Supplementary Material to "Inference on two-sample covariance difference for large-scale functional data"

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S1 Notations

To begin with, we outline the necessary notations used throughout this supplementary material. Given a generic vector $w = (w_1, \dots, w_p)' \in \mathbb{R}^p$, we represent its ℓ_q-norms as $\|w\|_q = (\sum_{l=1}^p |w_l|^q)^{1/q}$ for $1 \le q < \infty$, $\|w\|_0 = card\{l : w_l \ne 0\}$, and $\|w\|_\infty = \max_{l\le p} |w_l|$. For any two sequences a_n and b_n, we denote $a_n \lesssim b_n$ whenever $a_n \le c_0 b_n$ for some universal constant $c_0 > 0$. In a similar fashion, we use the notation $a_n \gtrsim b_n$ if $a_n \ge c_1 b_n$ for a constant $c_1 > 0$. Consequently, we express $a_n \asymp b_n$ on condition that $|a_n| \lesssim |b_n|$ and $|a_n| \gtrsim |b_n|$. We define a sequence of four-dimensional vectors as

$$
\begin{aligned}
\mathcal{L}_{d_n} =& \{(u_1, v_1, g_1, h_1), \dots, (u_{d_n}, v_{d_n}, g_{d_n}, h_{d_n})\} \\
=& \{1, \dots, p_n\} \times \{1, \dots, p_n\} \times \{1, \dots, s_n\} \times \{1, \dots, s_n\},
\end{aligned}
$$

which contains $d_n = p_n^2 s_n^2$ elements in total, and satisfies $u_k 10^3 + v_k 10^2 + g_k 10 + h_k < u_{k+1} 10^3 + v_{k+1} 10^2 + g_{k+1} 10 + h_{k+1}$ for any $1 \leq k \leq d_n - 1$. Based on the first truncated sample $\tilde{\mathcal{X}}^{n_1} = \{\theta_{1,i} \in \mathbb{R}^{p_n s_n} : i \leq n_1\}$, we define $\{\theta^\star_{1,i} \in \mathbb{R}^{d_n} : i \leq n_1\}$ as a collection of $i.i.d.$ random vectors, where $\theta^\star_{1,i} = (\theta^{\star'}_{1,i1}, \ldots, \theta^{\star'}_{1,id_n})'$, with each coordinate $\theta^\star_{1,ia} = (\theta_{1,iu_a g_a} - \eta_{1,u_a g_a})(\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}) - \sigma_{1,u_a v_a g_a h_a}$. Accordingly, we further denote the random vectors $\breve{\theta}^\star_{1,i} = (\breve{\theta}^{\star'}_{1,i1}, \ldots, \breve{\theta}^{\star'}_{1,id_n})'$ and $\hat{\theta}^\star_{1,i} = (\hat{\theta}^{\star'}_{1,i1}, \ldots, \hat{\theta}^{\star'}_{1,id_n})'$, with each $\breve{\theta}^\star_{1,ia} = (\theta_{1,iu_a g_a} - \hat{\eta}_{1,u_a g_a})(\theta_{1,iv_a h_a} - \hat{\eta}_{1,v_a h_a}) - \sigma_{1,u_a v_a g_a h_a}$ and $\hat{\theta}^\star_{1,ia} = (\theta_{1,iu_a g_a} - \hat{\eta}_{1,u_a g_a})(\theta_{1,iv_a h_a} - \hat{\eta}_{1,v_a h_a}) - \hat{\sigma}_{1,u_a v_a g_a h_a}$, respectively. We then write a covariance matrix M_1 as

$$M_1 = cov(\theta^\star_{1,i}) = (m_{1,ab})_{a \leq d_n, b \leq d_n} \in \mathbb{R}^{d_n \times d_n},$$

where the elements $m_{1,ab} = cov(\theta^\star_{1,ia}, \theta^\star_{1,ib})$ are estimated by

$$\begin{aligned} \hat{m}_{1,ab} =& n_1^{-1} \sum_{i=1}^{n_1} \{(\theta_{1,iu_a g_a} - \hat{\eta}_{1,u_a g_a})(\theta_{1,iv_a h_a} - \hat{\eta}_{1,v_a h_a}) - \hat{\sigma}_{1,u_a v_a g_a h_a}\} \times \\ & \{(\theta_{1,iu_b g_b} - \hat{\eta}_{1,u_b g_b})(\theta_{1,iv_b h_b} - \hat{\eta}_{1,v_b h_b}) - \hat{\sigma}_{1,u_b v_b g_b h_b}\}. \end{aligned}$$

Likewise, based on the second truncated sample $\tilde{\mathcal{Y}}^{n_1} = \{\theta_{2,i} \in \mathbb{R}^{p_n s_n} : i \leq n_2\}$, we define the quantities $\theta^\star_{2,i}$, $\breve{\theta}^\star_{2,i}$, $\hat{\theta}^\star_{2,i}$, $M_2 = cov(\theta^\star_{2,i}) = (m_{2,ab})_{a \leq d_n, b \leq d_n}$, and $\hat{m}_{2,ab}$. To this end, we denote $\{\varepsilon_i \in \mathbb{R}^{d_n} : i \leq n\}$ as a set of independent random vectors, such that $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{id_n})' = \theta^\star_{1,i} \cdot 1_{\{1 \leq i \leq n_1\}} - n_1 n_2^{-1} \theta^\star_{2,i-n_1} \cdot 1_{\{n_1+1 \leq i \leq n\}}$. Denoting the diagonal matrices $\Lambda = diag\{m_{1,aa} + n_1 n_2^{-1} m_{2,aa} : a \leq d_n\} \in \mathbb{R}^{d_n \times d_n}$ and $\hat{\Lambda} = diag\{\hat{m}_{1,aa} + n_1 n_2^{-1} \hat{m}_{2,aa} : a \leq d_n\} \in \mathbb{R}^{d_n \times d_n}$, we further denote three sets of random vectors as $\{\tilde{\varepsilon}_i \in \mathbb{R}^{d_n} : i \leq n\}$, $\{\breve{\varepsilon}_i \in \mathbb{R}^{d_n} : i \leq n\}$, and $\{\hat{\varepsilon}_i \in \mathbb{R}^{d_n} : i \leq n\}$. More precisely, we define

$$\tilde{\varepsilon}_i = (\tilde{\varepsilon}_{i1}, \ldots, \tilde{\varepsilon}_{id_n})' = (n_1 n^{-1}\Lambda)^{-1/2}\varepsilon_i = (n_1 n^{-1}\Lambda)^{-1/2}[\theta^\star_{1,i} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\theta^\star_{2,i-n_1} \cdot 1_{\{n_1+1\leq i\leq n\}}],$$

$$\tilde{\varepsilon}_{ia} = \{n_1 n^{-1}(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})\}^{-1/2}[\theta^\star_{1,ia} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\theta^\star_{2,(i-n_1)a} \cdot 1_{\{n_1+1\leq i\leq n\}}],$$

$$\breve{\varepsilon}_i = (\breve{\varepsilon}_{i1}, \ldots, \breve{\varepsilon}_{id_n})' = (n_1 n^{-1}\hat{\Lambda})^{-1/2}[\breve{\theta}^\star_{1,i} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\breve{\theta}^\star_{2,i-n_1} \cdot 1_{\{n_1+1\leq i\leq n\}}],$$

$$\breve{\varepsilon}_{ia} = \{n_1 n^{-1}(\hat{m}_{1,aa} + n_1 n_2^{-1}\hat{m}_{2,aa})\}^{-1/2}[\breve{\theta}^\star_{1,ia} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\breve{\theta}^\star_{2,(i-n_1)a} \cdot 1_{\{n_1+1\leq i\leq n\}}],$$

$$\hat{\varepsilon}_i = (\hat{\varepsilon}_{i1}, \ldots, \hat{\varepsilon}_{id_n})' = (n_1 n^{-1}\hat{\Lambda})^{-1/2}[\hat{\theta}^\star_{1,i} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\hat{\theta}^\star_{2,i-n_1} \cdot 1_{\{n_1+1\leq i\leq n\}}],$$

$$\hat{\varepsilon}_{ia} = \{n_1 n^{-1}(\hat{m}_{1,aa} + n_1 n_2^{-1}\hat{m}_{2,aa})\}^{-1/2}[\hat{\theta}^\star_{1,ia} \cdot 1_{\{1\leq i\leq n_1\}} - n_1 n_2^{-1}\hat{\theta}^\star_{2,(i-n_1)a} \cdot 1_{\{n_1+1\leq i\leq n\}}].$$

Accordingly, we express a series of random vectors as

$$\begin{aligned}
\tilde{T} &= (\tilde{T}_1, \ldots, \tilde{T}_{d_n})' = n^{-1/2}\sum_{i=1}^n \tilde{\varepsilon}_i, \quad \tilde{T}_a = n^{-1/2}\sum_{i=1}^n \tilde{\varepsilon}_{ia},\\
\breve{T} &= (\breve{T}_1, \ldots, \breve{T}_{d_n})' = n^{-1/2}\sum_{i=1}^n \breve{\varepsilon}_i, \quad \breve{T}_a = n^{-1/2}\sum_{i=1}^n \breve{\varepsilon}_{ia},\\
\tilde{T}_e &= (\tilde{T}_{e1}, \ldots, \tilde{T}_{ed_n})' = n^{-1/2}\sum_{i=1}^n e_i\tilde{\varepsilon}_i, \quad \tilde{T}_{ea} = n^{-1/2}\sum_{i=1}^n e_i\tilde{\varepsilon}_{ia},\\
\hat{T}_e &= (\hat{T}_{e1}, \ldots, \hat{T}_{ed_n})' = n^{-1/2}\sum_{i=1}^n e_i\hat{\varepsilon}_i, \quad \hat{T}_{ea} = n^{-1/2}\sum_{i=1}^n e_i\hat{\varepsilon}_{ia}.
\end{aligned}$$

It is not difficult to verify the following equations:

$$\hat{G}(\Sigma^X - \Sigma^Y) = \hat{G}(F_{\{b_k:k\leq s_n\}}(K^X - K^Y)) = \|\breve{T}\|_\infty, \quad \hat{G}_e = \|\hat{T}_e\|_\infty,$$

$$c_B(\alpha) = \inf\{t \in \mathbb{R} : P_e(\|\hat{T}_e\|_\infty \leq t) \geq 1-\alpha\}, \qquad \alpha \in (0,1),$$

$$\hat{H}_e(K^X - K^Y) = \|\hat{T}_e + \hat{J}(K^X - K^Y)\|_\infty, \tag{S1.1}$$

where $\hat{J}(K^X - K^Y) = [\hat{J}_1(K^X - K^Y), \ldots, \hat{J}_{d_n}(K^X - K^Y)]'$, with each coordinate function as $\hat{J}_a(K^X - K^Y) = (n_1^{-1}\hat{m}_{1,aa} + n_2^{-1}\hat{m}_{2,aa})^{-1/2}(\sigma_{1,u_a v_a g_a h_a} - \sigma_{2,u_a v_a g_a h_a})$. The subsequent section includes the auxiliary lemmas along with their corresponding proofs.

S2 Auxiliary Lemmas and Proofs

Lemma 1. *Suppose $X_1, \ldots, X_n$ are $i.i.d.$ random variables satisfying $E(X_i) = 0$ and $var(X_i) = 1$. Also assume that there exist universal constants $c_1, c_2, \gamma > 0$ such that for any $t > 0$,*

$$P(|X_i| \geq t) \leq c_1 \exp(-c_2 t^\gamma).$$

Then, there exist universal constants $c_3, c_4 > 0$ such that for any $t > 0$,

$$P(|n^{-1}\sum_{i=1}^{n} X_i| \geq t) \leq 2\exp(-nt^2/4) + c_3 \exp(-c_4 n^\gamma t^\gamma).$$

Proof. This is adapted from Theorem 6 of Delaigle et al. (2011). □

Lemma 2. *Under conditions (A1)–(A4), there exist universal constants $c_1, c_2 > 0$ such that:*

1) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j \leq p_n} \max_{k \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,jjkk}^{-1/2}(\theta_{1,ijk} - \eta_{1,jk})| \leq c_2 n^{-1/2}\{\log(np_n s_n)\}^{1/2}.$$

2) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1/2} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2}) -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1/2} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

3) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2 -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

4) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2 \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2 -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2 \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

5) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j \le p_n} \max_{k \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,jjkk}^{-1/2} (\theta_{1,ijk} - \hat{\eta}_{1,jk})| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

6) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \hat{\eta}_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1/2} (\theta_{1,ij_2k_2} - \hat{\eta}_{1,j_2k_2}) -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1/2} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

7) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \hat{\eta}_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \hat{\eta}_{1,j_2k_2})^2 -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1}) \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

8) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,ij_1k_1} - \hat{\eta}_{1,j_1k_1})^2 \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \hat{\eta}_{1,j_2k_2})^2 -$$

$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2 \sigma_{1,j_2j_2k_2k_2}^{-1} (\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

9) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j \le p_n} \max_{k \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,jjkk}^{-1/2} (\theta_{2,ijk} - \eta_{2,jk})| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

10) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1k_1} - \eta_{2,j_1k_1}) \sigma_{2,j_2j_2k_2k_2}^{-1/2} (\theta_{2,ij_2k_2} - \eta_{2,j_2k_2}) -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1k_1} - \eta_{2,j_1k_1}) \sigma_{2,j_2j_2k_2k_2}^{-1/2} (\theta_{2,ij_2k_2} - \eta_{2,j_2k_2})\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

11) *With probability at least* $1 - c_1 n^{-1}$, *we have:*

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1k_1} - \eta_{2,j_1k_1}) \sigma_{2,j_2j_2k_2k_2}^{-1} (\theta_{2,ij_2k_2} - \eta_{2,j_2k_2})^2 -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1k_1} - \eta_{2,j_1k_1}) \sigma_{2,j_2j_2k_2k_2}^{-1} (\theta_{2,ij_2k_2} - \eta_{2,j_2k_2})^2\}| \le c_2 n^{-1/2} \{\log(np_n s_n)\}^{1/2}.$$

12) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1} (\theta_{2,ij_1 k_1} - \eta_{2,j_1 k_1})^2 \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \eta_{2,j_2 k_2})^2 -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1} (\theta_{2,ij_1 k_1} - \eta_{2,j_1 k_1})^2 \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \eta_{2,j_2 k_2})^2\}| \le c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

13) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j \le p_n} \max_{k \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,jjkk}^{-1/2} (\theta_{2,ijk} - \hat{\eta}_{2,jk})| \le c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

14) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1 k_1} - \hat{\eta}_{2,j_1 k_1}) \sigma_{2,j_2 j_2 k_2 k_2}^{-1/2} (\theta_{2,ij_2 k_2} - \hat{\eta}_{2,j_2 k_2}) -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1 k_1} - \eta_{2,j_1 k_1}) \sigma_{2,j_2 j_2 k_2 k_2}^{-1/2} (\theta_{2,ij_2 k_2} - \eta_{2,j_2 k_2})\}| \le c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

15) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1 k_1} - \hat{\eta}_{2,j_1 k_1}) \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \hat{\eta}_{2,j_2 k_2})^2 -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{2,ij_1 k_1} - \eta_{2,j_1 k_1}) \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \eta_{2,j_2 k_2})^2\}| \le c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

16) With probability at least $1 - c_1 n^{-1}$, we have:

$$\max_{j_1 \le p_n} \max_{j_2 \le p_n} \max_{k_1 \le s_n} \max_{k_2 \le s_n} |n_2^{-1} \sum_{i=1}^{n_2} \sigma_{2,j_1 j_1 k_1 k_1}^{-1} (\theta_{2,ij_1 k_1} - \hat{\eta}_{2,j_1 k_1})^2 \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \hat{\eta}_{2,j_2 k_2})^2 -$$

$$E\{\sigma_{2,j_1 j_1 k_1 k_1}^{-1} (\theta_{2,ij_1 k_1} - \eta_{2,j_1 k_1})^2 \sigma_{2,j_2 j_2 k_2 k_2}^{-1} (\theta_{2,ij_2 k_2} - \eta_{2,j_2 k_2})^2\}| \le c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

Proof. To show part 4), for any $i \le n_1$, $j_1 \le p_n$, $j_2 \le p_n$, $k_1 \le s_n$, $k_2 \le s_n$, we denote the random variables $Z^{(4)}_{i,j_1j_2k_1k_2}$ as

$$Z^{(4)}_{i,j_1j_2k_1k_2} \tag{S2.2}$$
$$=\frac{(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2 - E\{(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2\}}{var^{1/2}\{(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2\}},$$

which satisfies $E(Z^{(4)}_{i,j_1j_2k_1k_2}) = 0$ and $var(Z^{(4)}_{i,j_1j_2k_1k_2}) = 1$. Denoting Δ_4 as

$$\Delta_4 = \max_{j_1\le p_n}\max_{j_2\le p_n}\max_{k_1\le s_n}\max_{k_2\le s_n}|n_1^{-1}\sum_{i=1}^{n_1}\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2 -$$
$$E\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2\}|,$$

it then follows from (S2.2) that

$$\max_{j_1\le p_n}\max_{j_2\le p_n}\max_{k_1\le s_n}\max_{k_2\le s_n}|n_1^{-1}\sum_{i=1}^{n_1}Z^{(4)}_{i,j_1j_2k_1k_2}|$$
$$=\max_{j_1\le p_n}\max_{j_2\le p_n}\max_{k_1\le s_n}\max_{k_2\le s_n}|n_1^{-1}\sum_{i=1}^{n_1}\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2 -$$
$$E\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2\}|\cdot$$
$$[var\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2})^2\}]^{-1/2}$$
$$\ge[\max_{j\le p_n}\max_{k\le s_n}E\{\sigma^{-4}_{1,jjkk}(\theta_{1,ijk}-\eta_{1,jk})^8\}]^{-1/2}\Delta_4 \ge c_1\Delta_4, \tag{S2.3}$$

for some universal constant $c_1 > 0$, where the last inequality holds from (A3).

For any $i \leq n_1$, $j_1 \leq p_n$, $j_2 \leq p_n$, $k_1 \leq s_n$, $k_2 \leq s_n$, $s > 0$, $t > 0$, we have

$$
\begin{aligned}
&P(|Z^{(4)}_{i,j_1j_2k_1k_2}| \geq t) \\
=&P\big[s|\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2 - \\
&E\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}|^{1/2} \geq \\
&var^{1/4}\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}st^{1/2}\big] \\
\leq&P\big[s|\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2 - \\
&E\{\sigma^{-1}_{1,j_1j_1k_1k_1}(\theta_{1,ij_1k_1} - \eta_{1,j_1k_1})^2\sigma^{-1}_{1,j_2j_2k_2k_2}(\theta_{1,ij_2k_2} - \eta_{1,j_2k_2})^2\}|^{1/2} \geq c_2st^{1/2}\big] \\
\leq&\exp(-c_2st^{1/2}) \cdot \exp\left(s\Big[\max_{j\leq p_n}\max_{k\leq s_n} E\{\sigma^{-2}_{1,jjkk}(\theta_{1,ijk} - \eta_{1,jk})^4\}\Big]^{1/2}\right)\cdot \\
&\max_{j\leq p_n}\max_{k\leq s_n} E[\exp\{s\sigma^{-1}_{1,jjkk}(\theta_{1,ijk} - \eta_{1,jk})^2\}] \\
\leq&\exp(-c_2st^{1/2} + c_3s)\cdot \max_{j\leq p_n}\max_{k\leq s_n} E[\exp\{s\sigma^{-1}_{1,jjkk}(\theta_{1,ijk} - \eta_{1,jk})^2\}], \qquad \text{(S2.4)}
\end{aligned}
$$

for some universal constant $c_2, c_3 > 0$, where the first equality is by (S2.2), the third last inequality holds from (A4), and the last inequality is based on (A3). Plugging $s = K_1$ into (S2.4), it follows from (A3) that there exist universal constant $c_4, c_5 > 0$ such that for any $t > 0$,

$$
P(|Z^{(4)}_{i,j_1j_2k_1k_2}| \geq t) \leq c_4 \exp(-c_5t^{1/2}). \qquad \text{(S2.5)}
$$

By combining (S2.5), (S2.2) with Lemma 1, it can be concluded that there exist universal constant $c_6, c_7 > 0$ such that for any $t > 0$,

$$
P(|n_1^{-1}\sum_{i=1}^{n_1} Z^{(4)}_{i,j_1j_2k_1k_2}| \geq t) \leq 2\exp(-n_1t^2/4) + c_6\exp(-c_7n_1^{1/2}t^{1/2}). \quad \text{(S2.6)}
$$

Therefore, we have that for any $t > 0$,

$$\begin{aligned}
&P(\Delta_4 \geq t) \\
\leq &P(\max_{j_1 \leq p_n} \max_{j_2 \leq p_n} \max_{k_1 \leq s_n} \max_{k_2 \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} Z^{(4)}_{i,j_1 j_2 k_1 k_2}| \geq c_1 t) \\
\leq &\sum_{j_1=1}^{p_n} \sum_{j_2=1}^{p_n} \sum_{k_1=1}^{s_n} \sum_{k_2=1}^{s_n} P(|n_1^{-1} \sum_{i=1}^{n_1} Z^{(4)}_{i,j_1 j_2 k_1 k_2}| \geq c_1 t) \\
\leq &2 p_n^2 s_n^2 \exp(-c_1^2 n_1 t^2/4) + c_6 p_n^2 s_n^2 \exp(-c_7 c_1^{1/2} n_1^{1/2} t^{1/2}), \qquad \text{(S2.7)}
\end{aligned}$$

where the first inequality is based on (S2.3), and the last inequality holds from (S2.6). Plugging $t = \max\{2c_1^{-1} n_1^{-1/2} \log^{1/2}(n_1 p_n^2 s_n^2), c_1^{-1} c_7^{-2} n_1^{-1} \log^2(n_1 p_n^2 s_n^2)\} = 2c_1^{-1} n_1^{-1/2} \log^{1/2}(n_1 p_n^2 s_n^2)$ into (S2.7) yields

$$P\{\Delta_4 \geq 2c_1^{-1} n_1^{-1/2} \log^{1/2}(n_1 p_n^2 s_n^2)\} \leq (2 + c_6) n_1^{-1}.$$

Together with (A1) and (A2) entails that there are universal constants $c_8, c_9 > 0$ such that

$$P\{\Delta_4 \leq c_8^{-1} n^{-1/2} \log^{1/2}(n p_n s_n)\} \geq 1 - c_9 n^{-1}.$$

which completes the proof of part 4). Similar arguments as those in part 4) lead to the conclusions in parts 1–3). To show part 5), first note that

$$\max_{j \leq p_n} \max_{k \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,jjkk}^{-1/2} (\theta_{1,ijk} - \hat{\eta}_{1,jk})| \leq 2 \max_{j \leq p_n} \max_{k \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,jjkk}^{-1/2} (\theta_{1,ijk} - \eta_{1,jk})|.$$

Together with part 1) implies part 5). Similar reasoning as part 5) leads to parts 6–8). Analogous to the proofs of parts 1–8), one can show parts 9–16). □

Lemma 3. *Under conditions (A1)–(A4), there exist universal constants* $c_1, c_2 > 0$ *such that:*

1) With probability at least $1 - c_1 n^{-1}$, *we have:*

$$\max_{a \leq d_n} \frac{|\hat{m}_{1,aa} - m_{1,aa}|}{\sigma_{1,u_a u_a g_a g_a} \sigma_{1,v_a v_a h_a h_a}} \leq c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

2) With probability at least $1 - c_1 n^{-1}$, *we have:*

$$\max_{a \leq d_n} \frac{|\hat{m}_{2,aa} - m_{2,aa}|}{\sigma_{2,u_a u_a g_a g_a} \sigma_{2,v_a v_a h_a h_a}} \leq c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}.$$

Proof. To show part 1), first note that

$$\max_{a \leq d_n} \frac{|\hat{m}_{1,aa} - m_{1,aa}|}{\sigma_{1,u_a u_a g_a g_a} \sigma_{1,v_a v_a h_a h_a}} \leq \Delta_1^* + 2\Delta_2^* + (\Delta_2^*)^2, \tag{S2.8}$$

where

$$\Delta_1^* = \max_{j_1 \leq p_n} \max_{j_2 \leq p_n} \max_{k_1 \leq s_n} \max_{k_2 \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,i j_1 k_1} - \hat{\eta}_{1,j_1 k_1})^2 \sigma_{1,j_2 j_2 k_2 k_2}^{-1} (\theta_{1,i j_2 k_2} - \hat{\eta}_{1,j_2 k_2})^2 -$$
$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1} (\theta_{1,i j_1 k_1} - \eta_{1,j_1 k_1})^2 \sigma_{1,j_2 j_2 k_2 k_2}^{-1} (\theta_{1,i j_2 k_2} - \eta_{1,j_2 k_2})^2\}|,$$
$$\Delta_2^* = \max_{j_1 \leq p_n} \max_{j_2 \leq p_n} \max_{k_1 \leq s_n} \max_{k_2 \leq s_n} |n_1^{-1} \sum_{i=1}^{n_1} \sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,i j_1 k_1} - \hat{\eta}_{1,j_1 k_1}) \sigma_{1,j_2 j_2 k_2 k_2}^{-1/2} (\theta_{1,i j_2 k_2} - \hat{\eta}_{1,j_2 k_2}) -$$
$$E\{\sigma_{1,j_1 j_1 k_1 k_1}^{-1/2} (\theta_{1,i j_1 k_1} - \eta_{1,j_1 k_1}) \sigma_{1,j_2 j_2 k_2 k_2}^{-1/2} (\theta_{1,i j_2 k_2} - \eta_{1,j_2 k_2})\}|.$$

By combining (S2.8) with Lemma 2 and (A2), it can be deduced that there exist universal constants $c_1, c_2 > 0$ such that

$$P\Big[\max_{a \leq d_n} \frac{|\hat{m}_{1,aa} - m_{1,aa}|}{\sigma_{1,u_a u_a g_a g_a} \sigma_{1,v_a v_a h_a h_a}} \leq c_2 n^{-1/2} \{\log(n p_n s_n)\}^{1/2}\Big] \geq 1 - c_1 n^{-1},$$

which finishes the proof of part 1). Similar reasoning leads to part 2). □

S3 Proofs of Main Theorems

Proof of Theorem 1. Recall the following four terms

$$\breve{T} = n^{-1/2}\sum_{i=1}^{n}\breve{\varepsilon}_i, \qquad \hat{T}_e = n^{-1/2}\sum_{i=1}^{n} e_i\hat{\varepsilon}_i,$$
$$\tilde{T} = n^{-1/2}\sum_{i=1}^{n}\tilde{\varepsilon}_i, \qquad \tilde{T}_e = n^{-1/2}\sum_{i=1}^{n} e_i\tilde{\varepsilon}_i,$$

where $\{\tilde{\varepsilon}_i \in \mathbb{R}^{d_n} : i \leq n\}$ represents a collection of centered independent random vectors that fulfills

$$\min_{a\leq d_n} n^{-1}\sum_{i=1}^{n} E(\tilde{\varepsilon}_{ia}^2) = \min_{a\leq d_n}\frac{\sum_{i=1}^{n_1} E(\theta_{1,ia}^{\star 2}) + n_1^2 n_2^{-2}\sum_{i=1}^{n_2} E(\theta_{2,ia}^{\star 2})}{n_1 m_{1,aa} + n_1^2 n_2^{-1} m_{2,aa}}$$
$$= \min_{a\leq d_n}\frac{n_1 m_{1,aa} + n_1^2 n_2^{-1} m_{2,aa}}{n_1 m_{1,aa} + n_1^2 n_2^{-1} m_{2,aa}} = 1. \tag{S3.9}$$

Based on (A1), (A3), and (A4), it can be deduced that for all $i \leq n$ and $a \leq d_n$

$$\tilde{\varepsilon}_{ia} \sim \text{sub-Exponential}(c_1), \tag{S3.10}$$

for a universal constant parameter $c_1 > 0$. By combining (S3.9), (S3.10), (A2) with Lemma 5 in Xue and Yao (2024), it can be concluded that

$$\lim_{n\to\infty}\sup_{A\in\mathcal{A}^{Re}}\left|P(\tilde{T}\in A) - P_e(\tilde{T}_e \in A)\right| = 0, \tag{S3.11}$$

where the set $\mathcal{A}^{Re}$ includes all hyperrectangles A of the form $A = \{\omega \in \mathbb{R}^{d_n} : \alpha_j \leq \omega_j \leq \beta_j, j \leq d_n\}$, satisfying $-\infty \leq \alpha_j \leq \beta_j \leq \infty$ for every $j \leq d_n$. To bound $\|\breve{T}-\tilde{T}\|_\infty$, denoting

$$\begin{aligned}
\tilde{\sigma}_{1,j_1j_2k_1k_2} &= n_1^{-1}\sum_{i=1}^{n_1}(\theta_{1,ij_1k_1}-\eta_{1,j_1k_1})(\theta_{1,ij_2k_2}-\eta_{1,j_2k_2}),\\
\hat{\sigma}_{1,j_1j_2k_1k_2} &= n_1^{-1}\sum_{i=1}^{n_1}(\theta_{1,ij_1k_1}-\hat{\eta}_{1,j_1k_1})(\theta_{1,ij_2k_2}-\hat{\eta}_{1,j_2k_2}),\\
\tilde{\sigma}_{2,j_1j_2k_1k_2} &= n_2^{-1}\sum_{i=1}^{n_2}(\theta_{2,ij_1k_1}-\eta_{2,j_1k_1})(\theta_{2,ij_2k_2}-\eta_{2,j_2k_2}),\\
\hat{\sigma}_{2,j_1j_2k_1k_2} &= n_2^{-1}\sum_{i=1}^{n_2}(\theta_{2,ij_1k_1}-\hat{\eta}_{2,j_1k_1})(\theta_{2,ij_2k_2}-\hat{\eta}_{2,j_2k_2}),
\end{aligned}$$

it can then be seen that

$$\begin{aligned}
&\|\breve{T}-\tilde{T}\|_\infty = \max_{a\le d_n}|n^{-1/2}\sum_{i=1}^{n}(\breve{\varepsilon}_{ia}-\tilde{\varepsilon}_{ia})|\\
=&\max_{a\le d_n}\Big|\frac{(\hat{\sigma}_{1,u_av_ag_ah_a}-\sigma_{1,u_av_ag_ah_a})-(\hat{\sigma}_{2,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a})}{(n_1^{-1}\hat{m}_{1,aa}+n_2^{-1}\hat{m}_{2,aa})^{1/2}}-\\
&\frac{(\tilde{\sigma}_{1,u_av_ag_ah_a}-\sigma_{1,u_av_ag_ah_a})-(\tilde{\sigma}_{2,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a})}{(n_1^{-1}m_{1,aa}+n_2^{-1}m_{2,aa})^{1/2}}\Big|\\
\le&\max_{a\le d_n}\Big|\frac{\tilde{\sigma}_{1,u_av_ag_ah_a}-\sigma_{1,u_av_ag_ah_a}}{(n_1^{-1}m_{1,aa}+n_2^{-1}m_{2,aa})^{1/2}}-\frac{\hat{\sigma}_{1,u_av_ag_ah_a}-\sigma_{1,u_av_ag_ah_a}}{(n_1^{-1}\hat{m}_{1,aa}+n_2^{-1}\hat{m}_{2,aa})^{1/2}}\Big|+\\
&\max_{a\le d_n}\Big|\frac{\tilde{\sigma}_{2,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a}}{(n_1^{-1}m_{1,aa}+n_2^{-1}m_{2,aa})^{1/2}}-\frac{\hat{\sigma}_{2,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a}}{(n_1^{-1}\hat{m}_{1,aa}+n_2^{-1}\hat{m}_{2,aa})^{1/2}}\Big|\\
\le&\Omega_1+\Omega_2+\Omega_3+\Omega_4, \qquad \text{(S3.12)}
\end{aligned}$$

where

$$\Omega_1 = \max_{a \le d_n} \left| \frac{\tilde{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} - \frac{\hat{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} \right|,$$

$$\Omega_2 = \max_{a \le d_n} \left| \frac{\hat{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} - \frac{\hat{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}}{(n_1^{-1} \hat{m}_{1,aa} + n_2^{-1} \hat{m}_{2,aa})^{1/2}} \right|,$$

$$\Omega_3 = \max_{a \le d_n} \left| \frac{\tilde{\sigma}_{2,u_a v_a g_a h_a} - \sigma_{2,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} - \frac{\hat{\sigma}_{2,u_a v_a g_a h_a} - \sigma_{2,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} \right|,$$

$$\Omega_4 = \max_{a \le d_n} \left| \frac{\hat{\sigma}_{2,u_a v_a g_a h_a} - \sigma_{2,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} - \frac{\hat{\sigma}_{2,u_a v_a g_a h_a} - \sigma_{2,u_a v_a g_a h_a}}{(n_1^{-1} \hat{m}_{1,aa} + n_2^{-1} \hat{m}_{2,aa})^{1/2}} \right|.$$

To bound Ω_1, notice that

$$\begin{aligned}
\Omega_1 =& \max_{a \le d_n} \left| \frac{\tilde{\sigma}_{1,u_a v_a g_a h_a} - \hat{\sigma}_{1,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} \right| \\
\le& n_1^{1/2} \max_{a \le d_n} \frac{|\tilde{\sigma}_{1,u_a v_a g_a h_a} - \hat{\sigma}_{1,u_a v_a g_a h_a}|}{m_{1,aa}^{1/2}} \\
=& n_1^{1/2} \max_{a \le d_n} \frac{|(\hat{\eta}_{1,u_a g_a} - \eta_{1,u_a g_a})(\hat{\eta}_{1,v_a h_a} - \eta_{1,v_a h_a})|}{m_{1,aa}^{1/2}} \\
=& n_1^{1/2} \max_{a \le d_n} \left| n_1^{-1} \sum_{i=1}^{n_1} \left(\frac{\theta_{1,iu_a g_a} - \eta_{1,u_a g_a}}{\sigma_{1,u_a u_a g_a g_a}^{1/2}} \right) \right| \cdot \left| n_1^{-1} \sum_{i=1}^{n_1} \left(\frac{\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}}{\sigma_{1,v_a v_a h_a h_a}^{1/2}} \right) \right| \cdot \\
& \left[var\left\{ \left(\frac{\theta_{1,iu_a g_a} - \eta_{1,u_a g_a}}{\sigma_{1,u_a u_a g_a g_a}^{1/2}} \right) \left(\frac{\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}}{\sigma_{1,v_a v_a h_a h_a}^{1/2}} \right) \right\} \right]^{-1/2}.
\end{aligned}$$

Together with (A1), (A4), and Lemma 2 yields that there are universal constants $c_1, c_2 > 0$ such that with probability at least $1 - c_1 n^{-1}$,

$$\Omega_1 \le c_2 n^{-1/2} \log(n p_n s_n). \tag{S3.13}$$

In a similar fashion, one can show that there exist universal constants $c_3, c_4 > 0$ such that with probability at least $1 - c_3 n^{-1}$,

$$\Omega_3 \le c_4 n^{-1/2} \log(np_n s_n). \tag{S3.14}$$

To bound Ω_2, first note that

$$\Omega_2 \le \Omega_1^* \Omega_2^*, \tag{S3.15}$$

where

$$\Omega_1^* = \max_{a \le d_n} \Big| \frac{\hat{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}}{(n_1^{-1} m_{1,aa} + n_2^{-1} m_{2,aa})^{1/2}} \Big|,$$
$$\Omega_2^* = \max_{a \le d_n} \Big| 1 - \frac{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}}{(\hat{m}_{1,aa} + n_1 n_2^{-1} \hat{m}_{2,aa})^{1/2}} \Big|.$$

To bound Ω_1^*, note that

$$\begin{aligned}\Omega_1^* \le & n_1^{1/2} \max_{a \le d_n} \frac{|\hat{\sigma}_{1,u_a v_a g_a h_a} - \sigma_{1,u_a v_a g_a h_a}|}{m_{1,aa}^{1/2}} \\ = & n_1^{1/2} \max_{a \le d_n} \Big| n_1^{-1} \sum_{i=1}^{n_1} \Big(\frac{\theta_{1,iu_a g_a} - \eta_{1,u_a g_a}}{\sigma_{1,u_a u_a g_a g_a}^{1/2}} \Big) \Big(\frac{\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}}{\sigma_{1,v_a v_a h_a h_a}^{1/2}} \Big) - \\ & E\Big\{ \Big(\frac{\theta_{1,iu_a g_a} - \eta_{1,u_a g_a}}{\sigma_{1,u_a u_a g_a g_a}^{1/2}} \Big) \Big(\frac{\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}}{\sigma_{1,v_a v_a h_a h_a}^{1/2}} \Big) \Big\} \Big| \cdot \\ & \Big[var\Big\{ \Big(\frac{\theta_{1,iu_a g_a} - \eta_{1,u_a g_a}}{\sigma_{1,u_a u_a g_a g_a}^{1/2}} \Big) \Big(\frac{\theta_{1,iv_a h_a} - \eta_{1,v_a h_a}}{\sigma_{1,v_a v_a h_a h_a}^{1/2}} \Big) \Big\} \Big]^{-1/2}.\end{aligned}$$

Together with (A1), (A4), and Lemma 2 yields that there are universal constants $c_5, c_6 > 0$ such that with probability at least $1 - c_5 n^{-1}$,

$$\Omega_1^* \le c_6 \{\log(np_n s_n)\}^{1/2}. \tag{S3.16}$$

To bound Ω_2^*, first note that

$$\begin{aligned}\Omega_2^* &\leq \max_{a\leq d_n}\left|1-\frac{m_{1,aa}+n_1n_2^{-1}m_{2,aa}}{\hat{m}_{1,aa}+n_1n_2^{-1}\hat{m}_{2,aa}}\right| \\ &\leq \max_{a\leq d_n}\left|1-\frac{m_{1,aa}}{\hat{m}_{1,aa}}\right|+\max_{a\leq d_n}\left|1-\frac{m_{2,aa}}{\hat{m}_{2,aa}}\right|. \qquad \text{(S3.17)}\end{aligned}$$

To bound $\max_{a\leq d_n}\left|1-\frac{m_{1,aa}}{\hat{m}_{1,aa}}\right|$, first notice that

$$\begin{aligned}&\max_{a\leq d_n}\left|1-\frac{m_{1,aa}}{\hat{m}_{1,aa}}\right| \\ =&\max_{a\leq d_n}\left|\Big(\frac{\hat{m}_{1,aa}-m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}\Big)\Big/\Big(\frac{m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}+\frac{\hat{m}_{1,aa}-m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}\Big)\right| \\ \leq&\Big(\max_{a\leq d_n}\left|\frac{\hat{m}_{1,aa}-m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}\right|\Big)\Big/\Big(\min_{a\leq d_n}\frac{m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}-\max_{a\leq d_n}\left|\frac{\hat{m}_{1,aa}-m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}}\right|\Big).\end{aligned}$$

Together with (A2), (A4), and Lemma 3 yields that there are universal constants $c_7, c_8 > 0$ such that with probability at least $1-c_7n^{-1}$,

$$\max_{a\leq d_n}\left|1-\frac{m_{1,aa}}{\hat{m}_{1,aa}}\right|\leq c_8n^{-1/2}\{\log(np_ns_n)\}^{1/2}. \qquad \text{(S3.18)}$$

In a similar fashion, it can be shown that there exist universal constants $c_9, c_{10} > 0$ such that with probability at least $1-c_9n^{-1}$,

$$\max_{a\leq d_n}\left|1-\frac{m_{2,aa}}{\hat{m}_{2,aa}}\right|\leq c_{10}n^{-1/2}\{\log(np_ns_n)\}^{1/2}. \qquad \text{(S3.19)}$$

By combining (S3.18) and (S3.19) with (S3.17), it can be concluded that there are universal constants $c_{11}, c_{12} > 0$ such that with probability at least $1-c_{11}n^{-1}$,

$$\Omega_2^*\leq c_{12}n^{-1/2}\{\log(np_ns_n)\}^{1/2}. \qquad \text{(S3.20)}$$

By combining (S3.20) and (S3.16) with (S3.15), it can be concluded that there are universal constants $c_{13}, c_{14} > 0$ such that with probability at least $1 - c_{13}n^{-1}$,

$$\Omega_2 \leq c_{14} n^{-1/2} \log(np_n s_n). \tag{S3.21}$$

In a similar fashion, it can be shown that there exist universal constants $c_{15}, c_{16} > 0$ such that with probability at least $1 - c_{15}n^{-1}$,

$$\Omega_4 \leq c_{16} n^{-1/2} \log(np_n s_n). \tag{S3.22}$$

By combining (S3.22), (S3.21), (S3.14), (S3.13) with (S3.12), it can be deduced that there are universal constants $c_{17}, c_{18} > 0$ such that with probability at least $1 - c_{17}n^{-1}$,

$$\|\breve{T} - \tilde{T}\|_\infty \leq c_{18} n^{-1/2} \log(np_n s_n). \tag{S3.23}$$

To bound $\|\hat{T}_e - \tilde{T}_e\|_\infty$, first note that for any $t > 0$,

$$\begin{aligned}
&P_e(\|\hat{T}_e - \tilde{T}_e\|_\infty \geq t) = P_e(\max_{a \leq d_n} |n^{-1/2} \sum_{i=1}^n e_i(\hat{\varepsilon}_{ia} - \tilde{\varepsilon}_{ia})| \geq t) \\
\leq &\sum_{a=1}^{d_n} P_e(|n^{-1/2} \sum_{i=1}^n e_i(\hat{\varepsilon}_{ia} - \tilde{\varepsilon}_{ia})| \geq t) \\
\leq &2 \sum_{a=1}^{d_n} \exp\Big\{ - \frac{t^2}{2n^{-1} \sum_{i=1}^n (\hat{\varepsilon}_{ia} - \tilde{\varepsilon}_{ia})^2} \Big\} \\
\leq &2d_n \exp\Big\{ - \frac{t^2}{2 \max_{a \leq d_n} n^{-1} \sum_{i=1}^n (\hat{\varepsilon}_{ia} - \tilde{\varepsilon}_{ia})^2} \Big\},
\end{aligned} \tag{S3.24}$$

where the first inequality is due to union bound inequality, and the second inequality holds from Hoeffding inequality. Plugging $t = \{2 \max_{a \leq d_n} n^{-1} \sum_{i=1}^n (\hat{\varepsilon}_{ia} -$

$\tilde{\varepsilon}_{ia})^2\}^{1/2}\{\log(nd_n)\}^{1/2}$ into (S3.24) yields

$$P_e\big[\|\hat{T}_e - \tilde{T}_e\|_\infty \geq \{2\max_{a\leq d_n} n^{-1}\sum_{i=1}^{n}(\hat{\varepsilon}_{ia}-\tilde{\varepsilon}_{ia})^2\}^{1/2}\{\log(nd_n)\}^{1/2}\big]$$

$$\leq 2n^{-1}. \tag{S3.25}$$

To bound $\max_{a\leq d_n} n^{-1}\sum_{i=1}^{n}(\hat{\varepsilon}_{ia}-\tilde{\varepsilon}_{ia})^2$, first note that for each $a \leq d_n$,

$$\sum_{i=1}^{n}(\hat{\varepsilon}_{ia}-\tilde{\varepsilon}_{ia})^2$$
$$\leq 2\sum_{i=1}^{n_1}\Big[\frac{(\theta_{1,iu_ag_a}-\eta_{1,u_ag_a})(\theta_{1,iv_ah_a}-\eta_{1,v_ah_a})-\sigma_{1,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}-$$
$$\frac{(\theta_{1,iu_ag_a}-\hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a}-\hat{\eta}_{1,v_ah_a})-\hat{\sigma}_{1,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}\Big]^2+$$
$$2\sum_{i=1}^{n_1}\Big[\frac{(\theta_{1,iu_ag_a}-\hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a}-\hat{\eta}_{1,v_ah_a})-\hat{\sigma}_{1,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}-$$
$$\frac{(\theta_{1,iu_ag_a}-\hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a}-\hat{\eta}_{1,v_ah_a})-\hat{\sigma}_{1,u_av_ag_ah_a}}{\{n_1n^{-1}(\hat{m}_{1,aa}+n_1n_2^{-1}\hat{m}_{2,aa})\}^{1/2}}\Big]^2+$$
$$2n_1^2n_2^{-2}\sum_{i=1}^{n_2}\Big[\frac{(\theta_{2,iu_ag_a}-\eta_{2,u_ag_a})(\theta_{2,iv_ah_a}-\eta_{2,v_ah_a})-\sigma_{2,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}-$$
$$\frac{(\theta_{2,iu_ag_a}-\hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a}-\hat{\eta}_{2,v_ah_a})-\hat{\sigma}_{2,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}\Big]^2+$$
$$2n_1^2n_2^{-2}\sum_{i=1}^{n_2}\Big[\frac{(\theta_{2,iu_ag_a}-\hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a}-\hat{\eta}_{2,v_ah_a})-\hat{\sigma}_{2,u_av_ag_ah_a}}{\{n_1n^{-1}(m_{1,aa}+n_1n_2^{-1}m_{2,aa})\}^{1/2}}-$$
$$\frac{(\theta_{2,iu_ag_a}-\hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a}-\hat{\eta}_{2,v_ah_a})-\hat{\sigma}_{2,u_av_ag_ah_a}}{\{n_1n^{-1}(\hat{m}_{1,aa}+n_1n_2^{-1}\hat{m}_{2,aa})\}^{1/2}}\Big]^2.$$

Together with (A1) yields that

$$\max_{a\leq d_n} n^{-1}\sum_{i=1}^{n}(\hat{\varepsilon}_{ia}-\tilde{\varepsilon}_{ia})^2 \lesssim \Pi_1+\Pi_2+\Pi_3+\Pi_4, \tag{S3.26}$$

where

$$\Pi_1 = \max_{a \le d_n} n_1^{-1} \sum_{i=1}^{n_1} \Big\{ \frac{(\theta_{1,iu_ag_a} - \eta_{1,u_ag_a})(\theta_{1,iv_ah_a} - \eta_{1,v_ah_a}) - \sigma_{1,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} - \frac{(\theta_{1,iu_ag_a} - \hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a} - \hat{\eta}_{1,v_ah_a}) - \hat{\sigma}_{1,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} \Big\}^2,$$

$$\Pi_2 = \max_{a \le d_n} n_1^{-1} \sum_{i=1}^{n_1} \Big\{ \frac{(\theta_{1,iu_ag_a} - \hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a} - \hat{\eta}_{1,v_ah_a}) - \hat{\sigma}_{1,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} - \frac{(\theta_{1,iu_ag_a} - \hat{\eta}_{1,u_ag_a})(\theta_{1,iv_ah_a} - \hat{\eta}_{1,v_ah_a}) - \hat{\sigma}_{1,u_av_ag_ah_a}}{(\hat{m}_{1,aa} + n_1 n_2^{-1} \hat{m}_{2,aa})^{1/2}} \Big\}^2,$$

$$\Pi_3 = \max_{a \le d_n} n_2^{-1} \sum_{i=1}^{n_2} \Big\{ \frac{(\theta_{2,iu_ag_a} - \eta_{2,u_ag_a})(\theta_{2,iv_ah_a} - \eta_{2,v_ah_a}) - \sigma_{2,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} - \frac{(\theta_{2,iu_ag_a} - \hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a} - \hat{\eta}_{2,v_ah_a}) - \hat{\sigma}_{2,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} \Big\}^2,$$

$$\Pi_4 = \max_{a \le d_n} n_2^{-1} \sum_{i=1}^{n_2} \Big\{ \frac{(\theta_{2,iu_ag_a} - \hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a} - \hat{\eta}_{2,v_ah_a}) - \hat{\sigma}_{2,u_av_ag_ah_a}}{(m_{1,aa} + n_1 n_2^{-1} m_{2,aa})^{1/2}} - \frac{(\theta_{2,iu_ag_a} - \hat{\eta}_{2,u_ag_a})(\theta_{2,iv_ah_a} - \hat{\eta}_{2,v_ah_a}) - \hat{\sigma}_{2,u_av_ag_ah_a}}{(\hat{m}_{1,aa} + n_1 n_2^{-1} \hat{m}_{2,aa})^{1/2}} \Big\}^2.$$

To bound Π_1, first note that

$$\Pi_1 \lesssim \Big(\min_{a \le d_n} \frac{m_{1,aa}}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}} \Big)^{-1} \cdot \Big\{ \max_{a \le d_n} \frac{(\hat{\sigma}_{1,u_av_ag_ah_a} - \sigma_{1,u_av_ag_ah_a})^2}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}} + \max_{a \le d_n} \frac{(\hat{\eta}_{1,u_ag_a} - \eta_{1,u_ag_a})^2 (\hat{\eta}_{1,v_ah_a} - \eta_{1,v_ah_a})^2}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}} + \max_{a \le d_n} \frac{(\hat{\eta}_{1,v_ah_a} - \eta_{1,v_ah_a})^2 (\hat{\sigma}_{1,u_au_ag_ag_a} - \sigma_{1,u_au_ag_ag_a})}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}} + \max_{a \le d_n} \frac{(\hat{\eta}_{1,u_ag_a} - \eta_{1,u_ag_a})^2 (\hat{\sigma}_{1,v_av_ah_ah_a} - \sigma_{1,v_av_ah_ah_a})}{\sigma_{1,u_au_ag_ag_a}\sigma_{1,v_av_ah_ah_a}} + \max_{a \le d_n} \frac{(\hat{\eta}_{1,v_ah_a} - \eta_{1,v_ah_a})^2}{\sigma_{1,v_av_ah_ah_a}} + \max_{a \le d_n} \frac{(\hat{\eta}_{1,u_ag_a} - \eta_{1,u_ag_a})^2}{\sigma_{1,u_au_ag_ag_a}} \Big\}.$$

Together with (A2), (A4), and Lemma 2 yields that there exist universal constants $c_{19}, c_{20} > 0$ such that with probability at least $1 - c_{19}n^{-1}$,

$$\Pi_1 \leq c_{20}n^{-1}\log(np_n s_n). \tag{S3.27}$$

In a similar fashion, it can be shown that there exist universal constants $c_{21}, c_{22} > 0$ such that with probability at least $1 - c_{21}n^{-1}$,

$$\Pi_3 \leq c_{22}n^{-1}\log(np_n s_n). \tag{S3.28}$$

To bound Π_2, first note that

$$\begin{aligned}\Pi_2 \leq & (\Omega_2^*)^2 \max_{a\leq d_n} \frac{\hat{m}_{1,aa}}{m_{1,aa}} \leq (\Omega_2^*)^2 \Big(1 + \max_{a\leq d_n} \frac{|\hat{m}_{1,aa} - m_{1,aa}|}{m_{1,aa}}\Big) \\ \leq & (\Omega_2^*)^2 \Big\{1 + \Big(\max_{a\leq d_n} \frac{|\hat{m}_{1,aa} - m_{1,aa}|}{\sigma_{1,u_a u_a g_a g_a}\sigma_{1,v_a v_a h_a h_a}}\Big) \Big/ \Big(\min_{a\leq d_n} \frac{m_{1,aa}}{\sigma_{1,u_a u_a g_a g_a}\sigma_{1,v_a v_a h_a h_a}}\Big)\Big\}.\end{aligned}$$

Together with (S3.20), (A2), (A4), and Lemma 3 yields that there exist universal constants $c_{23}, c_{24} > 0$ such that with probability at least $1 - c_{23}n^{-1}$,

$$\Pi_2 \leq c_{24}n^{-1}\log(np_n s_n). \tag{S3.29}$$

In a similar fashion, it can be shown that there exist universal constants $c_{25}, c_{26} > 0$ such that with probability at least $1 - c_{25}n^{-1}$,

$$\Pi_4 \leq c_{26}n^{-1}\log(np_n s_n). \tag{S3.30}$$

By combining (S3.30), (S3.29), (S3.28), (S3.27) with (S3.26), it can be concluded that there exist universal constants $c_{27}, c_{28} > 0$ such that with probability at least $1 - c_{27}n^{-1}$,

$$\max_{a \leq d_n} n^{-1} \sum_{i=1}^{n} (\hat{\varepsilon}_{ia} - \tilde{\varepsilon}_{ia})^2 \leq c_{28} n^{-1} \log(np_n s_n). \tag{S3.31}$$

Based on (S3.31) and (S3.25), it can be deduced that there exists a universal constant $c_{29} > 0$ such that

$$P_e(\|\hat{T}_e - \tilde{T}_e\|_\infty \geq c_{29} n^{-1/2} \log(np_n s_n)) \xrightarrow{p} 0.$$

Together with (S3.23) yields that there is a universal constant $c_{30} > 0$ such that

$$P(\|\breve{T} - \tilde{T}\|_\infty \geq a_n) \to 0,$$
$$P_e(\|\hat{T}_e - \tilde{T}_e\|_\infty \geq a_n) \xrightarrow{p} 0, \tag{S3.32}$$

where $a_n = c_{30} n^{-1/2} \log(np_n s_n)$. Based on (A2), it can be verified that

$$a_n^2 \max\{1, \log(d_n/a_n)\} \to 0. \tag{S3.33}$$

To this end, by combining (S3.33), (S3.32), (S3.11), (S3.10), (S3.9), (A2) with Lemma 5 in Xue and Yao (2024), it can be concluded that

$$\lim_{n\to\infty} \sup_{A \in \mathcal{A}^{Re}} \left| P(\breve{T} \in A) - P_e(\hat{T}_e \in A) \right|$$
$$= \lim_{n\to\infty} \sup_{t \geq 0} \left| P(\|\breve{T}\|_\infty \leq t) - P_e(\|\hat{T}_e\|_\infty \leq t) \right| = 0. \tag{S3.34}$$

Together with the notations in (S1.1) implies that

$$\lim_{n\to\infty} \sup_{t\geq 0} \left| P(\hat{G}(F_{\{b_k : k \leq s_n\}}(K^X - K^Y)) \leq t) - P_e(\hat{G}_e \leq t) \right| = 0,$$

which completes the proof. □

Proof of Theorem 2. Given the actual difference $K^X - K^Y$, one has

$$\lim_{n\to\infty} \big|power(K^X - K^Y) - power^*(K^X - K^Y)\big|$$
$$= \lim_{n\to\infty} \big|P\{\|\breve{T} + \hat{J}(K^X - K^Y)\|_\infty \le c_B(\alpha)\} - P_{e^*}\{\|\hat{T}_{e^*} + \hat{J}(K^X - K^Y)\|_\infty \le c_B(\alpha)\}\big|$$
$$\le \lim_{n\to\infty} \sup_{A\in\mathcal{A}^{Re}} \big|P(\breve{T} \in A) - P_{e^*}(\hat{T}_{e^*} \in A)\big| = 0,$$

where the last equality holds from (S3.34). This finishes the proof. □

Proof of Theorem 3. To begin with, the triangle inequality implies that

$$\begin{aligned} & power^*(K^X - K^Y) \\ =& 1 - P_{e^*}\{\|\hat{T}_{e^*} + \hat{J}(K^X - K^Y)\|_\infty \le c_B(\alpha)\} \\ \ge& 1 - P_{e^*}\{\|\hat{T}_{e^*}\|_\infty \ge \|\hat{J}(K^X - K^Y)\|_\infty - c_B(\alpha)\}. \end{aligned} \tag{S3.35}$$

In addition, we have that for any $t > 0$,

$$\begin{aligned} & P_{e^*}(\|\hat{T}_{e^*}\|_\infty \ge t) = P_{e^*}(\max_{a\le d_n} |n^{-1/2}\sum_{i=1}^n e_i^* \hat{\varepsilon}_{ia}| \ge t) \\ \le& \sum_{a=1}^{d_n} P_{e^*}(|n^{-1/2}\sum_{i=1}^n e_i^*\hat{\varepsilon}_{ia}| \ge t) \le 2\sum_{a=1}^{d_n} \exp\Big(-\frac{t^2}{2n^{-1}\sum_{i=1}^n \hat{\varepsilon}_{ia}^2}\Big) \\ \le& 2d_n \exp\Big(-\frac{t^2}{2\max_{a\le d_n} n^{-1}\sum_{i=1}^n \hat{\varepsilon}_{ia}^2}\Big), \end{aligned} \tag{S3.36}$$

where the first inequality is by union bound inequality, and the second inequality holds from Hoeffding inequality. Plugging $t = c_B(\alpha)$ into (S3.36) yields

$$c_B(\alpha) \le \{4\log(d_n) \cdot \max_{a\le d_n} n^{-1}\sum_{i=1}^n \hat{\varepsilon}_{ia}^2\}^{1/2}. \tag{S3.37}$$

By definition, it can be verified that

$$n^{-1}\sum_{i=1}^{n}\hat{\varepsilon}_{ia}^{2}=1, \quad \text{for all } \ a\le d_n.$$

Together with (S3.37) implies that

$$c_B(\alpha)\le 2\log^{1/2}(d_n). \tag{S3.38}$$

To bound $\|\hat{J}(K^X-K^Y)\|_\infty$, first note that

$$\begin{aligned}
\|\hat{J}(K^X-K^Y)\|_\infty &= n_1^{1/2}\max_{a\le d_n}\frac{|\sigma_{1,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a}|}{(\hat{m}_{1,aa}+n_1n_2^{-1}\hat{m}_{2,aa})^{1/2}}\\
=&n_1^{1/2}\max_{a\le d_n}\frac{|\sigma_{1,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a}|}{(m_{1,aa}+n_1n_2^{-1}m_{2,aa})^{1/2}}\cdot\frac{(m_{1,aa}+n_1n_2^{-1}m_{2,aa})^{1/2}}{(\hat{m}_{1,aa}+n_1n_2^{-1}\hat{m}_{2,aa})^{1/2}}\\
\ge&n_1^{1/2}(1-\Omega_2^*)\max_{a\le d_n}\frac{|\sigma_{1,u_av_ag_ah_a}-\sigma_{2,u_av_ag_ah_a}|}{(m_{1,aa}+n_1n_2^{-1}m_{2,aa})^{1/2}}.
\end{aligned}$$

Together with (A1), (A2), (S3.20), and the definition of $\mathcal{F}_n$, it is seen that there are universal constants $c_1,c_2>0$ such that with probability at least $1-c_1n^{-1}$,

$$\|\hat{J}(K^X-K^Y)\|_\infty\ge c_2K\log^{1/2}(np_ns_n). \tag{S3.39}$$

By choosing $K\ge 16c_2^{-1}$ in $\mathcal{F}_n$, it follows from (S3.38) and (S3.39) that with probability at least $1-c_1n^{-1}$,

$$\|\hat{J}(K^X-K^Y)\|_\infty-c_B(\alpha)\ge 2^{1/2}\log^{1/2}(np_n^2s_n^2). \tag{S3.40}$$

Plugging $t=\|\hat{J}(K^X-K^Y)\|_\infty-c_B(\alpha)$ into (S3.36) yields

$$\begin{aligned}
&P_{e^*}(\|\hat{T}_{e^*}\|_\infty\ge\|\hat{J}(K^X-K^Y)\|_\infty-c_B(\alpha))\\
\le&2p_n^2s_n^2\exp\big[-2^{-1}\{\|\hat{J}(K^X-K^Y)\|_\infty-c_B(\alpha)\}^2\big].
\end{aligned} \tag{S3.41}$$

By combining (S3.41), (S3.40) with (S3.35), it can be concluded that with probability at least $1 - c_1 n^{-1}$,

$$power^*(K^X - K^Y) \geq 1 - 2n^{-1} \rightarrow 1.$$

This completes the proof. $\square$

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