the fact that $T^r(t_k)$ is continuous, we have for any $\tau \in \Delta$,

$$\begin{aligned}
& Q_{T^r(t_k)|V(t_k),X>t_k}(\tau) = Q_{T^r(t_k)|X>t_k}(\tau) \\
\Leftrightarrow \quad & E\left\{I[T^r(t_k) \le Q_{T^r(t_k)|X>t_k}(\tau)] \mid V(t_k), X > t_k\right\} = \tau \\
\Leftrightarrow \quad & \mathrm{Cov}\left\{I[T^r(t_k) \le Q_{T^r(t_k)|X>t_k}(\tau)], I(V(t_k) \le v) \mid X > t_k\right\} = 0 \quad \forall v \in \Omega_{V(t_k)} \\
\Leftrightarrow \quad & \int_{\Omega_{V(t_k)}} \frac{\mathrm{Cov}^2\left\{I[T^r(t_k) \le Q_{T^r(t_k)|X>t_k}(\tau)], I(V(t_k) \le v) \mid X > t_k\right\}}{\tau(1-\tau)F_2(v;t_k)[1-F_2(v;t_k)]} d\mu_2(v) = 0 \\
\Leftrightarrow \quad & \int_{\Omega_{V(t_k)}} \frac{\mathrm{Cov}^2\left\{I(F_1(T^r(t_k); t_k) \le \tau), I(V(t_k) \le v) \mid X > t_k\right\}}{\tau(1-\tau)F_2(v;t_k)[1-F_2(v;t_k)]} d\mu_2(v) = 0.
\end{aligned}$$

This completes the proof. □

S3 Some Lemmas and the proofs

S3.1 Lemmas S1, S2, S3 and the proofs

We first introduce the stochastic equicontinuity Lemma S1.

Lemma S1. *Suppose Conditions S1 and S2 hold. Assume that the first derivative of $f_1(q_{1,t_k}(\tau))$ with respect to τ is bounded away from zero and infinity for $\tau \in \Delta$. Then we have*

$$\sup_{x \in [q_{1,t_k}(\tau_L), q_{1,t_k}(\tau_U)]} \sup_{|x-y| = O(n^{-1/2}\log^{1/2} n)} |F_{n,1}(x) - F_{n,1}(y) - F_1(x) + F_1(y)| = O(n^{-3/4}\log^{3/4} n)$$

almost surely.

Proof. The lines closely resemble the proof of Lemma 1 in Peng and Fine (2007). Specifically, following the lines in Lo and Singh (1986), to justify the results in Lemma S1, we only need to show that the two following results hold:
(i) $\mathrm{E}\{|\psi_1(x) - \psi_1(y)|\} = O(n^{-1/2}\log^{1/2} n)$;
(ii) $\mathrm{Var}\{|\psi_1(x) - \psi_1(y)|\} = O(n^{-1}\log n)$.
As mentioned in the proof of Lemma S2, for any $\tau_L \le \tau_1 < \tau_1' \le \tau_U$ and $q_{1,t_k}(\tau_1') - q_{1,t_k}(\tau_1) = O(n^{-1/2}\log^{1/2} n)$, we have from the definition

$$\begin{aligned}
\psi_1\{q_{1,t_k}(\tau_1)\} \quad = \quad & \left\{\frac{I[X_1^r(t_k) > q_{1,t_k}(\tau_1), X_1 > t_k]}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1)\}\pi_{t_k}} - (1-\tau_1)\right\} - \\
& \frac{1-\tau_1}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1)\}} \cdot \psi_{G,1}\{q_{1,t_k}(\tau_1)\} - \frac{1-\tau_1}{\pi_{t_k}}\{I[X_1 > t_k] - \pi_{t_k}\}
\end{aligned}$$

and

$$\begin{aligned}
\psi_1\{q_{1,t_k}(\tau_1')\} \quad = \quad & \left\{\frac{I[X_1^r(t_k) > q_{1,t_k}(\tau_1'), X_1 > t_k]}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1')\}\pi_{t_k}} - (1-\tau_1')\right\} - \\
& \frac{1-\tau_1'}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1')\}} \cdot \psi_{G,1}\{q_{1,t_k}(\tau_1')\} - \frac{1-\tau_1'}{\pi_{t_k}}\{I[X_1 > t_k] - \pi_{t_k}\}.
\end{aligned}$$

From the boundedness of f_1, we have

$$\tau_1' - \tau_1 = F_1\{q_{1,t_k}(\tau_1')\} - F_1\{q_{1,t_k}(\tau_1)\} = f_1\{q_{1,t_k}(\tau_1)\}\{q_{1,t_k}(\tau_1') - q_{1,t_k}(\tau_1)\} = O(n^{-1/2}\log^{1/2} n).$$

Also, from Conditions S1 and S2, we have the boundedness below by zero and above by some constant of $G^r_{C,t_k}\{q_{1,t_k}(\tau_1)\}$ and π_{t_k}. As X^r is continuous, we can conclude that the integrand in the above defined $\psi_{G,1}\{q_{1,t_k}(\tau_1)\}$ is uniformly bounded by a constant. This implies $|\psi_{G,1}\{q_{1,t_k}(\tau_1')\} - \psi_{G,1}\{q_{1,t_k}(\tau_1)\}| \le L(\tau_1' - \tau_1)$ for some constant L.

For (i), applying the above findings, we can write

$$\begin{aligned}
& \mathrm{E}\{|\psi_1\{q_{1,t_k}(\tau_1')\} - \psi_1\{q_{1,t_k}(\tau_1)\}|\} \\
\le\ & \mathrm{E}\left\{\left|\frac{I[q_{1,t_k}(\tau_1') > X_1^r(t_k) > q_{1,t_k}(\tau_1), X_1 > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}}\right|\right\} + \mathrm{E}\{|\tau_1 - \tau_1'|\} \\
& + \mathrm{E}\left\{\left|\frac{(1-\tau_1)L}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}} \cdot \{q_{1,t_k}(\tau_1') - q_{1,t_k}(\tau_1)\}\right|\right\} \text{(for some constant } L) \\
& + \mathrm{E}\left\{\left|\frac{\tau_1 - \tau_1'}{\pi_{t_k}}\{I[X_1 > t_k] - \pi_{t_k}\}\right|\right\} \\
\le\ & \frac{P[q_{1,t_k}(\tau_1') > X_1^r(t_k) > q_{1,t_k}(\tau_1) \mid X_1 > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}} + \mathrm{E}\{c_1(n^{-1/2}\log^{1/2} n)\} \\
=\ & \frac{\tau_1' - \tau_1}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}} + c_1(n^{-1/2}\log^{1/2} n) = c_2(n^{-1/2}\log^{1/2} n)
\end{aligned}$$

for some positive constant c_1 and c_2.

Next, for (ii), we can write

$$\begin{aligned}
& \mathrm{Var}\{|\psi_1\{q_{1,t_k}(\tau_1')\} - \psi_1\{q_{1,t_k}(\tau_1)\}|\} \\
\le\ & \mathrm{E}\left\{\left|\frac{I[q_{1,t_k}(\tau_1') > X_1^r(t_k) > q_{1,t_k}(\tau_1), X_1 > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}}\right|^2\right\} \\
& + c_1(n^{-1/2}\log^{1/2} n)\mathrm{E}\left\{\left|\frac{I[q_{1,t_k}(\tau_1') > X_1^r(t_k) > q_{1,t_k}(\tau_1), X_1 > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}}\right|\right\} \\
& + c_1^2(n^{-1}\log n) \\
\le\ & \frac{\tau_1' - \tau_1}{\{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}\}^2} + c_1(n^{-1/2}\log^{1/2} n) \cdot \frac{\tau_1' - \tau_1}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1')\}\pi_{t_k}} + c_1^2(n^{-1}\log n) \\
=\ & c_3(n^{-1}\log n)
\end{aligned}$$

for some positive constant c_3. This completes the proof of Lemma S1. □

Utilizing the results in Lemma S1, we can then introduce Lemma S2 for the asymptotic equivalence between $f_1\{q_{1,t_k}(\tau_1)\}\{\widehat{q}_{1,t_k}(\tau_1) - q_{1,t_k}(\tau_1)\}$ and $\tau_1 - F_{n,1}\{q_{1,t_k}(\tau_1)\}$.

Lemma S2. *Suppose Conditions S1 and S2 hold. Assume that the first derivative of $f_1(q_{1,t_k}(\tau_1))$ with respect to τ_1 is bounded away from zero and infinity for $\tau_1 \in \Delta$. Then we have with probability one*

$$\limsup_{n\to\infty}\left|\widehat{q}_{1,t_k}(\tau_1) - q_{1,t_k}(\tau_1) - \frac{\tau_1 - F_{n,1}\{q_{1,t_k}(\tau_1)\}}{f_1\{q_{1,t_k}(\tau_1)\}}\right| = o(n^{-1/2}).$$

Proof of Lemma S2. From Pepe (1991), we have

$$\begin{aligned}
n^{1/2}\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\} - G^r_{C,t_k}\{q_{1,t_k}(\tau)\} &\approx -n^{-1/2}\sum_{i=1}^n G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\int_k^{q_{1,t_k}(\tau)} y(s)^{-1} dM_i^{G^r_{C,t_k}}(s) \\
&\doteq n^{-1/2}\sum_{i=1}^n \psi_{G,i}\{q_{1,t_k}(\tau_1)\},
\end{aligned}$$

where $y(t) = P(X^r \ge t)$ and $M_i^{G^r_{C,t_k}}(\cdot)$ is the martingale process. Then from the definition, we can write

$$F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\} = n^{-1/2}\sum_{i=1}^n \psi_i\{q_{1,t_k}(\tau_1)\} + R_n\{q_{1,t_k}(\tau_1)\},$$

where $\sup_{\tau_1\in\Delta}|R_n\{q_{1,t_k}(\tau_1)\}| = o_p(n^{-1/2})$ and

$$\begin{aligned}\psi_i\{q_{1,t_k}(\tau_1)\} &= \left\{\frac{I[X_i^r(t_k) > q_{1,t_k}(\tau_1), X_i > t_k]}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1)\}\pi_{t_k}} - (1-\tau_1)\right\} - \\ &\quad \frac{1-\tau_1}{G_{C,t_k}^r\{q_{1,t_k}(\tau_1)\}}\cdot\psi_{G,i}\{q_{1,t_k}(\tau_1)\} - \frac{1-\tau_1}{\pi_{t_k}}\{I[X_i > t_k] - \pi_{t_k}\}\}\end{aligned}$$

Applying the functional law of the iterated logarithm (LIL) in Goodman et al. (1981), we have

$$\sup_{\tau_1\in\Delta}|F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}| = O(n^{-1/2}\log^{1/2}n), a.s.$$

By Taylor expansion, we can write

$$F_1\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\} = f_1\{q_{1,t_k}^*(\tau_1)\}\{\widehat{q}_{1,t_k}(\tau_1) - q_{1,t_k}(\tau_1)\}$$

for some $q_{1,t_k}^*(\tau_1)$ between $\widehat{q}_{1,t_k}(\tau_1)$ and $q_{1,t_k}(\tau_1)$.

Since f_1 is bounded away from zero and infinity on Δ, by Lemma S1 and utilizing the fact that $F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} = F_1\{q_{1,t_k}(\tau_1)\} = \tau_1$, we have

$$\begin{aligned}\sup_{\tau_1\in\Delta}|\widehat{q}_{1,t_k}(\tau_1) - q_{1,t_k}(\tau_1)| &= \sup_{\tau_1\in\Delta}\left|\frac{F_1\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}}{f_1\{q_{1,t_k}^*(\tau_1)\}}\right| \\ &\le c_1^{-1}\sup_{\tau_1\in\Delta}|F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} - F_{n,1}\{q_{1,t_k}(\tau_1)\}| \\ &= c_1^{-1}\sup_{\tau_1\in\Delta}|F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}| = O(n^{-1/2}\log^{1/2}n)\end{aligned}$$

almost surely for some positive constant c_1. The inequality in the second line is due to the fact that by Lemma S1, $F_1\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\} \approx F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} - F_{n,1}\{q_{1,t_k}(\tau_1)\}$, and f_1 is bounded away from zero and infinity on Δ. The equivalence in the third line utilizes $F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} = F_1\{q_{1,t_k}(\tau_1)\} = \tau_1$.

Following Lemma S1, note that we can write

$$\begin{aligned}F_1\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\} &= -[F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}] \\ &\quad - ([F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{\widehat{q}_{1,t_k}(\tau_1)\}] - [F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}]).\end{aligned}$$

Using Lemma S1 and by Taylor expansion, we have

$$\sup_{\tau_1\in\Delta}|[F_{n,1}\{\widehat{q}_{1,t_k}(\tau_1)\} - F_1\{\widehat{q}_{1,t_k}(\tau_1)\}] - [F_{n,1}\{q_{1,t_k}(\tau_1)\} - F_1\{q_{1,t_k}(\tau_1)\}]| = o(n^{-1/2}), \text{ a.s.}$$

This completes the proof of Lemma S2. □

Next, we introduce Lemma S3, which establishes the uniform consistency of $\widehat{q}_{1,t_k}(\tau)$ to $q_{1,t_k}(\tau)$ for $\tau \in \Delta$.

Lemma S3. *Suppose Conditions S1 and S2 hold. Assume that the first derivative of $f_1(q_{1,t_k}(\tau_1))$ with respect to τ_1 is bounded away from zero and infinity for $\tau_1 \in \Delta$. Then for any $c > 0$ and $0 < \zeta \le 1/2$, there exists positive constant c_1 such that for sufficiently large n,*

$$P\{\sup_{\tau\in\Delta}|\widehat{q}_{1,t_k}(\tau) - q_{1,t_k}(\tau)| > cn^{\zeta-1/2}\} \le O\{\exp[-c_1 n^{2\zeta} - \log(n^{\zeta-1/2})]\}.$$

Proof. From the condition, the first derivative of f_1 with respect to τ are bounded away from zero and infinity on Δ. Then there exist $w_0 \ge 0$, such that $|q_{1,t_k}(\tau_a) - q_{1,t_k}(\tau_b)| \le w_0|\tau_a - \tau_b|$. Given $0 < \delta < \min\{1, \frac{3w_0}{(\tau_U-\tau_L)}\}$, we can define a grid partition for $[\tau_L, \tau_U]$ as $\tau_L = \tau_0 < \tau_1 < \cdots < \tau_{N_\delta} = \tau_U$ with $N_\delta = \lfloor\frac{6w_0(\tau_U-\tau_L)}{\delta}\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer smaller than x. With this partition, the grid would be $|\tau_k - \tau_{k-1}| \le \frac{\delta}{3w_0}$. Then we have $|q_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_{k-1})| \le \frac{\delta}{3}$.

We learn from the definition for $\widehat{q}_{1,t_k}(\tau_k)$ and $q_{1,t_k}(\tau_k)$ that they are non-decreasing. Suppose $|\widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k)| \le \frac{\delta}{3}$ and $|\widehat{q}_{1,t_k}(\tau_{k-1}) - q_{1,t_k}(\tau_{k-1})| \le \frac{\delta}{3}$, then for $\tau_{k-1} \le x \le \tau_k$, we have

$$\widehat{q}_{1,t_k}(x) - q_{1,t_k}(x) \le \widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k) + q_{1,t_k}(\tau_k) - q_{1,t_k}(x) \le \frac{2\delta}{3} < \delta.$$

The other direction can be shown by the same arguments. Then we have $|\widehat{q}_{1,t_k}(x) - q_{1,t_k}(x)| < \delta$ for any $\tau_{k-1} \le x \le \tau_k$. Thus if $\sup_{\tau\in[\tau_L,\tau_U]} |\widehat{q}_{1,t_k}(\tau) - q_{1,t_k}(\tau)| > \delta$, there exist some $0 \le k \le N_\delta$ such that $|\widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k)| > \frac{\delta}{3}$. Let $\delta = cn^{\zeta-1/2}$. Then we have

$$P\left(\sup_{\tau\in[\tau_L,\tau_U]} |\widehat{q}_{1,t_k}(\tau) - q_{1,t_k}(\tau)| > cn^{\zeta-1/2}\right) \le N_\delta P\left(|\widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k)| > \frac{c}{3}n^{\zeta-1/2}\right)$$

From Breslow and Crowley (1974), we have the consistency and weak convergence of $\widehat{G}^r_{C,t_k}(s)$ to $G^r_{C,t_k}(s)$. By WLLN and CLT, we have $\widehat{\pi}_{t_k} \to_p \pi_{t_k}$ and $\sqrt{n}(\widehat{\pi}_{t_k} - \pi_{t_k}) = O_p(1)$. From the above and Condition S1 and S2, $\sqrt{n}[\widehat{G}^r_{C,t_k}(s)\widehat{\pi}_{t_k} - G^r_{C,t_k}(s)\pi_{t_k}] \approx \sqrt{n}\{G^r_{C,t_k}(s)(\widehat{\pi}_{t_k} - \pi_{t_k}) + [\widehat{G}^r_{C,t_k}(s) - G^r_{C,t_k}(s)]\pi_{t_k}\} = O_p(1)$.

Write

$$\begin{aligned}
& L_n(\tau) = \tau - F_{n,1}\{q_{1,t_k}(\tau)\} = \frac{1}{n}\sum_{i=1}^n \left\{\tau - \left\{1 - \frac{I[X_i^r(t_k) > q_{1,t_k}(\tau), X_i > t_k]}{\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\}\widehat{\pi}_{t_k}}\right\}\right\} \\
\approx\ & \frac{1}{n}\sum_{i=1}^n \left\{\tau - \left\{1 - \frac{I[X_i^r(t_k) > q_{1,t_k}(\tau), X_i > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\pi_{t_k}}\right\}\right\} \\
& -\frac{1-\tau}{G^r_{C,t_k}\{q_{1,t_k}(\tau)\}}\left\{\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\} - G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\right\} + \frac{1-\tau}{\pi_{t_k}}\{\widehat{\pi}_{t_k} - \pi_{t_k}\} \\
& +\frac{1}{n}\sum_{i=1}^n \left\{\frac{I[X_i^r(t_k) > q_{1,t_k}(\tau), X_i > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\pi_{t_k}} - (1-\tau)\right\}\left\{\frac{G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\pi_{t_k}}{\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\}\widehat{\pi}_{t_k}} - 1\right\} \\
\equiv\ & L_{n,1}(\tau) + L_{n,2}(\tau) + L_{n,3}(\tau) + o_p(n^{-1/2})
\end{aligned}$$

From Lemma S2, we have

$$\widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k) = \{f_1(q_{1,t_k}(\tau_k))\}^{-1}L_n(\tau_k) + R_n(\tau_k)$$

where $R_n(\tau_k) = o_p(n^{-1/2})$. Then we have for sufficient large n,

$$\begin{aligned}
& P\left(|\widehat{q}_{1,t_k}(\tau_k) - q_{1,t_k}(\tau_k)| > \frac{c}{3}n^{\zeta-1/2}\right) \\
\le\ & P\left(|L_n(\tau_k)| > \frac{\inf_{\tau\in[\tau_L,\tau_U]} f_1\{q_{1,t_k}(\tau)\}c}{6}n^{\zeta-1/2}\right) \ \left(\text{Let } c^* = \frac{\inf_{\tau\in[\tau_L,\tau_U]} f_1\{q_{1,t_k}(\tau)\}c}{6}\right) \\
\le\ & P\left(|L_{n,1}(\tau_k)| > \frac{c^*}{6}n^{\zeta-1/2}\right) + P\left(|L_{n,2}(\tau_k)| > \frac{c^*}{6}n^{\zeta-1/2}\right) + P\left(|L_{n,3}(\tau_k)| > \frac{c^*}{6}n^{\zeta-1/2}\right)
\end{aligned}$$

Since $I(X > t_k) \perp X^r(t_k)$, by Hoeffding's inequality and conditions S1 and S2, we have

$$\begin{aligned}
& P\left(|L_{n,1}(\tau_k)| > \frac{c^*}{6}n^{\zeta-1/2}\right) \\
=\ & P\left(\left|\left\{\frac{1}{n}\sum_{i=1}^n \frac{I[X_i^r(t_k) > q_{1,t_k}(\tau), X_i > t_k]}{G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\pi_{t_k}}\right\} - (1-\tau_k)\right| > \frac{c^*}{6}n^{\zeta-1/2}\right) \\
=\ & O[\exp(-c_{1,1}n^{2\zeta})]
\end{aligned}$$

and

$$\begin{aligned}
P\left(|L_{n,3}(\tau_k)| > \frac{c^*}{6}n^{\zeta-1/2}\right) &= P\left(\left|\left\{\frac{1}{n}\sum_{i=1}^n I(X_i > t_k)\right\} - \pi_{t_k}\right| > \frac{c^*\pi_{t_k}}{6(1-\tau_k)}n^{\zeta-1/2}\right) \\
&= O[\exp(-c_{1,3}n^{2\zeta})]
\end{aligned}$$

for some positive constant $c_{1,1}$ and $c_{1,3}$ as n is sufficiently large.

From Pepe (1991), we have

$$n^{1/2}|\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\} - G^r_{C,t_k}\{q_{1,t_k}(\tau)\}| \approx -n^{-1/2}\sum_{i=1}^n G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\int_k^{q_{1,t_k}(\tau)} y(s)^{-1}dM_i^{G^r_{C,t_k}}(s),$$

where $y(t) = pr(X^r \geq t)$ and $M_i^{G^r_{C,t_k}}(\cdot)$ is the martingale process. Then we have from Azuma–Hoeffding inequality

$$\begin{aligned} & P\left(|L_{n,2}(\tau_k)| > \frac{c^*}{6} n^{\zeta-1/2}\right) \\ = \ & P\left(\left|\widehat{G}^r_{C,t_k}\{q_{1,t_k}(\tau)\} - G^r_{C,t_k}\{q_{1,t_k}(\tau)\}\right| > \frac{c^* G^r_{C,t_k}\{q_{1,t_k}(\tau)\}}{6(1-\tau_k)} n^{\zeta-1/2}\right) \\ \leq \ & P\left(\left|\sum_{i=1}^n G^r_{C,t_k}\{q_{1,t_k}(\tau)\} \int_k^{q_{1,t_k}(\tau)} y(s)^{-1} dM_i^{G^r_{C,t_k}}(s) - 0\right| > \frac{c^* G^r_{C,t_k}\{q_{1,t_k}(\tau)\}}{12(1-\tau_k)} n^{\zeta-1/2}\right) \\ = \ & O[\exp(-c_{1,2} n^{2\zeta})] \end{aligned}$$

for some positive constant $c_{1,2}$ as n is sufficiently large.

These lead to

$$\begin{aligned} & P\left(\sup_{\tau \in [\tau_L, \tau_U]} |\widehat{q}_{1,t_k}(\tau) - q_{1,t_k}(\tau)| > c n^{\zeta-1/2}\right) \\ \leq \ & N_\delta O(\exp(-c_1 n^{2\zeta})) = O\{\exp[-c_1 n^{2\zeta} - \log(n^{\zeta-1/2})]\}. \end{aligned}$$

for a positive constant $c_1 = \min\{c_{1,1}, c_{1,2}, c_{1,3}\}$.

□

S3.2 Lemma S4 and the proof

Lemma S4. *Suppose Conditions S1 and S2 hold. Assume that the first derivative of $f_1(q_{1,t_k}(\tau_1))$ with respect to τ_1 is bounded away from zero and infinity for $\tau_1 \in \Delta$. Under the null hypothesis $H_{0,\mathcal{T}}$, we have*

$$\begin{aligned} & \sup_{x \in [q_{1,t_k}(\tau_L), q_{1,t_k}(\tau_U)]} \ \sup_{|x-y| = O(n^{-1/2}\log^{1/2} n)} |S_{n,12}(x,v) - S_{n,12}(y,v) - S_{12}(x,v) + S_{12}(y,v)| \\ = \ & O(n^{-3/4} \log^{3/4} n) \end{aligned}$$

almost surely.

Proof. This lemma can be derived following the similar idea as in the proof of Lemma S1. Specifically, we can first write from the definition

$$S_{n,12}\{q_{1,t_k}(\tau_1, v)\} - S_{12}\{q_{1,t_k}(\tau_1, v)\} = n^{-1/2} \sum_{i=1}^n \upsilon_i\{q_{1,t_k}(\tau_1)\} + R'_n\{q_{1,t_k}(\tau_1)\},$$

where $\sup_{\tau_1 \in \Delta} |R'_n\{q_{1,t_k}(\tau_1)\}| = o_p(n^{-1/2})$ and

$$\begin{aligned} \upsilon_i\{q_{1,t_k}(\tau_1)\} = \ & \left\{\frac{I(X_i^r(t_k) > q_{1,t_k}(\tau_1), V_i(t_k) \leq v, X_i > t_k)}{G^r_{C,t_k}(q_{1,t_k}(\tau_1))\pi_{t_k}} - S_{12}\{q_{1,t_k}(\tau_1), v\}\right\} - \\ & \frac{S_{12}\{q_{1,t_k}(\tau_1), v\}}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1)\}} \cdot \psi_{G,i}\{q_{1,t_k}(\tau_1)\} - \frac{S_{12}\{q_{1,t_k}(\tau_1), v\}}{\pi_{t_k}} \{I[X_i > t_k] - \pi_{t_k}\} \\ = \ & \left\{\frac{I(X_i^r(t_k) > q_{1,t_k}(\tau_1), V_i(t_k) \leq v, X_i > t_k)}{G^r_{C,t_k}(q_{1,t_k}(\tau_1))\pi_{t_k}} - (1-\tau_1) F_2(v; t_k)\right\} - \\ & \frac{(1-\tau_1) F_2(v; t_k)}{G^r_{C,t_k}\{q_{1,t_k}(\tau_1)\}} \cdot \psi_{G,i}\{q_{1,t_k}(\tau_1)\} - \frac{(1-\tau_1) F_2(v; t_k)}{\pi_{t_k}} \{I[X_i > t_k] - \pi_{t_k}\}. \end{aligned}$$

The second equivalence holds as we have $S_{12}(q_{1,t_k}(\tau_1), v) = (1-\tau_1) F_2(v; t_k)$ under the null hypothesis. Then we can follow similar lines as in the proof of Lemma S1 to show that
(i) $\mathrm{E}\{|\upsilon_1\{q_{1,t_k}(\tau_1')\} - \upsilon_1\{q_{1,t_k}(\tau_1)\}|\} = O(n^{-1/2} \log^{1/2} n)$;
(ii) $\mathrm{Var}\{|\upsilon_1\{q_{1,t_k}(\tau_1')\} - \upsilon_1\{q_{1,t_k}(\tau_1)\}|\} = O(n^{-1} \log n)$
for any $\tau_L \leq \tau_1 < \tau_1' \leq \tau_U$ and $q_{1,t_k}(\tau_1') - q_{1,t_k}(\tau_1) = O(n^{-1/2} \log^{1/2} n)$. This completes the proof of Lemma S4. □

S4 Proof of Theorem 1

We first establish the distribution of $\frac{n^{1/2}\widehat{c}(\tau_1, v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))}}$ for any τ_1 and v assuming the null hypothesis holds.
Denote

$$W_{n,12}(\tau_1, v; t_k) = \frac{\widehat{c}(\tau_1, v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))}},$$

and

$$W_{12}(\tau_1, v; t_k) = \frac{F_{12}(q_{1,t_k}(\tau_1), v) - \tau_1 F_2(v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}.$$

From the definition of $\widehat{c}(\tau_1, v; t_k)$ and the fact that $F_{n,1}(\widehat{q}_{1,t_k}(\tau_1)) = \tau_1$, we have

$$\begin{aligned} W_{n,12}(\tau_1, v; t_k) &= \frac{\widehat{c}(\tau_1, v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))}} \\ &= \frac{F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{n,1}(\widehat{q}_{1,t_k}(\tau_1)) \cdot F_{n,2}(v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))}} \\ &= \frac{F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - \tau_1 F_{n,2}(v; t_k)}{\sqrt{\tau_1(1-\tau_1)F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))}}. \end{aligned}$$

We can write

$$\begin{aligned} & n^{1/2}\left\{W_{n,12}(\tau_1, v; t_k) - W_{12}(\tau_1, v; t_k)\right\} \\ \approx\ & n^{1/2}\left\{\frac{F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{12}(q_{1,t_k}(\tau_1), v)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}} - \frac{\tau_1[F_{n,2}(v;t_k) - F_2(v;t_k)]}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right. \\ & \left. - \frac{\sqrt{\tau_1(1-\tau_1)}[F_{12}(q_{1,t_k}(\tau_1), v) - \tau_1 F_2(v;t_k)]}{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}\left[\sqrt{F_{n,2}(v;t_k)(1-F_{n,2}(v;t_k))} - \sqrt{F_2(v;t_k)(1-F_2(v;t_k))}\right]\right\} \\ =\ & n^{1/2}\left\{\frac{F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{12}(q_{1,t_k}(\tau_1), v)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}} - \frac{\tau_1[F_{n,2}(v;t_k) - F_2(v;t_k)]}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right\}. \end{aligned}$$

The second equivalence holds since we have $F_{12}(q_{1,t_k}(\tau_1), v) = \tau_1 F_2(v; t_k)$ for any τ_1 and v under the null hypothesis.

For $F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{12}(q_{1,t_k}(\tau_1), v)$, we can express it as

$$\begin{aligned} & F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{12}(q_{1,t_k}(\tau_1), v) \\ =\ & F_{n,2}(v;t_k) - S_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - [F_2(v;t_k) - S_{12}(q_{1,t_k}(\tau_1), v)] \\ =\ & [F_{n,2}(v;t_k) - F_2(v;t_k)] - [S_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - S_{12}(q_{1,t_k}(\tau_1), v)] \\ =\ & [F_{n,2}(v;t_k) - F_2(v;t_k)] - [S_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - S_{n,12}(q_{1,t_k}(\tau_1), v)] - [S_{n,12}(q_{1,t_k}(\tau_1), v) - S_{12}(q_{1,t_k}(\tau_1), v)] \end{aligned}$$

This leads to

$$\begin{aligned} & n^{1/2}\left\{W_{n,12}(\tau_1, v; t_k) - W_{12}(\tau_1, v; t_k)\right\} \\ \approx\ & n^{1/2}\left\{\frac{F_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - F_{12}(q_{1,t_k}(\tau_1), v)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}} - \frac{\tau_1[F_{n,2}(v;t_k) - F_2(v;t_k)]}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right\} \\ =\ & n^{1/2}\left\{\frac{(1-\tau_1)[F_{n,2}(v;t_k) - F_2(v;t_k)]}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right\} - n^{1/2}\left\{\frac{S_{n,12}(\widehat{q}_{1,t_k}(\tau_1), v) - S_{n,12}(q_{1,t_k}(\tau_1), v)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right\} \\ & -n^{1/2}\left\{\frac{S_{n,12}(q_{1,t_k}(\tau_1), v) - S_{12}(q_{1,t_k}(\tau_1), v)}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}\right\} \\ =\ & \frac{I_1 - I_2 - I_3}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)(1-F_2(v;t_k))}}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= n^{1/2}\left\{(1-\tau_1)[F_{n,2}(v;t_k)-F_2(v;t_k)]\right\}, \\ I_2 &= n^{1/2}\left\{S_{n,12}(\widehat{q}_{1,t_k}(\tau_1),v)-S_{n,12}(q_{1,t_k}(\tau_1),v)\right\}, \\ I_3 &= n^{1/2}\left\{S_{n,12}(q_{1,t_k}(\tau_1),v)-S_{12}(q_{1,t_k}(\tau_1),v)\right\}. \end{aligned}$$

From the definition, we have

$$\begin{aligned} I_1 &= n^{-1/2}\sum_{i=1}^{n}(1-\tau_1)\left\{\frac{I(V_i(t_k)\le v, X_i>t_k)}{\widehat{\pi}_{t_k}}-F_2(v;t_k)\right\} \\ &= n^{-1/2}\sum_{i=1}^{n}\frac{(1-\tau_1)I(X_i>t_k)}{\widehat{\pi}_{t_k}}\left\{I(V_i(t_k)\le v)-F_2(v;t_k)\right\} \\ &\approx n^{-1/2}\sum_{i=1}^{n}\frac{(1-\tau_1)I(X_i>t_k)}{\pi_{t_k}}\left\{I(V_i(t_k)\le v)-F_2(v;t_k)\right\} \\ &\quad -n^{-1/2}(\widehat{\pi}_{t_k}-\pi_{t_k})\sum_{i=1}^{n}\frac{(1-\tau_1)I(X_i>t_k)}{\pi_{t_k}^2}\left\{I(V_i(t_k)\le v)-F_2(v;t_k)\right\} \\ &\approx n^{-1/2}\sum_{i=1}^{n}\frac{(1-\tau_1)I(X_i>t_k)}{\pi_{t_k}}\left\{I(V_i(t_k)\le v)-F_2(v;t_k)\right\}, \end{aligned}$$

where the second equivalence holds since $\widehat{\pi}_{t_k}=\frac{1}{n}\sum_{i=1}^{n} I(X_i>t_k)$, and the fourth asymptotically equivalence holds from the definition of $F_2(v;t_k)$. From the condition, we learn that the derivative of $f_1(q_{1,t_k}(\tau_1))$ is finite and bounded below by 0. Notably, the null hypothesis implies the first derivative of $S_{12}(s,v)$, $S_{12}^{(1)}(s,v)=f_1(s)F_2(v;t_k)$. By Lemma S2, Lemma S3, Lemma S4, and Taylor expansion, we have

$$\begin{aligned} I_2 &\approx n^{1/2}\left\{S_{12}(\widehat{q}_{1,t_k}(\tau_1),v)-S_{12}(q_{1,t_k}(\tau_1),v)\right\} \\ &\approx n^{1/2}S_{12}^{(1)}(q_{1,t_k}(\tau_1),v)\left\{\widehat{q}_{1,t_k}(\tau_1)-q_{1,t_k}(\tau_1)\right\} \\ &= n^{1/2}F_2(v;t_k)f_1(q_{1,t_k}(\tau_1))\left\{\widehat{q}_{1,t_k}(\tau_1)-q_{1,t_k}(\tau_1)\right\} \\ &\approx n^{1/2}F_2(v;t_k)\left\{\tau_1-F_{n,1}(q_{1,t_k}(\tau_1))\right\} \\ &\approx -n^{-1/2}\sum_{i=1}^{n}F_2(v;t_k)\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1), X_i>t_k)}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))\widehat{\pi}_{t_k}}-(1-\tau_1)\right\} \\ &= -n^{-1/2}\sum_{i=1}^{n}\frac{F_2(v;t_k)I(X_i>t_k)}{\widehat{\pi}_{t_k}}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1))}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))}-(1-\tau_1)\right\} \\ &\approx -n^{-1/2}\sum_{i=1}^{n}\frac{F_2(v;t_k)I(X_i>t_k)}{\pi_{t_k}}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1))}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))}-(1-\tau_1)\right\} \end{aligned}$$

We also have from the definition that

$$\begin{aligned} I_3 &= n^{-1/2}\sum_{i=1}^{n}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1), V_i(t_k)\le v, X_i>t_k)}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))\widehat{\pi}_{t_k}}-S_{12}(q_{1,t_k}(\tau_1),v)\right\} \\ &= n^{-1/2}\sum_{i=1}^{n}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1), V_i(t_k)\le v, X_i>t_k)}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))\widehat{\pi}_{t_k}}-(1-\tau_1)F_2(v;t_k)\right\} \\ &= n^{-1/2}\sum_{i=1}^{n}\frac{I(X_i>t_k)}{\widehat{\pi}_{t_k}}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1), V_i(t_k)\le v)}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))}-(1-\tau_1)F_2(v;t_k)\right\} \\ &\approx n^{-1/2}\sum_{i=1}^{n}\frac{I(X_i>t_k)}{\pi_{t_k}}\left\{\frac{I(X_i^r(t_k)>q_{1,t_k}(\tau_1), V_i(t_k)\le v)}{\widehat{G}_{C,t_k}^r(q_{1,t_k}(\tau_1))}-(1-\tau_1)F_2(v;t_k)\right\} \end{aligned}$$

where the second equivalence holds under the null hypothesis.

From the above, combining I_2 and I_3, we have

$$\begin{aligned}
I_2 + I_3 &\approx n^{-1/2} \sum_{i=1}^{n} \frac{I(X_i > t_k)}{\pi_{t_k}} \left\{ \frac{I(X_i^r(t_k) > q_{1,t_k}(\tau_1), V_i(t_k) \le v)}{\widehat{G}^r_{C,t_k}(q_{1,t_k}(\tau_1))} - (1-\tau_1) F_2(v; t_k) \right\} \\
&\quad - n^{-1/2} \sum_{i=1}^{n} \frac{F_2(v; t_k) I(X_i > t_k)}{\pi_{t_k}} \left\{ \frac{I(X_i^r(t_k) > q_{1,t_k}(\tau_1))}{\widehat{G}^r_{C,t_k}(q_{1,t_k}(\tau_1))} - (1-\tau_1) \right\} \\
&= n^{-1/2} \sum_{i=1}^{n} \frac{I(X_i > t_k, X_i^r(t_k) > q_{1,t_k}(\tau_1))}{\widehat{G}^r_{C,t_k}(q_{1,t_k}(\tau_1)) \pi_{t_k}} \{ I(V_i(t_k) \le v) - F_2(v; t_k) \} \\
&\approx n^{-1/2} \sum_{i=1}^{n} \frac{I(X_i > t_k, X_i^r(t_k) > q_{1,t_k}(\tau_1))}{G^r_{C,t_k}(q_{1,t_k}(\tau_1)) \pi_{t_k}} \{ I(V_i(t_k) \le v) - F_2(v; t_k) \}
\end{aligned}$$

All the above leads to

$$n^{1/2} W_{n,12}(\tau_1, v; t_k) \approx -n^{-1/2} \sum_{i=1}^{n} \frac{\xi_i(\tau_1, v; t_k)}{\sqrt{\tau_1(1-\tau_1)}}$$

holds uniformly for any (τ_1, v), where

$$\begin{aligned}
\xi_i(\tau_1, v; t_k) &= \frac{I(X_i > t_k)}{\pi_{t_k} \sqrt{F_2(v; t_k)(1 - F_2(v; t_k))}} \times \\
&\quad \{ I[X_i^r(t_k) > q_{1,t_k}(\tau_1)][I(V_i(t_k) \le v) - F_2(v; t_k)] / G^r_{C,t_k}(q_{1,t_k}(\tau_1)) \\
&\quad - (1-\tau_1)[I(V_i(t_k) \le v) - F_2(v; t_k)] \}.
\end{aligned}$$

Define $\mathcal{F} = \{ \frac{\xi_i(\tau_1, v; t_k)}{\sqrt{\tau_1(1-\tau_1)}}, \tau_1 \in \Delta, v \in \Omega_{V(t_k)}, t_k \in \mathcal{T} \}$. The function class $\mathcal{F}$ is Donsker and thus Glivenko-Cantelli (van der Vaart et al., 1996) since the class of indicator functions is Donsker and τ_1, $F_2(v; t_k)$, $1/\pi_{t_k}$ and $1/G^r_{C,t_k}(q_{1,t_k}(\tau_1))$ are uniformly bounded. As a result of Donsker theorem,

$$n^{1/2} W_{n,12}(\tau_1, v; t_k) \to \chi(\tau_1, v; t_k),$$

where $\chi(\tau_1, v; t_k)$ for any $t_k \in \mathcal{T}$ is a separable Gaussian process depending on (τ_1, v) for $(\tau_1, v) \in \Delta \otimes \Omega_{V(t_k)}$ with $E\{\chi(\tau_1, v; t_k)\} = 0$ and covariance matrix

$$E\{\chi(\tau_1, v; t_k)\chi(\tau_1', v'; t_k)\} = \frac{\{\min(\tau_1, \tau_1') - \tau_1\tau_1'\}\{\min(F_2(v; t_k), F_2(v'; t_k)) - F_2(v; t_k)F_2(v'; t_k)\}}{\sqrt{\tau_1(1-\tau_1)F_2(v; t_k)[1 - F_2(v; t_k)]\tau_1'(1-\tau_1')F_2(v'; t_k)[1 - F_2(v'; t_k)]}}.$$

Then, by the extended continuous mapping theorem (Theorem 1.11.1 in van der Vaart et al. (1996)), under the null hypothesis, we have

$$\int_{\Delta} \int_{\Omega_{V(t_k)}} \frac{nc^2(\tau_1, v; t_k)}{\tau_1(1-\tau_1)F_{n,2}(v; t_k)(1 - F_{n,2}(v; t_k))} d\mu_1(\tau_1) d\mu_2(v) \to_d \int_{\Delta} \int_{\Omega_{V(t_k)}} \chi^2(\tau_1, v; t_k) d\mu_1(\tau_1) d\mu_2(v).$$

Since $\mathcal{T}$ is a finite set, by continuous mapping theorem, we establish the limiting null distribution as

$$\begin{aligned}
n\widehat{q}_{max}(T, \bar{\boldsymbol{Z}}; \Delta, \mathcal{T}) &\to_d \max_{t_k \in \mathcal{T}} \int_{\Delta} \int_{\Omega_{V(t_k)}} \chi^2(\tau_1, v; t_k) d\mu_1(\tau_1) d\mu_2(v); \\
n\widehat{q}_{sum}(T, \bar{\boldsymbol{Z}}; \Delta, \mathcal{T}) &\to_d \sum_{t_k \in \mathcal{T}} \int_{\Delta} \int_{\Omega_{V(t_k)}} \chi^2(\tau_1, v; t_k) d\mu_1(\tau_1) d\mu_2(v).
\end{aligned}$$

This completes the proof of Theorem 1.

S5 Proof of Theorem 2

We first investigate the asymptotic limit of $n\widehat{q}_{max}(T, \bar{\boldsymbol{Z}}; \Delta, \mathcal{T})$ under the alternative hypothesis $H_{a,max}$. Let $\gamma_0 = q_{max}(T, \bar{\boldsymbol{Z}}; \Delta, \mathcal{T})$. Under the alternative hypothesis $H_{a,max}$, we have $\gamma_0 > 0$. Since $an^{-1} = o_p(1)$, we have $P(an^{-1} > \gamma_0/2) \to 0$ as $n \to \infty$. Under the alternative hypothesis $H_{a,max}$, we have $\gamma_0 > 0$ and thus

$$P(\widehat{q}_{max}(T, \bar{\boldsymbol{Z}}; \Delta, \mathcal{T}) > \gamma_0/2) \to P(\gamma_0 > \gamma_0/2) = 1.$$

This suggests that for any a,

$$\begin{aligned} & P(n\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T}) > a) = P(\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T}) > an^{-1}) \\ \geq \quad & P(\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T}) > \gamma_0/2) - P(an^{-1} > \gamma_0/2) \end{aligned}$$

It then follows that $P(n\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T}) > a) \to 1$ as $n \to \infty$ under $H_{a,max}$. Denote $C_{\max,\alpha}$ as the α-level critical value determined upon the limit null distribution of $n\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$, which is greater than 0. Then we have

$$P(n\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T}) > C_{\max,\alpha}) \to 1$$

as $n \to \infty$ given $H_{a,max}$ holds. This implies that $n\widehat{q}_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$ is a consistent test against $H_{a,max}$.

Follow similar lines, we can show that $n\widehat{q}_{sum}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$ is a consistent test against $H_{a,sum}$. This completes the proof of Theorem 2.

S6 Justification of the Resampling Procedure

Firstly, as justified in the Proof of Theorem 1, there is weak convergence of $n^{1/2}W_{n,12}$ to a mean zero Gaussian process with covariance process

$$E\{\chi(\tau_1,v;t_k)\chi(\tau_1',v';t_k)\} = \frac{\{\min(\tau_1,\tau_1') - \tau_1\tau_1'\}\{\min(F_2(v;t_k),F_2(v';t_k)) - F_2(v;t_k)F_2(v';t_k)\}}{\sqrt{\tau_1(1-\tau_1)F_2(v;t_k)[1-F_2(v;t_k)]\tau_1'(1-\tau_1')F_2(v';t_k)[1-F_2(v';t_k)]}}.$$

Next, given that $\{\iota_i^b\}_{i=1}^n$ are i.i.d. random variables following a standard normal distribution. Conditional on the observed data $\{(X_i^r(t_k),V_i(t_k),\delta_i,I(X_i > t_k))\}_{i=1}^n$, we have the asymptotic covariance matrix of $\frac{n^{-1/2}\sum_{i=1}^n \widehat{\xi}_i(\tau_1,v;t_k)\iota_i^b}{\sqrt{\tau_1(1-\tau_1)}}$ as

$$\begin{aligned} & E\left\{ \frac{n^{-1/2}\sum_{i=1}^n \widehat{\xi}_i(\tau_1',v';t_k)\iota_i^b}{\sqrt{\tau_1'(1-\tau_1')}} \cdot \frac{n^{-1/2}\sum_{i=1}^n \widehat{\xi}_i(\tau_1,v;t_k)\iota_i^b}{\sqrt{\tau_1(1-\tau_1)}} \middle| \{(X_i^r(t_k),V_i(t_k),\delta_i,I(X_i > t_k))\}_{i=1}^n \right\} \\ = \quad & \frac{n^{-1}\sum_{i=1}^n \widehat{\xi}_i(\tau_1',v';t_k)\widehat{\xi}_i(\tau_1,v;t_k)}{\sqrt{\tau_1'(1-\tau_1')\tau_1(1-\tau_1)}}, \end{aligned}$$

which converges in probability to $E\{\chi(\tau_1,v;t_k)\chi(\tau_1',v';t_k)\}$.

Following the arguments in Lin et al. (1993), conditional on $\{(X_i^r(t_k),V_i(t_k),\delta_i,I(X_i > t_k))\}_{i=1}^n$, we can show that $\frac{n^{-1/2}\sum_{l=1}^n \widehat{\xi}_i(\tau_1,v;t_k)\iota_i^b}{\sqrt{\tau_1(1-\tau_1)}}$ is zero-mean Gaussian with covariance function converging to the same limit as $W_{n,12}$. Applying the extended continuous mapping theorem as in the proof of Theorem 1, we have that under the null hypothesis, the conditional distribution of $n\widehat{q}_{max,b}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$ (or $n\widehat{q}_{sum,b}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$) given the observed data is asymptotically equivalent to the unconditional distributions of $nq_{max}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$ (or $nq_{sum}(T,\bar{\boldsymbol{Z}};\Delta,\mathcal{T})$). This completes the justification for the resampling procedure in Section 2.

S7 Additional Table and Figure

Table S1: Sensitivity analysis of the FIRST data with p values from the proposed GIQC(MAX) and GIQC(SUM) tests under alternative landmark time grids, $\mathcal{T} = \{1.5,2,3,4,5,6,7,9,11,13\}$, $\mathcal{T}_1 = \{1.5,2,3,4,5,6,7\}$, and $\mathcal{T}_2 = \{2,4,6,9,13\}$. The quantile level interval is fixed at $\Delta = [0.1,0.4]$.

		GIQC(MAX)			GIQC(SUM)		
Landmark set	Description	P(B)	P(F)	P(P)	P(B)	P(F)	P(P)
$\mathcal{T}$	Primary grid	0.097	0.391	0.329	0.170	0.479	0.404
$\mathcal{T}_1$	Truncated grid	0.092	0.365	0.270	0.158	0.380	0.325
$\mathcal{T}_2$	Coarser grid	0.082	0.412	0.247	0.180	0.457	0.342

Table S2: Sensitivity analysis of the FIRST data with p values from the proposed GIQC(MAX) and GIQC(SUM) tests under alternative landmark time grids, $\mathcal{T} = \{1.5, 2, 3, 4, 5, 6, 7, 9, 11, 13\}$, $\mathcal{T}_1 = \{1.5, 2, 3, 4, 5, 6, 7\}$, and $\mathcal{T}_2 = \{2, 4, 6, 9, 13\}$. The quantile interval is fixed at $\Delta_1 = [0.1, 0.3]$.

		GIQC(MAX)			GIQC(SUM)		
Landmark set	Description	P(B)	P(F)	P(P)	P(B)	P(F)	P(P)
$\mathcal{T}$	Primary grid	0.109	0.523	0.223	0.131	0.560	0.318
$\mathcal{T}_1$	Truncated grid	0.106	0.487	0.188	0.127	0.467	0.253
$\mathcal{T}_2$	Coarser grid	0.092	0.475	0.182	0.141	0.542	0.286

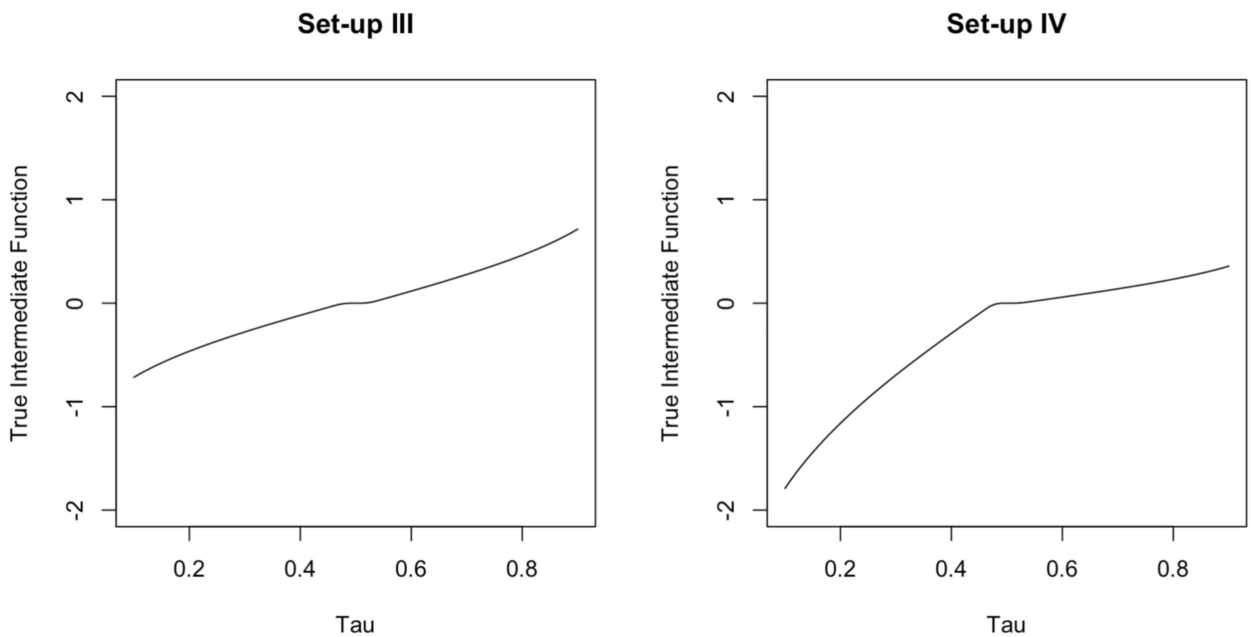

Figure S1: The true intermediate function $\eta(\tau)$ used for simulation set-ups III and IV.

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