Supplementary Materials

This supplementary file includes selected graphs for the simulation example, and proof and detailed derivations for the theorems and remarks.

Selected figures from the simulation examples.

Figure S1: This figure presents the histograms of the estimator of off-diagonal (panel (A)) and diagonal (panel (B)) components of the sample matrix $\hat{\tau}_{\mathbf{X}}$, $\widehat{\tau}_{X_1,X_2}$ and those for $\hat{\tau}^*_{\mathbf{X}}$, $\widehat{\tau}^*_{X_1,X_2}$ (panels (C), (D)), of a random sample from sαs bivariate random vector with independent components and parameters $\alpha = 0.7,$ and $\sigma_{11} = (0.5, 0.75)$.

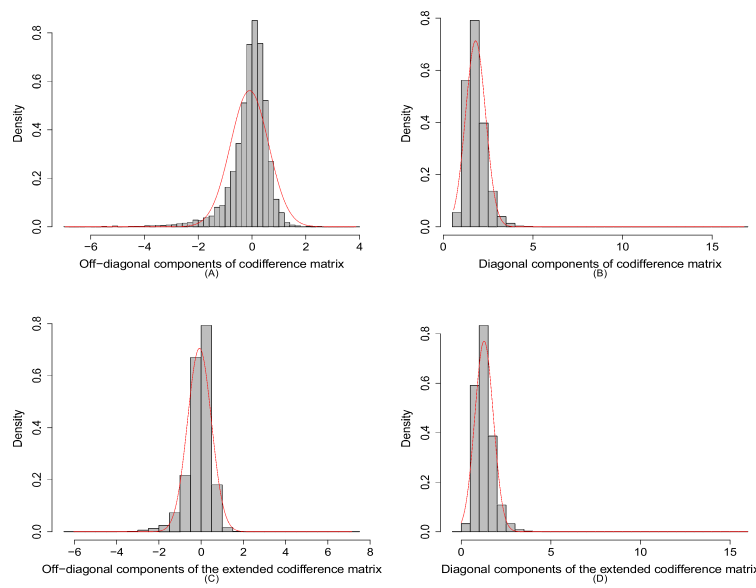


Figure S2: This figure presents the histograms of estimator of codifference, i.e. $\hat{\tau}_{X_1,X_2}$, (panel A) and extended codifference, i.e. $\hat{\tau}^*_{X_1,X_2}$, (panel B), of a random sample from sαs bivariate random vector with independent components and parameters $\alpha = 1.1$, and $\sigma_{11} = (0.5, 0.75)$.

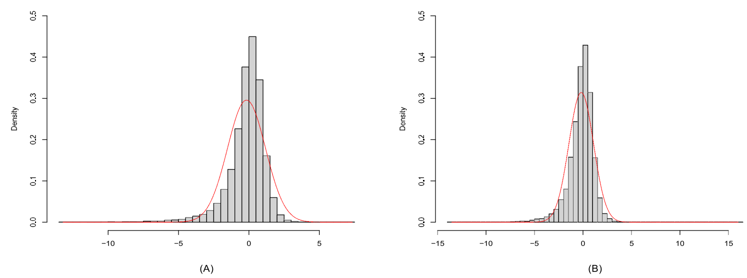


Figure S3: Histograms of off-diagonal (panel (A)) and diagonal (panel (B)) components of the sample matrix $\widehat{\tau}_{X_1,X_2}$, and corresponding components of $\widehat{\tau}^*_{X_1,X_2}$ (panels (C) and (D)) for sub-Gaussian random vectors.

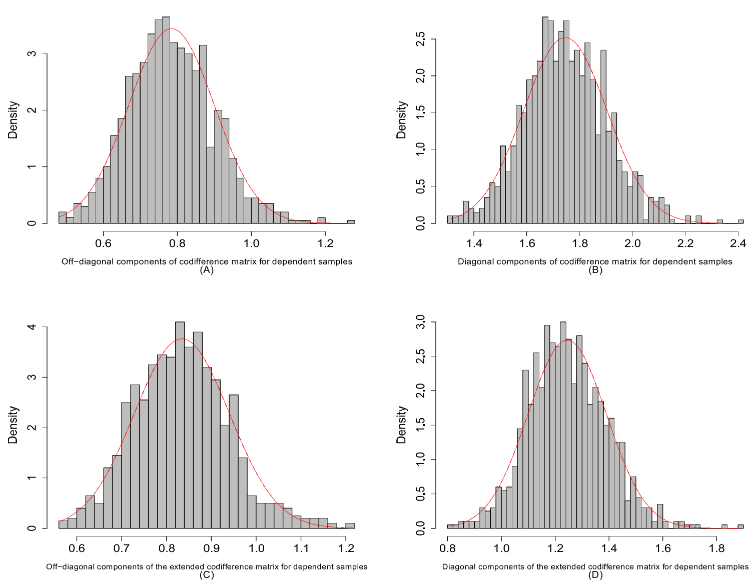


Figure S4: Confidence region for the estimation of the parameters (β_1, β_2): panel (A) corresponds to $\gamma_{12} = -0.3$, and panel (B) to $\gamma_{12} = 0$.

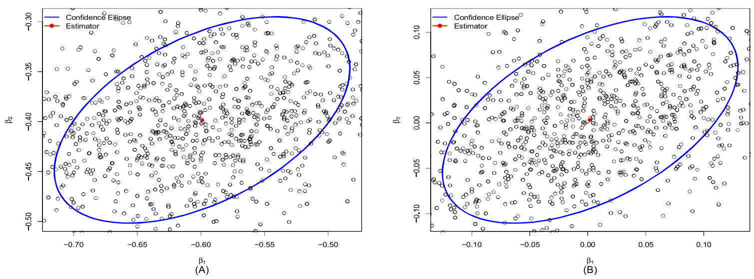
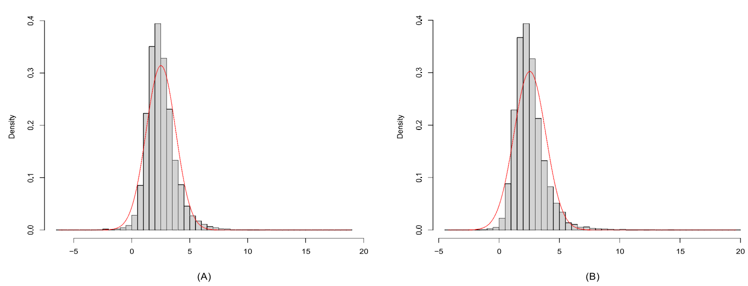


Figure S5: (A): $\widehat{\tau}_{X_1,X_2}$, and (B): $\widehat{\tau}^*_{X_1,X_2}$, for sub-Gaussian sample



Proof of results of Section 2

Proof of Theorem 2.9.

Proof. By Definition

$$
\begin{aligned}
&\|X_1 - X_2\|_\alpha^\alpha + \|X_1 + X_2\|_\alpha^\alpha \\
&= \int_{SP_2} |s_1 - s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) + \int_{SP_2} |s_1 + s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) \,.
\end{aligned}
$$

Suppose that X and Y are independent, then by Example 2.3.5 in Samorodnitsky and Taqqu (1994), $s_1 \, s_2 = 0$, Γ_{X_1,X_2} a.e., therefore

$$
\begin{aligned}
&\|X_1 - X_2\|_\alpha^\alpha + \|X_1 + X_2\|_\alpha^\alpha \\
&= 2 \left(\int_{SP_2} |s_1|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) + \int_{SP_2} |s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) \right) \\
&= 2 \left(\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha \right) \,.
\end{aligned}
$$

This shows that if (X_1, X_2) are two independent SαS random variables, then $\tau^*_{X_1,X_2} = 0$. Now, suppose that $\tau^*_{X_1,X_2} = 0$ and $0 < \alpha < 2$, then by (2.7.9) in Samorodnitsky and Taqqu (1994),

$$
\begin{aligned}
&\|X_1 - X_2\|_\alpha^\alpha + \|X_1 + X_2\|_\alpha^\alpha \\
&= \int_{SP_2} |s_1 - s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) + \int_{SP_2} |s_1 + s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) \\
&\leq 2 \left(\int_{SP_2} |s_1|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) + \int_{SP_2} |s_2|^\alpha \, \Gamma_{X_1,X_2}(d\mathbf{s}) \right) \\
&= 2 \left(\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha \right) \,.
\end{aligned}
$$

The equality holds if and only if $s_1 s_2 = 0$, a.e.-Γ_{X_1,X_2}. Therefore, again by Example 2.3.5 in Samorodnitsky and Taqqu (1994) we can conclude that X is independent from Y.

□

Proof of Theorem 2.11. Let us first provide more details on the properties imposed in Definition 2.7, then we prove the result. If X_1, X_2 are two independent SαS random variables, then -as shown in Example 2.3.5- the corresponding spectral measure concentrated on 4 points $(0,1), (1,0), (0,-1)$, and $(-1,0)$ and therefore we expect that the spectral covariance function becomes zero, i.e.

$$\int_{SP_2} f(s_1, s_2)\,\Gamma(d\mathbf{s}) = 0\,.$$

Equivalently for non-negative constants a_i, $i = 1, \ldots, 4$,

$$a_1\, f(0,1) + a_2\, f(1,0) + a_3\, f(0,-1) + a_4\, f(-1,0) = 0\,. \tag{6.31}$$

For $i = 1, 2$, let β_i and σ_i denote the skewness and scale parameters of X_i, respectively. Then the coefficients a_i, for $i = 1, \ldots, 4$, as given in equation (6.31), can be reduced to:

$$a_1 = \sigma_1^\alpha \frac{1+\beta_1}{2}\,,\; a_2 = \sigma_1^\alpha \frac{1-\beta_1}{2}\,,\; a_3 = \sigma_2^\alpha \frac{1+\beta_2}{2}\,,\; a_4 = \sigma_2^\alpha \frac{1-\beta_2}{2}\,.$$

Now, let us consider the property (iii) of Definition 2.7 in more detail. Random variables $X_1, \ldots, X_n$ are said to be associated if, for any functions $f, g : \mathbb{R}^n \to \mathbb{R}$ which are non-decreasing in each argument,

$$Cov\,(f(X_1,\dots,X_n), g(X_1,\dots,X_n)) \geq 0\,,$$

whenever the covariance exists. From Theorem 4.6.1 in Samorodnitsky and Taqqu (1994), if $X_1,\dots,X_n$ are jointly α-stable random variables, $0<\alpha<2$, with spectral measure Γ, on the unit sphere SP_n in $\mathbb{R}^n$, then $X_1,\dots,X_n$ are associated if and only if $\Gamma(SP_n^-)=0$, where

$$SP_n^- = \{(s_1,\dots,s_n)\in SP_n:\ \text{for some } i,j\in\{1,\dots,n\}, s_i>0, s_j<0\}\ .$$

Therefore, from the above arguments and Theorem 4.6.3. in Samorodnitsky and Taqqu (1994), property (iii) holds if $\int_{SP_2} f(s_1,s_2)\,\Gamma(d\mathbf{s})$ is the measure of dependence of the bivariate SαS random vector $\mathbf{X}$. Therefore,

$$\Gamma(SP_n^-)=0\ \Rightarrow\ \int_{SP_2} f(s_1,s_2)\,\Gamma(d\mathbf{s})\geq 0$$

and

$$\Gamma(SP_n^+)=0\ \Rightarrow\ \int_{SP_2} f(s_1,s_2)\,\Gamma(d\mathbf{s})\leq 0\,, \tag{6.32}$$

where

$$SP_n^+ = \{(s_1,\dots,s_n)\in SP_n:\ \text{for some } i,j\in\{1,\dots,n\}, s_i\,s_j>0\}\ .$$

Now, we provide the proof for Theorem 2.11.

Proof. The first two measures obviously satisfy the conditions (2.2) and (6.32). Since for $\alpha=2$, these measures of dependence reduce to $[X_1,X_2]_2=\frac{1}{2}\,Cov(X_1,X_2)$, and condition (ii) does not hold for them. Then, an argument similar to Example 2.7.12 in Samorodnitsky and Taqqu (1994) can be applied to show that the condition (6.31) does not hold for these measures of dependence.

Obviously, the covariation norm neither satisfies (2.2) nor (6.31). See Example 2.7.12 in Samorodnitsky and Taqqu (1994). Since $[X_1, X_2]_2 = \frac{1}{2}Cov(X_1, X_2)$, the condition (ii) does not hold but satisfies (6.32).

We now consider the (extended) codifference as a measure of dependence. Obviously, Theorem 2.9 implies (6.31). Since the codifference reduces to the covariance function for the case $\alpha = 2$, condition (ii) holds. Based on Equations (2.7.7) and (2.7.9) in Samorodnitsky and Taqqu (1994), $\tau^*_{X_1,X_2} \geq 0$, therefore condition (6.32) does not hold for this case.

□

7. Proof of results given in Section 3

Proof of Theorem 3.1.

Proof. Obviously, $\frac{\sqrt{n}}{n}\sum_{j=1}^{n}\left(\cos(X_j) - \mu_{1;\tau})\right) \sim N(0, \sigma^2_{1;\tau})$ for sufficiently large n, where $\mu_{1;\tau} = \mathbb{E}(\cos(X))$ and $\sigma^2_{1;\tau} = V(\cos(X))$. Moreover, $\mathbb{E}\left(\cos X\right) = \mathbb{E}\left(\exp(iX)\right) = \exp(-\sigma^\alpha)$, and

$$\exp(-2^\alpha\sigma^\alpha) = \mathbb{E}\left(\exp(2iX)\right) = 2\mathbb{E}\left(\cos^2 X\right) - 1 .$$

Therefore $V(\cos X) = \frac{1}{2}(\exp(-2^\alpha\sigma^\alpha)+1) - \exp(-2\sigma^\alpha)$. Again using CLT,

$$\sqrt{n}\left(\begin{pmatrix} \frac{1}{n}\sum_{l=1}^n \cos(X_{1l}) \\ \frac{1}{n}\sum_{l=1}^n \cos(2X_{1l}) \end{pmatrix} - \begin{pmatrix} \mu_{1;\tau} \\ \mu_{*1;\tau} \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}_2\left(\mathbf{0}, \mathbf{\Sigma}_{2;\tau}\right) ,$$

where

$$\mathbf{\Sigma}_{2;\tau} = \begin{pmatrix} \sigma_{1;\tau} & \sigma_{12;\tau} \\ \sigma_{12;\tau} & \sigma_{22;\tau} \end{pmatrix} , \tag{7.33}$$

in which

$$\sigma_{22;\tau} = \frac{1}{2}\left(\exp(-4^\alpha\sigma^\alpha)+1\right) - \exp(-2(2\sigma)^\alpha)$$

and

$$\sigma_{12;\tau} = \frac{1}{2}\left(\exp(-3^\alpha\sigma^\alpha)+\exp(-\sigma^\alpha)\right) - \exp\left(-(2\sigma)^\alpha - \sigma^\alpha\right) .$$

The last equality follows from the following argument.

$$E(\cos(3X)) = E(\exp(iX)\exp(i2X)) = E(\cos(X)\cos(2X)) - E(\sin(X)\sin(2X))$$
$$= E(\cos(X)\cos(2X)) - \frac{1}{2}E(\cos(X)) + \frac{1}{2}E(\cos(3X)) .$$

Obviously, for large values of n , $\frac{1}{n}\sum_{l=1}^n \cos(X_{1l})$ and $\frac{1}{n}\sum_{l=1}^n \cos(2X_{1l})$ are both eventually non-negative. Therefore (3.14) and (3.15) follow using the Delta method.

$$\frac{\sqrt{n}}{\sigma_{1;\tau}}(\hat{\tau}_{X,X} - \tau_{X,X}) \xrightarrow{d} \mathcal{N}(0,1) ,$$
$$\frac{\sqrt{n}}{\sigma_{2;\tau}}(\hat{\tau}^*_{X,X} - \tau^*_{X,X}) \xrightarrow{d} \mathcal{N}(0,1) ,$$

where

$$\sigma_{1;\tau} = \frac{4\sigma_{11;\tau}}{\mu^2_{1;\tau}} , \quad \sigma_{2;\tau} = \boldsymbol{\sigma}_{2;\tau}\mathbf{\Sigma}_{2;\tau}\boldsymbol{\sigma}^\top_{2;\tau} , \tag{7.34}$$

$\mu_{1;\tau} = \exp(-\sigma^\alpha)$, $\sigma^2_{11;\tau} = \frac{1}{2}(\exp(-2^\alpha\sigma^\alpha)+1) - \exp(-2\sigma^\alpha)$, $\boldsymbol{\sigma}_{2;\tau} = (-2\mu^{-1}_{1;\tau}, \frac{1}{2}\mu^{-1}_{*1;\tau})$, $\mu_{*1;\tau} = \exp(-2^\alpha\sigma^\alpha)$, and $\boldsymbol{\Sigma}_{2;\tau}$ is given in (7.33). This completes the proof.

□

Proof of Theorem 3.2

Proof. Let

$$A_n = \frac{1}{n}\sum_{l=1}^n \cos X_{1l}\,, \ \ B_n = \frac{1}{n}\sum_{l=1}^n \cos X_{2l}\,,$$
$$C_n = \frac{1}{n}\sum_{l=1}^n \cos(X_{1l}-X_{2l})\,, \quad D_n = \frac{1}{n}\sum_{l=1}^n \cos(X_{1l}+X_{2l})\,.$$

Then by CLT, $\sqrt{n}\left((A_n,B_n,C_n,D_n)^\top - \boldsymbol{\mu}_\tau^\top\right) \xrightarrow{d} \mathcal{N}_4\left(\mathbf{0},\Sigma_\tau\right)$, where

$$\boldsymbol{\mu}_\tau^\top = (\mu_1,\mu_2,\mu_3,\mu_4) = \left(e^{-\sigma^\alpha_{X_1}}, e^{-\sigma^\alpha_{X_2}}, e^{-\sigma^\alpha_{X_1-X_2}}, e^{-\sigma^\alpha_{X_1+X_2}}\right)\,. \tag{7.35}$$

$$\Sigma_\tau = (\sigma_{ij;\tau})_{i,j=1,\ldots,4}\,, \tag{7.36}$$
$$\sigma_{ii;\tau} = \frac{1}{2}\left(\exp(-2^\alpha\sigma^\alpha_{ii})+1\right) - \exp(-2\sigma^\alpha_{ii})\,, \quad i=1,\ldots,4\,,$$
$$\sigma_{12;\tau} = \sigma_{21;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{X_1+X_2}) + \exp(-\sigma^\alpha_{X_1-X_2})\right) - \exp(-\sigma^\alpha_{X_1} - \sigma^\alpha_{X_2})\,,$$
$$\sigma_{13;\tau} = \sigma_{31;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{2X_1-X_2}) + \exp(-\sigma^\alpha_{X_2})\right) - \exp(-\sigma^\alpha_{X_1} - \sigma^\alpha_{X_1-X_2})\,,$$
$$\sigma_{14;\tau} = \sigma_{41;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{2X_1+X_2}) + \exp(-\sigma^\alpha_{X_2})\right) - \exp(-\sigma^\alpha_{X_1} - \sigma^\alpha_{X_1+X_2})\,,$$
$$\sigma_{23;\tau} = \sigma_{32;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{2X_2-X_1}) + \exp(-\sigma^\alpha_{X_1})\right) - \exp(-\sigma^\alpha_{X_2} - \sigma^\alpha_{X_1-X_2})\,,$$
$$\sigma_{24;\tau} = \sigma_{42;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{2X_2+X_1}) + \exp(-\sigma^\alpha_{X_1})\right) - \exp(-\sigma^\alpha_{X_2} - \sigma^\alpha_{X_1+X_2})\,,$$
$$\sigma_{34;\tau} = \sigma_{43;\tau} = \frac{1}{2}\left(\exp(-\sigma^\alpha_{2X_2}) + \exp(-\sigma^\alpha_{2X_1})\right) - \exp(-\sigma^\alpha_{X_1-X_2} - \sigma^\alpha_{X_1+X_2})\,,$$
$$\sigma_{11} = \sigma_X = \int_{S_1} |s_1|\Gamma(d\mathbf{s})\,, \quad \sigma_{22} = \sigma_{X_2} = \int_{S_1} |s_2|\Gamma(d\mathbf{s})\,,$$
$$\sigma_{33} = \sigma_{X_1-X_2} = \int_{S_1} |s_1-s_2|\Gamma(d\mathbf{s})\,, \quad \sigma_{44} = \sigma_{X_1+X_2} = \int_{S_1} |s_1+s_2|\Gamma(d\mathbf{s})\,,$$

Using the Delta method, we get,

$$\frac{\sqrt{n}}{\sigma_{3;\tau}}\left(\ln\left(\frac{C_n}{A_nB_n}\right)-\tau_{X_1,X_2}\right)\xrightarrow{d}\mathcal{N}(0,1)\,,$$

and

$$\frac{\sqrt{n}}{\sigma_{4;\tau}}\left(\frac{1}{2}\ln\left(\frac{C_nD_n}{(A_nB_n)^2}\right)-\tau^*_{X_1,X_2}\right)\xrightarrow{d}\mathcal{N}(0,1)\,,$$

where

$$\begin{aligned}
&\sigma^2_{3;\tau}=\boldsymbol{\sigma}_{3;\tau}\Sigma_\tau\boldsymbol{\sigma}^\top_{3;\tau}\,,\quad \sigma^2_{4;\tau}=\boldsymbol{\sigma}_{4;\tau}\Sigma_\tau\boldsymbol{\sigma}^\top_{4;\tau}\,,\\
&\boldsymbol{\sigma}_{3;\tau}=\left(-\mu_1^{-1},-\mu_2^{-1},\mu_3^{-1},0\right)\\
&\boldsymbol{\sigma}_{4;\tau}=\left(-\mu_1^{-1},-\mu_2^{-1},\frac{1}{2}\mu_3^{-1},\frac{1}{2}\mu_4^{-1}\right)\,.
\end{aligned}\tag{7.37}$$

This completes the proof.

□

Proof of Theorem 3.3.

Proof. Let $\hat{\vartheta}_{ij;-}=\frac{1}{n}\sum_{k=1}^n\cos(X_{ki}-X_{kj})$, and $\hat{\vartheta}_i=\frac{1}{n}\sum_{k=1}^n\cos(X_{ki})$, then by CLT,

$$\sqrt{n}\left(\begin{pmatrix}\hat{\vartheta}_1\\ \vdots\\ \hat{\vartheta}_p\\ \hat{\vartheta}_{12;-}\\ \vdots\\ \hat{\vartheta}_{p-1,p;-}\end{pmatrix}-\begin{pmatrix}\exp(-\sigma_1^\alpha)\\ \vdots\\ \exp(-\sigma_p^\alpha)\\ \exp(-\sigma^\alpha_{1,-2})\\ \vdots\\ \exp(-\sigma^\alpha_{p-1,-p})\end{pmatrix}\right)\xrightarrow{d}\mathcal{N}_{\frac{1}{2}(p^2+p)}\left(\mathbf{0},\boldsymbol{\Sigma}_{1;\vartheta}\right)\,,\tag{7.38}$$

where $\sigma_i^\alpha = \int_{\mathcal{S}_d} |u_i|^\alpha \Gamma(d\mathbf{u})$, $\sigma_{i,\pm j}^\alpha = \int_{\mathcal{S}_d} |u_i \pm u_j|^\alpha \Gamma(d\mathbf{u})$,

$$\boldsymbol{\Sigma}_{1;\vartheta} = \begin{pmatrix} \boldsymbol{\Sigma}_{11;\vartheta} & \boldsymbol{\Sigma}_{12;\vartheta} \\ \boldsymbol{\Sigma}_{12;\vartheta}^\top & \boldsymbol{\Sigma}_{22;\vartheta} \end{pmatrix}, \tag{7.39}$$

in which $\boldsymbol{\Sigma}_{11;\vartheta}$ is a $p \times p$ matrix whose i-th diagonal components is $\frac{1}{2}(e^{-2^\alpha \sigma_i^\alpha} + 1) - e^{-2\sigma_i^\alpha}$ and (i,j)-th off-diagonal components equal $\frac{1}{2}(e^{-\sigma_{i,+j}^\alpha} + e^{-\sigma_{i,-j}^\alpha}) - e^{-\sigma_i^\alpha - \sigma_j^\alpha}$. To introduce the $\frac{1}{2}(p^2 - p) \times \frac{1}{2}(p^2 - p)$ matrix $\boldsymbol{\Sigma}_{22;\vartheta}$, we need to introduce the notation $\sigma_{ai,bj,cl}^\alpha = \int_{\mathcal{S}_d} |au_i + bu_j + cu_l|^\alpha \Gamma(d\mathbf{u})$. Using this notation the i-th components of the diagonal of $\boldsymbol{\Sigma}_{22;\vartheta}$ which corresponds to $\hat{\vartheta}_{lk;-}$ equals $\frac{1}{2}(e^{-2^\alpha \sigma_{l,-k}^\alpha} + 1) - e^{-2\sigma_{l,-k}^\alpha}$, and the (i,j)-th entries of off-diagonal components correspond to $\hat{\vartheta}_{l_1 k_1;-}$ and $\hat{\vartheta}_{l_2 k_2;-}$ is a_{l_1,k_1,l_2,k_2}, that can be expressed as

$$a_{l_1,k_1,l_2,k_2} = \begin{cases} 0, & \text{if } \; l_1 \neq l_2,\; \& \; k_1 \neq k_2, \\ \frac{1}{2}\left(e^{-\sigma_{2l_1,-k_1,-k_2}^\alpha + \sigma_{k_1,-k_2}^\alpha}\right) - e^{-\sigma_{l_1,-k_1}^\alpha - \sigma_{l_1,-k_2}^\alpha}, & \text{if } \; l_1 = l_2,\; \& \; k_1 \neq k_2, \\ \frac{1}{2}\left(e^{-\sigma_{2k_1,-l_1,-l_2}^\alpha + \sigma_{l_1,-l_2}^\alpha}\right) - e^{-\sigma_{l_1,-k_1}^\alpha - \sigma_{l_2,-k_1}^\alpha}, & \text{if } \; l_1 \neq l_2,\; \& \; k_1 = k_2. \end{cases}$$

In addition, the (i,j)-th components of $\boldsymbol{\Sigma}_{12;\vartheta}$, corresponding to $\hat{\vartheta}_i$ and $\hat{\vartheta}_{jk;-}$ may be represented by a_{ijk}, as follows

$$a_{i,j,k} = \begin{cases} 0, & \text{if } \; i \neq j,\; \& \; i \neq k, \\ \frac{1}{2}(e^{-\sigma_i^\alpha} + e^{-\sigma_{2k,-i}^\alpha}) - e^{-\sigma_k^\alpha - \sigma_{i,-k}^\alpha}, & \text{if } \; i = j,\; \& \; i \neq k, \\ \frac{1}{2}(e^{-\sigma_i^\alpha} + e^{-\sigma_{2j,-i}^\alpha}) - e^{-\sigma_j^\alpha - \sigma_{i,-j}^\alpha}, & \text{if } \; i \neq j,\; \& \; i = k. \end{cases}$$

Now, introduce the function

$$\begin{aligned} & g_1\left(\hat{\vartheta}_1, \ldots, \hat{\vartheta}_p, \hat{\vartheta}_{12;-}, \ldots, \hat{\vartheta}_{p-1,p;-}\right) \\ & = \left(\ln \hat{\vartheta}_1^{-2}, \ldots, \ln \hat{\vartheta}_p^{-2}, \ln \frac{\hat{\vartheta}_{12;-}}{\hat{\vartheta}_1 \hat{\vartheta}_2}, \ldots, \ln \frac{\hat{\vartheta}_{p-1,p;-}}{\hat{\vartheta}_{p-1} \hat{\vartheta}_p}\right). \end{aligned}$$

The gradient of g_1 is

$$\nabla g_1(\boldsymbol{\varpi}) = \begin{pmatrix} -2\hat{\vartheta}_1^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -2\hat{\vartheta}_p^{-1} & 0 & \cdots & 0 \\ -\hat{\vartheta}_1^{-1} & -\hat{\vartheta}_2^{-1} & 0 & \cdots & 0 & 0 & \hat{\vartheta}_{1p;-}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\hat{\vartheta}_{p-1}^{-1} & -\hat{\vartheta}_p^{-1} & 0 & \cdots & \hat{\vartheta}_{p-1,p;-}^{-1} \end{pmatrix}.$$

Therefore, by the Delta method

$$\sqrt{n}\left(\widehat{\boldsymbol{\vartheta}}_{\mathbf{X}} - \boldsymbol{\tau}_{\mathbf{X}}\right) \xrightarrow{d} \mathcal{N}_{\frac{1}{2}(p^2+p)}(\mathbf{0}, \nabla g_1 \boldsymbol{\Sigma}_{1;\vartheta} (\nabla g_1)^\top)\,, \tag{7.40}$$

and $\nabla g_1 = \nabla g_1(\boldsymbol{\mu}_{1;\vartheta})$ with $\boldsymbol{\mu}_{1;\vartheta}$ is the mean vector given in (7.38).

Now, we consider the asymptotic distribution of $\widehat{\boldsymbol{\vartheta}}^*_{\mathbf{X}}$. For this case, we need to define the new notation $\hat{\vartheta}_{i;2} = \frac{1}{n}\sum_{k=1}^{n}\cos(2X_{ki})$. Now from CLT,

$$\sqrt{n}\left(\begin{pmatrix}\hat{\vartheta}_1\\ \vdots\\ \hat{\vartheta}_p\\ \hat{\vartheta}_{12;-}\\ \vdots\\ \hat{\vartheta}_{p-1,p;-}\\ \hat{\vartheta}_{1;2}\\ \vdots\\ \hat{\vartheta}_{p;2}\\ \hat{\vartheta}_{12;+}\\ \vdots\\ \hat{\vartheta}_{p-1,p;+}\end{pmatrix} - \begin{pmatrix}\exp(-\sigma_1^\alpha)\\ \vdots\\ \exp(-\sigma_p^\alpha)\\ \exp(-\sigma_{1,-2}^\alpha)\\ \vdots\\ \exp(-\sigma_{p-1,-p}^\alpha)\\ \exp(-2^\alpha\sigma_1^\alpha)\\ \vdots\\ \exp(-2^\alpha\sigma_p^\alpha)\\ \exp(-\sigma_{1,2}^\alpha)\\ \vdots\\ \exp(-\sigma_{p-1,p}^\alpha)\end{pmatrix}\right) \xrightarrow{d} \mathcal{N}_{p^2+p}\left(\mathbf{0}, \mathbf{\Sigma}_{2;\vartheta}\right), \tag{7.41}$$

where $\sigma_i^\alpha = \int_{\mathcal{S}_d}|u_i|^\alpha\Gamma(d\mathbf{u})$, $\sigma_{i,\pm j}^\alpha = \int_{\mathcal{S}_d}|u_i \pm u_j|^\alpha\Gamma(d\mathbf{u})$,

$$\mathbf{\Sigma}_{2;\vartheta} = \begin{pmatrix}\mathbf{\Sigma}_{11;\vartheta} & \mathbf{\Sigma}_{12;\vartheta} & \mathbf{\Sigma}_{13;\vartheta} & \mathbf{\Sigma}_{14;\vartheta}\\ \mathbf{\Sigma}_{12;\vartheta}^\top & \mathbf{\Sigma}_{22;\vartheta} & \mathbf{\Sigma}_{23;\vartheta} & \mathbf{\Sigma}_{24;\vartheta}\\ \mathbf{\Sigma}_{13;\vartheta}^\top & \mathbf{\Sigma}_{23;\vartheta}^\top & \mathbf{\Sigma}_{33;\vartheta} & \mathbf{\Sigma}_{34;\vartheta}\\ \mathbf{\Sigma}_{14;\vartheta}^\top & \mathbf{\Sigma}_{24;\vartheta}^\top & \mathbf{\Sigma}_{34;\vartheta}^\top & \mathbf{\Sigma}_{44;\vartheta}\end{pmatrix}, \tag{7.42}$$

where $\mathbf{\Sigma}_{ij;\vartheta}$ can be obtained as before. Now, we introduce the following transformation to obtain the joint limiting distribution of the codifference matrix,

$$g_2\left(\hat{\vartheta}_1,\ldots,\hat{\vartheta}_p,\hat{\vartheta}_{12;-},\ldots,\hat{\vartheta}_{p-1,p;-},\hat{\vartheta}_{1;2},\ldots,\hat{\vartheta}_{p;2},\hat{\vartheta}_{1p;+},\ldots,\hat{\vartheta}_{p-1,p;+}\right)$$
$$=\left(\frac{1}{2}\ln\frac{\hat{\vartheta}_{1;2}}{\hat{\vartheta}_1^4},\ldots,\frac{1}{2}\ln\frac{\hat{\vartheta}_{p;2}}{\hat{\vartheta}_p^4},\ln\frac{\left(\hat{\vartheta}_{12;-}\hat{\vartheta}_{12;+}\right)^{\frac{1}{2}}}{\hat{\vartheta}_1\hat{\vartheta}_2},\ldots,\ln\frac{\left(\hat{\vartheta}_{p-1,p;-}\hat{\vartheta}_{p-1,p;+}\right)^{\frac{1}{2}}}{\hat{\vartheta}_{p-1}\hat{\vartheta}_p}\right).$$

Thus

$$\sqrt{n}\left(\widehat{\boldsymbol{\vartheta}}^*_{\mathbf{X}}-\boldsymbol{\tau}^*_{\mathbf{X}}\right)\xrightarrow{d}\mathcal{N}_{\frac{1}{2}(p^2+p)}(\mathbf{0},\nabla g_2\boldsymbol{\Sigma}_{2;\vartheta}(\nabla g_2)^\top)\,, \tag{7.43}$$

where $\nabla g_2=\nabla g_2(\boldsymbol{\mu}_{2;\vartheta})$ and $\boldsymbol{\mu}_{2;\vartheta}$ is the mean vector given in (7.41).

□

Proof of Theorem 3.5.

Proof. Let $u_i=\sqrt{n}(\beta_i-\hat{\beta}_{i:n})$, $i=1,2$, where $\hat{\beta}_{i:n}$ is the M-estimator of β_i defined in (3.24). Then the minimizing argument given in (3.24) can be rephrased by the following objective function:

$$A_n(u):=\sum_{j=1}^{n}\left(\left(\rho_1\left(\frac{X_{2j}}{X_{1j}}-n^{-\frac{1}{2}}u_1\right)-\rho_1\left(\frac{X_{2j}}{X_{1j}}\right)\right)+\left(\rho_2\left(\frac{X_{1j}}{X_{2j}}-n^{-\frac{1}{2}}u_2\right)-\rho_2\left(\frac{X_{1j}}{X_{2j}}\right)\right)\right)$$
$$=n^{-\frac{1}{2}}\sum_{j=1}^{n}\left(u_1\psi_1\left(\frac{X_{2j}}{X_{1j}}\right)+u_2\psi_2\left(\frac{X_{1j}}{X_{2j}}\right)\right)+\frac{1}{2n}\sum_{j=1}^{n}\left(u_1^2\psi_1'(c_{j1})+u_2^2\psi_2'(c_{j2})\right)\,,$$

where c_{ji} , $i=1,2, j=1,\ldots,n$, are constants such that c_{j1} is between $\frac{X_{2j}}{X_{1j}}-n^{-\frac{1}{2}}u_1$ and $\frac{X_{2j}}{X_{1j}}$ and c_{j2} is between $\frac{X_{1j}}{X_{2j}}-n^{-\frac{1}{2}}u_2$ and $\frac{X_{1j}}{X_{2j}}$. We have

$$n^{-1}\sum_{j=1}^{n}u_1^2\left|\psi'(c_{j1})-\psi_1'\left(\frac{X_{2j}}{X_{1j}}\right)\right|\leq n^{-1}\sum_{j=1}^{n}u_1^2|n^{-\frac{1}{2}}u_1|\to 0\,.$$

Similarly $n^{-1}\sum_{j=1}^{n}u_2^2\left|\psi'(c_{j2})-\psi'\left(\frac{X_{1j}}{X_{2j}}\right)\right|$ vanishes almost surely as $n\to\infty$ .CLT implies:

$$A_n\xrightarrow{d}-\mathbf{u}^\top\mathbf{Z}+\frac{1}{2}\mathbf{u}^\top\mathbf{C}\mathbf{u}\,,$$

where $\mathbf{u}^\top = (u_1, u_2)$, and $\mathbf{Z}$ is a bivariate Gaussian random vector with zero mean and variance-covariance matrix $\boldsymbol{\Sigma}_M$. By using Lemma 2.1 in Sohrabi and Zarepour (2018) and setting $A_n(\mathbf{u}) = 0$ and solving for (u_1, u_2), the results follows.

□

Supporting derivation for Remark 3.7

This appendix provides additional explanations and supplementary material related to Remark 3.7. The aim is to offer further context and background for readers interested in a deeper understanding of the subject. Although not essential for following the main results, these details may help clarify the assumptions, methodology, and reasoning that underlie our approach.

Asymptotic Distribution of $\hat{\beta}_1\hat{\beta}_2$ and Hypothesis Testing

Consider the function $h(\boldsymbol{\beta}) = \beta_1\beta_2$, where $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top$. We apply the *delta method* to derive the asymptotic distribution of $h(\hat{\boldsymbol{\beta}}_n) = \hat{\beta}_1\hat{\beta}_2$. From Theorem 3.5, we know that

$$\sqrt{n}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Upsilon), \quad \text{where } \Upsilon = \mathbf{c}^{-1}\boldsymbol{\Sigma}_M\mathbf{c}.$$

The gradient of h at $\boldsymbol{\beta}$ is

$$\nabla h(\boldsymbol{\beta}) = \begin{pmatrix} \beta_2 \\ \beta_1 \end{pmatrix}.$$

Thus, by the delta method,

$$\sqrt{n}\left(\hat{\beta}_1\hat{\beta}_2 - \beta_1\beta_2\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2_{\beta_1\beta_2}),$$

with asymptotic variance

$$\sigma^2_{\beta_1\beta_2} = \nabla h(\boldsymbol{\beta})^\top \Upsilon \nabla h(\boldsymbol{\beta}) = \beta_2^2\Upsilon_{11} + 2\beta_1\beta_2\Upsilon_{12} + \beta_1^2\Upsilon_{22},$$

where Υ_{ij} denotes the (i,j)-th entry of Υ.

Test for $H_0 : \beta_1\beta_2 = 0$

We propose the test statistic

$$T_n = \frac{\hat\beta_1\hat\beta_2}{\sqrt{\widehat{\sigma^2_{\beta_1\beta_2}}/n}},$$

where

$$\widehat{\sigma^2_{\beta_1\beta_2}} = \hat\beta_2^2\hat\Upsilon_{11} + 2\hat\beta_1\hat\beta_2\hat\Upsilon_{12} + \hat\beta_1^2\hat\Upsilon_{22}.$$

Under H_0, for sufficiently large n, we have

$$T_n \approx \mathcal{N}(0,1),$$

so the null hypothesis is rejected at significance level α whenever $|T_n| > z_{1-\alpha/2}$.

Estimating $\hat\Upsilon_{11}, \hat\Upsilon_{12}, \hat\Upsilon_{22}$

Let $\hat{\boldsymbol{\Sigma}}_M$ denote the estimator of $\boldsymbol{\Sigma}_M$ from Remark 3.6, and define $\hat{\mathbf{c}} = \mathrm{diag}(\hat c_1, \hat c_2)$, where

$$\hat c_1 = \frac{1}{n}\sum_{i=1}^n \psi_1'\left(\frac{X_{2i}}{X_{1i}}\right), \quad \hat c_2 = \frac{1}{n}\sum_{i=1}^n \psi_2'\left(\frac{X_{1i}}{X_{2i}}\right).$$

Define $\hat\Upsilon = \hat{\mathbf{c}}^{-1}\hat{\boldsymbol{\Sigma}}_M\hat{\mathbf{c}}$. Denote by $\hat\Upsilon_{ij}$ the (i,j)-th entry of $\hat\Upsilon$. The plug-in estimators are then given by

$$\hat{\Upsilon}_{11} = \frac{1}{\hat{c}_1^2}\widehat{V}\left(\psi_1\left(\tfrac{X_2}{X_1}\right)\right),$$
$$\hat{\Upsilon}_{22} = \frac{1}{\hat{c}_2^2}\widehat{V}\left(\psi_2\left(\tfrac{X_1}{X_2}\right)\right),$$
$$\hat{\Upsilon}_{12} = \hat{\Upsilon}_{21} = \frac{1}{\hat{c}_1\hat{c}_2}\widehat{Cov}\left(\psi_1\left(\tfrac{X_2}{X_1}\right), \psi_2\left(\tfrac{X_1}{X_2}\right)\right).$$

Hence, all components of the test statistic T_n can be consistently estimated from the sample.