Supplementary material for "Upper- and Lower-quantile Functions with an Emphasis on Discrete Populations"

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S1 Bootstrapping Sample Upper- and Lower-quantiles and Their Asymptotic Results

In this section, we consider the bootstrapping inferential procedure for upper- and lower-quantiles themselves for practical purposes.

It is well understood that standard bootstrap fails for sample classical quantiles under discrete populations, because neither the CDF nor the quantile function of a discrete distribution is differentiable (DasGupta, 2008; Jentsch and Leucht, 2016). Although our upper- and lower-quantiles functions are continuous, except for one point, they are not differentiable at the supporting points of X. Therefore, the standard bootstrap may not deliver

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satisfactory performance for upper- and lower-quantiles. This section investigates the m-out-of-n bootstrap approach, which has been successfully used in the inference for mid-quantile (Jentsch and Leucht, 2016).

Given the independent and identically distributed sample $X_1, X_2, \cdots, X_n$ from the discrete population X, suppose that $X_1^\dagger, X_2^\dagger, \cdots, X_m^\dagger$ are independent and identically distributed bootstrap replicates, where bootstrap sample size m is less than n. For $l = 1, 2, \cdots, L$, denote by $r_{l,m}^\dagger$ the frequency of $X_1^\dagger, X_2^\dagger, \cdots, X_m^\dagger$ taking the value x_l. Furthermore, let $\hat{p}_{l,m}^\dagger = r_{l,m}^\dagger / m$, and $\hat{p}_{l,m}^{*\dagger} = \sum_{i=1}^{l} \hat{p}_{i,m}^\dagger$, $l = 1, 2, \cdots, L$. Based on this bootstrap sample, definitions of upper- and lower-quantiles in Section 4.1, the bootstrap counterparts could be easily written as

$$\hat{Q}_{U,m}^\dagger(\alpha) = \begin{cases} x_1, & 0 < \alpha < \hat{p}_{1,m}^{*\dagger}; \\ x_l, & \alpha = \hat{p}_{l,m}^{*\dagger}, l = 1, 2, \cdots, L-1; \\ \omega x_l + (1-\omega) x_{l+1}, & \hat{p}_{l,m}^{*\dagger} < \alpha = \omega \hat{p}_{l,m}^{*\dagger} + (1-\omega)\hat{p}_{l+1,m}^{*\dagger} < \hat{p}_{l+1,m}^{*\dagger}; \\ & 0 < \omega < 1, l = 1, 2, \cdots, L-1, \end{cases}$$

and

$$\hat{Q}^{\dagger}_{L,m}(\alpha) = \begin{cases} x_l, & \alpha = \hat{p}^{*\dagger}_{l-1,m}, l = 2, \cdots, L; \\ \omega x_l + (1-\omega)x_{l+1}, & \hat{p}^{*\dagger}_{l-1,m} < \alpha = \omega\hat{p}^{*\dagger}_{l-1,m} + (1-\omega)\hat{p}^{*\dagger}_{l,m} < \hat{p}^{*\dagger}_{l,m}; \\ & 0 < \omega < 1, l = 1, 2, \cdots, L-1. \\ x_L, & \hat{p}^{*\dagger}_{L-1,m} < \alpha < 1, \end{cases}$$

respectively. In addition, in the subsequent description, we continue to adopt the notations in Sections 2, 3, 4.1, and 4.2.

The following theorem states the asymptotic properties of m-out-of-n bootstrap for sample upper-quantile.

Theorem S1. *Define $x_0 = x_1$, $r^{\dagger}_{0,m} = 0$. Let $0 < \alpha < 1$, as $m = o(n)$, $m \to \infty$, the following results hold:*

(1) *If $\alpha < p_1$, then $\hat{Q}^{\dagger}_{U,m}(\alpha)$ converges to x_1 in probability.*

(2) *If $\alpha = \omega p^*_l + (1-\omega)p^*_{l+1}$ for some $1 \le l \le L-1$ and $0 < \omega < 1$, then*

$$\sqrt{m}[\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)]f_U[Q_U(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]$$

in probability, where $\hat{Q}_{U,n}(\alpha)$ is described in Section 4 and $f_U(x)$ is defined in Theorem 2.

(3) *If $\alpha = p^*_l$ for some $2 \le l \le L-1$, then*

$$\sqrt{m}[\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)]f_U[\hat{Q}^{\dagger}_{U,m}(\alpha), Q_U(\alpha)] \xrightarrow{L} N[0, \alpha(1-\alpha)]$$

in probability and $f_U(x, x_l)$ is defined in Theorem 2.

(4) *If $\alpha = p_1$, then*

$$\sqrt{m}[\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)]f_{U,+}(x_1) \xrightarrow{L} ZI(Z > 0)$$

in probability, where $Q_U(\alpha) = x_1$, $Z \sim N[0, \alpha(1-\alpha)]$ and $f_{U,+}(x_1) = p_2/(x_2 - x_1)$ is defined in Theorem 2.

Theorem S1 shows that the m-out-of-n bootstrap works well for all cases considered in Theorem 2. The differing results arise from the distinct local differentiability properties of the upper-CDF, with the result in Case 4 additionally reflecting a boundary effect. Another notable distinction lies in the centering: Case 2 differs from Cases 3 and 4, which share the same centering. In Case 2, $\hat{Q}^{\dagger}_{U,m}(\alpha)$ is centered at $\hat{Q}_{U,n}(\alpha)$, whereas in Cases 3 and 4, it is centered at $Q_U(\alpha)$.

Similar to Theorem S1, we could get access to the bootstrap asymptotic properties for sample lower-quantile, which are depicted in the following Theorem S2.

Theorem S2. *Define $x_{L+1} = x_L$, $r^{\dagger}_{L+1,m} = 0$. Let $0 < \alpha < 1$, as $m = o(n)$, $m \to \infty$, the following results hold:*

(1) *If $\alpha > p^*_{L-1}$, then $\hat{Q}^{\dagger}_{L,m}(\alpha)$ converges to x_L in probability.*

(2) *If* $\alpha = \omega p_{l-1}^* + (1-\omega)p_l^*$ *for some* $1 \le l \le L-1$ *and* $0 < \omega < 1$*, then*

$$\sqrt{m}[\hat{Q}_{L,m}^{\dagger}(\alpha) - \hat{Q}_{L,n}(\alpha)]f_L[Q_L(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha) + \omega^2 p_l - \omega p_l\big]$$

in probability, where $\hat{Q}_{L,n}(\alpha)$ *is described in Section 4 and* $f_L(x)$ *is defined in Theorem 3.*

(3) *If* $\alpha = p_{l-1}^*$ *for some* $2 \le l \le L-1$*, then*

$$\sqrt{m}[\hat{Q}_{L,m}^{\dagger}(\alpha) - Q_L(\alpha)]f_L[\hat{Q}_{L,m}^{\dagger}(\alpha), Q_L(\alpha)] \xrightarrow{L} N[0, \alpha(1-\alpha)]$$

in probability, where $Q_L(\alpha) = x_l$ *and* $f_L(x, x_l)$ *is defined in Theorem 3.*

(4) *If* $\alpha = p_{L-1}^*$*, then*

$$\sqrt{m}[\hat{Q}_{L,m}^{\dagger}(\alpha) - Q_L(\alpha)]f_{L,-}(x_L) \xrightarrow{L} ZI(Z < 0)$$

in probability, where $Q_L(\alpha) = x_L$*,* $Z \sim N[0, \alpha(1-\alpha)]$ *and* $f_{L,-}(x_L) = p_{L-1}/(x_L - x_{L-1})$ *is defined in Theorem 3.*

The results in Theorem S2 are much like those in Theorem S1, and thus similar remarks could be composed. To avoid repetition, it is not described again.

Notwithstanding the asymptotic results for m-out-of-n bootstrap are achieved in Theorems S1 and S2, they remain futile. To the end of real applications, we prove that the m-out-of-n bootstrap asymptotics is still valid when $\hat{Q}^{\dagger}_{U,m}(\alpha)$ is centered around $\hat{Q}_{U,n}(\alpha)$ and $\hat{Q}^{\dagger}_{L,m}(\alpha)$ is centered around $\hat{Q}_{L,n}(\alpha)$ for all scenarios, respectively. These conclusions are presented in the following corollaries.

Corollary S1. *For $0 < \alpha < 1$, if $m = o(n)$, $m \to \infty$, then it holds that*

$$d_{KS}\Big(\sqrt{m}\big(\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\big), \sqrt{n}\big(\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\big)\Big) \longrightarrow 0$$

in probability as $n \to \infty$, where $d_{KS}(Y, Z)$ is the Kolmogorov-Smirnov distance between the CDFs of Y and Z.

Corollary S2. *For $0 < \alpha < 1$, if $m = o(n)$, $m \to \infty$, then it holds that*

$$d_{KS}\Big(\sqrt{m}\big(\hat{Q}^{\dagger}_{L,m}(\alpha) - \hat{Q}_{L,n}(\alpha)\big), \sqrt{n}\big(\hat{Q}_{L,n}(\alpha) - Q_L(\alpha)\big)\Big) \longrightarrow 0$$

in probability as $n \to \infty$, where $d_{KS}(Y, Z)$ is the Kolmogorov-Smirnov distance between the CDFs of Y and Z.

Corollary S1 states that the asymptotic distribution of $\sqrt{n}\big[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\big]$ could be well approximated by the bootstrap asymptotic distribution of $\sqrt{m}\big[\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\big]$ for any $\alpha \in (0, 1)$. This corollary makes the inference for $Q_U(\alpha)$ feasible. For the same reason, Corollary S2 provides the theoretical foundations for the use of bootstrap inference of lower-sample $Q_L(\alpha)$. For the m-out-of-n bootstrap, the choice of bootstrap sample size m is frequently important. However, in our current context, the bootstrap is robust to the choice of order of m based on our limited experience. We have tried $m = n^{2/3}, n^{3/4}, n^{4/5}$, and all these options perform very satisfactorily in our simulation studies. So, for the sake of space, we do not further investigate this matter here.

S2 Width Inflation Problem

Following the notations in Sections 2 and 3, the results about the width inflation of prediction intervals obtained through upper- and lower-quantiles relative to mid-quantile are summarized in the following proposition.

Proposition S1. *For any $0 < \alpha < 1$, the following results hold:*

(1) *If there exists an integer $0 \leq l_1 < L$ such that $p^*_{l_1} \leq \alpha \leq p^*_{l_1} + (p_{l_1+1})/2$, then, $Q_U(\alpha) = \omega_1 x_{l_1} + (1-\omega_1)x_{l_1+1}$ and $Q_M(\alpha) = \omega_2 x_{l_1} + (1-\omega_2)x_{l_1+1}$, where $0 \leq \omega_2 \leq \omega_1 \leq 1$. Furthermore, the difference of endpoints of the right-sided prediction intervals is*

$$Q_M(\alpha) - Q_U(\alpha) = \begin{cases} 0, & l_1 = 0, \\ (\omega_1 - \omega_2)(x_{l_1+1} - x_{l_1}), & 1 \leq l_1 < L. \end{cases}$$

(2) *If there exists an integer $0 < l_2 \leq L$ such that $p^*_{l_2-1} + p_{l_2}/2 \leq 1 - \alpha \leq p^*_{l_2}$, then $Q_M(1-\alpha) = \omega_3 x_{l_2} + (1-\omega_3)x_{l_2+1}$ and $Q_L(1-\alpha) = \omega_4 x_{l_2} + (1-\omega_4)x_{l_2+1}$, where $0 \leq \omega_4 \leq \omega_3 \leq 1$. Furthermore, the difference of*

endpoints of the left-sided prediction intervals is

$$Q_L(1-\alpha) - Q_M(1-\alpha) = \begin{cases} (\omega_3 - \omega_4)(x_{l_2+1} - x_{l_2}), & 1 \le l_2 < L, \\ 0, & l_2 = L. \end{cases}$$

(3) *If there exist two integers* $0 \le l_1 < l_2 \le L$ *such that* $p^*_{l_1} \le \alpha/2 \le p^*_{l_1} + p_{l_1+1}/2$ *and* $p^*_{l_2-1} + p_{l_2}/2 \le 1-\alpha/2 \le p^*_{l_2}$, *then* $Q_U(\alpha/2) = \omega_1 x_{l_1} + (1-\omega_1)x_{l_1+1}$, $Q_M(\alpha/2) = \omega_2 x_{l_1} + (1-\omega_2)x_{l_1+1}$, $Q_M(1-\alpha/2) = \omega_3 x_{l_2} + (1-\omega_3)x_{l_2+1}$ *and* $Q_L(1-\alpha/2) = \omega_4 x_{l_2} + (1-\omega_4)x_{l_2+1}$, *where* $0 \le \omega_2 \le \omega_1 \le 1$ *and* $0 \le \omega_4 \le \omega_3 \le 1$. *Furthermore, the increase of prediction interval width is*

$$\begin{aligned} & \{Q_L(1-\alpha/2) - Q_U(\alpha/2)\} - \{Q_M(1-\alpha/2) - Q_M(\alpha/2)\} \\ = & \begin{cases} 0, & l_1 = 0,\ l_2 = L, \\ (\omega_3 - \omega_4)(x_{l_2+1} - x_{l_2}), & l_1 = 0,\ 1 \le l_2 < L, \\ (\omega_1 - \omega_2)(x_{l_1+1} - x_{l_1}), & 1 \le l_1 < L,\ l_2 = L, \\ (\omega_1 - \omega_2)(x_{l_1+1} - x_{l_1}) + (\omega_3 - \omega_4)(x_{l_2+1} - x_{l_2}), & 1 \le l_1 < l_2 < L. \end{cases} \end{aligned}$$

The proof of this proposition is deferred to the next section.

S3 Proofs of the Propositions, Theorems and Corollaries

Proof of Proposition 2: (1) When there exists $0 \le l_1 < L$ such that $p^*_{l_1} \le \alpha \le p^*_{l_1} + p_{l_1+1}/2$, we consider the following two cases:

(a) If $l_1 = 0$, that is $0 < \alpha \le p_1/2$, then $Q_M(\alpha) = x_1$ and $\Pr(X < Q_M(\alpha)) = 0 \le \alpha$.

(b) If $0 < l_1$, according to the definition of mid-distribution, we have $x_{l_1} < Q_M(\alpha) \le x_{l_1+1}$. Thus $\Pr(X < Q_M(\alpha)) = \Pr(X < x_{l_1+1}) = p^*_{l_1} \le \alpha$.

When there exists l_1 such that $p^*_{l_1} + p_{l_1+1}/2 < \alpha < p^*_{l_1+1}$, $x_{l_1+1} < Q_M(\alpha) < x_{l_1+2}$. Thus, $\Pr(X < Q_M(\alpha)) = \Pr(X \le x_{l_1+1}) = p^*_{l_1+1} > \alpha$.

(2) When there exists l_2 such that $p^*_{l_2-1} + p_{l_2}/2 \le 1 - \alpha \le p^*_{l_2}$, we consider the following two cases:

(2a) If $l_2 = L$, that is $p^*_{L-1} + p_L/2 \le 1 - \alpha \le 1$, then $Q_M(1-\alpha) = x_L$ and $\Pr(X \le Q_M(1-\alpha)) = \Pr(X \le x_L) = 1 \ge 1 - \alpha$.

(2b) If $l_2 < L$, according to the definition of mid-distribution, we have $\Pr(X \le Q_M(1-\alpha)) = F(Q_M(1-\alpha)) \ge F_M(Q_M(1-\alpha)) = 1 - \alpha$.

When there exists l_2 such that $p^*_{l_2-1} < 1 - \alpha < p^*_{l_2-1} + p_{l_2}/2$, $\Pr(X \le Q_M(1-\alpha)) = F(Q_M(1-\alpha)) < F_M(Q_M(1-\alpha)) = 1 - \alpha$.

(3) This result could be concluded directly from (1) and (2). $\square$

Proof of Proposition 3: (1) If $\alpha \leq p_1$, note that $Q_U(\alpha) = x_1$. So $\Pr(X < Q_U(\alpha)) = \Pr(X < x_1) = 0 < \alpha$. If $\alpha > p_1$, according to the definition of $F_U(x)$, we have $\Pr(X < Q_U(\alpha)) \leq F(Q_U(\alpha)) \leq F_U(Q_U(\alpha)) = \alpha$.

(2) If $p_1 + p_2 + \cdots + p_{L-1} \leq 1-\alpha$, it concludes that $Q_L(1-\alpha) = x_L$. Thus $\Pr(X \leq Q_L(1-\alpha)) = F(x_L) = 1 \geq 1-\alpha$. If $p_1 + p_2 + \cdots + p_{L-1} > 1-\alpha$, according to definition of $F_L(x)$, we can obtain that $\Pr(X \leq Q_L(1-\alpha)) = F(Q_L(1-\alpha)) \geq F_L(Q_L(1-\alpha)) = 1-\alpha$.

(3) It is noted that $\Pr(Q_U(\alpha/2) \leq X \leq Q_L(1-\alpha/2)) = \Pr(X \leq Q_L(1-\alpha/2)) - \Pr(X < Q_U(\alpha/2))$. Thus, the conclusion follows from the results in (1) and (2). $\square$

Proof of Theorem 1: Here, we only prove the result of lower-quantile in this Theorem, while that of upper-quantile could be obtained in the similar way.

If $p^*_{L-1} < \alpha$, $Q_L(\alpha) = x_L$. As n goes to ∞, $\hat{p}^*_{L-1,n}$ converges to p^*_{L-1} with probability 1. Thus, as n goes to ∞, it holds that $\hat{p}^*_{L-1,n} < \alpha$ with probability tending to 1. Consequently, $\hat{Q}_{L,n}(\alpha) = x_L$ with probability tending to 1, as n goes to ∞. Therefore, $\hat{Q}_{L,n}(\alpha)$ converges to $Q_L(\alpha)$ in probability, as n goes to ∞.

If $0 < \alpha < p^*_{L-1}$, and $\alpha \neq p^*_{l-1}$, $l = 2, 3, \cdots, L$, then there exist $1 \leq l < L$ and $0 < \omega < 1$ such that $\alpha = \omega p^*_{l-1} + (1-\omega)p^*_l$. So $Q_L(\alpha) =$

$\omega x_l + (1-\omega)x_{l+1}$. As n goes to ∞, $\hat{p}^*_{l-1,n}$ and $\hat{p}^*_{l,n}$ converge to p^*_{l-1} and p^*_l with probability 1, respectively. Thus, as n goes to ∞, it holds that $\hat{p}^*_{l-1,n} < \alpha < \hat{p}^*_{l,n}$ with probability tending to 1, that is there exists $0 < \hat{\omega}_n < 1$ so that $\alpha = \hat{\omega}_n \hat{p}^*_{l-1,n} + (1-\hat{\omega}_n)\hat{p}^*_{l,n}$ with probability tending to 1. Therefore, $\hat{Q}_{L,n}(\alpha) = \hat{\omega}_n x_l + (1-\hat{\omega}_n)x_{l+1}$ with probability tending to 1 as n goes to ∞. From the equation

$$\hat{\omega}_n \hat{p}^*_{l-1,n} + (1-\hat{\omega}_n)\hat{p}^*_{l,n} = \omega p^*_{l-1} + (1-\omega)p^*_l,$$

we have that

$$\hat{\omega}_n = [p^*_l - \hat{p}^*_{l,n} + \omega(p^*_{l-1} - p^*_l)]/(\hat{p}^*_{l-1,n} - \hat{p}^*_{l,n}).$$

This expression implies that $\hat{\omega}_n$ converges to ω with probability 1 as n goes to ∞. Thus, we arrive at that $\hat{Q}_{L,n}(\alpha)$ converges to $Q_L(\alpha)$ in probability as n goes to ∞.

At last, we consider the scenarios of $\alpha = p^*_{l-1}$, $l = 2, 3, \cdots, L$. Note that $\hat{p}^*_{l-2,n}$ and $\hat{p}^*_{l,n}$ converge to p^*_{l-2} and p^*_l almost surely, as n goes to ∞. Thus, as n goes to ∞, it holds that $\hat{p}^*_{l-2,n} < \alpha < \hat{p}^*_{l,n}$ with probability tending to 1, that is there exist $0 \leq \hat{\omega}_{n,1}, \hat{\omega}_{n,2} < 1$ such that $\alpha = \hat{\omega}_{n,1}\hat{p}^*_{l-2,n} + (1-\hat{\omega}_{n,1})\hat{p}^*_{l-1,n}$ or $\alpha = (1-\hat{\omega}_{n,2})\hat{p}^*_{l-1,n} + \hat{\omega}_{n,2}\hat{p}^*_{l,n}$ with probability tending to 1. As in the proof in the last paragraph, we could find that $\hat{\omega}_{n,1}$ and $\hat{\omega}_{n,2}$ converge to 0 with probability 1 as n goes to ∞. After some simple algebraic calculations,

α could be equivalently expressed as

$$\alpha = \hat{p}^*_{l-1,n} + I(\alpha > \hat{p}^*_{l-1,n})\hat{\omega}_{n,2}(\hat{p}^*_{l,n} - \hat{p}^*_{l-1,n}) - I(\alpha \leq \hat{p}^*_{l-1,n})\hat{\omega}_{n,1}(\hat{p}^*_{l-1,n} - \hat{p}^*_{l-2,n}).$$

Because $\hat{p}^*_{l-1,n}$, $\hat{p}^*_{l-2,n}$ and $\hat{p}^*_{l,n}$ converge to p^*_{l-1}, p^*_{l-2} and p^*_l, we finally could conclude that $\hat{Q}_{L,n}(\alpha)$ converges to $Q_L(\alpha)$ in probability as n goes to ∞. $\square$

Proof of Corollary 1: Note that

$$\begin{aligned} & \Pr_{(\boldsymbol{X},X)}\{X \notin [\hat{Q}_{U,n}(\alpha/2), \hat{Q}_{L,n}(1-\alpha/2)]\} \\ = \; & \Pr_{(\boldsymbol{X},X)}\{X < \hat{Q}_{U,n}(\alpha/2)\} + \Pr_{(\boldsymbol{X},X)}\{X > \hat{Q}_{L,n}(1-\alpha/2)\}. \end{aligned} \tag{S1}$$

Firstly, we deal with the first term of (S1). For any positive constant ϵ, we have

$$\begin{aligned} & \Pr_{(\boldsymbol{X},X)}\{X < \hat{Q}_{U,n}(\alpha/2)\} \\ \leq \; & \Pr_{(\boldsymbol{X},X)}\{X < \hat{Q}_{U,n}(\alpha/2), \hat{Q}_{U,n}(\alpha/2) < Q_U(\alpha/2) + \epsilon\} \\ & + \Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{U,n}(\alpha/2) - Q_U(\alpha/2)| \geq \epsilon\} \\ \leq \; & \Pr_{(\boldsymbol{X},X)}\{X \leq Q_U(\alpha/2) + \epsilon\} + \Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{U,n}(\alpha/2) - Q_U(\alpha/2)| \geq \epsilon\}. \end{aligned} \tag{S2}$$

From Theorem 1, it is easy to know that $\hat{Q}_{U,n}(\alpha/2)$ converges to $Q_U(\alpha/2)$ in probability as n goes to ∞. Thus,

$$\Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{U,n}(\alpha/2) - Q_U(\alpha/2)| \geq \epsilon\} = o(1), \tag{S3}$$

as n goes to infinity.

As for the first term of (S2), $\Pr_{(\boldsymbol{X},X)}\{X \leq Q_U(\alpha/2)+\epsilon\} = F(Q_U(\alpha/2)+\epsilon)$, where F is the cumulative distribution function of X. From the definition of upper-CDF and the assumption in this theorem, it could be obtained that the $F(Q_U(\alpha/2)+\epsilon) \leq \alpha/2$ for sufficiently small positive number ϵ. This result, together with (S2) and (S3), leads to

$$\Pr_{(\boldsymbol{X},X)}\{X < \hat{Q}_{U,n}(\alpha/2)\} \leq \alpha/2 + o(1). \tag{S4}$$

Secondly, we deal with the second term of (S1). For any positive constant ϵ,

$$\begin{aligned} & \Pr_{(\boldsymbol{X},X)}\{X > \hat{Q}_{L,n}(1-\alpha/2)\} \\ \leq\ & \Pr_{(\boldsymbol{X},X)}\{X > \hat{Q}_{L,n}(1-\alpha/2), \hat{Q}_{L,n}(1-\alpha/2) > Q_L(1-\alpha/2)-\epsilon\} \\ & \quad +\Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{L,n}(1-\alpha/2) - Q_L(1-\alpha/2)| \geq \epsilon\} \\ \leq\ & \Pr_{(\boldsymbol{X},X)}\{X > Q_L(1-\alpha/2)-\epsilon\} \\ & +\Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{L,n}(1-\alpha/2) - Q_L(1-\alpha/2)| \geq \epsilon\}. \end{aligned} \tag{S5}$$

From Theorem 1, $\hat{Q}_{L,n}(1-\alpha/2)$ converges to $Q_L(1-\alpha/2)$ in probability as n goes to ∞. Thus,

$$\Pr_{(\boldsymbol{X},X)}\{|\hat{Q}_{L,n}(1-\alpha/2) - Q_L(1-\alpha/2)| \geq \epsilon\} = o(1). \tag{S6}$$

For the first term of (S5),

$$\Pr_{(\boldsymbol{X},X)}\{X > Q_L(1-\alpha/2)-\epsilon\} = 1 - F(Q_L(1-\alpha/2)-\epsilon). \tag{S7}$$

From the definition of lower-distribution and the assumption in this theorem, it could be obtained that the $F(Q_L(1-\alpha/2)-\epsilon) \geq 1-\alpha/2$ for sufficiently small positive number ϵ. This result, together with (S5) to (S7), produces that

$$\Pr_{(\boldsymbol{X},X)}\{X > \hat{Q}_{L,n}(1-\alpha/2)\} \leq \alpha/2 + o(1). \tag{S8}$$

From (S1), (S4) and (S8), we arrive at

$$\Pr_{(\boldsymbol{X},X)}\{X \notin [\hat{Q}_{U,n}(\alpha/2), \hat{Q}_{L,n}(1-\alpha/2)]\} \leq \alpha + o(1).$$

Eventually, it is concluded the result of this theorem, that is

$$\begin{aligned}
& \Pr_{(\boldsymbol{X},X)}\{X \in [\hat{Q}_{U,n}(\alpha/2), \hat{Q}_{L,n}(1-\alpha/2)]\} \\
= \;& 1 - \Pr_{(\boldsymbol{X},X)}\{X \notin [\hat{Q}_{U,n}(\alpha/2), \hat{Q}_{L,n}(1-\alpha/2)]\} \\
\geq \;& 1 - \alpha + o(1).
\end{aligned}$$

$\square$

Proof of Theorem 2: (1) When n is sufficiently large, with probability to 1, we can find $0 < \hat{\omega}_n < 1$ such that $\alpha = \hat{\omega}_n(r_{0,n}/n) + (1-\hat{\omega}_n)(r_{1,n}/n)$. Hence, as n goes to ∞, with probability tending to 1, $\hat{Q}_{U,n}(\alpha) = \hat{\omega}_n x_0 + (1-\hat{\omega}_n)x_1 = x_1$.

(2) When n is sufficiently large, with probability to 1, we can find $0 < \hat{\omega}_n < 1$ such that

$$\alpha = \hat{\omega}_n \Big(\frac{r_{1,n} + \cdots + r_{l,n}}{n} \Big) + (1 - \hat{\omega}_n) \Big(\frac{r_{1,n} + \cdots + r_{l+1,n}}{n} \Big).$$

Thus

$$\hat{\omega}_n = \frac{(r_{1,n} + \cdots + r_{l+1,n})/n - \alpha}{r_{l+1,n}/n}, \quad \hat{Q}_{U,n}(\alpha) = \hat{\omega}_n x_l + (1 - \hat{\omega}_n) x_{l+1}.$$

Furthermore,

$$\hat{Q}_{U,n}(\alpha) - Q_U(\alpha) = \hat{Q}_{U,n}(\alpha) - \omega x_l - (1 - \omega) x_{l+1} = (\hat{\omega}_n - \omega)(x_l - x_{l+1}),$$

where

$$\hat{\omega}_n - \omega = \frac{(r_{1,n} + \cdots + r_{l+1,n})/n - \alpha}{r_{l+1,n}/n} - \frac{(p_1 + p_2 + \cdots + p_{l+1}) - \alpha}{p_{l+1}}.$$

Because $x_l - x_{l+1}$ is constant, we only need to investigate the asymptotic normality of $\hat{\omega}_n - \omega$.

Note that the random vector $(r_{1,n}, r_{2,n}, \cdots, r_{l+1,n})^T$ follows the multinomial distribution with the mean vector $(np_1, np_2, \cdots, np_{l+1})^T$ and the covariance matrix $\Sigma_n = (\sigma_{n,m_1,m_2})_{(l+1)\times(l+1)}$, where $\sigma_{n,m,m} = np_m(1 - p_m)$ for $m = 1, 2, \cdots, l+1$ and $\sigma_{n,m_1,m_2} = -np_{m_1}p_{m_2}$ for $m_1 \neq m_2, m_1, m_2 = 1, 2, \cdots, l+1$. So as n goes to ∞,

$$\sqrt{n}[(r_{1,n}, r_{2,n}, \cdots, r_{l+1,n})^T/n - (p_1, p_2, \cdots, p_{l+1})^T] \longrightarrow N(\mathbf{0}_{l+1}, \Sigma_{(l+1)\times(l+1)})$$

in distribution, where $\mathbf{0}_{l+1}$ is a $(l+1)$-dimensional vector with every component being 0 and $\Sigma_{(l+1)\times(l+1)} = (\sigma_{i,j})_{(l+1)\times(l+1)}$ is the covariance matrix whose diagonal entries are $\sigma_{m,m} = p_m(1-p_m)$ and whose off-diagonal entries are $\sigma_{m_1,m_2} = -p_{m_1}p_{m_2}$ for $m_1 \neq m_2$.

Through the delta method, after some straightforward calculations, we can obtain that as n goes to ∞,

$$\sqrt{n}(\hat{\omega}_n - \omega) \longrightarrow N(0, \sigma_1^2)$$

in distribution, where $\sigma_1^2 = p_{l+1}^{-2}\big[\alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]$.

It is noted that

$$\begin{aligned}\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_U[Q_U(\alpha)] &= \sqrt{n}(\hat{\omega}_n - \omega)(x_l - x_{l+1})f_U[Q_U(\alpha)] \\ &= \sqrt{n}(\hat{\omega}_n - \omega)(-p_{l+1}),\end{aligned}$$

Therefore, as n goes to ∞,

$$\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)]f_U[Q_U(\alpha)] \longrightarrow N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]$$

in distribution.

(3) Let $s_l^2 = p_l(1-p_l)$, for $l = 1, 2, \cdots, L$. Note that $(r_{1,n} + \cdots + r_{l,n})/n$ converges almost surely to $p_1 + p_2 + \cdots + p_l = p_l^* = \alpha$ as n goes to ∞.

Hence, for n sufficiently large n, we can get $0 \leq \hat{\omega}_{n,1}, \hat{\omega}_{n,2} < 1$ such that

$$\alpha = \begin{cases} \hat{\omega}_{n,1} \frac{r_{1,n}+\cdots+r_{l-1,n}}{n} + (1-\hat{\omega}_{n,1}) \frac{r_{1,n}+\cdots+r_{l,n}}{n}, & \frac{r_{1,n}+\cdots+r_{l,n}}{n} \geq \alpha; \\ (1-\hat{\omega}_{n,2}) \frac{r_{1,n}+\cdots+r_{l,n}}{n} + \hat{\omega}_{n,2} \frac{r_{1,n}+\cdots+r_{l+1,n}}{n}, & \frac{r_{1,n}+\cdots+r_{l,n}}{n} < \alpha. \end{cases}$$

We consolidate the two cases into the one form as

$$\begin{aligned} \alpha &= \frac{r_{1,n}+\cdots+r_{l,n}}{n} - I\Big(\frac{r_{1,n}+\cdots+r_{l,n}}{n} \geq \alpha\Big) \hat{\omega}_{n,1} \frac{r_{l,n}}{n} \\ &\quad + I\Big(\frac{r_{1,n}+\cdots+r_{l,n}}{n} < \alpha\Big) \hat{\omega}_{n,2} \frac{r_{l+1,n}}{n}. \end{aligned}$$

Define $U_n = \sqrt{n}\Big(\alpha - \sum_{m=1}^{l} r_{m,n}/n\Big)$. By some simple algebraic manipulations, we have

$$\begin{aligned} & \hat{Q}_{U,n}(\alpha) \\ = \; & I(U_n \leq 0)\big[\hat{\omega}_{n,1} x_{l-1} + (1-\hat{\omega}_{n,1}) x_l\big] + I(U_n > 0)\big[(1-\hat{\omega}_{n,2}) x_l + \hat{\omega}_{n,2} x_{l+1}\big] \\ = \; & x_l + \frac{U_n}{\sqrt{n}} \Big\{ \frac{x_l - x_{l-1}}{r_{l,n}/n} I(U_n \leq 0) + \frac{x_{l+1} - x_l}{r_{l+1,n}/n} I(U_n > 0) \Big\}. \end{aligned}$$

Thus, for any $a \leq 0$, we have

$$\Pr\{\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_l] \leq a\} = \Pr\Big\{U_n \frac{x_l - x_{l-1}}{r_{l,n}/n} \leq a\Big\} = \Pr\Big\{\frac{U_n}{r_{l,n}/n} \leq \frac{a}{x_l - x_{l-1}}\Big\}.$$

From the definition of U_n, the central limit theorem and the law of large numbers, note that

$$\frac{U_n}{r_{l,n}/n} = \frac{U_n}{p_l} + o_p(1).$$

In addition, denote $Y_l = \sqrt{n}(r_{l,n}/n - p_l)/s_l$, for $l = 1, 2, \cdots, L$. Then

$$U_n = \sqrt{n}\Big(\sum_{m=1}^{l} p_m - \sum_{m=1}^{l} \frac{r_{m,n}}{n}\Big) = \sum_{m=1}^{l} \sqrt{n}\Big(p_m - \frac{r_{m,n}}{n}\Big) = -\sum_{m=1}^{l} s_m Y_m.$$

Thus

$$\begin{aligned}
& \lim_{n\to\infty} \Pr\left\{\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_l] \le a\right\} \\
= & \lim_{n\to\infty} \Pr\left(\frac{U_n}{r_{l,n}/n} \le \frac{a}{x_l - x_{l-1}}\right) \\
= & \lim_{n\to\infty} \Pr\left\{\frac{-\sum_{m=1}^{l} s_m Y_m}{p_l} \le \frac{a}{x_l - x_{l-1}}\right\}.
\end{aligned}$$

Similar to the proof in (2), it could be shown that n goes to ∞,

$$-\sum_{m=1}^{l} s_m Y_m / p_l \longrightarrow N(0, \sigma_2^2)$$

in distribution, where

$$\begin{aligned}
\sigma_2^2 &= p_l^{-2}\left(\sum_{m=1}^{l} s_m^2 - 2\sum_{\substack{m_1,m_2=1\\ m_1<m_2}}^{l} p_{m_1} p_{m_2}\right) \\
&= p_l^{-2}\left[\sum_{m=1}^{l} p_m(1-p_m) - 2\sum_{\substack{m_1,m_2=1\\ m_1<m_2}}^{l} p_{m_1} p_{m_2}\right] \\
&= p_l^{-2}\alpha(1-\alpha).
\end{aligned}$$

Therefore, we obtain that

$$\lim_{n\to\infty} \Pr\left\{\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_l] \le a\right\} = \Phi\left[\frac{a}{\sigma_2(x_l - x_{l-1})}\right]$$

for any $a \le 0$.

For any $a > 0$, we have

$$\begin{aligned}
& \lim_{n\to\infty} \Pr\left\{\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_l] \le a\right\} \\
= & \lim_{n\to\infty} \Pr\left\{U_n \le 0 \text{ or } 0 < U_n \frac{x_{l+1} - x_l}{r_{l+1,n}/n} \le a\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Pr\left\{U_n \frac{x_{l+1}-x_l}{r_{l+1,n}/n} \leq a\right\} \\
&= \lim_{n\to\infty} \Pr\left\{\frac{-\sum_{m=1}^{l} s_m Y_m}{p_{l+1}} \leq \frac{a}{x_{l+1}-x_l}\right\}.
\end{aligned}
$$

Similar to the proof in (2), it could be shown that n goes to ∞,

$$
-\sum_{m=1}^{l} s_m Y_m / p_{l+1} \longrightarrow N(0, \sigma_3^2)
$$

in distribution, where $\sigma_3^2 = p_{l+1}^{-2}[\alpha(1-\alpha)]$. Thus, we have

$$
\lim_{n\to\infty} \Pr\left\{\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_l] \leq a\right\} = \Phi\Big[\frac{a}{\sigma_3(x_{l+1}-x_l)}\Big].
$$

(4) Define $U_{n,1} = \sqrt{n}\{\alpha - r_{1,n}/n\}$. Since $\alpha = p_1$, by the central limit theorem, $U_{n,1} \xrightarrow{L} N[0, \alpha(1-\alpha)]$. Moreover, $r_{2,n}/n \xrightarrow{p} p_2$ and $\Pr\Big\{\alpha < \frac{r_{1,n}+r_{2,n}}{n}\Big\} \to 1$. Hence, with probability tending to 1,

$$
\sqrt{n}[\hat{Q}_{U,n}(\alpha) - x_1] = U_{n,1} \frac{x_2 - x_1}{r_{2,n}/n} I(U_{n,1} > 0).
$$

Therefore, by the continuous mapping theorem and the law of large numbers,

$$
\sqrt{n}[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)] \xrightarrow{L} \frac{x_2 - x_1}{p_2} Z I(Z > 0),
$$

where $Z \sim N[0, \alpha(1-\alpha)]$. □

Proof of Theorem 3: The proof of Theorem 3 is similar to that of Theorem 2, and is therefore omitted.

Proof of Theorem 4: (1) When n is sufficiently large, with probability to 1, we can find $0 < \hat{\omega}_n < 1$ and $l \geq 1$ such that

$$\begin{aligned}\alpha &= \hat{\omega}_n \frac{r_{1,n} + \cdots + r_{l,n}}{n} + (1-\hat{\omega}_n)\frac{r_{1,n} + \cdots + r_{l+1,n}}{n} \\ &= \hat{\omega}_n \hat{F}_{U,n}(Z_l) + (1-\hat{\omega}_n)\hat{F}_{U,n}(Z_{l+1}).\end{aligned}$$

Hence, $\hat{Q}_{U,n}(\alpha) = \hat{\omega}_n Z_l + (1-\hat{\omega}_n)Z_{l+1}$. For an arbitrary constant a, we have

$$\Pr\{\hat{Q}_{U,n}(\alpha) \leq Q(\alpha) + n^{-\frac{1}{2}}a\} = \Pr\{F[\hat{Q}_{U,n}(\alpha)]) \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\}.$$

Because $Z_l \leq \hat{Q}_{U,n}(\alpha) \leq Z_{l+1}$, $F(Z_l) \leq F[\hat{Q}_{U,n}(\alpha)] \leq F(Z_{l+1})$. Therefore, we have

$$\begin{aligned}\Pr\left\{F(Z_{l+1}) \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\} &\leq \Pr\left\{F[\hat{Q}_{U,n}(\alpha)] \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\} \\ &\leq \Pr\left\{F(Z_l) \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\}.\end{aligned}$$

In the following, we prove that

$$\lim_{n\to\infty} \Pr\left\{F(Z_m) \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\} = \Phi\left\{\frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}}\right\}$$

for $m = l$ and $l+1$.

Let $N = \sum_{i=1}^{n} I\{F(X_i) \leq F[Q(\alpha) + n^{-\frac{1}{2}}a]\}$. Then N follows the binomial distribution with parameters n and $F[Q(\alpha)+n^{-\frac{1}{2}}a]$. By the Liapounov central limit theorem, it could be shown that as n goes to ∞,

$$\frac{N - nc_n}{\sqrt{nc_n(1-c_n)}} \longrightarrow N(0,1)$$

in distribution, where $c_n = F[Q(\alpha) + n^{-\frac{1}{2}} a]$.

It is noted that

$$\begin{aligned} & \Pr[F(Z_m) \leq c_n] \\ = & \Pr[N \geq r_{1,n} + \cdots + r_{m,n}] \\ = & \Pr\left[\frac{N - nc_n}{\sqrt{nc_n(1-c_n)}} \geq \frac{r_{1,n} + \cdots + r_{m,n} - nc_n}{\sqrt{nc_n(1-c_n)}}\right]. \end{aligned}$$

From the assumption in the theorem, it holds that

$$c_n = \alpha + n^{-\frac{1}{2}} a\{f[Q(\alpha)] + \varepsilon_n\}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Then

$$\frac{nc_n - (r_{1,n} + \cdots + r_{m,n})}{\sqrt{nc_n(1-c_n)}} = \frac{n\alpha - (r_{1,n} + \cdots + r_{m,n})}{\sqrt{nc_n(1-c_n)}} + \frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} + o(1).$$

It is easily verified that $|n\alpha - (r_{1,n} + \cdots + r_{m,n})| < r_{l,n} + r_{l+1,n}$ for both $m = l$ and $l + 1$. In addition, because $F(x)$ is differentiable and $f(x)$ is positive in the neighborhood of $Q(\alpha)$, $r_{l,n} = 1$ and $r_{l+1,n} = 1$ with probability 1. So $\{n\alpha - (r_{1,n} + \cdots + r_{m,n})\}/\sqrt{n} = o_p(1)$. Thus,

$$\frac{nc_n - (r_{1,n} + \cdots + r_{m,n})}{\sqrt{nc_n(1-c_n)}} = \frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} + o_p(1).$$

Therefore,

$$\begin{aligned} & \lim_{n\to\infty} \Pr[F(Z_m) \leq c_n] \\ = & \lim_{n\to\infty} \Pr\left[\frac{N - nc_n}{\sqrt{nc_n(1-c_n)}} \geq -\frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} + o_p(1)\right] \end{aligned}$$

$$= \Phi\left\{\frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}}\right\}.$$

Finally, we obtain that

$$\begin{aligned}
& \lim_{n\to\infty} \Pr\{\hat{Q}_{U,n}(\alpha) \le Q(\alpha) + n^{-\frac{1}{2}}a\} \\
= & \lim_{n\to\infty} \Pr\left\{F[\hat{Q}_{U,n}(\alpha)] \le F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\} \\
= & \Phi\left\{\frac{af[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}}\right\}.
\end{aligned}$$

This completes the proof for (1).

(2) When n is sufficiently large, with probability to 1, we can find $0 < \hat{\omega}_n < 1$ and $l \ge 1$ such that

$$\begin{aligned}
\alpha &= \hat{\omega}_n \frac{r_{1,n} + \cdots + r_{l,n}}{n} + (1-\hat{\omega}_n)\frac{r_{1,n} + \cdots + r_{l+1,n}}{n} \\
&= \hat{\omega}_n \hat{F}_{U,n}(Z_l) + (1-\hat{\omega}_n)\hat{F}_{U,n}(Z_{l+1}).
\end{aligned}$$

Hence, $\hat{Q}_{U,n}(\alpha) = \hat{\omega}_n Z_l + (1-\hat{\omega}_n) Z_{l+1}$.

For an arbitrary constant $a \le 0$, we have

$$\Pr\{\hat{Q}_{U,n}(\alpha) \le Q(\alpha) + n^{-\frac{1}{2}}a\} = \Pr\left\{F[\hat{Q}_{U,n}(\alpha)] \le F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\}.$$

Following the proof in (1) with $f[Q(\alpha)]$ replaced by $f_-[Q(\alpha)]$, we have that

$$\lim_{n\to\infty} \Pr\{F(Z_m) \le c_n\} = \Phi\left\{\frac{af_-[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}}\right\}$$

for $m = l$ and $l+1$. Therefore,

$$\lim_{n\to\infty} \Pr\{f[\hat{Q}_{U,n}(\alpha), Q(\alpha)]\hat{Q}_{U,n}(\alpha) \le f[\hat{Q}_{U,n}(\alpha), Q(\alpha)][Q(\alpha) + n^{-\frac{1}{2}}a]\}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Pr\{\hat{Q}_{U,n}(\alpha) \le Q(\alpha) + n^{-\frac{1}{2}}a\} \\
&= \lim_{n\to\infty} \Pr\left\{F[\hat{Q}_{U,n}(\alpha)] \le F[Q(\alpha) + n^{-\frac{1}{2}}a]\right\} \\
&= \Phi\left\{\frac{af_-[Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}}\right\},
\end{aligned}
$$

or equivalently,

$$\lim_{n\to\infty} \Pr\left\{\frac{\sqrt{n}f[\hat{Q}_{U,n}(\alpha), Q(\alpha)][\hat{Q}_{U,n}(\alpha) - Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} \le a\right\} = \Phi(a)$$

For an arbitrary constant $a > 0$, we have

$$
\begin{aligned}
& \lim_{n\to\infty} \Pr\left\{\frac{\sqrt{n}f[\hat{Q}_{U,n}(\alpha), Q(\alpha)][\hat{Q}_{U,n}(\alpha) - Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} \le a\right\} \\
= & \lim_{n\to\infty} \Pr\left\{\frac{\sqrt{n}f[\hat{Q}_{U,n}(\alpha), Q(\alpha)][\hat{Q}_{U,n}(\alpha) - Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} \le 0\right\} \\
& + \lim_{n\to\infty} \Pr\left\{0 < \frac{\sqrt{n}f[\hat{Q}_{U,n}(\alpha), Q(\alpha)][\hat{Q}_{U,n}(\alpha) - Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} \le a\right\} \\
= & \lim_{n\to\infty} \Pr\left\{\frac{\sqrt{n}f_+[Q(\alpha)][\hat{Q}_{U,n}(\alpha) - Q(\alpha)]}{\sqrt{\alpha(1-\alpha)}} \le a\right\} \\
= & \Phi(a),
\end{aligned}
$$

where the last equality is derived by following exactly the same procedure as for the situation of $a \le 0$, and replacing $f_-[Q(\alpha)]$ by $f_+[Q(\alpha)]$. $\square$

Proof of Theorem 5: The proof of Theorem 5 follows from the similar arguments as that of Theorem 4, and is therefore omitted.

Proof of Theorem S1: By the laws of large numbers for the sample mean and the bootstrap mean (Csörgo, 1992), it holds that $r_{l,n}/n$ and $r^{\dagger}_{l,m}/m$

converge to p_l almost surely for $l = 1, 2, \cdots, L$ as n and m go to ∞.

(1) When n and m are sufficiently large, with probability tending to 1, we can find $0 < \hat{\omega}_m^{\dagger} < 1$ such that $\alpha = \hat{\omega}_m^{\dagger}(r_{0,m}^{\dagger}/m) + (1 - \hat{\omega}_m^{\dagger})(r_{1,m}^{\dagger}/m)$. Hence as n and m go to ∞, with probability tending to 1, $\hat{Q}_{U,m}^{\dagger}(\alpha) = \hat{\omega}_m^{\dagger} x_0 + (1 - \hat{\omega}_m^{\dagger}) x_1 = x_1$.

(2) When n and m are sufficiently large, with probability tending to 1, we can find $0 < \hat{\omega}_n, \hat{\omega}_m^{\dagger} < 1$ such that

$$\begin{aligned} \alpha &= \hat{\omega}_n \Big(\frac{r_{1,n} + \cdots + r_{l,n}}{n} \Big) + (1 - \hat{\omega}_n) \Big(\frac{r_{1,n} + \cdots + r_{l+1,n}}{n} \Big) \\ &= \hat{\omega}_m^{\dagger} \Big(\frac{r_{1,m}^{\dagger} + \cdots + r_{l,m}^{\dagger}}{m} \Big) + (1 - \hat{\omega}_m^{\dagger}) \Big(\frac{r_{1,m}^{\dagger} + \cdots + r_{l+1,m}^{\dagger}}{m} \Big). \end{aligned}$$

Thus

$$\hat{\omega}_n = \frac{(r_{1,n} + \cdots + r_{l+1,n})/n - \alpha}{r_{l+1,n}/n}, \quad \hat{\omega}_m^{\dagger} = \frac{(r_{1,m}^{\dagger} + \cdots + r_{l+1,m}^{\dagger})/m - \alpha}{r_{l+1,m}^{\dagger}/m},$$

and

$$\hat{Q}_{U,n}(\alpha) = \hat{\omega}_n x_l + (1 - \hat{\omega}_n) x_{l+1}, \quad \hat{Q}_{U,m}^{\dagger}(\alpha) = \hat{\omega}_m^{\dagger} x_l + (1 - \hat{\omega}_m^{\dagger}) x_{l+1}.$$

Furthermore,

$$\hat{Q}_{U,m}^{\dagger}(\alpha) - \hat{Q}_{U,n}(\alpha) = \hat{\omega}_m^{\dagger} x_l + (1 - \omega_m^{\dagger}) x_{l+1} - [\hat{\omega}_n x_l + (1 - \hat{\omega}_n) x_{l+1}] = (\hat{\omega}_m^{\dagger} - \hat{\omega}_n)(x_l - x_{l+1}),$$

where

$$\hat{\omega}_m^{\dagger} - \hat{\omega}_n = \frac{(r_{1,m}^{\dagger} + \cdots + r_{l+1,m}^{\dagger})/m - \alpha}{r_{l+1,m}^{\dagger}/m} - \frac{(r_{1,n} + \cdots + r_{l+1,n})/n - \alpha}{r_{l+1,n}/n}.$$

Because $x_l - x_{l+1}$ is constant, we only need to investigate the asymptotic normality of $\hat{\omega}_m^\dagger - \hat{\omega}_n$.

In the proof of Theorem 2, it has been shown that as n goes to ∞,

$$\sqrt{n}[(r_{1,n}, r_{2,n}, \cdots, r_{l+1,n})^T/n - (p_1, p_2, \cdots, p_{l+1})^T] \longrightarrow N(\mathbf{0}_{l+1}, \Sigma_{(l+1)\times(l+1)}),$$

in distribution, where $\mathbf{0}_{l+1}$ and $\Sigma_{(l+1)\times(l+1)}$ are specified in the proof of Theorem 2. Furthermore, by Theorem 2 of Bickel and Freedman (1981), we have

$$\sqrt{m}[(r^\dagger_{1,m}, r^\dagger_{2,m}, \cdots, r^\dagger_{l+1,m})^T/m - (r_{1,n}, r_{2,n}, \cdots, r_{l+1,n})^T/n] \xrightarrow{L} N(\mathbf{0}_{l+1}, \Sigma_{(l+1)\times(l+1)}),$$

in probability as n and m go to ∞. Once more, in the proof of Theorem 2, we have

$$\sqrt{n}(\hat{\omega}_n - \omega) \longrightarrow N\Big[0, p_{l+1}^{-2}\big[\alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]\Big],$$

in distribution as n goes to ∞. Therefore, by the theorem of delta method for bootstrap (van Der Vaart, 1998), we obtain that

$$\sqrt{m}(\hat{\omega}^\dagger_m - \hat{\omega}_n) \xrightarrow{L} N\Big[0, p_{l+1}^{-2}\big[\alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]\Big]$$

in probability as n and m go to ∞.

Thus, as n and m go to ∞,

$$\sqrt{m}[\hat{Q}^\dagger_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)] f_U[Q_U(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]$$

almost surely.

(3) Define $U_m^\dagger = \sqrt{m}\{\hat{F}_{U,n}(x_l) - \hat{F}_{U,m}^\dagger(x_l)\}$. Equivalently, $U_m^\dagger = -\frac{1}{\sqrt{m}}\sum_{i=1}^m \{I(X_i^\dagger \leq x_l) - \hat{F}_{U,n}(x_l)\}$. Similar to the proof in (2), we get that

$$U_m^\dagger \xrightarrow{L} N(0, \alpha(1-\alpha)),$$

in probability. Define $\tilde{U}_m^\dagger = \sqrt{m}\{\alpha - \hat{F}_{U,m}^\dagger(x_l)\}$. Then

$$\tilde{U}_m^\dagger = U_m^\dagger + \sqrt{m}\{\alpha - \hat{F}_{U,n}(x_l)\} = U_m^\dagger - \sqrt{\frac{m}{n}}\sqrt{n}\{\hat{F}_{U,n}(x_l) - F_U(x_l)\}.$$

Since $\sqrt{n}\{\hat{F}_{U,n}(x_l) - F_U(x_l)\} = O_p(1)$ and $m = o(n)$, it follows that $\sqrt{m}\{\alpha - \hat{F}_{U,n}(x_l)\} = o_p(1)$. Consequently, $\tilde{U}_m^\dagger \xrightarrow{L} N(0, \alpha(1-\alpha))$ in probability.

Because $F_U(x_l) - F_U(x_{l-1}) = p_l > 0$ and $F_U(x_{l+1}) - F_U(x_l) = p_{l+1} > 0$, the bootstrap law of large numbers implies that $\Pr^\dagger\{\hat{F}_{U,m}^\dagger(x_{l-1}) < \alpha < \hat{F}_{U,m}^\dagger(x_{l+1})\} \xrightarrow{p} 1$. Moreover,

$$\hat{p}_{l,m}^\dagger = \frac{r_{l,m}^\dagger}{m} \xrightarrow{p} p_l, \quad \hat{p}_{l+1,m}^\dagger = \frac{r_{l+1,m}^\dagger}{m} \xrightarrow{p} p_{l+1}.$$

On the event $\hat{F}_{U,m}^\dagger(x_{l-1}) < \alpha < \hat{F}_{U,m}^\dagger(x_{l+1})$, the definition of the bootstrap upper-quantile gives

$$\sqrt{m}\{\hat{Q}_{U,m}^\dagger(\alpha) - x_l\} = \tilde{U}_m^\dagger \frac{x_l - x_{l-1}}{\hat{p}_{l,m}^\dagger} I(\tilde{U}_m^\dagger \leq 0) + \tilde{U}_m^\dagger \frac{x_{l+1} - x_l}{\hat{p}_{l+1,m}^\dagger} I(\tilde{U}_m^\dagger > 0).$$

The function $g_l(z) = \frac{x_l - x_{l-1}}{p_l} zI(z \leq 0) + \frac{x_{l+1} - x_l}{p_{l+1}} zI(z > 0)$ is continuous. Therefore, the continuous mapping theorem and the bootstrap law of large numbers yield

$$\sqrt{m}\{\hat{Q}_{U,m}^\dagger(\alpha) - x_l\} \xrightarrow{L} \frac{x_l - x_{l-1}}{p_l} ZI(Z \leq 0) + \frac{x_{l+1} - x_l}{p_{l+1}} ZI(Z > 0)$$

in probability, where $Z \sim N[0, \alpha(1-\alpha)]$.

Finally, on the negative and positive sides of x_l, respectively,

$$f_{U,-}(x_l) = \frac{p_l}{x_l - x_{l-1}}, \quad f_{U,+}(x_l) = \frac{p_{l+1}}{x_{l+1} - x_l}.$$

Hence,

$$\sqrt{m}\{\hat{Q}^{\dagger}_{U,m}(\alpha) - x_l\} f_U[\hat{Q}^{\dagger}_{U,m}(\alpha), x_l] = \tilde{U}^{\dagger}_m \left\{ \frac{p_l}{\hat{p}^{\dagger}_{l,m}} I(\tilde{U}^{\dagger}_m \leq 0) + \frac{p_{l+1}}{\hat{p}^{\dagger}_{l+1,m}} I(\tilde{U}^{\dagger}_m > 0) \right\}.$$

Then we have

$$\sqrt{m}\{\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)\} f_U\left[\hat{Q}^{\dagger}_{U,m}(\alpha), Q_U(\alpha)\right] \xrightarrow{L} N(0, \alpha(1-\alpha))$$

in probability.

(4) Define $\tilde{U}^{\dagger}_{m,1} = \sqrt{m}\{\alpha - \hat{F}^{\dagger}_{U,m}(x_1)\}$. As in the proof of (3),

$$\tilde{U}^{\dagger}_{m,1} \xrightarrow{L} N[0, \alpha(1-\alpha)]$$

in probability. Moreover, the bootstrap law of large numbers implies that

$$\Pr{}^{\dagger}\{\alpha < \hat{F}^{\dagger}_{U,m}(x_2)\} \xrightarrow{p} 1, \quad \hat{p}^{\dagger}_{2,m} = \frac{r^{\dagger}_{2,m}}{m} \xrightarrow{p} p_2.$$

Hence, with conditional probability tending to 1,

$$\sqrt{m}[\hat{Q}^{\dagger}_{U,m}(\alpha) - x_1] = \tilde{U}^{\dagger}_{m,1} \frac{x_2 - x_1}{\hat{p}^{\dagger}_{2,m}} I(\tilde{U}^{\dagger}_{m,1} > 0).$$

Therefore,

$$\sqrt{m}[\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)] \xrightarrow{L} \frac{x_2 - x_1}{p_2} Z I(Z > 0)$$

in probability, where $Z \sim N[0, \alpha(1-\alpha)]$. □

Proof of Theorem S2: The proofs of Theorem S2 is similar to that of Theorem S1, and is therefore omitted.

Proof of Corollary S1: We prove the result in this corollary by considering different situations separately.

(1) In the first case, we consider the situation $\alpha < p_1$. By the formula of $\hat{Q}_{U,n}(\alpha)$ and $\hat{Q}^{\dagger}_{U,m}(\alpha)$ and the strong law of large numbers, we have that $\hat{Q}_{U,n}(\alpha)$ converges to $Q_U(\alpha)$ almost surely as n goes to ∞ and $\hat{Q}^{\dagger}_{U,m}(\alpha)$ strongly converges to $\hat{Q}_{U,n}(\alpha)$ almost surely as m goes to ∞. Then

$$\begin{aligned}
& \sup_{x\in R}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\le x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\le x\right)\right| \\
\le\ & \sup_{x<0}\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\le x\right)+\sup_{x<0}\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\le x\right) \\
& +\sup_{x\ge 0}\left|1-\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\le x\right)\right|+\sup_{x\ge 0}\left|1-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\le x\right)\right| \\
\le\ & \Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)<0\right)+\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)<0\right) \\
& +1-\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\le 0\right)+1-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\le 0\right) \\
=\ & \Pr{}^{\dagger}\left(\hat{Q}^{\dagger}_{U,m}(\alpha)<\hat{Q}_{U,n}(\alpha)\right)+\Pr\left(\hat{Q}_{U,n}(\alpha)<Q_U(\alpha)\right)+\Pr{}^{\dagger}\left(\hat{Q}^{\dagger}_{U,m}(\alpha)>\hat{Q}_{U,n}(\alpha)\right) \\
& +\Pr\left(\hat{Q}_{U,n}(\alpha)>Q_U(\alpha)\right) \\
=\ & o_p(1).
\end{aligned}$$

(2) Secondly, let's check the situation that $\alpha = p_1$. Note that

$$
\begin{aligned}
& \sup_{x\in R}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right| \\
\leq\ & \sup_{x\leq 0}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right| \\
& +\sup_{x>0}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right|.
\end{aligned}
$$

Besides, under $m = o(n)$, we have

$$
\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-Q_U(\alpha)\Big)-\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)=\sqrt{\frac{m}{n}}\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\right)=o_p(1),
$$

which means that $\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-Q_U(\alpha)\Big)$ and $\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)$ have the same limiting distribution in probability as m goes to ∞. With the aid of this fact, similar to the proof of Pólya's theorem, we could show that

$$
\sup_{x>0}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right|=o_p(1).
$$

Furthermore,

$$
\begin{aligned}
& \sup_{x\leq 0}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right| \\
\leq\ & \sup_{x<0}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq x\right)\right| \\
& +\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq 0\right)-\frac{1}{2}+\frac{1}{2}-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq 0\right)\right| \\
\leq\ & \Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big)\leq 0\right)+\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big)\leq 0\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2} - \Pr^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\Big) \leq 0\right) + \frac{1}{2} - \Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\Big) \leq 0\right) \\
\leq\ & \Pr^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\Big) < 0\right) + \Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\Big) < 0\right) + o_p(1) \\
=\ & \Pr^{\dagger}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha) < \hat{Q}_{U,n}(\alpha)\Big) + \Pr\Big(\hat{Q}_{U,n}(\alpha) < Q_U(\alpha)\Big) + o_p(1) \\
=\ & o_p(1).
\end{aligned}
$$

Thus, we could conclude that

$$
\sup_{x\in R}\left|\Pr^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big) \leq x\right) - \Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big) \leq x\right)\right| = o_p(1).
$$

(3) Thirdly, we show the scenario that $\alpha = \omega p^*_l + (1-\omega)p^*_{l+1}$ for some $1 \leq l \leq L-1$ and $0 < \omega < 1$. In Theorems 2 and S1, we have proven that

$$
\sqrt{n}\Big[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\Big] f_U[Q_U(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]
$$

as n goes to ∞, and

$$
\sqrt{m}\Big[\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\Big] f_U[Q_U(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha) + \omega^2 p_{l+1} - \omega p_{l+1}\big]
$$

in probability as n and m goes to ∞. Therefore, by the Pólya's theorem, we get

$$
\sup_{x\in R}\left|\Pr^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big) \leq x\right) - \Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big) \leq x\right)\right| = o_p(1).
$$

(4) Finally, we consider the case that $\alpha = p^*_l$ for some $2 \leq l \leq L$. Once more, according to Theorems 2 and S1, we have that

$$
\sqrt{n}\Big[\hat{Q}_{U,n}(\alpha) - Q_U(\alpha)\Big] f_U[\hat{Q}_{U,n}(\alpha), Q_U(\alpha)] \xrightarrow{L} N\big[0, \alpha(1-\alpha)\big]
$$

as n goes to ∞, and

$$\sqrt{m}\Big[\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)\Big] f_U(\hat{Q}^{\dagger}_{U,m}(\alpha), \hat{Q}_{U,n}(\alpha)) \xrightarrow{L} N\big[0, \alpha(1-\alpha)\big]$$

in probability as n and m goes to ∞. Recall the fact that $\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha) - Q_U(\alpha)\Big)$ and $\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha) - \hat{Q}_{U,n}(\alpha)\Big)$ have the same limiting distribution in probability as $m = o(n)$, n and m goes to ∞. This fact, together with the Pólya's theorem, indicates that

$$\sup_{x\in R}\left|\Pr{}^{\dagger}\left(\sqrt{m}\Big(\hat{Q}^{\dagger}_{U,m}(\alpha)-\hat{Q}_{U,n}(\alpha)\Big) \le x\right)-\Pr\left(\sqrt{n}\Big(\hat{Q}_{U,n}(\alpha)-Q_U(\alpha)\Big) \le x\right)\right| = o_p(1).$$

□

Proof of Corollary S2: The proof of this corollary is similar to that of Corollary S1. Thus, we simply omit it here. □

Proof of Proposition S1: (1) If $l_1 = 0$, it is easy to see that $Q_U(\alpha) = Q_M(\alpha) = x_1$. Therefore, the additional leftward extension is zero. If $1 \le l_1 < L$, from the definition of upper-quantile and mid-quantile, we have that both $Q_U(\alpha)$ and $Q_M(\alpha)$ lie in $[x_{l_1}, x_{l_1+1}]$. To be specific, $Q_U(\alpha) = \{\omega_1 x_{l_1} + (1-\omega_1)x_{l_1+1}\}$, $Q_M(\alpha) = \{\omega_2 x_{l_1} + (1-\omega_2)x_{l_1+1}\}$, where $0 \le \omega_1 \le 1$ and $0 \le \omega_2 \le 1$. Moreover, from the fact that $Q_U(\alpha) \le Q_M(\alpha)$, we obtain that $\omega_1 \ge \omega_2$. Therefore, the additional leftward extension is $Q_M(\alpha) - Q_U(\alpha) = (\omega_1 - \omega_2)(x_{l_1+1} - x_{l_1})$.

(2) If $l_2 = L$, it is easy to see that $Q_M(1-\alpha) = Q_L(1-\alpha) = x_L$.

Therefore, the additional rightward extension is zero. If $1 \leq l_2 < L$, from the definition of lower-quantile and mid-quantile, we have that both $Q_M(1-\alpha)$ and $Q_L(1-\alpha)$ lie in $[x_{l_2}, x_{l_2+1}]$. To be specific, $Q_M(1-\alpha) = \{\omega_3 x_{l_2} + (1-\omega_3)x_{l_2+1}\}$ and $Q_L(1-\alpha) = \{\omega_4 x_{l_2} + (1-\omega_4)x_{l_2+1}\}$, where $0 \leq \omega_4 \leq 1$ and $0 \leq \omega_3 \leq 1$. Moreover, from the fact that $Q_M(1-\alpha) \leq Q_L(1-\alpha)$, we obtain that $\omega_3 \geq \omega_4$. Therefore, the additional rightward extension is $Q_L(1-\alpha) - Q_M(1-\alpha) = (\omega_3 - \omega_4)(x_{l_2+1} - x_{l_2})$.

(3) It is noted that $\{Q_L(1-\alpha/2) - Q_U(\alpha/2)\} - \{Q_M(1-\alpha/2) - Q_M(\alpha/2)\} = \{Q_M(\alpha/2) - Q_U(\alpha/2)\} + \{Q_L(1-\alpha/2) - Q_M(1-\alpha/2)\}$. Then, the conclusion follows immediately by substituting α with $\alpha/2$ in the (1) and (2). $\square$

S4 Additional illustrations and simulation studies

S4.1 Illustration of Propositions 2 and 3

In Propositions 2 and 3, we have proved that under some circumstances the mid-quantile could not be used to construct prediction interval for X with guaranteed coverage probability, while the upper- and lower-quantiles could overcome this problem. The following example presents three concrete cases to illustrate Propositions 2 and 3, and highlight the importance of our upper- and lower-quantiles. For simplicity, we only display the results for two-sided intervals and focus on the failures of mid-quantile, that is, circumstances in which mid-quantiles could not provide intervals with guaranteed coverage probabilities. In all these cases, we aim to construct one interval with 95% coverage probability.

Example S1 (a) The first case is an artificial discrete distribution. Suppose that X is a discrete random variable with possible values and corresponding probability masses $1, 2, 3, 4, 5, 6, 7$ and $0.01, 0.025, 0.31, 0.31, 0.31, 0.025, 0.01$, respectively. After some tedious calculations, we obtain $Q_M(0.025) = 2.0149$, $Q_M(0.975) = 5.9851$, $Q_U(0.025) = 1.6$, and $Q_L(0.975) = 6.4$ with

$$\Pr(Q_M(0.025) \leq X \leq Q_M(0.975)) = 0.93 < 0.95$$

and

$$\Pr(Q_U(0.025) \leq X \leq Q_L(0.975)) = 0.98 > 0.95.$$

(b) The second case is a Binomial distribution. Assume that the random variable X follows the Binomial distribution $B(16, 0.5)$. Then we could acquire $Q_M(0.025) \approx 4.0101$, $Q_M(0.975) \approx 11.9899$, $Q_U(0.025) \approx 3.5172$, and $Q_L(0.975) \approx 12.4828$ with

$$\Pr(Q_M(0.025) \leq X \leq Q_M(0.975)) \approx 0.9232 < 0.95$$

and

$$\Pr(Q_U(0.025) \leq X \leq Q_L(0.975)) \approx 0.9787 > 0.95.$$

(c) The third case concerns a Poisson distribution. Suppose that X has a Poisson distribution with mean 5. Direct calculations lead to $Q_M(0.025) \approx 1.0337$, $Q_M(0.975) \approx 9.9177$, $Q_U(0.025) \approx 0.5421$, and $Q_L(0.975) \approx 10.3766$ with

$$\Pr(Q_M(0.025) \leq X \leq Q_M(0.975)) \approx 0.9277 < 0.95$$

and

$$\Pr(Q_U(0.025) \leq X \leq Q_L(0.975)) \approx 0.9796 > 0.95.$$

From the results in Example S1, we could clearly see that the mid-quantile may not produce an interval with guaranteed coverage probability, while our upper- and lower-quantiles could.

S4.2 Additional simulation studies to evaluate of the established theoretical results

In this subsection, we provide two additional numerical examples to evaluate the theoretical results in both the main body of this article and the supplementary material.

Example S2 In the first example, we revisit Example S1 to compare the empirical coverage probabilities of prediction intervals based on mid-

Table S1: Empirical coverage probabilities in Example S2.

	Case (a)	Case (b)	Case (c)
Mid	0.9346	0.9303	0.9277
UL	0.9800	0.9787	0.9696

quantiles, upper- and lower-quantiles. In these three cases, we set the sample size $n = 5,000$, 100,000 and 5,000, respectively. Based on 500 replications, we obtain empirical 95% coverage probabilities. The results are listed in Table S1.

Table S1 implies that the empirical coverage probabilities of intervals based on mid-quantiles, upper- and lower-quantiles are consistent with the corresponding theoretical coverage probabilities presented in Example S1.

In this example, the mid-quantiles $Q_M(0.025)$s and $Q_M(0.975)$s are very close to the support points of the discrete random variables. For example, in Case (a), $Q_M(0.025) = 2.0149$ is very close to support point 2. Besides, the CDFs are discontinuous at the support points. Therefore, in order to estimate the coverage probabilities as accurately as possible, we need to acquire rather accurate estimators of $Q_M(0.025)$s and $Q_M(0.975)$s. This is the reason why the sample sizes are very large in this example.

In the next example, we examine the performances of the bootstrapping procedure and evaluate its theoretical properties.

Example S3 In this example, we aim to construct 95% confidence interval for upper- and lower-medians of binomial distribution $\mathrm{Bin}(\mathrm{L}, 0.5)$ with $L = 2, 3, 6, 7, 10, 11$ via the bootstrapping procedure. $L = 2, 6, 10$ are corresponding to Case 2 of Theorems 2 and 3, while $L = 3, 7, 11$ belong to Case 3. For the m-out-of-n bootstrap, we choose $m = n^{2/3}, n^{3/4}, n^{4/5}$, which all satisfy $m = o(n)$. Moreover, the sample size n is set to be 500, 1000 and 2000. All simulation results are based on 1000 data repetitions with 1000 bootstrap replications in each repetition.

The empirical 95% coverage probabilities for upper-medians are listed in Table S2 and imply that the empirical coverage probabilities fluctuate around the nominal confidence level 95%, and the bootstrap method behaves equally well as the asymptotically normal approximation. We do not exhibit the empirical coverage probabilities for lower-medians here. This is because that these results are almost the same as those of upper-medians.

Table S2: Results for the bootstrap confidence intervals for upper-median in Example S3.

m	n 500	1000	2000	500	1000	2000
		$L=2$			$L=3$	
$n^{2/3}$	0.958	0.970	0.957	0.908	0.933	0.934
$n^{3/4}$	0.957	0.966	0.956	0.912	0.934	0.943
$n^{4/5}$	0.958	0.968	0.955	0.917	0.939	0.946
CLT approx.	0.951	0.966	0.955	0.943	0.960	0.964
		$L=6$			$L=7$	
$n^{2/3}$	0.970	0.959	0.964	0.933	0.921	0.933
$n^{3/4}$	0.967	0.957	0.963	0.933	0.923	0.940
$n^{4/5}$	0.964	0.958	0.962	0.938	0.929	0.942
CLT approx.	0.956	0.953	0.962	0.955	0.942	0.957
		$L=10$			$L=11$	
$n^{2/3}$	0.971	0.977	0.975	0.943	0.943	0.926
$n^{3/4}$	0.968	0.971	0.971	0.944	0.947	0.928
$n^{4/5}$	0.968	0.970	0.967	0.939	0.949	0.931
CLT approx.	0.960	0.964	0.965	0.952	0.959	0.942

[1] CLT approx. represents the method of asymptotically normal approximation.

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