Supplementary Material of "Two-Way Factor Model Framework for High-Dimensional Panel Time Series"

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1 Lemmas and Proofs

Lemma 1. *Let* $cov(\eta_t) = \Sigma_\epsilon$*, then*

$$\Omega_E^{-1} = \begin{bmatrix} \Gamma_1 & \Gamma_4 & \mathbf{0} & \cdots & \mathbf{0} \\ \Gamma_4^\tau & \Gamma_2 & \Gamma_4 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \Gamma_4^\tau & \Gamma_2 & \Gamma_4 \\ \mathbf{0} & \cdots & \mathbf{0} & \Gamma_4^\tau & \Gamma_3 \end{bmatrix},$$

where $\Gamma_3 = (\Sigma_E - \Phi\Sigma_E\Phi^\tau)^{-1}$, $\Gamma_1 = \Sigma_E^{-1} + \Phi^\tau\Gamma_3\Phi$, $\Gamma_2 = \Gamma_3 + \Phi^\tau\Gamma_3\Phi$, $\Gamma_4 = -\Phi^\tau\Gamma_3$ *and* $\Sigma_E = \Phi\Sigma_E\Phi^\tau + \Sigma_\epsilon$.

Proof of Lemma 1. For further details, please refer to Proposition 2.1 in **?**. The proof is omitted.

□

Lemma 2. *Assuming that Assumption 2.1 holds, the inverse matrix of* Ω_Y *is*

$$\begin{aligned} \Omega_Y^{-1} &= \sigma_\varepsilon^{-2}\Big\{\Big[I_T - \frac{1}{T}\sigma_\varepsilon^{-2}LL^\tau + \frac{1}{T^2}\sigma_\varepsilon^{-2}L\Sigma_F^{-1}L^\tau\Big] \otimes \Big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\Big)\Big\} \\ &\quad + \frac{1}{N^2}\sigma_\varepsilon^{-4}\Big(I_T \otimes \Lambda\Big)\Omega_E^{-1}\Big(I_T \otimes \Lambda^\tau\Big), \end{aligned}$$

where $\mathbf{1}_m$ *denotes a* $m \times m$ *matrix with each element being 1.*

Proof of Lemma 2. Let $A_1 = (L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T) \otimes I_c$, $A_2 = (L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T) \otimes I_N$, $B = I_T \otimes \Lambda$, by $\Lambda^\tau \Lambda = \sigma_\varepsilon^2 N I_c$, we have $B^\tau A_2^{-1} B = N\sigma_\varepsilon^2 A_1^{-1}$. Then,

$$
\begin{aligned}
\Omega_Y^{-1} &= \Big(\big(L\Sigma_F L^\tau\big) \otimes I_N + \big(I_T \otimes \Lambda\big)\Omega_E\big(I_T \otimes \Lambda^\tau\big) + \sigma_\varepsilon^2 I_{TN} \Big)^{-1} \\
&= \Big(\big(L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T\big) \otimes I_N + \big(I_T \otimes \Lambda\big)\Omega_E\big(I_T \otimes \Lambda^\tau\big) \Big)^{-1} \\
&= \Big(A_2 + B\Omega_E B^\tau \Big)^{-1} = A_2^{-1} - A_2^{-1} B \Big(B^\tau A_2^{-1} B + \Omega_E^{-1} \Big)^{-1} B^\tau A_2^{-1} \\
&= A_2^{-1} - A_2^{-1} B \Big(N\sigma_\varepsilon^2 A_1^{-1} + \Omega_E^{-1} \Big)^{-1} B^\tau A_2^{-1} \\
&= A_2^{-1} - A_2^{-1} B \frac{1}{N\sigma_\varepsilon^2} \Big\{ A_1 + A_1 \frac{\Omega_E^{-1}}{N\sigma_\varepsilon^2} \sum_{k=1}^{\infty} (-1)^k \Big(A_1 \frac{\Omega_E^{-1}}{N\sigma_\varepsilon^2} \Big)^{k-1} A_1 \Big\} B^\tau A_2^{-1} \\
&= A_2^{-1} - \frac{1}{N\sigma_\varepsilon^2} \Big(L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T \Big)^{-1} \otimes \big(\Lambda\Lambda^\tau\big) - B \frac{\Omega_E^{-1}}{N^2\sigma_\varepsilon^4} \sum_{k=1}^{\infty} (-1)^k \Big(A_1 \frac{\Omega_E^{-1}}{N\sigma_\varepsilon^2} \Big)^{k-1} B^\tau \\
&= A_2^{-1} - \frac{1}{N\sigma_\varepsilon^2} \Big(L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T \Big)^{-1} \otimes \big(\Lambda\Lambda^\tau\big) + B \frac{\Omega_E^{-1}}{N^2\sigma_\varepsilon^4} B^\tau + B \frac{\Omega_E^{-1}}{N^2\sigma_\varepsilon^4} \sum_{k=1}^{\infty} (-1)^k \Big(A_1 \frac{\Omega_E^{-1}}{N\sigma_\varepsilon^2} \Big)^{k} B^\tau .
\end{aligned}
$$

By Lemma 1, it is easy to know that Ω_E^{-1} is block tridiagonal. A_1 is also a sparse block matrix. Thus, according to the structure of A_1 and Ω_E^{-1}, it is easy to see that each element in $A_1 \Omega_E^{-1}$ is $\mathbf{O}(1)$. Hence, for any $k \geq 1$, $\Big(A_1 \frac{\Omega_E^{-1}}{N\sigma_\varepsilon^2} \Big)^k = \mathbf{O}\big(T^{k-1} N^{-k}\big)$.

Furthermore, by IC: $L^\tau L = \sigma_\varepsilon^2 T I_r$, $\Lambda^\tau \Lambda = \sigma_\varepsilon^2 N I_c$, Σ_F and Σ_E are diagonal, we have under $\mathbf{O}\big(T^{k-1} N^{-k}\big) = \mathbf{o}(1)$, and

$$
\begin{aligned}
\Omega_Y^{-1} &= A_2^{-1} - \frac{1}{N\sigma_\varepsilon^2} \Big(L\Sigma_F L^\tau + \sigma_\varepsilon^2 I_T \Big)^{-1} \otimes \big(\Lambda\Lambda^\tau\big) + B \frac{\Omega_E^{-1}}{N^2\sigma_\varepsilon^4} B^\tau + \mathbf{o}\Big(\frac{1}{N^2}\Big) \mathbf{1}_{TN} \\
&= \Big\{ \Big[\sigma_\varepsilon^{-2} I_T - \frac{1}{T} \sigma_\varepsilon^{-4} L L^\tau + \frac{1}{T^2} \sigma_\varepsilon^{-4} L \Sigma_F^{-1} L^\tau + \mathbf{o}\Big(\frac{1}{T^2}\Big) \mathbf{1}_T \Big] \otimes I_N \Big\} \\
&\quad - \Big\{ \Big[\sigma_\varepsilon^{-2} I_T - \frac{1}{T} \sigma_\varepsilon^{-4} L L^\tau + \frac{1}{T^2} \sigma_\varepsilon^{-4} L \Sigma_F^{-1} L^\tau + \mathbf{o}\Big(\frac{1}{T^2}\Big) \mathbf{1}_T \Big] \otimes \big(\Lambda\Lambda^\tau\big) \Big\} + \mathbf{o}\Big(\frac{1}{N^2}\Big) \mathbf{1}_{TN} \\
&= \sigma_\varepsilon^{-2} \Big\{ \Big[I_T - \frac{1}{T} \sigma_\varepsilon^{-2} L L^\tau + \frac{1}{T^2} \sigma_\varepsilon^{-2} L \Sigma_F^{-1} L^\tau \Big] \otimes \Big(I_N - \frac{1}{N} \sigma_\varepsilon^{-2} \Lambda\Lambda^\tau \Big) \Big\} \\
&\quad + \frac{1}{N^2} \sigma_\varepsilon^{-4} \Big(I_T \otimes \Lambda \Big) \Omega_E^{-1} \Big(I_T \otimes \Lambda^\tau \Big) + \Big\{ \mathbf{o}\Big(\frac{1}{T^2}\Big) + \mathbf{o}\Big(\frac{1}{N^2}\Big) \Big\} \mathbf{1}_{TN} \\
&\approx \sigma_\varepsilon^{-2} \Big\{ \Big[I_T - \frac{1}{T} \sigma_\varepsilon^{-2} L L^\tau + \frac{1}{T^2} \sigma_\varepsilon^{-2} L \Sigma_F^{-1} L^\tau \Big] \otimes \Big(I_N - \frac{1}{N} \sigma_\varepsilon^{-2} \Lambda\Lambda^\tau \Big) \Big\} \\
&\quad + \frac{1}{N^2} \sigma_\varepsilon^{-4} \Big(I_T \otimes \Lambda \Big) \Omega_E^{-1} \Big(I_T \otimes \Lambda^\tau \Big).
\end{aligned}
$$

□

Lemma 3. *Assuming that both Assumptions 2.1 and 2.2 hold, then*

(1) $\Omega_Y^{-1} \Big(I_T \otimes \Lambda \Big) \Omega_E = \frac{1}{N} \sigma_\varepsilon^{-2} \Big(I_T \otimes \Lambda \Big) + \Big\{ \mathbf{o}\Big(\frac{1}{T^2}\Big) + \mathbf{o}\Big(\frac{1}{N^2}\Big) \Big\} \mathbf{1}_{Tc}$;

(2) $\Omega_E \Big(I_T \otimes \Lambda^\tau \Big) \Omega_Y^{-1} \Big(I_T \otimes \Lambda \Big) = I_{Tc} + \Big\{ \mathbf{o}\Big(\frac{N}{T^2}\Big) + \mathbf{o}\Big(\frac{1}{N}\Big) \Big\} \mathbf{1}_{TN,Tc}$;

(3) $\Omega_Y^{-1}\Big(\big(L\Sigma_F\big)\otimes I_N\Big)=\frac{1}{T}\sigma_\varepsilon^{-2}\Big(L\otimes I_N\Big)+\Big\{\mathbf{O}\Big(\frac{1}{NT}\Big)+\mathbf{O}\Big(\frac{1}{N^2}\Big)\Big\}\mathbf{1}_{TN,rN}$;

(4) $\Omega_Y^{-1}\Big(L\otimes I_N\Big)=\frac{1}{T}\sigma_\varepsilon^{-2}\Big(\big(L\Sigma_F^{-1}\big)\otimes I_N\Big)+\Big\{\mathbf{O}\Big(\frac{1}{NT}\Big)+\mathbf{O}\Big(\frac{1}{N^2}\Big)\Big\}\mathbf{1}_{TN,rN}$;

(5) $\Big(L^\tau\otimes I_N\Big)\Omega_Y^{-1}\Big(L\otimes I_N\Big)=\Big(\Sigma_F^{-1}\otimes I_N\Big)+\Big\{\mathbf{O}\Big(\frac{T}{N^2}\Big)+\mathbf{O}\Big(\frac{1}{N}\Big)\Big\}\mathbf{1}_{Nr}$.

Proof of Lemma 3. It can be easily obtained using IC and Lemma 2. Further details have been omitted for brevity.

□

Lemma 4. (1)

$$\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}vec(\Sigma_E)}=\begin{pmatrix}-\Sigma_E^{-1}\otimes\Sigma_E^{-1}-(\Phi^\tau\Gamma_3)\otimes(\Phi^\tau\Gamma_3)+(\Phi^\tau\Gamma_3\Phi)\otimes(\Phi^\tau\Gamma_3\Phi)\\ -\Gamma_3\otimes\Gamma_3+(\Gamma_3\Phi)\otimes(\Gamma_3\Phi)-(\Phi^\tau\Gamma_3)\otimes(\Phi^\tau\Gamma_3)+(\Phi^\tau\Gamma_3\Phi)\otimes(\Phi^\tau\Gamma_3\Phi)\\ -\Gamma_3\otimes\Gamma_3+(\Gamma_3\Phi)\otimes(\Gamma_3\Phi)\\ \Gamma_3\otimes(\Phi^\tau\Gamma_3)-(\Gamma_3\Phi)\otimes(\Phi^\tau\Gamma_3\Phi)\end{pmatrix};$$

(2)

$$\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}vec(\Phi)}=\begin{pmatrix}(K_{cc}+I_{c^2})\big(I_c\otimes(\Phi^\tau\Gamma_3)+(\Phi^\tau\Gamma_3\Phi\Sigma_E)\otimes(\Phi^\tau\Gamma_3)\big)\\ (K_{cc}+I_{c^2})\big((\Gamma_3\Phi\Sigma_E)\otimes\Gamma_3+I_c\otimes(\Phi^\tau\Gamma_3)+(\Phi^\tau\Gamma_3\Phi\Sigma_E)\otimes(\Phi^\tau\Gamma_3)\big)\\ (K_{cc}+I_{c^2})\big((\Gamma_3\Phi\Sigma_E)\otimes\Gamma_3\big)\\ -K_{cc}(I_c\otimes\Gamma_3)-\big((\Gamma_3\Phi\Sigma_E)\otimes(\Phi^\tau\Gamma_3)\big)-K_{cc}\big((\Phi^\tau\Gamma_3\Phi\Sigma_E)\otimes\Gamma_3\big)\end{pmatrix}.$$

Proof of Lemma 4. It can be obtained by direct calculation, although the details are not mentioned.

□

Proof of Theorem 2.1. The derivative of the negative log-likelihood function can be calculated as follows:

$$\boldsymbol{d}\mathbb{L}=\frac{1}{2}tr\big[(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1})\boldsymbol{d}\Omega_Y\big],$$

where

$$\begin{aligned}\boldsymbol{d}\Omega_Y &= \boldsymbol{d}\sigma_\varepsilon^2I_{TN}+(\boldsymbol{d}L\Sigma_FL^\tau)\otimes I_N+(L\boldsymbol{d}\Sigma_FL^\tau)\otimes I_N+(L\Sigma_F\boldsymbol{d}L^\tau)\otimes I_N\\ &\quad+(I_T\otimes\boldsymbol{d}\Lambda)\Omega_E(I_T\otimes\Lambda^\tau)+(I_T\otimes\Lambda)\boldsymbol{d}\Omega_E(I_T\otimes\Lambda^\tau)+(I_T\otimes\Lambda)\Omega_E(I_T\otimes\boldsymbol{d}\Lambda^\tau).\end{aligned}$$

We begin by examining the total differential of $\mathbb{L}$, which can be expressed as follows:

$$\begin{aligned}\boldsymbol{d}\mathbb{L} &= \frac{1}{2}\big\{tr\big[\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)(\boldsymbol{d}\Omega_Y)\big]\big\}\\ &= \frac{1}{2}\Big\{tr\Big[\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\Big(\boldsymbol{d}\sigma_\varepsilon^2I_{TN}+(\boldsymbol{d}L\Sigma_FL^\tau)\otimes I_N\\ &\quad+(L\Sigma_F\boldsymbol{d}L^\tau)\otimes I_N+(L\boldsymbol{d}\Sigma_FL^\tau)\otimes I_N+(I_T\otimes\boldsymbol{d}\Lambda)\Omega_E(I_T\otimes\Lambda^\tau)\\ &\quad+(I_T\otimes\Lambda)\Omega_E(I_T\otimes\boldsymbol{d}\Lambda^\tau)+(I_T\otimes\Lambda)\boldsymbol{d}\Omega_E(I_T\otimes\Lambda^\tau)\Big)\Big]\Big\}\\ &= \frac{1}{2}tr\big[\boldsymbol{d}\sigma_\varepsilon^2\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\big]+tr\Big[\big(\boldsymbol{d}L^\tau\otimes I_N\big)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\Big(\big(L\Sigma_F\big)\otimes I_N\Big)\Big]\\ &\quad+\frac{1}{2}\Big\{tr\Big[\big(\boldsymbol{d}\Sigma_F\otimes I_N\big)\big(L^\tau\otimes I_N\big)\Big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\Big)\big(L\otimes I_N\big)\Big]\Big\}\end{aligned}$$

$$
\begin{aligned}
&+tr\Big[(I_T\otimes \boldsymbol{d}\Lambda^\tau)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\big(I_T\otimes \boldsymbol{d}\Lambda\big)\Omega_E\Big]\\
&-\frac{1}{2}\Big\{tr\Big[\boldsymbol{d}\Omega_E^{-1}\Omega_E\big(I_T\otimes\Lambda^\tau\big)\Big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\Big)\big(I_T\otimes\Lambda\big)\Omega_E\Big]\Big\}\\
= \;&\frac{1}{2}\boldsymbol{d}\sigma_\varepsilon^2 tr\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)+tr\Big[\boldsymbol{d}L^\tau \mathrm{trs}\Big(K_{NT}\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)K_{TN}\Big)L\Sigma_F\Big]\\
&+\frac{1}{2}\Big\{tr\Big[\boldsymbol{d}\Sigma_F L^\tau \mathrm{trs}\Big(K_{NT}\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)K_{TN}\Big)L\Big]\Big\}\\
&+tr\Big[\boldsymbol{d}\Lambda^\tau \mathrm{trs}\Big(\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\big(I_T\otimes\Lambda\big)\Omega_E\Big)\Big]\\
&-\frac{1}{2}\Big\{tr\Big[\boldsymbol{d}\Omega_E^{-1}\Big(\Omega_E\big(I_T\otimes\Lambda^\tau\big)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\big(I_T\otimes\Lambda\big)\Omega_E\Big)\Big]\Big\}. \qquad \text{(A.1)}
\end{aligned}
$$

Using the property $tr(A^\tau B) = \mathrm{vec}^\tau(A)\mathrm{vec}(B)$ for any two matrices A and B, and Lemma 3, we obtain:

$$
\begin{aligned}
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\sigma_\varepsilon^2} &= \frac{1}{2}tr\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big),\\
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\mathrm{vec}(L)} &= \mathrm{vec}\Big\{\mathrm{trs}\big[K_{NT}\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)K_{TN}\big]L\Sigma_F\Big\}\\
&= \mathrm{vec}\Big\{NL/(T\sigma_\varepsilon^2)-\mathrm{trs}\big(K_{TN}\Omega_Y^{-1}SK_{TN}\big)L/(T\sigma_\varepsilon^2)\Big\},\\
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\mathrm{vec}(\Lambda)} &= \mathrm{vec}\Big\{\mathrm{trs}\big[\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)(I_T\otimes\Lambda)\Omega_E\big]\Big\}\\
&= \mathrm{vec}\Big\{T\Lambda/(N\sigma_\varepsilon^2)-\mathrm{trs}\big(\Omega_Y^{-1}S\big)\Lambda/(N\sigma_\varepsilon^2)\Big\},\\
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_F)} &= \frac{1}{2}\mathrm{vec}\Big\{\mathrm{trs}\big[K_{Nr}(L^\tau\otimes I_N)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)\big](L\otimes I_N)K_{rN}\Big\}\\
&= \frac{1}{2}\mathrm{vec}\Big\{N\Sigma_F^{-1}-\Sigma_F^{-1}L^\tau\mathrm{trs}\big(K_{TN}SK_{NT}\big)L\Sigma_F^{-1}/(T^2\sigma_\varepsilon^4)\Big\},\\
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} &= -\frac{1}{2}\Upsilon_1^\tau\mathrm{vec}\Big\{\Omega_E(I_T\otimes\Lambda^\tau)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)(I_T\otimes\Lambda)\Omega_E\Big\}\\
&= -\frac{1}{2}\Upsilon_1^\tau\mathrm{vec}\Big\{\Omega_E-\Lambda^\tau\mathrm{trs}\big(S\big)\Lambda/(N^2\sigma_\varepsilon^4)\Big\},\\
\frac{\boldsymbol{\partial}\mathbb{L}}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} &= -\frac{1}{2}\Upsilon_2^\tau\mathrm{vec}\Big\{\Omega_E(I_T\otimes\Lambda^\tau)\big(\Omega_Y^{-1}-\Omega_Y^{-1}S\Omega_Y^{-1}\big)(I_T\otimes\Lambda)\Omega_E\Big\}\\
&= -\frac{1}{2}\Upsilon_2^\tau\mathrm{vec}\Big\{\Omega_E-\Lambda^\tau\mathrm{trs}\big(S\big)\Lambda/(N^2\sigma_\varepsilon^4)\Big\}.
\end{aligned}
$$

□

Proof of Theorem 2.2. The second derivative of the negative log likelihood function is

$$
\boldsymbol{d}^2\mathbb{L}=\frac{1}{2}tr\Big(-\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Omega_Y^{-1}\boldsymbol{d}\Omega_Y+\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Omega_Y^{-1}S\Omega_Y^{-1}\boldsymbol{d}\Omega_Y+\Omega_Y^{-1}S\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Big). \text{(A.2)}
$$

Then, we have

$$
\mathbb{E}\Big(\boldsymbol{d}^2\mathbb{L}\Big)
$$

$$
\begin{aligned}
&= \frac{1}{2}tr\Big(\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Big)\\
&= \frac{1}{2}tr\Big\{\Omega_Y^{-1}\boldsymbol{d}\sigma_\varepsilon^2 I_{TN} + \Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big] + \Omega_Y^{-1}\Big[\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\\
&\quad + \Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big] + \Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\\
&\quad + \Omega_Y^{-1}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big) + \Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\boldsymbol{d}\Lambda^\tau\Big)\Big\}^2\\
&= \frac{1}{2}tr\Big[\Big(\boldsymbol{d}\sigma_\varepsilon^2\Big)^2\Omega_Y^{-2}\Big] + 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Big\} + tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Big\}\\
&\quad + tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\} + 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(\boldsymbol{d}L\Sigma_F L^\tau + L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau + \boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau + \boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau + \boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\boldsymbol{d}\Lambda^\tau\Big)\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau + \boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
&\quad + \frac{1}{2}tr\Big\{\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Big\}^2 + tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\boldsymbol{d}\Lambda^\tau\Big)\Big\}^2\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\boldsymbol{d}\Lambda^\tau\Big)\Big\}\\
&\quad + 2tr\Big\{\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
&\quad + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
&\quad + 2tr\Big\{\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big\}\\
&\quad + \frac{1}{2}tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}^2. \qquad \text{(A.3)}
\end{aligned}
$$

Thus, we can obtain the block of $\mathbb{I}(\Theta)$ accordingly. By $\frac{1}{2}tr\Big[\Big(\boldsymbol{d}\sigma_\varepsilon^2\Big)^2\Omega_Y^{-2}\Big]$ in (A.3), we can easily obtain

$$
\mathcal{I}_{\sigma_\varepsilon^2,\sigma_\varepsilon^2} = \frac{1}{2T}tr\big(\Omega_Y^{-2}\big).
$$

By $tr(A^\tau B) = \mathrm{vec}^\tau(A)\mathrm{vec}(B)$ for any two matrices A and B, we have in (A.3)

$$
\begin{aligned}
& 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Big\}\\
&= 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big[K_{TN}\Big(I_N\otimes L\Sigma_F\boldsymbol{d}L^\tau\Big)K_{NT}\Big]\Big\}\\
&= 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Big[\Big(K_{NT}\Omega_Y^{-2}K_{TN}\Big)\Big(I_N\otimes L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]\Big\}\\
&= 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Big[\mathrm{trs}\Big(K_{NT}\Omega_Y^{-2}K_{TN}\Big)L\Sigma_F\boldsymbol{d}L^\tau\Big]\Big\}\\
&= 2\boldsymbol{d}\sigma_\varepsilon^2\mathrm{vec}^\tau\Big[\mathrm{trs}\Big(K_{NT}\Omega_Y^{-2}K_{TN}\Big)L\Sigma_F\Big]\mathrm{vec}\big(\boldsymbol{d}L^\tau\big).
\end{aligned}
$$

By Lemma 2, $\Omega_Y^{-2} \approx \sigma_\varepsilon^{-4}\Big\{\Big[I_T - \frac{1}{T}\sigma_\varepsilon^{-2}LL^\tau + \frac{1}{T^3}\sigma_\varepsilon^{-2}L\Sigma_F^{-2}L^\tau\Big] \otimes \Big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\Big)\Big\} + \frac{1}{N^3}\sigma_\varepsilon^{-6}\Big(I_T \otimes \Lambda\Big)\Omega_E^{-2}\Big(I_T \otimes \Lambda^\tau\Big)$. Then,

$$\begin{aligned}
\mathcal{I}_{\sigma_\varepsilon^2,L} &= \frac{1}{T}\mathrm{vec}^\tau\Big[\mathrm{trs}\Big(K_{NT}\Omega_Y^{-2}K_{TN}\Big)L\Sigma_F\Big] \\
&= \frac{1}{T}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-4}\mathrm{trs}\Big[\big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big) \otimes \Big(I_T - \frac{1}{T}\sigma_\varepsilon^{-2}LL^\tau + \frac{1}{T^3}\sigma_\varepsilon^{-2}L\Sigma_F^{-2}L^\tau\Big)\Big]L\Sigma_F\Big\} \\
&\quad + \frac{1}{TN^3}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-6}\mathrm{trs}\Big[K_{NT}\Big(I_T \otimes \Lambda\Big)\Omega_E^{-2}\Big(I_T \otimes \Lambda^\tau\Big)K_{TN}\Big]L\Sigma_F\Big\} \\
&= \frac{N}{T^3}\sigma_\varepsilon^{-4}\mathrm{vec}^\tau\Big(L\Sigma_F^{-1}\Big) + \frac{1}{TN^2}\sigma_\varepsilon^{-6}\lambda_{\max}(\Omega_E^{-2})\mathrm{vec}^\tau\Big[tr\Big(\Lambda\Lambda^\tau\Big)L\Sigma_F\Big] \\
&= \mathbf{O}\Big(\frac{N}{T^3}\Big)\mathrm{vec}^\tau\big(\mathbf{1}_{rT}\big) + \mathbf{O}\Big(\frac{1}{TN^2}\Big)\mathrm{vec}^\tau\big(\mathbf{1}_{rT}\big) = \mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_{rT}\big).
\end{aligned}$$

In (A.3),

$$\begin{aligned}
& tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big) \otimes I_N\Big]\Big\} \\
=\ & \boldsymbol{d}\sigma_\varepsilon^2 tr\Big\{\Big[K_{Nr}\big(L^\tau \otimes I_N\big)\Omega_Y^{-2}\big(L \otimes I_N\big)K_{rN}\Big]\big(I_N \otimes \boldsymbol{d}\Sigma_F\big)\Big\} \\
=\ & \boldsymbol{d}\sigma_\varepsilon^2 tr\Big\{\mathrm{trs}\Big[K_{Nr}\big(L^\tau \otimes I_N\big)\Omega_Y^{-2}\big(L \otimes I_N\big)K_{rN}\Big]\boldsymbol{d}\Sigma_F\Big\} \\
=\ & \boldsymbol{d}\sigma_\varepsilon^2 \mathrm{vec}^\tau\Big\{\mathrm{trs}\Big[K_{Nr}\big(L^\tau \otimes I_N\big)\Omega_Y^{-2}\big(L \otimes I_N\big)K_{rN}\Big]\Big\}\mathrm{vec}\big(\boldsymbol{d}\Sigma_F\big).
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{I}_{\sigma_\varepsilon^2,\Sigma_F} &= \frac{1}{2T}\mathrm{vec}^\tau\Big\{\mathrm{trs}\Big[K_{Nr}\big(L^\tau \otimes I_N\big)\Omega_Y^{-2}\big(L \otimes I_N\big)K_{rN}\Big]\Big\} \\
&= \frac{1}{2T^2}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-2}\mathrm{trs}\Big[K_{Nr}\Big(\Sigma_F^{-2} \otimes \big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big)\Big)K_{rN}\Big]\Big\} \\
&\quad + \frac{1}{2TN^3}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-6}\mathrm{trs}\Big[K_{Nr}\Big(L^\tau \otimes \Lambda\Big)\Omega_E^{-2}\big(L \otimes \Lambda^\tau\big)K_{rN}\Big]\Big\} \\
&= \frac{1}{2T^2}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-2}\mathrm{trs}\Big(\big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big) \otimes \Sigma_F^{-2}\Big)\Big\} \\
&\quad + \frac{1}{2TN^3}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-6}\mathrm{trs}\Big[K_{Nr}\big(L^\tau \otimes \Lambda\big)\Omega_E^{-2}\big(L \otimes \Lambda^\tau\big)K_{rN}\Big]\Big\} \\
&= \frac{N}{2T^2}\mathrm{vec}^\tau\Big(\sigma_\varepsilon^{-2}\Sigma_F^{-2}\Big) + \mathbf{O}\Big(\frac{1}{N^3}\Big)\mathrm{vec}^\tau\Big(\mathbf{1}_{rr}\Big) \\
&= \frac{N}{2T^2}\mathrm{vec}^\tau\Big(\sigma_\varepsilon^{-2}\Sigma_F^{-2}\Big) = \mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_r\big).
\end{aligned}$$

In (A.3),

$$\begin{aligned}
& 2tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big(I_T \otimes \boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T \otimes \Lambda^\tau\Big)\Big\} \\
=\ & 2\boldsymbol{d}\sigma_\varepsilon^2 tr\Big\{\Big[\Omega_E\Big(I_T \otimes \Lambda^\tau\Big)\Omega_Y^{-2}\Big(I_T \otimes \boldsymbol{d}\Lambda\Big)\Big]\Big\} \\
=\ & 2\frac{1}{N^2}\boldsymbol{d}\sigma_\varepsilon^2 tr\Big\{\sigma_\varepsilon^{-4}\Big[\Omega_E^{-1}\Big(I_T \otimes \Lambda^\tau\Big)\Big(I_T \otimes \boldsymbol{d}\Lambda\Big)\Big]\Big\}
\end{aligned}$$

$$= 2\frac{1}{N^2}\boldsymbol{d}\sigma_\varepsilon^2\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-4}\Big[\mathrm{trs}\Big(\Omega_E^{-1}\Big)\Lambda^\tau\Big]\Big\}\mathrm{vec}\big(\boldsymbol{d}\Lambda\big).$$

Then

$$\begin{aligned}\mathcal{I}_{\sigma_\varepsilon^2,\Lambda} &= \frac{1}{TN^2}\mathrm{vec}^\tau\Big\{\sigma_\varepsilon^{-4}\Big[\mathrm{trs}\Big(\Omega_E^{-1}\Big)\Lambda^\tau\Big]\Big\}\\ &= \mathbf{O}\Big(\frac{1}{N^2}\Big)\mathrm{vec}^\tau\big(\mathbf{1}_{cN}\big) = \mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_{cN}\big).\end{aligned}$$

In (A.3),

$$\begin{aligned}& tr\Big\{\boldsymbol{d}\sigma_\varepsilon^2\Omega_Y^{-2}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\ =\ & \boldsymbol{d}\sigma_\varepsilon^2 tr\Big\{\Big[\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big]\Big[\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Big)\Big]\Big\}\\ =\ & \boldsymbol{d}\sigma_\varepsilon^2\mathrm{vec}^\tau\Big\{\Big[\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big]\Big[\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big)\Big]\Big\}\mathrm{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big).\\ =\ & \boldsymbol{d}\sigma_\varepsilon^2\mathrm{vec}^\tau\Big(\frac{1}{N}\sigma_\varepsilon^{-2}I_{TN}\Big)\mathrm{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big).\end{aligned}$$

Then, by $\mathrm{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big) = \Upsilon_1\mathrm{vec}\big(\boldsymbol{d}\Sigma_E\big)$, $\mathrm{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big) = \Upsilon_2\mathrm{vec}\big(\boldsymbol{d}\Phi\big)$,

$$\begin{aligned}\mathcal{I}_{\sigma_\varepsilon^2,\Sigma_E} &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\mathrm{vec}^\tau\Big(I_{TN}\Big)\Upsilon_1,\\ &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\mathrm{vec}^\tau\Big(I_{TN}\Big)\Upsilon\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)}\\ &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\Big[\Upsilon^\tau\mathrm{vec}\Big(I_{TN}\Big)\Big]^\tau\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)}\\ &= \frac{1}{2N}\sigma_\varepsilon^{-2}\boldsymbol{\Delta}\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} = \mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_{r^2}\big),\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{\sigma_\varepsilon^2,\Phi} &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\mathrm{vec}^\tau\Big(I_{TN}\Big)\Upsilon_2,\\ &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\mathrm{vec}^\tau\Big(I_{TN}\Big)\Upsilon\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Phi)}\\ &= \frac{1}{2TN}\sigma_\varepsilon^{-2}\Big[\Upsilon^\tau\mathrm{vec}\Big(I_{TN}\Big)\Big]^\tau\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Phi)}\\ &= \frac{1}{2N}\sigma_\varepsilon^{-2}\boldsymbol{\Delta}^\tau\frac{\boldsymbol{\partial}\Gamma}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} = \mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_{r^2}\big),\end{aligned}$$

where $\boldsymbol{\Delta} = (\mathbf{0}, I_c, \mathbf{0}, \mathbf{0})$.

In (A.3), by $tr\big(A^\tau BCD\big) = \big[\mathrm{vec}(A)\big]^\tau\big(D^\tau\otimes B\big)\mathrm{vec}(C)$ for any matrices A, B, C and D,

$$\begin{aligned}& tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(\boldsymbol{d}L\Sigma_F L^\tau + L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Big\}\\ =\ & tr\Big\{\Big[\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big\}\\ & + tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\otimes I_N\Big]\Big\}\end{aligned}$$

$$
\begin{aligned}
&= tr\Big\{K_{TN}\Big[I_N\otimes\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}\Big\}\\
&\quad+tr\Big\{\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Big\}\\
&= tr\Big\{\Big[I_N\otimes\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big\}\\
&\quad+tr\Big\{\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big\}\\
&= tr\Big\{\Big(I_N\otimes\boldsymbol{d}L\Big)\Big[K_{Nr}\Big(\Sigma_F L^\tau\otimes I_N\Big)\Omega_Y^{-1}\Big(L\Sigma_F\otimes I_N\Big)K_{rN}\Big]\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big(K_{NT}\Omega_Y^{-1}K_{TN}\Big)\Big\}\\
&\quad+tr\Big\{\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big[K_{NT}\Omega_Y^{-1}\Big(L\Sigma_F\otimes I_N\Big)K_{rN}\Big]\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big[K_{NT}\Omega_Y^{-1}\Big(L\Sigma_F\otimes I_N\Big)K_{rN}\Big]\Big\}\\
&= tr\Big\{\Big(I_N\otimes\boldsymbol{d}L\Big)\Big(I_N\otimes\Sigma_F\Big)\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big(K_{NT}\Omega_Y^{-1}K_{TN}\Big)\Big\}\\
&\quad+\frac{1}{T^2}\sigma_\varepsilon^{-4}tr\Big\{\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big(I_N\otimes L\Big)\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\Big(I_N\otimes L\Big)\Big\}\\
&= tr\Big\{\boldsymbol{d}L\Sigma_F\boldsymbol{d}L^\tau\mathrm{trs}\Big(K_{NT}\Omega_Y^{-1}K_{TN}\Big)\Big\}+\frac{N}{T^2}\sigma_\varepsilon^{-4}tr\Big(\boldsymbol{d}L^\tau L\boldsymbol{d}L^\tau L\Big).
\end{aligned}
$$

The above equality is due to that $\frac{1}{N^2}\sigma_\varepsilon^{-4}\Big(K_{NT}(I_T\otimes\Lambda)\Omega_E^{-1}(L\Sigma_F\otimes\Lambda^\tau)K_{rN}\Big)=\mathbf{O}\Big(\frac{1}{N^2}\Big)\mathbf{1}_{NT,rN}$. Then, we have

$$
\begin{aligned}
\mathcal{I}_{L,L} &= \frac{1}{T}\Sigma_F\otimes\mathrm{trs}\Big(K_{NT}\Omega_Y^{-1}K_{TN}\Big)+\mathbf{O}\Big(\frac{N}{T^3}\Big)\mathbf{1}_{rT}\\
&= \frac{1}{T}\Sigma_F\otimes\mathrm{trs}\Big(K_{NT}\Omega_Y^{-1}K_{TN}\Big).
\end{aligned}
$$

In (A.3),

$$
\begin{aligned}
&tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau+\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\otimes I_N\Big]\Big\}\\
&= tr\Big\{\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\Sigma_F\boldsymbol{d}L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\Big]K_{NT}\Big\}\\
&\quad+tr\Big\{\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\Big]K_{NT}\Omega_Y^{-1}K_{TN}\Big[I_N\otimes\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big)\Big]K_{NT}\Big\}\\
&= tr\Big\{K_{Nr}\Big(L^\tau\otimes I_N\Big)\Omega_Y^{-1}\Big(L\Sigma_F\otimes I_N\Big)K_{rN}\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)K_{NT}\Omega_Y^{-1}\Big(L\otimes I_N\Big)K_{rN}\Big(I_N\otimes\boldsymbol{d}\Sigma_F\Big)\Big\}\\
&\quad+tr\Big\{K_{Nr}\Big(L^\tau\otimes I_N\Big)\Omega_Y^{-1}K_{TN}\Big(I_N\otimes\boldsymbol{d}L\Big)K_{Nr}\Big(\Sigma_F L^\tau\otimes I_N\Big)\Omega_Y^{-1}\Big(L\otimes I_N\Big)K_{rN}\Big(I_N\otimes\boldsymbol{d}\Sigma_F\Big)\Big\}\\
&= 2\sigma_\varepsilon^{-2}tr\Big\{K_{Nr}\Big[\Big(I_r\otimes\big(I_N-\frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big)\Big)+\frac{1}{N^2}\sigma_\varepsilon^{-4}\Big(L^\tau\otimes\Lambda\Big)\Omega_E^{-1}\Big(L\Sigma_F\otimes\Lambda^\tau\Big)\Big]K_{rN}\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\\
&\quad K_{NT}\Big[\Big(\frac{1}{T}L\Sigma_F^{-1}\otimes\big(I_N-\frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big)\Big)+\frac{1}{N^2}\sigma_\varepsilon^{-4}\Big(I_T\otimes\Lambda\Big)\Omega_E^{-1}\Big(L\otimes\Lambda^\tau\Big)\Big]K_{rN}\Big(I_N\otimes\boldsymbol{d}\Sigma_F\Big)\Big\}\\
&= 2\sigma_\varepsilon^{-2}tr\Big\{K_{Nr}\Big[\Big(I_r\otimes\big(I_N-\frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big)\Big)+\mathbf{O}\Big(\frac{T}{N^2}\Big)\mathbf{1}_{rN}\Big]K_{rN}\Big(I_N\otimes\boldsymbol{d}L^\tau\Big)\\
&\quad K_{NT}\Big[\Big(\frac{1}{T}L\Sigma_F^{-1}\otimes\big(I_N-\frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\big)\Big)+\mathbf{O}\Big(\frac{1}{N^2}\Big)\mathbf{1}_{TN,rN}\Big]K_{rN}\Big(I_N\otimes\boldsymbol{d}\Sigma_F\Big)\Big\}\\
&= 2\sigma_\varepsilon^{-2}\frac{N}{T}tr\Big\{\boldsymbol{d}L^\tau L\Sigma_F^{-1}\boldsymbol{d}\Sigma_F\Big\}.
\end{aligned}
$$

Then, we have

$$
\mathcal{I}_{L,\Sigma_F} = \sigma_\varepsilon^{-2}\frac{N}{T^2}I_r\otimes\big(L\Sigma_F^{-1}\big)=\mathbf{o}(1)\mathrm{vec}^\tau\big(\mathbf{1}_{rT,r^2}\big).
$$

In (A.3),

$$
\begin{aligned}
& tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau+\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
& +tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau+\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\boldsymbol{d}\Lambda^\tau\Big)\Big\}\\
= & \ tr\Big\{\boldsymbol{d}\Big[\Omega_Y^{-1}\Big(L\Sigma_F L^\tau\otimes I_N\Big)\Big]\Big|_L\boldsymbol{d}\Big[\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big]\Big|_\Lambda\Big\}\\
= & \ \frac{1}{NT}\sigma_\varepsilon^{-4}tr\Big\{\boldsymbol{d}\Big[\Big(LL^\tau\otimes I_N\Big)+\Big\{\mathbf{O}\Big(\frac{1}{NT}\Big)+\mathbf{O}\Big(\frac{1}{N^2}\Big)\Big\}\mathbf{1}_{TN}\Big]\Big|_L\\
& \qquad \boldsymbol{d}\Big[\Big(I_T\otimes\Lambda\Lambda^\tau\Big)+\Big\{\mathbf{o}\Big(\frac{1}{T^2}\Big)+\mathbf{o}\Big(\frac{1}{N^2}\Big)\Big\}\mathbf{1}_{TN}\Big]\Big|_\Lambda\Big\}\\
\approx & \ \frac{1}{NT}\sigma_\varepsilon^{-4}tr\Big(\boldsymbol{d}LL^\tau+L\boldsymbol{d}L^\tau\Big)tr\Big(\boldsymbol{d}\Lambda\Lambda^\tau+\Lambda\boldsymbol{d}\Lambda^\tau\Big)\\
= & \ \frac{4}{NT}\sigma_\varepsilon^{-4}\text{vec}^\tau\Big(\boldsymbol{d}L\Big)\text{vec}\Big(L\Big)\text{vec}^\tau\Big(\Lambda\Big)\text{vec}\Big(\boldsymbol{d}\Lambda\Big).
\end{aligned}
$$

By Assumption 2.2, we then have

$$
\mathcal{I}_{L,\Lambda} \ = \ \sigma_\varepsilon^{-4}\frac{2}{NT^2}\text{vec}\Big(L\Big)\text{vec}^\tau\Big(\Lambda\Big)=\mathbf{O}\Big(\frac{1}{NT^2}\Big)\mathbf{1}_{Tr,Nc}=\mathbf{o}(1)\mathbf{1}_{Tr,Nc}.
$$

In (A.3),

$$
\begin{aligned}
& tr\Big\{\Omega_Y^{-1}\Big[\Big(L\Sigma_F\boldsymbol{d}L^\tau+\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}\\
= & \ 2tr\Big\{\Big[\Big(\boldsymbol{d}L\Sigma_F L^\tau\Big)\otimes I_N\Big]\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big\}\\
= & \ 2tr\Big\{\Big(I_N\otimes\boldsymbol{d}L\Big)K_{Nr}\Big(\Sigma_F L^\tau\otimes I_N\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}K_{TN}\Big\}\\
= & \ \frac{2\sigma_\varepsilon^{-4}}{N^2}tr\Big\{\Big(I_N\otimes\boldsymbol{d}L\Big)\Big(I_N\otimes\Sigma_F L^\tau\Big)K_{NT}\Big(I_T\otimes\Lambda\Big)\boldsymbol{d}\Omega_E^{-1}\Big(I_T\otimes\Lambda^\tau\Big)K_{TN}\Big\}\\
= & \ \frac{2\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}tr\Big\{\boldsymbol{d}L\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,1}\Lambda\boldsymbol{d}\Gamma_1\Lambda^\tau\Big(K_{TN}\Big)_{1,t_1}\Big\}\\
& +\frac{2\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=2}^{T-1}tr\Big\{\boldsymbol{d}L\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\boldsymbol{d}\Gamma_2\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1}\Big\}\\
& +\frac{2\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}tr\Big\{\boldsymbol{d}L\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,T}\Lambda\boldsymbol{d}\Gamma_3\Lambda^\tau\Big(K_{TN}\Big)_{T,t_1}\Big\}\\
& +\frac{2\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}tr\Big\{\boldsymbol{d}L\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\boldsymbol{d}\Gamma_4\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1+1}\Big\}\\
& +\frac{2\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}tr\Big\{\boldsymbol{d}L\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1+1,t_2}\Lambda\boldsymbol{d}\Gamma_4^\tau\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1}\Big\}.
\end{aligned}
$$

Then, by using the sparse structure of K_{NT} and K_{TN}, we have

$$
\mathcal{I}_{L,\Sigma_E} \ = \ \frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\Big\{\Big[\Big(K_{TN}\Big)_{1,t_1}^\tau\Lambda\Big]\otimes\Big[\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,1}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_1)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)}
$$

$$
\begin{aligned}
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=2}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_2)}{\boldsymbol{\partial} \mathrm{vec}(\Sigma_E)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \left\{ \left[\left(K_{TN} \right)^\tau_{T,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,T} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_3)}{\boldsymbol{\partial} \mathrm{vec}(\Sigma_E)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1+1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_4)}{\boldsymbol{\partial} \mathrm{vec}(\Sigma_E)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1+1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_4^\tau)}{\boldsymbol{\partial} \mathrm{vec}(\Sigma_E)} \\
= \; & \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} \\
& + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} = \mathbf{o}(1) \mathbf{1}_{Tr,Nc},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{L,\Phi} = \; & \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \left\{ \left[\left(K_{TN} \right)^\tau_{1,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,1} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_1)}{\boldsymbol{\partial} \mathrm{vec}(\Phi)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=2}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_2)}{\boldsymbol{\partial} \mathrm{vec}(\Phi)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \left\{ \left[\left(K_{TN} \right)^\tau_{T,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,T} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_3)}{\boldsymbol{\partial} \mathrm{vec}(\Phi)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1+1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_4)}{\boldsymbol{\partial} \mathrm{vec}(\Phi)} \\
& + \frac{\sigma_\varepsilon^{-4}}{TN} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T-1} \left\{ \left[\left(K_{TN} \right)^\tau_{t_2,t_1} \Lambda \right] \otimes \left[\Sigma_F L^\tau \left(K_{NT} \right)_{t_1+1,t_2} \Lambda \right] \right\} \frac{\boldsymbol{\partial} \mathrm{vec}(\Gamma_4^\tau)}{\boldsymbol{\partial} \mathrm{vec}(\Phi)} \\
= \; & \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} \\
& + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} + \mathbf{O}\Big(\frac{1}{TN}\Big) \mathbf{1}_{Tr,Nc} = \mathbf{o}(1) \mathbf{1}_{Tr,Nc}.
\end{aligned}
$$

In (A.3),

$$
\begin{aligned}
& \frac{1}{2} tr \Big\{ \Omega_Y^{-1} \Big[\Big(L \boldsymbol{d} \Sigma_F L^\tau \Big) \otimes I_N \Big] \Big\}^2 \\
= \; & \frac{1}{2} tr \Big\{ \Omega_Y^{-1} \Big(L \otimes I_N \Big) K_{rN} \Big(I_N \otimes \boldsymbol{d} \Sigma_F \Big) K_{Nr} \Big(L^\tau \otimes I_N \Big) \Big\}^2 \\
= \; & \frac{1}{2} tr \Big\{ \Big(I_N \otimes \boldsymbol{d} \Sigma_F \Big) K_{Nr} \Big(L^\tau \otimes I_N \Big) \Omega_Y^{-1} \Big(L \otimes I_N \Big) K_{rN} \\
& \Big(I_N \otimes \boldsymbol{d} \Sigma_F \Big) K_{Nr} \Big(L^\tau \otimes I_N \Big) \Omega_Y^{-1} \Big(L \otimes I_N \Big) K_{rN} \Big\} \\
= \; & \frac{1}{2} tr \Big\{ \Big(I_N \otimes \boldsymbol{d} \Sigma_F \Big) K_{Nr} \Big[\Sigma_F^{-1} \otimes \Big(I_N - \frac{1}{N} \sigma_\varepsilon^{-2} \Lambda \Lambda^\tau \Big) + \frac{1}{N^2} \sigma_\varepsilon^{-4} \Big(L^\tau \otimes \Lambda \Big) \Omega_E^{-1} \Big(L \otimes \Lambda^\tau \Big) \Big] K_{rN}
\end{aligned}
$$

$$
\begin{aligned}
& \Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big) K_{Nr}\Big[\Sigma_F^{-1} \otimes \Big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau\Big) + \frac{1}{N^2}\sigma_\varepsilon^{-4}\Big(L^\tau \otimes \Lambda\Big)\Omega_E^{-1}\Big(L \otimes \Lambda^\tau\Big)\Big] K_{rN}\Big\} \\
= \; & \frac{1}{2} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big) K_{Nr}\Big[\Sigma_F^{-1} \otimes \Big(I_N - \mathbf{O}\Big(\frac{1}{N}\Big)\mathbf{1}_N\Big) + \mathbf{O}\Big(\frac{T}{N^2}\Big)\mathbf{1}_{rN}\Big] K_{rN} \\
& \Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big) K_{Nr}\Big[\Sigma_F^{-1} \otimes \Big(I_N - \mathbf{O}\Big(\frac{1}{N}\Big)\mathbf{1}_N\Big) + \mathbf{O}\Big(\frac{T}{N^2}\Big)\mathbf{1}_{rN}\Big] K_{rN}\Big\} \\
= \; & \frac{1}{2} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes \Sigma_F^{-1}\Big)\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes \Sigma_F^{-1}\Big)\Big\} \\
= \; & \frac{N}{2} tr\Big(\boldsymbol{d}\Sigma_F \Sigma_F^{-1} \boldsymbol{d}\Sigma_F \Sigma_F^{-1}\Big).
\end{aligned}
$$

Then,

$$
\mathcal{I}_{\Sigma_F,\Sigma_F} = \frac{N}{2T}\Big(\Sigma_F^{-1} \otimes \Sigma_F^{-1}\Big).
$$

In (A.3),

$$
\begin{aligned}
& 2tr\Big\{\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big) \otimes I_N\Big]\Omega_Y^{-1}\Big(I_T \otimes \boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T \otimes \Lambda^\tau\Big)\Big\} \\
= \; & 2tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big) K_{Nr}\Big(L^\tau \otimes I_N\Big)\Omega_Y^{-1}\Big(I_T \otimes \Lambda\Big)\Omega_E\Big(I_T \otimes \boldsymbol{d}\Lambda^\tau\Big)\Omega_Y^{-1}\Big(L \otimes I_N\Big) K_{rN}\Big\} \\
= \; & \frac{2\sigma_\varepsilon^{-4}}{N} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes L^\tau\Big) K_{TN}\Big(I_T \otimes \Lambda\Big)\Big(I_T \otimes \boldsymbol{d}\Lambda^\tau\Big) \\
& \Big[\frac{1}{T}\Big((L\Sigma_F^{-1}) \otimes (I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda\Lambda^\tau)\Big) + \frac{1}{N^2}\sigma_\varepsilon^{-2}\Big(I_T \otimes \Lambda\Big)\Omega_E^{-1}\Big(L \otimes \Lambda^\tau\Big)\Big] K_{rN}\Big\} \\
= \; & \frac{2\sigma_\varepsilon^{-4}}{N} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes L^\tau\Big) K_{TN}\Big(I_T \otimes \Lambda\Big)\Big(I_T \otimes \boldsymbol{d}\Lambda^\tau\Big) \\
& \Big[\frac{1}{T}\Big((L\Sigma_F^{-1}) \otimes (I_N - \mathbf{O}\Big(\frac{1}{N}\Big)\mathbf{1}_N)\Big) + \mathbf{O}\Big(\frac{1}{N^2}\Big)\mathbf{1}_{TN,rN}\Big] K_{rN}\Big\} \\
= \; & \frac{2\sigma_\varepsilon^{-4}}{NT} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes L^\tau\Big) K_{TN}\Big(I_T \otimes \Lambda\Big)\Big(I_T \otimes \boldsymbol{d}\Lambda^\tau\Big) K_{TN}\Big(I_N \otimes (L\Sigma_F^{-1})\Big)\Big\} \\
= \; & \frac{2\sigma_\varepsilon^{-2}}{N} tr\Big\{\Big((\Lambda\boldsymbol{d}\Lambda^\tau) \otimes (\boldsymbol{d}\Sigma_F\Sigma_F^{-1})\Big)\Big\} = \frac{2\sigma_\varepsilon^{-2}}{N}\mathrm{vec}^\tau(d\Lambda)\mathrm{vec}(\Lambda)\mathrm{vec}^\tau(\Sigma_F^{-1})\mathrm{vec}(\boldsymbol{d}\Sigma_F).
\end{aligned}
$$

Then, by Assumption 2.2, we have

$$
\mathcal{I}_{\Sigma_F,\Lambda} \;=\; \sigma_\varepsilon^{-2}\frac{1}{NT}\mathrm{vec}(\Lambda)\mathrm{vec}^\tau(\Sigma_F^{-1}) = \mathbf{O}\Big(\frac{1}{NT}\Big)\mathbf{1}_{Nr,rc} = \mathbf{o}(1)\mathbf{1}_{Nr,rc}.
$$

In (A.3),

$$
\begin{aligned}
& tr\Big\{\Omega_Y^{-1}\Big[\Big(L\boldsymbol{d}\Sigma_F L^\tau\Big) \otimes I_N\Big]\Omega_Y^{-1}\Big(I_T \otimes \Lambda\Big)\Omega_E \boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T \otimes \Lambda^\tau\Big)\Big\} \\
= \; & tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big) K_{Nr}\Big(L^\tau \otimes I_N\Big)\Omega_Y^{-1}\Big(I_T \otimes \Lambda\Big)\Omega_E \boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T \otimes \Lambda^\tau\Big)\Omega_Y^{-1}\Big(L \otimes I_N\Big) K_{rN}\Big\} \\
= \; & \frac{\sigma_\varepsilon^{-4}}{N^2} tr\Big\{\Big(I_N \otimes \boldsymbol{d}\Sigma_F\Big)\Big(I_N \otimes L^\tau\Big) K_{NT}\Big(I_T \otimes \Lambda\Big)\boldsymbol{d}\Omega_E^{-1}\Big(I_T \otimes \Lambda^\tau\Big) K_{TN}\Big(I_N \otimes L\Big)\Big\} \\
= \; & \frac{\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T} tr\Big\{\boldsymbol{d}\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,1}\Lambda\boldsymbol{d}\Gamma_1\Lambda^\tau\Big(K_{TN}\Big)_{1,t_1} L\Big\}
\end{aligned}
$$

$$+\frac{\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=2}^{T-1} tr\Big\{\boldsymbol{d}\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\boldsymbol{d}\Gamma_2\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1}L\Big\}$$

$$+\frac{\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T} tr\Big\{\boldsymbol{d}\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,T}\Lambda\boldsymbol{d}\Gamma_3\Lambda^\tau\Big(K_{TN}\Big)_{T,t_1}L\Big\}$$

$$+\frac{\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1} tr\Big\{\boldsymbol{d}\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\boldsymbol{d}\Gamma_4\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1+1}L\Big\}$$

$$+\frac{\sigma_\varepsilon^{-4}}{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1} tr\Big\{\boldsymbol{d}\Sigma_F L^\tau\Big(K_{NT}\Big)_{t_1+1,t_2}\Lambda\boldsymbol{d}\Gamma_4^\tau\Lambda^\tau\Big(K_{TN}\Big)_{t_2,t_1}L\Big\}.$$

Then, by using the sparse structure of K_{NT} and K_{TN}, we have

$$\begin{aligned}
\mathcal{I}_{\Sigma_F,\Sigma_E} &= \frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{1,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,1}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_1)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=2}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_2)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{T,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,T}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_3)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1+1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_4)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1+1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_4^\tau)}{\boldsymbol{\partial}\text{vec}(\Sigma_E)} \\
&= \mathbf{O}\Big(\frac{1}{TN}\Big)\mathbf{1}_{T^2,c^2}=\mathbf{o}(1)\mathbf{1}_{T^2,c^2},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{\Sigma_F,\Phi} &= \frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{1,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,1}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_1)}{\boldsymbol{\partial}\text{vec}(\Phi)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=2}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_2)}{\boldsymbol{\partial}\text{vec}(\Phi)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{T,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,T}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_3)}{\boldsymbol{\partial}\text{vec}(\Phi)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1+1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_4)}{\boldsymbol{\partial}\text{vec}(\Phi)} \\
&\quad+\frac{\sigma_\varepsilon^{-4}}{TN}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T-1}\Big\{\Big[L^\tau\Big(K_{TN}\Big)^\tau_{t_2,t_1}\Lambda\Big]\otimes\Big[L^\tau\Big(K_{NT}\Big)_{t_1+1,t_2}\Lambda\Big]\Big\}\frac{\boldsymbol{\partial}\text{vec}(\Gamma_4^\tau)}{\boldsymbol{\partial}\text{vec}(\Phi)}
\end{aligned}$$

$$= \mathbf{O}\Big(\frac{1}{TN}\Big)\mathbf{1}_{T^2,c^2} = \mathbf{o}(1)\mathbf{1}_{T^2,c^2}.$$

In (A.3),

$$\begin{aligned}
& tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes \boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E \\
& \qquad \Big(I_T\otimes \boldsymbol{d}\Lambda^\tau\Big)\Big\} + tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big(I_T\otimes \boldsymbol{d}\Lambda^\tau\Big)\Big\}^2 \\
= \ & tr\Big\{\Big(I_T\otimes \boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes \boldsymbol{d}\Lambda^\tau\Big)\Omega_Y^{-1}\Big\} + \frac{T}{N^2}\sigma_\varepsilon^{-4}tr\Big(\boldsymbol{d}\Lambda^\tau\Lambda\boldsymbol{d}\Lambda^\tau\Lambda\Big) \\
= \ & tr\Big\{\sum_{t_2=1}^T\sum_{t_1=1}^T \boldsymbol{d}\Lambda\Big(\Omega_E\Big)_{t_1,t_2}\boldsymbol{d}\Lambda^\tau\Big(\Omega_Y^{-1}\Big)_{t_2,t_1}\Big\} + \frac{T}{N^2}\sigma_\varepsilon^{-4}tr\Big(\boldsymbol{d}\Lambda^\tau\Lambda\boldsymbol{d}\Lambda^\tau\Lambda\Big).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{I}_{\Lambda,\Lambda} &= \frac{1}{T}\sum_{t_2=1}^T\sum_{t_1=1}^T\Big(\Omega_E\Big)^\tau_{t_1,t_2}\otimes\Big(\Omega_Y^{-1}\Big)_{t_2,t_1} + \frac{1}{N^2}K_{Nc}\Big(\Lambda^\tau\otimes\Lambda\Big)\Big] \\
&= \frac{1}{T}\sum_{t_2=1}^T\sum_{t_1=1}^T\Big(\Omega_E\Big)^\tau_{t_1,t_2}\otimes\Big(\Omega_Y^{-1}\Big)_{t_1,t_2} + \mathbf{o}(1)\mathbf{1}_{Nc} \approx \Sigma_E\otimes I_N.
\end{aligned}$$

In (A.3),

$$\begin{aligned}
& 2tr\Big\{\Big(I_T\otimes \boldsymbol{d}\Lambda\Big)\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big\} \\
= \ & 2\sigma_\varepsilon^{-2}\frac{1}{N}tr\Big\{\Big(I_T\otimes \boldsymbol{d}\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Big(I_T\otimes\Lambda^\tau\Big)\Big\} \\
= \ & 2\sigma_\varepsilon^{-2}\Big[\frac{1}{N}tr\Big(\boldsymbol{d}\Lambda\Sigma_E\boldsymbol{d}\Gamma_1\Lambda^\tau\Big) + \frac{T-2}{N}tr\Big(\boldsymbol{d}\Lambda\Sigma_E\boldsymbol{d}\Gamma_2\Lambda^\tau\Big) + \frac{1}{N}tr\Big(\boldsymbol{d}\Lambda\Sigma_E\boldsymbol{d}\Gamma_3\Lambda^\tau\Big) \\
& + \frac{T-1}{N}tr\Big(\boldsymbol{d}\Lambda\Sigma_E\Phi^\tau\boldsymbol{d}\Gamma_4\Lambda^\tau\Big) + \frac{T-1}{N}tr\Big(\boldsymbol{d}\Lambda\Phi\Sigma_E\boldsymbol{d}\Gamma_4^\tau\Lambda^\tau\Big)\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{I}_{\Lambda,\Sigma_E} &= \sigma_\varepsilon^{-2}\Big[\frac{1}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_1)}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} + \frac{T-2}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_2)}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} \\
&\quad + \frac{1}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_3)}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} + \frac{T-1}{TN}tr\Big(\Lambda\otimes\big(\Sigma_E\Phi^\tau\big)\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_4)}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)} \\
&\quad + \frac{T-1}{TN}tr\Big(\Lambda\otimes\big(\Phi\Sigma_E\big)\Big)K_{cc}\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_1)}{\boldsymbol{\partial}\mathrm{vec}(\Sigma_E)}\Big] = \mathbf{o}(1)\mathbf{1}_{c^2},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{\Lambda,\Phi} &= \sigma_\varepsilon^{-2}\Big[\frac{1}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_1)}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} + \frac{T-2}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_2)}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} \\
&\quad + \frac{1}{TN}\Big(\Lambda\otimes\Sigma_E\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_3)}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} + \frac{T-1}{TN}tr\Big(\Lambda\otimes\big(\Sigma_E\Phi^\tau\big)\Big)\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_4)}{\boldsymbol{\partial}\mathrm{vec}(\Phi)} \\
&\quad + \frac{T-1}{TN}tr\Big(\Lambda\otimes\big(\Phi\Sigma_E\big)\Big)K_{cc}\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma_1)}{\boldsymbol{\partial}\mathrm{vec}(\Phi)}\Big] = \mathbf{o}(1)\mathbf{1}_{c^2}.
\end{aligned}$$

In (A.3),

$$
\begin{aligned}
& \frac{1}{2}tr\Big\{\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Big\}^2 \\
= & \frac{1}{2}tr\Big\{\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big(I_T\otimes\Lambda^\tau\Big)\Omega_Y^{-1}\Big(I_T\otimes\Lambda\Big)\Omega_E\Big\} \\
= & \frac{1}{2}tr\Big\{\boldsymbol{d}\Omega_E^{-1}\Omega_E\boldsymbol{d}\Omega_E^{-1}\Omega_E\Big\} \\
= & \frac{1}{2}\text{vec}^\tau\Big(\boldsymbol{d}\Omega_E^{-1}\Big)\Big(\Omega_E\otimes\Omega_E\Big)\text{vec}\Big(\boldsymbol{d}\Omega_E^{-1}\Big)\Big\}.
\end{aligned}
$$

Then, by Lemma 4, $\text{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big)=\Upsilon_1\text{vec}\big(\boldsymbol{d}\Sigma_E\big)$ and $\text{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big)=\Upsilon_2\text{vec}\big(\boldsymbol{d}\Phi\big)$,

$$
\begin{aligned}
\mathcal{I}_{\Sigma_E,\Sigma_E} &= \frac{1}{2T}\Upsilon_1^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon_1 \\
&= \frac{1}{2T}\Big[\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}(\Sigma_E)}\Big]^\tau\Upsilon^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}(\Sigma_E)},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{\Phi,\Phi} &= \frac{1}{2T}\Upsilon_2^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon_2 \\
&= \frac{1}{2T}\Big[\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}(\Phi)}\Big]^\tau\Upsilon^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}^\tau(\Phi)},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_{\Sigma_E,\Phi} &= \frac{1}{2T}\Upsilon_1^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon_2 \\
&= \frac{1}{2T}\Big[\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}(\Sigma_E)}\Big]^\tau\Upsilon^\tau\Big(\Omega_E\otimes\Omega_E\Big)\Upsilon\frac{\boldsymbol{\partial}\Gamma^\tau}{\boldsymbol{\partial}\text{vec}(\Phi)}.
\end{aligned}
$$

□

Lemma 5. *The time series $\{X_t : X_t=(x_{1,t},x_{2,t},\ldots,x_{d,t})^\tau, 1\le t\le T\}$ is d-dimension stationary with $d\ll T$ and α-mixing, and satisfying with Assumption 2.3. Then it holds that*

$$
\mathbb{E}\Big\{\frac{1}{T}\sum_{t=1}^{T}\Big(x_{i,t}x_{j,t}-\mathbb{E}(x_{i,t}x_{j,t})\Big)\Big\}^2=\mathbf{O}\big(T^{-1}\big) \tag{A.4}
$$

and

$$
\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\Big(x_{i,t}-\mathbb{E}(x_{i,t})\Big)\to\mathbf{N}(0,\sigma^2), \tag{A.5}
$$

where $\sigma^2=\text{var}(\sum_{t=1}^{T}x_{i,t})$.

Proof of Lemma 5. By Assumption 2.3 and Theorem 3 of **?**, we have for i,j

$$
\mathbb{E}\Big\{\frac{1}{T}\sum_{t=1}^{T}x_{i,t}x_{j,t}-\mathbb{E}(x_{i,t}x_{j,t})\Big\}^2
$$

$$
\begin{aligned}
&= \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{Var}(x_{i,t}x_{j,t}) + \frac{1}{T^2}\sum_{t_1\neq t_2}\mathrm{Cov}(x_{i,t_1}x_{j,t_1}, x_{i,t_2}x_{j,t_2}) \\
&\leq \frac{1}{T^2}\sum_{t=1}^{T}\big\{\mathbb{E}(x_{i,t}^4)\big\}^{1/2}\big\{\mathbb{E}(x_{j,t}^4)\big\}^{1/2} \\
&\quad +\frac{1}{T^2}\sum_{t_1\neq t_2}\alpha^{1/2}(|t_1-t_2|)\big\{\mathbb{E}(x_{i,t_1})^4\mathbb{E}(x_{j,t_1})^4\big\}^{1/4}\big\{\mathbb{E}(x_{i,t_2})^4\mathbb{E}(x_{j,t_2})^4\big\}^{1/4} \\
&= \mathbf{O}\big(T^{-1}\big).
\end{aligned}
$$

The inequality in question is a result of both the Cauchy-Schwarz inequality and Davydov inequality, thus, validating the Equation (A.4).

Moreover, according to Theorem 2.21 in **?** concerning the central limit theorem (CLT) of α-mixing processes, we can derive equation (A.5). Therefore, Lemma 5 is established.

□

Lemma 6. *Defining the true value of Θ as $\Theta^\dagger = (\sigma_\varepsilon^{\dagger 2}, L^\dagger, \Lambda^\dagger, \Sigma_F^\dagger, \Sigma_E^\dagger, \Phi^\dagger)$, and assuming the conditions of Assumption 2.1 and 2.3, it holds that*

$$
\begin{aligned}
&tr\big(\Omega_Y^\dagger - S\big) = \mathbf{O}_P\big(TN^{1/2}\big) + \mathbf{O}_P\big(T^{1/2}N\big), \\
&tr\Big(\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big) = \mathbf{O}_P\big(N^{1/2}T^{1/2}\big).
\end{aligned}
$$

Proof of Lemma 6. Let $\Omega_Y^\dagger$ denote the true value of Ω_Y, then $\Omega_Y^\dagger = (L^\dagger\Sigma_F^\dagger L^{\dagger\tau})\otimes I_N + (I_T\otimes\Lambda)\Omega_E^\dagger(I_T\otimes\Lambda^{\dagger\tau}) + \sigma_\varepsilon^{\dagger 2}I_{TN}$. Since

$$
\begin{aligned}
&S = \mathrm{vec}(Y)\mathrm{vec}(Y)^\tau \\
&= \big[(L^\dagger\otimes I_N)\mathrm{vec}(F) + (I_T\otimes\Lambda^\dagger)\mathrm{vec}(E^\tau) + \mathrm{vec}(\varepsilon)\big]\big[(L^\dagger\otimes I_N)\mathrm{vec}(F) + (I_T\otimes\Lambda^\dagger)\mathrm{vec}(E^\tau) + \mathrm{vec}(\varepsilon)\big]^\tau \\
&= (L^\dagger\otimes I_N)\mathrm{vec}(F)\mathrm{vec}(F)^\tau(L^\dagger\otimes I_N)^\tau + (I_T\otimes\Lambda^\dagger)\mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau(I_T\otimes\Lambda^\dagger)^\tau + \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau \\
&\quad +(L^\dagger\otimes I_N)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau(I_T\otimes\Lambda^\dagger)^\tau + (I_T\otimes\Lambda^\dagger)\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau(L^\dagger\otimes I_N)^\tau \\
&\quad +(L^\dagger\otimes I_N)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau + \mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau(L^\dagger\otimes I_N)^\tau \\
&\quad +(I_T\otimes\Lambda^\dagger)\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau + \mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau(I_T\otimes\Lambda^\dagger)^\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Omega_Y^\dagger - S &= \big(L^\dagger\otimes I_N\big)\big[\Sigma_F^\dagger\otimes I_N - \mathrm{vec}(F)\mathrm{vec}(F)^\tau\big]\big(L^\dagger\otimes I_N\big)^\tau \\
&\quad +\big(I_T\otimes\Lambda^\dagger\big)\big[\Omega_E^\dagger - \mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\big]\big(I_T\otimes\Lambda^\dagger\big)^\tau + \big[\sigma_\varepsilon^{\dagger 2}I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big] \\
&\quad -\big(I_T\otimes\Lambda^\dagger\big)\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\big(L^\dagger\otimes I_N\big)^\tau - \big(L^\dagger\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\big(I_T\otimes\Lambda^\dagger\big)^\tau \\
&\quad -\big(L^\dagger\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau - \mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\big(L^\dagger\otimes I_N\big)^\tau \\
&\quad -\big(I_T\otimes\Lambda^\dagger\big)\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau - \mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\big(I_T\otimes\Lambda^\dagger\big)^\tau. \qquad \text{(A.6)}
\end{aligned}
$$

By Lemma 5 for $\frac{1}{cT}\sum_{k=1}^{c}\sum_{t=1}^{T}\big(\sigma_{E_{k\cdot}}^{\dagger 2} - E_{kt}^2\big)$, $\frac{1}{rcNT}\sum_{j=1}^{r}\sum_{k=1}^{c}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{jt}\lambda_{ik}f_{ij}e_{kt}$ and $\frac{1}{cNT}\sum_{k=1}^{c}\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_{ik}e_{kt}\varepsilon_{it}$, respectively, we thus have

$$
tr\big(\Omega_Y^\dagger - S\big)
$$

$$
\begin{aligned}
&= T\cdot tr\Big\{\big[\Sigma_F^\dagger\otimes I_N-\mathrm{vec}(F)\mathrm{vec}(F)^\tau\big]\Big\}+N\cdot tr\Big\{\big[\Omega_E^\dagger-\mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\big]\Big\}\\
&\quad +tr\big[\sigma_\varepsilon^{\dagger 2}I_{TN}-\mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big]-2tr\Big[\big(I_T\otimes\Lambda^\dagger\big)\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\big(L^\dagger\otimes I_N\big)^\tau\Big]\\
&\quad -2tr\Big[\big(L^\dagger\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau\Big]-2tr\Big[\big(I_T\otimes\Lambda^\dagger\big)\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau\Big]\\
&= T\sum_{j=1}^{r}\sum_{i=1}^{N}\big(\Sigma_{F_{jj}}^\dagger-f_{ij}^2\big)+N\sum_{k=1}^{c}\sum_{t=1}^{T}\big(\Sigma_{E_{kk}}^\dagger-e_{kt}^2\big)+\sum_{i=1}^{N}\sum_{t=1}^{T}\big(\sigma_\varepsilon^{\dagger 2}-\varepsilon_{it}^2\big)\\
&\quad -2\sum_{j=1}^{r}\sum_{k=1}^{c}\sum_{i=1}^{N}\sum_{t=1}^{T}l_{jt}\lambda_{ik}f_{ij}e_{kt}-2\sum_{j=1}^{r}\sum_{i=1}^{N}\sum_{t=1}^{T}l_{jt}f_{ij}\varepsilon_{it}-2\sum_{k=1}^{c}\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_{ik}e_{kt}\varepsilon_{it}\\
&= \mathbf{O}_P\big(TN^{1/2}\big)+\mathbf{O}_P\big(T^{1/2}N\big).
\end{aligned}
$$

Furthermore, by Lemma 2, we have

$$
\begin{aligned}
\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger-S\big)\Omega_Y^{\dagger-1} \approx\ & \frac{1}{T^2}\sigma_\varepsilon^{\dagger-4}\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)\big[\Sigma_F^\dagger\otimes I_N-\mathrm{vec}(F)\mathrm{vec}(F)^\tau\big]\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)^\tau\\
&+\frac{1}{N^2}\sigma_\varepsilon^{\dagger-4}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^{\dagger-1}\big[\Omega_E^\dagger-\mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\big]\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^\dagger\big)^\tau\\
&-\frac{1}{NT}\sigma_\varepsilon^{\dagger-4}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^{\dagger-1}\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)^\tau\\
&-\frac{1}{NT}\sigma_\varepsilon^{\dagger-4}\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^\dagger\big)^\tau\\
&-\frac{1}{T}\sigma_\varepsilon^{\dagger-2}\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau\Omega_Y^{\dagger-1} \qquad\qquad (A.7)\\
&-\frac{1}{N}\sigma_\varepsilon^{\dagger-2}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^{\dagger-1}\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau\Omega_Y^{\dagger-1}\\
&-\frac{1}{T}\sigma_\varepsilon^{\dagger-2}\Omega_Y^{\dagger-1}\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\Big(\big(L^\dagger\Sigma_F^{\dagger-1}\big)\otimes I_N\Big)^\tau\\
&-\frac{1}{N}\sigma_\varepsilon^{\dagger-2}\Omega_Y^{\dagger-1}\mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^\dagger\big)^\tau\\
&+\Omega_Y^{\dagger-1}\big[\sigma_\varepsilon^{\dagger 2}I_{TN}-\mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big]\Omega_Y^{\dagger-1}.
\end{aligned}
$$

Hence, similar to the proof of $tr\big(\Omega_Y^\dagger-S\big)$, we have

$$
tr\Big(\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger-S\big)\Omega_Y^{\dagger-1}\Big)=\mathbf{O}_P\big(N^{1/2}T^{1/2}\big).
$$

□

Lemma 7. *At the true values $\Sigma_E^\dagger,\Phi^\dagger$, denote $\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta,\Theta^\dagger)$ for Σ_E and Φ, that is*

$$
\Omega_E^{-1}\big(\Sigma_E,\Phi\big)-\Omega_E^{-1}\big(\Sigma_E^\dagger,\Phi^\dagger\big)=\begin{bmatrix}\triangle\Gamma_1 & \triangle\Gamma_4 & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \triangle\Gamma_4^\tau & \triangle\Gamma_2 & \triangle\Gamma_4 & \cdots & \boldsymbol{0}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ \boldsymbol{0} & \cdots & \triangle\Gamma_4^\tau & \triangle\Gamma_2 & \triangle\Gamma_4\\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \triangle\Gamma_4^\tau & \triangle\Gamma_3\end{bmatrix}.
$$

Then

$$
\triangle\Gamma_1 = \Sigma_E^{-1}\big(\Sigma_E-\Sigma_E^\dagger\big)\Sigma_E^{-1}+\big(\Phi-\Phi^\dagger\big)^\tau\Sigma_\epsilon^{-1}\Phi+\Phi^\tau\Sigma_\epsilon^{-1}\big(\Phi-\Phi^\dagger\big)+\Phi^\tau\Sigma_\epsilon^{-1}
$$

$$
\begin{aligned}
& \Big(\big(\Sigma_E-\Sigma_E^{\dagger}\big)-\big(\Phi-\Phi^{\dagger}\big)\Sigma_E\Phi^{\tau}-\Phi\Sigma_E\big(\Phi-\Phi^{\dagger}\big)^{\tau}-\Phi\big(\Sigma_E-\Sigma_E^{\dagger}\big)\Phi^{\tau}\Big)\Sigma_{\epsilon}^{-1}\Phi, \\
\triangle\Gamma_2 \;=\; & \Sigma_{\epsilon}^{-1}\Big(\big(\Sigma_E-\Sigma_E^{\dagger}\big)-\big(\Phi-\Phi^{\dagger}\big)\Sigma_E\Phi^{\tau}-\Phi\Sigma_E\big(\Phi-\Phi^{\dagger}\big)^{\tau}-\Phi\big(\Sigma_E-\Sigma_E^{\dagger}\big)\Phi^{\tau}\Big)\Sigma_{\epsilon}^{-1} \\
& +\big(\Phi-\Phi^{\dagger}\big)^{\tau}\Sigma_{\epsilon}^{-1}\Phi+\Phi^{\tau}\Sigma_{\epsilon}^{-1}\big(\Phi-\Phi^{\dagger}\big)+\Phi^{\tau}\Sigma_{\epsilon}^{-1}\Big(\big(\Sigma_E-\Sigma_E^{\dagger}\big) \\
& -\big(\Phi-\Phi^{\dagger}\big)\Sigma_E\Phi^{\tau}-\Phi\Sigma_E\big(\Phi-\Phi^{\dagger}\big)^{\tau}-\Phi\big(\Sigma_E-\Sigma_E^{\dagger}\big)\Phi^{\tau}\Big)\Sigma_{\epsilon}^{-1}\Phi, \\
\triangle\Gamma_3 \;=\; & \Sigma_{\epsilon}^{-1}\Big(\big(\Sigma_E-\Sigma_E^{\dagger}\big)-\big(\Phi-\Phi^{\dagger}\big)\Sigma_E\Phi^{\tau}-\Phi\Sigma_E\big(\Phi-\Phi^{\dagger}\big)^{\tau}-\Phi\big(\Sigma_E-\Sigma_E^{\dagger}\big)\Phi^{\tau}\Big)\Sigma_{\epsilon}^{-1}, \\
\triangle\Gamma_4 \;=\; & -\big(\Phi-\Phi^{\dagger}\big)\Sigma_{\epsilon}^{-1}-\Phi\Big[\Sigma_{\epsilon}^{-1}\Big(\big(\Sigma_E-\Sigma_E^{\dagger}\big)-\big(\Phi-\Phi^{\dagger}\big)\Sigma_E\Phi^{\tau} \\
& -\Phi\Sigma_E\big(\Phi-\Phi^{\dagger}\big)^{\tau}-\Phi\big(\Sigma_E-\Sigma_E^{\dagger}\big)\Phi^{\tau}\Big)\Sigma_{\epsilon}^{-1}\Big],
\end{aligned}
$$

where $\Sigma_{\epsilon}=\Sigma_E-\Phi\Sigma_E\Phi^{\tau}$.

Proof of Lemma 7. With Lemma 1 and (A.1), the proof of this statement becomes straightforward.

□

Proof of Theorem 2.3. Recall $\Theta=(\sigma_{\varepsilon}^2, L, \Lambda, \Sigma_F, \Sigma_E, \Phi)$, the following neighborhoods are needed. For any given constant $C_1>0$, define

$$
\begin{aligned}
& \mathbb{V}_{\sigma_{\varepsilon}^2}=\big\{\sigma_{\varepsilon}^2:|\sigma_{\varepsilon}^2-\sigma_{\varepsilon}^{\dagger 2}|\le C_1T^{-1}\big\}, \\
& \mathbb{V}_L=\Big\{L:\Big\{\frac{1}{T}tr\Big[\Big(L-L^{\dagger}\Big)\Big(L-L^{\dagger}\Big)^{\tau}\Big]\Big\}^{1/2}\le C_1T^{-1/2}\Big\}, \\
& \mathbb{V}_{\Lambda}=\Big\{\Lambda:\Big\{\frac{1}{N}tr\Big[\Big(\Lambda-\Lambda^{\dagger}\Big)\Big(\Lambda-\Lambda^{\dagger}\Big)^{\tau}\Big]\Big\}^{1/2}\le C_1T^{-1/2}\Big\}, \\
& \mathbb{V}_{\Sigma_F}=\Big\{\Sigma_F:\Big\{tr\Big[\Big(\Sigma_F-\Sigma_F^{\dagger}\Big)\Big(\Sigma_F-\Sigma_F^{\dagger}\Big)^{\tau}\Big]\Big\}^{1/2}\le C_1T^{-1/2}\Big\}, \\
& \mathbb{V}_{\Sigma_E}=\Big\{\Sigma_E:\Big\{tr\Big[\Big(\Sigma_E-\Sigma_E^{\dagger}\Big)\Big(\Sigma_E-\Sigma_E^{\dagger}\Big)^{\tau}\Big]\Big\}^{1/2}\le C_1T^{-1/2}\Big\}, \\
& \mathbb{V}_{\Phi}=\big\{\Phi:\|\mathrm{vec}\big(\Phi-\Phi^{\dagger}\big)\|\le C_1T^{-1/2}\big\}, \\
& \mathbb{V}_{\Theta}=\mathbb{V}_{\sigma_{\varepsilon}^2}\cap\mathbb{V}_L\cap\mathbb{V}_{\Lambda}\cap\mathbb{V}_{\Sigma_F}\cap\mathbb{V}_{\Sigma_E}\cap\mathbb{V}_{\Phi},
\end{aligned}
$$

and

$$
\mathbb{Q}(x|\Theta,\Theta^{\dagger})=\mathbb{L}\Big(\Theta^{\dagger}+x\big(\Theta-\Theta^{\dagger}\big)\Big).
$$

We then have $\mathbb{Q}(0|\Theta,\Theta^{\dagger})=\mathbb{L}(\Theta^{\dagger})$ and $\mathbb{Q}(1|\Theta,\Theta^{\dagger})=\mathbb{L}(\Theta)$. Meanwhile, the Taylor expansion of $\mathbb{Q}(1|\Theta,\Theta^{\dagger})$ at 0 is

$$
\mathbb{Q}(1|\Theta,\Theta^{\dagger})=\mathbb{Q}(0|\Theta,\Theta^{\dagger})+\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta,\Theta^{\dagger})+\frac{1}{2}\frac{\boldsymbol{d}^2\mathbb{Q}}{\boldsymbol{d}x^2}(0|\Theta,\Theta^{\dagger})+\frac{1}{6}\frac{\boldsymbol{d}^3\mathbb{Q}}{\boldsymbol{d}x^3}(u|\Theta,\Theta^{\dagger}), \qquad (A.8)
$$

for $u\in(0,1)$. Note that $\mathbb{V}_{\Theta}$ is an elliptical region that includes the points corresponding to the true parameter $\Theta^{\dagger}$ as interior points. To establish the consistency of the MLE of Θ, we need to show that for all $\Theta\in\eth\mathbb{V}_{\Theta}$ (i.e., on the boundary of $\mathbb{V}_{\Theta}$), the likelihood function $\mathbb{L}(\Theta)$ is smaller than the likelihood function at the true parameter $\Theta^{\dagger}$, denoted by $\mathbb{L}(\Theta^{\dagger})$.

Specifically, we need to prove that $\mathbb{Q}(1|\Theta, \Theta^\dagger) > \mathbb{Q}(0|\Theta, \Theta^\dagger)$ with probability one. Here, $\eth\mathbb{V}_\Theta$ refers to the boundary of $\mathbb{V}_\Theta$.

We show that the first order differential in (A.8) is

$$\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta, \Theta^\dagger) = \mathbf{O}_P\big(T^{1/2}\big).$$

Recall $\Omega_Y^\dagger = (L^\dagger \Sigma_F^\dagger L^{\dagger\tau}) \otimes I_N + (I_T \otimes \Lambda)\Omega_E^\dagger (I_T \otimes \Lambda^{\dagger\tau}) + \sigma_\varepsilon^{\dagger 2} I_{TN}$. Accordingly, it's the same for $\Omega_Y^{\dagger-1}$. By (A.1), we can obtain that

$$\begin{aligned}
\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta, \Theta^\dagger) \;=\;& \frac{1}{2}\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big) tr\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) \\
&+ tr\Big[\big(L - L^\dagger\big)^\tau \mathrm{trs}\Big(K_{NT}\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) K_{TN}\Big) L^\dagger \Sigma_F^\dagger\Big] \\
&+ \frac{1}{2}\Big\{tr\Big[\big(\Sigma_F - \Sigma_F^\dagger\big) L^{\dagger\tau} \mathrm{trs}\Big(K_{NT}\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) K_{TN}\Big) L^\dagger\Big]\Big\} \\
&+ tr\Big[\big(\Lambda - \Lambda^\dagger\big)^\tau \mathrm{trs}\Big(\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big)\big(I_T \otimes \Lambda^\dagger\big)\Omega_E^\dagger\Big)\Big] \\
&- \frac{1}{2}\Big\{tr\Big[\big(\Omega_E^{-1} - \Omega_E^{\dagger-1}\big)\Big(\Omega_E^\dagger\big(I_T \otimes \Lambda^{\dagger\tau}\big)\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big)\big(I_T \otimes \Lambda^\dagger\big)\Omega_E^\dagger\Big)\Big]\Big\} \\
=\;& DQ_1 + DQ_2 + DQ_3 + DQ_4 + DQ_5,
\end{aligned}$$

where $\Omega_E^{-1} - \Omega_E^{\dagger-1} = \Omega_E^{-1}\big(\Sigma_E, \Phi\big) - \Omega_E^{-1}\big(\Sigma_E^\dagger, \Phi^\dagger\big)$ and

$$\begin{aligned}
DQ_1 &= \frac{1}{2}\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big) tr\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big), \\
DQ_2 &= tr\Big[\big(L - L^\dagger\big)^\tau \mathrm{trb}\Big(K_{NT}\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) K_{TN}\Big) L^\dagger \Sigma_F^\dagger\Big], \\
DQ_3 &= \frac{1}{2}\Big\{tr\Big[\big(\Sigma_F - \Sigma_F^\dagger\big) L^{\dagger\tau} \mathrm{trb}\Big(K_{NT}\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) K_{TN}\Big) L^\dagger\Big]\Big\}, \\
DQ_4 &= tr\Big[\big(\Lambda - \Lambda^\dagger\big)^\tau \mathrm{trb}\Big(\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big)\big(I_T \otimes \Lambda^\dagger\big)\Omega_E^\dagger\Big)\Big], \\
DQ_5 &= -\frac{1}{2}\Big\{tr\Big[\big(\Omega_E^{-1} - \Omega_E^{\dagger-1}\big)\Big(\Omega_E^\dagger\big(I_T \otimes \Lambda^{\dagger\tau}\big)\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big)\big(I_T \otimes \Lambda^\dagger\big)\Omega_E^\dagger\Big)\Big]\Big\}.
\end{aligned}$$

By Lemma 2, we have $\Omega_Y^{\dagger-1} = \sigma_\varepsilon^{\dagger-2}\Big\{\Big[I_T - \frac{1}{T}\sigma_\varepsilon^{\dagger-2} L^\dagger L^{\dagger\tau}\Big] \otimes \Big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda^\dagger\Lambda^{\dagger\tau}\Big)\Big\} + \Big\{\mathbf{o}\Big(\frac{1}{T^2}\Big) + \mathbf{o}\Big(\frac{1}{N^2}\Big)\Big\} \cdot \mathbf{1}_{TN}$. Then, $\Omega_Y^{\dagger-2} = \sigma_\varepsilon^{\dagger-4}\Big\{\Big[I_T - \frac{1}{T}\sigma_\varepsilon^{\dagger-2} L^\dagger L^{\dagger\tau}\Big] \otimes \Big(I_N - \frac{1}{N}\sigma_\varepsilon^{-2}\Lambda^\dagger\Lambda^{\dagger\tau}\Big)\Big\} + \Big\{\mathbf{o}\Big(\frac{1}{T^2}\Big) + \mathbf{o}\Big(\frac{1}{N^2}\Big)\Big\} \cdot \mathbf{1}_{TN}$. By Lemma 6, we can obtain that

$$\begin{aligned}
DQ_1 \;&=\; \frac{1}{2}\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big) tr\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) \\
&=\; \frac{1}{2}\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big) tr\big[\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\big] \\
&=\; \mathbf{O}_P(1).
\end{aligned}$$

Since

$$DQ_2 \;=\; tr\Big[\big(L - L^\dagger\big)^\tau \mathrm{trs}\Big(K_{NT}\big(\Omega_Y^{\dagger-1} - \Omega_Y^{\dagger-1} S \Omega_Y^{\dagger-1}\big) K_{TN}\Big) L^\dagger \Sigma_F^\dagger\Big]$$

$$
\begin{aligned}
&= tr\Big\{\Big((L-L^\dagger)^\tau \otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\Omega_Y^\dagger - S\Big)\Omega_Y^{\dagger-1}\Big((L^\dagger\Sigma_F^\dagger)\otimes I_N\Big)\Big\} \\
&= tr\Big\{\Omega_Y^{\dagger-1}\Big(\Omega_Y^\dagger - S\Big)\Omega_Y^{\dagger-1}\Big((L^\dagger\Sigma_F^\dagger)\otimes I_N\Big)\Big((L-L^\dagger)^\tau \otimes I_N\Big)\Big\} \\
&\approx \frac{1}{T}\sigma_\varepsilon^{\dagger-2} tr\Big[\Big(I_N \otimes (L-L^\dagger)^\tau(L^\dagger\Sigma_F^{\dagger-1})\Big)\Big(I_N\otimes\Sigma_F^\dagger - \mathrm{vec}(F^\tau)\mathrm{vec}(F^\tau)^\tau\Big)\Big] \\
&\quad + \frac{1}{N}\sigma_\varepsilon^{\dagger-2} tr\Big[\Big((L^\dagger\Sigma_F^\dagger)(L-L^\dagger)^\tau \otimes I_c\Big)\Omega_E^{\dagger-1}\Big(\Omega_E^\dagger - \mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\Big)\Omega_E^{\dagger-1}\Big] \\
&\quad - \frac{1}{NT}\sigma_\varepsilon^{\dagger-4} tr\Big[\Big((L-L^\dagger)^\tau(L^\dagger\Sigma_F^{\dagger-1})\otimes I_N\Big)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\Big((L^\dagger\Sigma_F^\dagger)\otimes\Lambda^{\dagger\tau}\Big)\Big] \\
&\quad - \frac{1}{T^2}\sigma_\varepsilon^{\dagger-4} tr\Big[\Big((L-L^\dagger)^\tau(L^\dagger\Sigma_F^{\dagger-1})\otimes I_N\Big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau(L^\dagger\otimes I_N)\Big] \\
&\quad - \frac{1}{N}\sigma_\varepsilon^{\dagger-2} tr\Big[\Big((L-L^\dagger)^\tau\otimes\Lambda^\dagger\Big)\Omega_E^{\dagger-1}\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\Big] \\
&\quad - \frac{1}{NT}\sigma_\varepsilon^{\dagger-4} tr\Big[\Big((L-L^\dagger)^\tau\otimes\Lambda^\dagger\Big)\Omega_E^{\dagger-1}\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau(L^\dagger\otimes I_N)\Big] \\
&\quad - \frac{1}{N}\sigma_\varepsilon^{\dagger-2} tr\Big[\Big((L-L^\dagger)^\tau\otimes I_N\Big)\Omega_Y^{\dagger-1}\mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\Big((L^\dagger\Sigma_F^{\dagger-1})\otimes\Lambda^{\dagger\tau}\Big)\Big] \\
&\quad + \frac{1}{T}\sigma_\varepsilon^{\dagger-2} tr\Big[\Big((L-L^\dagger)^\tau\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\sigma_\varepsilon^{\dagger 2} I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\Big)\Big((L^\dagger\Sigma_F^{\dagger-1})\otimes I_N\Big)\Big] \\
&\quad - tr\Big[\Big((L-L^\dagger)^\tau\otimes I_N\Big)\Omega_Y^{\dagger-1}\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\Big].
\end{aligned}
$$

By Lemma 3 and Lemma 6, we examine the above terms carefully and find them to be controlled by $\mathbf{O}_P(1)$ excluding the last term. For the last term, we have

$$
\begin{aligned}
tr\Big[\Big((L-L^\dagger)^\tau\otimes I_N\Big)\Omega_Y^{\dagger-1}\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\Big] &\approx tr\Big[\Big((L-L^\dagger)^\tau\otimes I_N\Big)\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\Big] \\
&= \sum_{j=1}^r\sum_{i=1}^N\sum_{t=1}^T (l_{jt}-l_{jt}^\dagger)f_{ij}\varepsilon_{it} = \mathbf{O}_P(T^{1/2}).
\end{aligned}
$$

Thus, we have $DQ_2 = \mathbf{O}_P(T^{1/2})$. Follow the same spirt above, it is easily to obtain that

$$
\begin{aligned}
2DQ_3 &= \Big\{tr\Big[(\Sigma_F-\Sigma_F^\dagger)L^{\dagger\tau}\mathrm{trs}\Big(K_{NT}(\Omega_Y^{\dagger-1}-\Omega_Y^{\dagger-1}S\Omega_Y^{\dagger-1})K_{TN}\Big)L^\dagger\Big]\Big\} \\
&= tr\Big\{\Big((\Sigma_F-\Sigma_F^\dagger)\otimes I_N\Big)\Big(L^{\dagger\tau}\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\Omega_Y^\dagger - S\Big)\Omega_Y^{\dagger-1}\Big(L^\dagger\otimes I_N\Big)\Big\} \\
&= \frac{1}{T^2}\sigma_\varepsilon^{\dagger-4} tr\Big\{\Big(\Omega_Y^\dagger - S\Big)\Big((L^\dagger\Sigma_F^{\dagger-1})\otimes I_N\Big)\Big((\Sigma_F-\Sigma_F^\dagger)\otimes I_N\Big)\Big((L^\dagger\Sigma_F^{\dagger-1})^\tau\otimes I_N\Big)\Big\} \\
&= \mathbf{O}_P(N^{1/2}T^{-1/2}+NT^{-1}) = \mathbf{O}_P(1).
\end{aligned}
$$

By (A.6) in Lemma 6, we have $tr\big[\Omega_Y^{\dagger-1}(\Omega_Y^\dagger - S)\big] = O_p(T^{1/2}N^{1/2})$. Then,

$$
\begin{aligned}
DQ_4 &= tr\Big[(\Lambda-\Lambda^\dagger)^\tau\mathrm{trs}\Big((\Omega_Y^{\dagger-1}-\Omega_Y^{\dagger-1}S\Omega_Y^{\dagger-1})(I_T\otimes\Lambda^\dagger)\Omega_E^\dagger\Big)\Big] \\
&= tr\Big[\Big(I_T\otimes(\Lambda-\Lambda^\dagger)^\tau\Big)\Omega_Y^{\dagger-1}(\Omega_Y^\dagger - S)\Omega_Y^{\dagger-1}(I_T\otimes\Lambda^\dagger)\Omega_E^\dagger\Big)\Big] \\
&= \frac{1}{N}\sigma_\varepsilon^{\dagger-2} tr\Big[\Omega_Y^{\dagger-1}(\Omega_Y^\dagger - S)(I_T\otimes\Lambda^\dagger)\Big(I_T\otimes(\Lambda-\Lambda^\dagger)^\tau\Big)\Big] \\
&= \mathbf{O}_P(T^{1/2}).
\end{aligned}
$$

For $\Omega_E^{-1}-\Omega_E^{\dagger-1}$ in DQ_5, by Lemma 7, $\mathbb{V}_{\Sigma_E}$ and $\mathbb{V}_\Phi$, we can obtain $tr\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)=O_p(1)$.

$$
\begin{aligned}
2DQ_5 &= tr\Big[\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Big(\Omega_E^{\dagger}\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(\Omega_Y^{\dagger-1}-\Omega_Y^{\dagger-1}S\Omega_Y^{\dagger-1}\big)\big(I_T\otimes\Lambda^{\dagger}\big)\Omega_E^{\dagger}\Big)\Big]\\
&= \frac{1}{N^2}\sigma_\varepsilon^{\dagger-4}tr\Big[\big(\Omega_Y^{\dagger}-S\big)\Big(\big(I_T\otimes\Lambda^{\dagger}\big)\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\big(I_T\otimes\Lambda^{\dagger\tau}\big)\Big)\Big]\\
&= \mathbf{O}_P\big(T^{1/2}+TN^{-1/2}\big)=\mathbf{O}_P\big(T^{1/2}\big).
\end{aligned}
$$

From the results of $DQ_1\sim DQ_5$, with probability going to one, we have

$$
\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta,\Theta^\dagger)\le\big(1+\sqrt{r}\big)C_1/\sqrt{y}T^{1/2}, \tag{A.9}
$$

by Assumption 2.1-(A1).

Next, we show that the second order differential in (A.2) is

$$
\frac{\boldsymbol{d}^2\mathbb{Q}}{\boldsymbol{d}x^2}(0|\Theta,\Theta^\dagger)=\mathbf{O}_P\big(T^{1/2}\big).
$$

By (A.2) and (A.3), we have

$$
\begin{aligned}
&-\frac{\boldsymbol{d}^2\mathbb{Q}}{\boldsymbol{d}x^2}(0|\Theta,\Theta^\dagger)\\
=\ & tr\Big[\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger2}\big)^2\Omega_Y^{\dagger-2}\Psi^\dagger\Big]\\
&+4tr\Big\{\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Big\}\\
&+2tr\Big\{\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
&+2tr\Big\{\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+4tr\Big\{\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(\big(L-L^\dagger\big)\Sigma_F^\dagger L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
&+8tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Omega_Y^{\dagger-1}\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+2tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&+tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\},
\end{aligned}
$$

where $\Psi^\dagger = \Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big) + \frac{1}{2}I_{TN}$. The above sum has 15 terms, which are marked by $DSQ_i, i = 1, \ldots, 15$. Because $\frac{1}{2}I_{TN}$ in $\Psi^\dagger$ is nearly negligible for these terms, we only need to show $\Psi^\dagger \approx \Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)$ in them. By Lemma 3, Lemma 6 and $tr^2(AB) \le tr(A^\tau A)tr(B^\tau B)$ for any matrices A and B, we have

$$\begin{aligned}
DSQ_1 &= tr\Big[\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)^2 \Omega_Y^{\dagger-2}\Psi^\dagger\Big] \\
&\approx \big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)^2 tr\Big[\Omega_Y^{\dagger-1}\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big] \\
&= \mathbf{O}_P\big(N^{1/2}T^{-3/2}\big) = \mathbf{O}_P\big(T^{-1}\big), \\
DSQ_2 &= 4tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\Big) \otimes I_N\Big]\Big\} \\
&\approx \frac{4}{T}\sigma_\varepsilon^{\dagger-2} tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(L - L^\dagger\big)^\tau\Big) \otimes I_N\Big]\Big\} \\
&= \mathbf{O}_P\big(N^{1/2}T^{-1}\big) = \mathbf{O}_P\big(T^{-1/2}\big), \\
DSQ_3 &= 2tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big[\Big(L^\dagger\big(\Sigma_F - \Sigma_F^\dagger\big)L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\} \\
&\approx \frac{2}{T}\sigma_\varepsilon^{\dagger-2} tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^{\dagger-1}\big(\Sigma_F - \Sigma_F^\dagger\big)^\tau L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\} \\
&= \mathbf{O}_P\big(N^{1/2}T^{-1}\big) = \mathbf{O}_P\big(T^{-1/2}\big), \\
DSQ_4 &= 2tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big(I_T \otimes \Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1} - \Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T \otimes \Lambda^{\dagger\tau}\Big)\Big\} \\
&\approx \frac{2}{N^2}\sigma_\varepsilon^{\dagger-2} tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big(I_T \otimes \Lambda^\dagger\Big)\big(\Omega_E^{-1} - \Omega_E^{\dagger-1}\big)\Big(I_T \otimes \Lambda^{\dagger\tau}\Big)\Big\} \\
&= \mathbf{O}_P\big(N^{-1/2}T^{-1}\big) = \mathbf{O}_P\big(T^{-1/2}\big), \\
DSQ_5 &= 4tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\Psi^\dagger\Omega_Y^{\dagger-2}\Big(I_T \otimes \big(\Lambda - \Lambda^\dagger\big)\Big)\Omega_E^\dagger\Big(I_T \otimes \Lambda^{\dagger\tau}\Big)\Big\} \\
&\approx \frac{4}{N}\sigma_\varepsilon^{\dagger-2} tr\Big\{\big(\sigma_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big)\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-2}\Big(I_T \otimes \big(\Lambda - \Lambda^\dagger\big)\Big)\Big(I_T \otimes \Lambda^{\dagger\tau}\Big)\Big\} \\
&= \mathbf{O}_P\big(T^{-1/2}\big).
\end{aligned}$$

By (A.7), we have

$$\begin{aligned}
DSQ_6 &= 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\Big) \otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(\big(L - L^\dagger\big)\Sigma_F^\dagger L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\} \\
&= 4tr\Big\{\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\Big) \otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(\big(L - L^\dagger\big)\Sigma_F^\dagger L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\} \\
&\approx 4tr\Big[\Big(\Sigma_F^\dagger \otimes I_N - \mathrm{vec}(F)\mathrm{vec}(F)^\tau\Big)\Big(\big(L - L^\dagger\big)^\tau \otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(L - L^\dagger\big) \otimes I_N\Big)\Big] \\
&\quad + \frac{4}{N}\sigma_\varepsilon^{\dagger-2} tr\Big\{\Omega_E^{\dagger-1}\Big(\Omega_E^\dagger - \mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\Big)\Omega_E^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\big(L - L^\dagger\big)\Sigma_F^\dagger L^{\dagger\tau}\Big) \otimes I_c\Big]\Big\} \\
&\quad - \frac{8}{N}\sigma_\varepsilon^{\dagger-2} tr\Big\{\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\big(I_T \otimes \Lambda^\dagger\big)^\tau\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\big(L - L^\dagger\big)\Big) \otimes I_N\Big]\Big\} \\
&\quad - \frac{8}{T}\sigma_\varepsilon^{\dagger-2} tr\Big\{\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau\Big[\Big(L^\dagger\big(L - L^\dagger\big)^\tau\big(L - L^\dagger\big)\Big) \otimes I_N\Big]\Big\} \\
&\quad - \frac{8}{NT}\sigma_\varepsilon^{\dagger-2} tr\Big\{\mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger-1}\big(I_T \otimes \Lambda^\dagger\big)^\tau\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L - L^\dagger\big)^\tau\big(L - L^\dagger\big)L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\} \\
&\quad + \frac{4}{T^2}\sigma_\varepsilon^{\dagger-4} tr\Big\{\Big(\sigma_\varepsilon^{\dagger 2}I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\Big)\Big[\Big(L^\dagger\big(L - L^\dagger\big)^\tau\big(L - L^\dagger\big)L^{\dagger\tau}\Big) \otimes I_N\Big]\Big\}.
\end{aligned}$$

Following the proof of Lemma 6, the first term of the above six terms is

$$
\begin{aligned}
& 4tr\Big[\Big(\Sigma_F^\dagger\otimes I_N-\mathrm{vec}(F)\mathrm{vec}(F)^\tau\Big)\Big(\big(L-L^\dagger\big)^\tau\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(L-L^\dagger\big)\otimes I_N\Big)\Big]\\
\approx\ & 4tr\Big[\Big(I_N\otimes\Sigma_F^\dagger-\mathrm{vec}(F^\tau)\mathrm{vec}(F^\tau)^\tau\Big)\Big(I_N\otimes\big(L-L^\dagger\big)^\tau\big(L-L^\dagger\big)\Big)\Big]\\
=\ & 4tr\Big[I_N\otimes\Big(\big(L-L^\dagger\big)\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)-\mathrm{vec}\Big(\big(L-L^\dagger\big)F^\tau\Big)\mathrm{vec}\Big(\big(L-L^\dagger\big)F^\tau\Big)^\tau\Big]\\
=\ & 4\sum_{t=1}^{T}\Big[\sum_{i=1}^{N}\sum_{j=1}^{r}(l_{jt}-l_{jt}^\dagger)^2\Sigma_{F_{jj}}^\dagger-\sum_{i=1}^{N}\Big(\sum_{j=1}^{r}(l_{jt}-l_{jt}^\dagger)f_{ij}\Big)^2\Big]\\
=\ & 4\sum_{t=1}^{T}(l_{\cdot t}-l_{\cdot t}^\dagger)^2\Big\{\sum_{i=1}^{N}\Big[\sum_{j=1}^{r}\Sigma_{F_{jj}}^\dagger-\Big(\sum_{j=1}^{r}f_{ij}\Big)^2\Big]\Big\}=\mathbf{O}_P\big(T^{1/2}\big).
\end{aligned}
$$

The others are similar to the first term, thus $DSQ_6=\mathbf{O}_P\big(T^{1/2}\big)$. Furthermore, by (A.6) and (A.7), we obtain

$$
\begin{aligned}
DSQ_7\ =\ & 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
\approx\ & \frac{4}{T^3}\sigma_\varepsilon^{\dagger-6}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Big[\Big(L^\dagger\Sigma_F^{\dagger-1}\big(\Sigma_F-\Sigma_F^\dagger\big)\Sigma_F^{\dagger-1}L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
\approx\ & \frac{4}{T}\sigma_\varepsilon^{\dagger-2}tr\Big\{\Big(I_N\otimes\Sigma_F^\dagger-\mathrm{vec}(F^\tau)\mathrm{vec}(F^\tau)^\tau\Big)\Big[I_N\otimes\Big(\big(L-L^\dagger\big)^\tau L^\dagger\Sigma_F^{\dagger-1}\big(\Sigma_F-\Sigma_F^\dagger\big)\Sigma_F^{\dagger-1}\Big)\Big]\Big\}\\
=\ & \mathbf{O}_P\big(1\big),\\
DSQ_8\ =\ & 8tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
\approx\ & \frac{8}{NT}\sigma_\varepsilon^{\dagger-4}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\big(L-L^\dagger\big)^\tau\otimes I_N\Big]\Big[I_T\otimes\Big(\big(\Lambda-\Lambda^\dagger\big)\Lambda^{\dagger\tau}\Big)\Big]\Big\}\\
\approx\ & 8tr\Big\{\big(\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\big)\Big(\big(L-L^\dagger\big)^\tau\otimes I_N\Big)\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Big)\Big\}\\
=\ & \mathbf{O}_P\big(1\big),\\
DSQ_9\ =\ & 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\Sigma_F^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
\approx\ & \frac{4}{N^2T}\sigma_\varepsilon^{\dagger-6}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\big(L-L^\dagger\big)^\tau\Big)\otimes I_N\Big]\Big(I_T\otimes\Lambda^\dagger\Big)\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
=\ & \mathbf{O}_P\big(1\big),\\
DSQ_{10}\ =\ & tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
\approx\ & tr\Big\{\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger-S\big)\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger\big(\Sigma_F-\Sigma_F^\dagger\big)\Sigma_F^{\dagger-1}\big(\Sigma_F-\Sigma_F^\dagger\big)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
\approx\ & \frac{1}{T^2}\sigma_\varepsilon^{\dagger-4}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\Sigma_F^{\dagger-1}\big(\Sigma_F-\Sigma_F^\dagger\big)\Sigma_F^{\dagger-1}\big(\Sigma_F-\Sigma_F^\dagger\big)\Sigma_F^{\dagger-1}L^{\dagger\tau}\Big)\otimes I_N\Big]\Big\}\\
=\ & \mathbf{O}_P\big(N^{1/2}T^{-1}+N^{-1}T^{1/2}\big)=\mathbf{O}_P\big(T^{-1/2}\big),\\
DSQ_{11}\ =\ & 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Big\}\\
\approx\ & \frac{4}{N^2}\sigma_\varepsilon^{\dagger-4}tr\Big\{\Omega_Y^{\dagger-1}\big(\Omega_Y^\dagger-S\big)\Big(I_T\otimes\Lambda^\dagger\big(\Lambda-\Lambda^\dagger\big)^\tau\Lambda^\dagger\big(\Lambda-\Lambda^\dagger\big)^\tau\Big)\Big\}\\
=\ & \mathbf{O}_P\big(N^{-1/2}T^{1/2}\big)=\mathbf{O}_P\big(1\big),
\end{aligned}
$$

$$
\begin{aligned}
DSQ_{12} &= 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Omega_Y^{\dagger-1}\Big(I_T\otimes(\Lambda-\Lambda^\dagger)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\\
&\approx \frac{4}{N^3}\sigma_\varepsilon^{\dagger-6}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big(I_T\otimes\Lambda^\dagger\Big)(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big(I_T\otimes(\Lambda-\Lambda^\dagger)\Big)\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&= \mathbf{O}_P\big(N^{-1/2}T^{1/2}\big)=\mathbf{O}_P\big(1\big),\\
DSQ_{13} &= 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger(\Sigma_F-\Sigma_F^\dagger)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes(\Lambda-\Lambda^\dagger)\Big)\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&\approx \frac{4}{NT^2}\sigma_\varepsilon^{\dagger-6}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\Sigma_F^{\dagger-1}(\Sigma_F-\Sigma_F^\dagger)\Sigma_F^{\dagger-1}L^{\dagger\tau}\Big)\otimes I_N\Big]\Big(I_T\otimes(\Lambda-\Lambda^\dagger)\Big)\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&= \mathbf{O}_P\big(N^{1/2}T^{-1}\big)=\mathbf{O}_P\big(T^{-1/2}\big),\\
DSQ_{14} &= 4tr\Big\{\Psi^\dagger\Omega_Y^{\dagger-1}\Big[\Big(L^\dagger(\Sigma_F-\Sigma_F^\dagger)L^{\dagger\tau}\Big)\otimes I_N\Big]\Omega_Y^{\dagger-1}\Big(I_T\otimes\Lambda^\dagger\Big)\Omega_E^\dagger(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Omega_E^\dagger\Big(I_T\otimes\Lambda^{\dagger\tau}\Big)\Big\}\\
&\approx \frac{4}{N^2T}\sigma_\varepsilon^{\dagger-6}tr\Big\{\big(\Omega_Y^\dagger-S\big)\Big[\Big(L^\dagger\Sigma_F^{\dagger-1}(\Sigma_F-\Sigma_F^\dagger)L^{\dagger\tau}\Big)\otimes I_N\Big]\Big(I_T\otimes\Lambda^\dagger\Big)(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Big(I_T\otimes\Lambda^{\dagger\tau}\\
&= \mathbf{O}_P\big(1\big),\\
DSQ_{15} &= tr\Big(\Psi^\dagger\Omega_Y^{\dagger-1}(I_T\otimes\Lambda^\dagger)\Omega_E^\dagger(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Omega_E^\dagger(\Omega_E^{-1}-\Omega_E^{\dagger-1})\Omega_E^\dagger(I_T\otimes\Lambda^{\dagger\tau})\Big)\\
&\approx \frac{1}{N^2}\sigma_\varepsilon^{\dagger-4}tr\Big(\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(\Omega_Y^\dagger-S\big)\big(I_T\otimes\Lambda^\dagger\big)\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Big)\\
&\approx tr\Big(\big(\Omega_E^\dagger-\mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\big)\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Omega_E^\dagger\big(\Omega_E^{-1}-\Omega_E^{\dagger-1}\big)\Big)\\
&= \mathbf{O}_P\big(T^{1/2}\big).
\end{aligned}
$$

From the results of $DSQ_1\sim DSQ_{15}$, with probability going to one, we have

$$
\frac{\boldsymbol{d}^2\mathbb{Q}}{\boldsymbol{d}x^2}(0|\Theta,\Theta^\dagger)\le 4C_1^2/\sqrt{y}T^{1/2}. \tag{A.10}
$$

Finally, for any $u\in(0,1)$, we show that the third order differential in (A.8) is

$$
\frac{\boldsymbol{d}^3\mathbb{Q}}{\boldsymbol{d}x^3}(u|\Theta,\Theta^\dagger)=\mathbf{O}_P\big(N^{1/2}T^{-1/2}\big)+\mathbf{O}_P\big(NT^{-1}\big)=\mathbf{O}_P\big(1\big). \tag{A.11}
$$

Denote $\Theta^\S=\Theta^\dagger+u\big(\Theta-\Theta^\dagger\big)$, we then have

$$
\begin{aligned}
\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Big|_{\Theta^\S} \approx\ & \sigma_\varepsilon^{\S-2}\boldsymbol{d}\sigma_\varepsilon^2\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\\
&+\sigma_\varepsilon^{\S-2}\Big[\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\boldsymbol{d}L\Sigma_F^\S L^{\S\tau}\Big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big]\\
&+\sigma_\varepsilon^{\S-2}\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Omega_E^\S\big(I_T\otimes\Lambda^{\S\tau}\big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Big(\Omega_Y^{-1}\boldsymbol{d}\Omega_Y\Big|_{\Theta^\S}\Big)^3 \approx\ & \sigma_\varepsilon^{\S-6}\big(\boldsymbol{d}\sigma_\varepsilon^2\big)^3\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\\
&+\sigma_\varepsilon^{\S-6}\big(\boldsymbol{d}\sigma_\varepsilon^2\big)^2\Big[\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\boldsymbol{d}L\Sigma_F^\S L^{\S\tau}\Big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big]\\
&+\sigma_\varepsilon^{\S-6}\big(\boldsymbol{d}\sigma_\varepsilon^2\big)^2\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Omega_E^\S\big(I_T\otimes\Lambda^{\S\tau}\big).
\end{aligned}
$$

Similar to the first and second order differential, we can obtain the third order differential in (A.2) as follows:

$$
\begin{aligned}
& \frac{\boldsymbol{d}^3\mathbb{Q}}{\boldsymbol{d}x^3}(u|\Theta,\Theta^\dagger) \\
\approx\ & -3\sigma_\varepsilon^{\S-6}\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger 2}\big)^3 tr\Big[\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\big(\Omega_Y^\S-S\big)\Big] \\
& -\frac{3}{N}\sigma_\varepsilon^{\S-8}\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger 2}\big)^2 tr\Big\{\Big[\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\big(L-L^\dagger\big)L^{\S\tau}\Big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big]\big(\Omega_Y^\S-S\big)\Big\} \\
& -\frac{3}{N}\sigma_\varepsilon^{\S-8}\big(\sigma_\varepsilon^2-\sigma_\varepsilon^{\dagger 2}\big)^2 tr\Big[\Big(\big(I_T-\frac{1}{T}L^\S L^{\S\tau}\big)\otimes\big(I_N-\frac{1}{N}\Lambda^\S\Lambda^{\S\tau}\big)\Big)\Big(I_T\otimes\big(\Lambda-\Lambda^\dagger\big)\Lambda^{\S\tau}\Big)\big(\Omega_Y^\S-S\big)\Big] \\
=\ & \mathbf{O}_P\big(N^{1/2}T^{-3/2}\big)+\mathbf{O}_P\big(NT^{-2}\big).
\end{aligned}
$$

By combining equations (A.9), (A.10), and (A.11) with (A.8), it can be shown that, with probability approaching one, a constant C_1 exists such that $C_1 > (1+\sqrt{r})/4$, and that

$$
\frac{\boldsymbol{d}^2\mathbb{Q}}{\boldsymbol{d}x^2}(0|\Theta,\Theta^\dagger) > \left(\frac{\boldsymbol{d}\mathbb{Q}}{\boldsymbol{d}x}(0|\Theta,\Theta^\dagger)\vee\frac{\boldsymbol{d}^3\mathbb{Q}}{\boldsymbol{d}x^3}(u|\Theta,\Theta^\dagger\right).
$$

These final results demonstrate that if we choose a sufficiently large radius for $\mathbb{V}_\Theta$, then $\mathbb{L}(\Theta) > \mathbb{L}(\Theta^\dagger)$ for all $\Theta\in\eth\mathbb{V}_\Theta$ with probability one. As a result, we can prove the consistency of the maximum likelihood estimate of Θ.

□

Proof of Theorem 2.4. Recall $\boldsymbol{\nabla}\mathbb{Q}(x|\widehat{\Theta},\Theta^\dagger)=\boldsymbol{\nabla}\mathbb{L}\Big(\Theta^\dagger+x\big(\widehat{\Theta}-\Theta^\dagger\big)\Big)$, here $\Theta^\dagger$ denotes the true value of Θ. Consider the second order Taylor expansion of the score function at $x=1$ around $x=0$

$$
\mathbf{0}=\boldsymbol{\nabla}\mathbb{Q}(1|\widehat{\Theta},\Theta^\dagger)=\boldsymbol{\nabla}\mathbb{Q}(0|\widehat{\Theta},\Theta^\dagger)+\frac{\boldsymbol{d\nabla}\mathbb{Q}}{\boldsymbol{d}x}(0|\widehat{\Theta},\Theta^\dagger)+\frac{1}{2}\frac{\boldsymbol{d}^2\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x^2}(u|\widehat{\Theta},\Theta^\dagger),
$$

for $u\in(0,1)$. Then we have

$$
\begin{aligned}
\widehat{\Theta}-\Theta^\dagger\ =\ & -\mathbb{I}^{-1}(\Theta^\dagger)\boldsymbol{\nabla}\mathbb{Q}(0|\widehat{\Theta},\Theta^\dagger)-\mathbb{I}^{-1}(\Theta^\dagger)\Big(\widehat{\mathbb{I}}(\Theta^\dagger)-\mathbb{I}(\Theta^\dagger)\Big)\big(\widehat{\Theta}-\Theta^\dagger\big) \\
& -\frac{1}{2}\mathbb{I}^{-1}(\Theta^\dagger)\frac{\boldsymbol{d}^2\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x^2}(u|\widehat{\Theta},\Theta^\dagger), \qquad \text{(A.12)}
\end{aligned}
$$

where $\widehat{\mathbb{I}}(\Theta^\dagger)$ is the Hessian matrix. According to (A.12), we show that the finite dimension distribution of its first term

$$
\mathbb{I}^{-1}(\Theta^\dagger)\boldsymbol{\nabla}\mathbb{Q}(0|\widehat{\Theta},\Theta^\dagger)
$$

is asymptotically normal with mean zero and covariance matrix

$$
\Sigma=\begin{pmatrix}
\mathrm{Cov}(\widehat{\sigma}_\varepsilon^2) & & & & \\
& \mathrm{Cov}\big(\mathrm{vec}(\widehat{L})\big) & & \mathbf{0} & \\
& & \mathrm{Cov}\big(\mathrm{diag}(\widehat{\Sigma}_F)\big) & & \\
& \mathbf{0} & & \mathrm{Cov}\big(\mathrm{vec}(\widehat{\Lambda})\big) & \\
& & & & \mathrm{Cov}\big(\widehat{\Sigma}_E,\widehat{\Phi}\big)
\end{pmatrix}.
$$

Here, $\mathrm{Cov}\big(\widehat{\Sigma}_E, \widehat{\Phi}\big)$ is the asymptotic covariance matrix of the MLE of $\big(\mathrm{diag}^\tau(\Sigma_E), \mathrm{vec}^\tau(\Phi)\big)^\tau$. According to $\mathcal{I}_\Theta$, we can find that there are uncorrelated among $\widehat{\sigma}_\varepsilon^2$, $\widehat{L}$, $\widehat{\Lambda}$ and $\widehat{\Sigma}_F$, which are also uncorrelated with $\widehat{\Sigma}_E$ and $\widehat{\Phi}$. However, $\widehat{\Sigma}_E$ and $\widehat{\Phi}$ are correlated, we should be careful about them.

By (A.7) in Lemma 6, we can easily obtain the asymptotic normality of all parameters. For $\widehat{\sigma}_\varepsilon^2$, we can obtain

$$\begin{aligned}
\widehat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2} &= -tr^{-1}\big(\Omega_Y^{\dagger -2}\big) tr\big(\Omega_Y^{\dagger -1} - \Omega_Y^{\dagger -1} S \Omega_Y^{\dagger -1}\big) \\
&= -tr^{-1}\big(\Omega_Y^{\dagger -2}\big) tr\Big(\Omega_Y^{\dagger -1}\big(\Omega_Y^{\dagger} - S\big)\Omega_Y^{\dagger -1}\Big) \\
&\approx -tr^{-1}\big(\Omega_Y^{\dagger -2}\big) tr\big[\Omega_Y^{\dagger -1}\big(\sigma_\varepsilon^{\dagger 2} I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big)\Omega_Y^{\dagger -1}\big] \\
&\approx -tr^{-1}\big(\Omega_Y^{\dagger -2}\big) tr\big[\Omega_Y^{\dagger -1}\big(\sigma_\varepsilon^{\dagger 2} I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big)\Omega_Y^{\dagger -1}\big] \\
&\approx \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\big(\varepsilon_{it}^2 - \sigma_\varepsilon^{\dagger 2}\big).
\end{aligned}$$

Then, we have $\sqrt{NT}\big(\widehat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^{\dagger 2}\big) \to \mathbf{N}\Big(0, \mathbb{E}(\varepsilon_{it}^4) - \sigma_\varepsilon^{\dagger 4}\Big)$.

For $\widehat{L}$, we can obtain

$$\begin{aligned}
& \mathrm{vec}\big(\widehat{L} - L^\dagger\big) \\
= & -\big[\mathrm{trs}\big(K_{NT}\Omega_Y^{\dagger -1}K_{TN}\big)\otimes\Sigma_F^\dagger\big]^{-1}\mathrm{vec}\big\{\Sigma_F^\dagger L^{\dagger\tau}\mathrm{trs}\big[K_{NT}\big(\Omega_Y^{\dagger -1} - \Omega_Y^{\dagger -1}S\Omega_Y^{\dagger -1}\big)K_{TN}\big]\big\} \\
= & -\mathrm{vec}\big\{L^{\dagger\tau}\mathrm{trs}\big[K_{NT}\big(\Omega_Y^{\dagger -1} - \Omega_Y^{\dagger -1}S\Omega_Y^{\dagger -1}\big)K_{TN}\big]\mathrm{trs}^{-1}\big(K_{NT}\Omega_Y^{\dagger -1}K_{TN}\big)\big\} \\
\approx & -\frac{1}{N}\sigma_\varepsilon^{\dagger 2}\mathrm{vec}\big\{\mathrm{trs}\big[K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\Omega_Y^{\dagger -1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger -1}K_{TN}\big]\big\} \\
\approx & -\frac{1}{NT}\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(I_N\otimes\Sigma_F^{\dagger -1}\big)\Big(I_N\otimes\Sigma_F^\dagger - \mathrm{vec}(F^\tau)\mathrm{vec}(F^\tau)^\tau\Big)\Big(I_N\otimes\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\Big)\Big]\Big\} \\
& - \frac{1}{N^3}\sigma_\varepsilon^{\dagger -2}\mathrm{vec}\Big\{\mathrm{trs}\Big[K_{Nr}\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\Big(\Omega_E^\dagger - \mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\Big)\Omega_E^{\dagger -1}\big(I_T\otimes\Lambda^\dagger\big)^\tau K_{TN}\Big]\Big\} \\
& + \frac{1}{N^2T}\sigma_\varepsilon^{\dagger -2}\mathrm{vec}\Big\{\mathrm{trs}\Big[K_{Nr}\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\mathrm{vec}(E^\tau)\mathrm{vec}(F^\tau)^\tau\Big(I_N\otimes\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\Big)\Big]\Big\} \\
& + \frac{1}{N^2}\mathrm{vec}\Big[\mathrm{trs}\Big(K_{Nr}\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger -1}\big(I_T\otimes\Lambda^\dagger\big)^\tau K_{TN}\Big)\Big] \\
& + \frac{1}{N^2}\mathrm{vec}\Big[\mathrm{trs}\Big(K_{Nr}\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau\Omega_Y^{\dagger -1}K_{TN}\Big)\Big] \\
& + \frac{1}{N^2T}\sigma_\varepsilon^{\dagger -2}\mathrm{vec}\Big\{\mathrm{trs}\Big[K_{Nr}\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)\otimes I_N\Big)\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\otimes I_N\Big)K_{TN}\Big]\Big\} \\
& + \frac{1}{NT^2}\sigma_\varepsilon^{\dagger -2}\Big\{\mathrm{trs}\Big[K_{Nr}\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)\otimes I_N\Big)\mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger -1}\big(I_T\otimes\Lambda^\dagger\big)^\tau K_{TN}\Big]\Big\} \\
& - \frac{1}{NT}\Big\{\mathrm{trs}\Big[\Big(I_N\otimes\big(\Sigma_F^{\dagger -1}L^{\dagger\tau}\big)\Big)\Big(\sigma_\varepsilon^{\dagger 2}I_{TN} - \mathrm{vec}(\varepsilon^\tau)\mathrm{vec}(\varepsilon^\tau)^\tau\Big)K_{NT}\Omega_Y^{\dagger -1}K_{TN}\Big]\Big\} \\
& + \frac{1}{N}\sigma_\varepsilon^{\dagger 2}\mathrm{vec}\Big[\mathrm{trs}\Big(\big(I_N\otimes\Sigma_F^{\dagger -1}\big)\mathrm{vec}(F^\tau)\mathrm{vec}(\varepsilon^\tau)^\tau K_{NT}\Omega_Y^{\dagger -1}K_{TN}\Big)\Big] \\
= & \frac{1}{N}\sigma_\varepsilon^{\dagger 2}\mathrm{vec}\Big[\mathrm{trs}\Big(\big(I_N\otimes\Sigma_F^{\dagger -1}\big)\mathrm{vec}(F^\tau)\mathrm{vec}(\varepsilon^\tau)^\tau K_{NT}\Omega_Y^{\dagger -1}K_{TN}\Big)\Big] + \mathbf{o}_P(N^{-1/2})\mathbf{1}_{1,T} \\
\approx & \mathrm{vec}\Big(\Sigma_F^{\dagger -1}\big(\frac{1}{N}\sum_{i=1}^{N}F_i\varepsilon_{i.}^\tau\big)\Big) = \big(I_T\otimes\Sigma_F^{\dagger -1}\big)\mathrm{vec}\Big(\frac{1}{N}\sum_{i=1}^{N}F_i\varepsilon_{i.}^\tau\Big),
\end{aligned}$$

where $\varepsilon_{i\cdot} = (\varepsilon_{i1}, \varepsilon_{i2}, \ldots, \varepsilon_{iT})^\tau$. Since

$$\begin{aligned}
\mathrm{cov}\Big[\mathrm{vec}\Big(\sum_{i=1}^N F_i\varepsilon_{i\cdot}^\tau\Big)\Big] &= \mathrm{cov}\Big[\sum_{i=1}^N \mathrm{vec}\Big(F_i\varepsilon_{i\cdot}^\tau\Big)\Big] = \mathrm{cov}\Big(\sum_{i=1}^N \big(I_T\otimes F_i\big)\mathrm{vec}\big(\varepsilon_{i\cdot}^\tau\big)\Big) \\
&= \mathbb{E}\Big[\Big(\sum_{i=1}^N \big(I_T\otimes F_i\big)\varepsilon_{i\cdot}\Big)\Big(\sum_{i=1}^N \big(I_T\otimes F_i\big)\varepsilon_{i\cdot}\Big)^\tau\Big] \\
&= \mathbb{E}\Big[\sum_{i_1,i_2=1}^N \mathbb{E}\Big(\big(I_T\otimes F_{i_1}\big)\varepsilon_{i_1\cdot}\varepsilon_{i_2\cdot}^\tau\big(I_T\otimes F_{i_2}^\tau\big)\big|_{F_{i_1},F_{i_2}}\Big)\Big] \\
&= \mathbb{E}\Big(\sigma_\varepsilon^{\dagger 2}\sum_{i=1}^N I_T\otimes\big(F_iF_i^\tau\big)\Big) = N\sigma_\varepsilon^{\dagger 2}\big(I_T\otimes\Sigma_F^\dagger\big).
\end{aligned}$$

We then have for $t = 1, \ldots, T$

$$\sqrt{N}\mathrm{vec}\big(\widehat{L}_t - L_t^\dagger\big) \to \mathbf{N}_r\Big(\mathbf{0}, \sigma_\varepsilon^{\dagger 2}\Sigma_F^{\dagger -1}\Big).$$

For $\widehat{\Sigma}_F$, we can obtain

$$\begin{aligned}
&\mathrm{vec}\big(\widehat{\Sigma}_F - \Sigma_F^\dagger\big) \\
= &-\frac{1}{N}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big[\mathrm{trs}\Big((L^\dagger\otimes I_N)^\tau\Omega_Y^{\dagger -1}\big(\Omega_Y^\dagger - S\big)\Omega_Y^{\dagger -1}(L^\dagger\otimes I_N)\Big)\Big] \\
\approx &-\frac{1}{N}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\big(\Sigma_F^\dagger\otimes I_N - \mathrm{vec}(F)\mathrm{vec}(F)^\tau\big)\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\Big)\Big] \\
&-\frac{1}{N^3}\sigma_\varepsilon^{\dagger -4}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\big(\Omega_E^\dagger - \mathrm{vec}(E^\tau)\mathrm{vec}(E^\tau)^\tau\big)\Omega_E^{\dagger -1}\big(L^\dagger\otimes\Lambda^\dagger\big)^\tau\Big)\Big] \\
&+\frac{1}{N^2}\sigma_\varepsilon^{\dagger -4}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\mathrm{vec}(E^\tau)\mathrm{vec}(F)^\tau\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\Big)\Big] \\
&+\frac{1}{N^2}\sigma_\varepsilon^{\dagger -4}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger -1}\big(L^\dagger\otimes\Lambda^{\dagger\tau}\big)\Big)\Big] \\
&+\frac{1}{NT}\sigma_\varepsilon^{\dagger -2}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^\tau\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)\otimes I_N\Big)\Big]\Big\} \\
&+\frac{1}{N^2T}\sigma_\varepsilon^{\dagger -4}\big(\Sigma_F^\dagger\otimes\Sigma_F^\dagger\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(L^{\dagger\tau}\otimes\Lambda^\dagger\big)\Omega_E^{\dagger -1}\mathrm{vec}(E^\tau)\mathrm{vec}(\varepsilon)^\tau\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)\otimes I_N\Big)\Big]\Big\} \\
&+\frac{1}{NT}\sigma_\varepsilon^{\dagger -2}\mathrm{vec}\Big\{\mathrm{trs}\Big[\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\otimes I_N\Big)\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^\tau\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\Big]\Big\} \\
&+\frac{1}{N^2T}\sigma_\varepsilon^{\dagger -4}\mathrm{vec}\Big\{\mathrm{trs}\Big[\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\otimes I_N\Big)\mathrm{vec}(\varepsilon)\mathrm{vec}(E^\tau)^\tau\Omega_E^{\dagger -1}\big(L^\dagger\otimes\Lambda^{\dagger\tau}\big)\Big]\Big\} \\
&-\frac{1}{NT^2}\sigma_\varepsilon^{\dagger -4}\mathrm{vec}\Big\{\mathrm{trs}\Big[\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)^\tau\otimes I_N\Big)\big(\sigma_\varepsilon^{\dagger 2}I_{TN} - \mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^\tau\big)\Big(\big(L^\dagger\Sigma_F^{\dagger -1}\big)\otimes I_N\Big)\Big]\Big\} \\
= &-\frac{1}{N}\mathrm{vec}\Big[\mathrm{trs}\Big(I_N\otimes\Sigma_F^\dagger - \mathrm{vec}(F^\tau)\mathrm{vec}(F^\tau)^\tau\Big)\Big] + \mathbf{o}_P(T^{-1/2})\mathbf{1}_{1,N} \\
\approx &\ \mathrm{vec}\Big(\frac{1}{N}\sum_{i=1}^N F_iF_i^\tau - \Sigma_F^\dagger\Big).
\end{aligned}$$

Then we have

$$
\begin{aligned}
\mathrm{diag}\big(\widehat{\Sigma}_F-\Sigma^{\dagger}\big) &= \frac{1}{N}\sum_{i=1}^{N}\mathrm{diag}\big(F_iF_i^{\tau}-\Sigma_F^{\dagger}\big)\\
&= \Big(\frac{1}{N}\sum_{i=1}^{N}\big(f_{i1}^2-\sigma_{F_1}^{\dagger 2}\big),\ldots,\frac{1}{N}\sum_{i=1}^{N}\big(f_{ir}^2-\sigma_{F_r}^{\dagger 2}\big)\Big)^{\tau}.
\end{aligned}
$$

Thus, $\sqrt{N}\mathrm{diag}\big(\widehat{\Sigma}_F-\Sigma^{\dagger}\big)\rightarrow \mathbf{N}_r\Big(\mathbf{0},\mathbb{E}\big(\mathrm{diag}(F_iF_i^{\tau})^2\big)-\Sigma_F^{\dagger 2}\Big)$.

For $\widehat{\Lambda}$, we can obtain

$$
\begin{aligned}
&\mathrm{vec}\big(\widehat{\Lambda}-\Lambda^{\dagger}\big)\\
= &-\Big[\sum_{t_2=1}^{T}\sum_{t_1=1}^{T}\big(\Omega_Y^{-1}\big)_{t_1,t_2}\otimes\big(\Omega_E\big)_{t_1,t_2}\Big]^{-1}\mathrm{vec}\big\{\mathrm{trs}\big[\Omega_E^{\dagger}(I_T\otimes\Lambda^{\dagger\tau})\big(\Omega_Y^{\dagger-1}-\Omega_Y^{\dagger-1}S\Omega_Y^{\dagger-1}\big)\big]\big\}\\
\approx &-\frac{1}{T}\big(I_N\otimes\Sigma_E^{\dagger}\big)^{-1}\mathrm{vec}\big\{\mathrm{trs}\big[\Omega_E^{\dagger}(I_T\otimes\Lambda^{\dagger\tau})\Omega_Y^{\dagger-1}\big(\Omega_Y^{\dagger}-S\big)\Omega_Y^{\dagger-1}\big]\big\}\\
\approx &-\frac{1}{T^3}\sigma_{\varepsilon}^{\dagger-4}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\Omega_E^{\dagger}\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)\otimes\Lambda^{\dagger\tau}\Big)\Big(\Sigma_F^{\dagger}\otimes I_N-\mathrm{vec}(F)\mathrm{vec}(F)^{\tau}\Big)\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)^{\tau}\otimes I_N\\
&-\frac{1}{NT}\sigma_{\varepsilon}^{\dagger-2}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\Big(\Omega_E^{\dagger}-\mathrm{vec}(E^{\tau})\mathrm{vec}(E^{\tau})^{\tau}\Big)\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\Big]\Big\}\\
&+\frac{1}{T^2}\sigma_{\varepsilon}^{\dagger-2}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\mathrm{vec}(E^{\tau})\mathrm{vec}(F)^{\tau}\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)^{\tau}\otimes I_N\Big)\Big]\Big\}\\
&+\frac{1}{NT^2}\sigma_{\varepsilon}^{\dagger-4}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\Omega_E^{\dagger}\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)\otimes\Lambda^{\dagger\tau}\Big)\mathrm{vec}(F)\mathrm{vec}(E^{\tau})^{\tau}\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\Big]\Big\}\\
&+\frac{1}{T^2}\sigma_{\varepsilon}^{\dagger-2}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\Omega_E^{\dagger}\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)\otimes\Lambda^{\dagger\tau}\Big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^{\tau}\Omega_Y^{\dagger-1}\Big]\Big\}\\
&+\frac{1}{NT^2}\sigma_{\varepsilon}^{\dagger-4}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^{\tau}\Big(\big(L^{\dagger}\Sigma_F^{\dagger-1}\big)^{\tau}\otimes I_N\Big)\Big]\Big\}\\
&+\frac{1}{N^2T}\sigma_{\varepsilon}^{\dagger-4}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\mathrm{vec}(\varepsilon)\mathrm{vec}(E^{\tau})^{\tau}\Omega_E^{\dagger-1}\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\Big]\Big\}\\
&-\frac{1}{NT}\sigma_{\varepsilon}^{\dagger-2}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big\{\mathrm{trs}\Big[\big(I_T\otimes\Lambda^{\dagger}\big)^{\tau}\Big(\sigma_{\varepsilon}^{\dagger 2}I_{TN}-\mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^{\tau}\Big)\Omega_Y^{\dagger-1}\Big]\Big\}\\
&+\frac{1}{T}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\mathrm{vec}(E^{\tau})\mathrm{vec}(\varepsilon)^{\tau}\Omega_Y^{\dagger-1}\Big)\Big]\\
= &\ \frac{1}{T}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\mathrm{vec}(E^{\tau})\mathrm{vec}(\varepsilon)^{\tau}\Omega_Y^{\dagger-1}\Big)\Big]+\mathbf{o}_p(T^{-1/2})\mathbf{1}_{1,N}\\
\approx &\ \frac{1}{T}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big[\mathrm{trs}\Big(\mathrm{vec}(E^{\tau})\mathrm{vec}(\varepsilon)^{\tau}\Big)\Big]=\frac{1}{T}\big(I_N\otimes\Sigma_E^{\dagger-1}\big)\mathrm{vec}\Big(\sum_{t=1}^{T}E_t\varepsilon_{\cdot t}^{\tau}\Big),
\end{aligned}
$$

where $\varepsilon_{\cdot t}=(\varepsilon_{1t},\varepsilon_{2t},\ldots,\varepsilon_{Nt})^{\tau}$. Since

$$
\begin{aligned}
\mathrm{cov}\Big[\mathrm{vec}\Big(\sum_{t=1}^{T}E_t\varepsilon_{\cdot t}^{\tau}\Big)\Big] &= \mathrm{cov}\Big[\sum_{t=1}^{T}\mathrm{vec}\Big(E_t\varepsilon_{\cdot t}^{\tau}\Big)\Big]=\mathrm{cov}\Big(\sum_{t=1}^{T}\big(I_N\otimes E_t\big)\mathrm{vec}\big(\varepsilon_{\cdot t}^{\tau}\big)\Big)\\
&= \mathbb{E}\Big[\Big(\sum_{t=1}^{T}\big(I_N\otimes E_t\big)\varepsilon_{\cdot t}\Big)\Big(\sum_{t=1}^{T}\big(I_N\otimes E_t\big)\varepsilon_{\cdot t}\Big)^{\tau}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\sum_{t_1,t_2=1}^{T}\mathbb{E}\Big(\big(I_N\otimes E_{t_1}\big)\varepsilon_{\cdot t_1}\varepsilon_{\cdot t_2}^{\tau}\big(I_N\otimes E_{t_2}^{\tau}\big)\big|_{E_{t_1},E_{t_2}}\Big)\Big]\\
&= \mathbb{E}\Big(\sigma_{\varepsilon}^{\dagger 2}\sum_{t=1}^{T} I_N\otimes\big(E_tE_t^{\tau}\big)\Big) = T\sigma_{\varepsilon}^{\dagger 2}\big(I_N\otimes\Sigma_E^{\dagger}\big).
\end{aligned}
$$

We then have for $i = 1,\ \ldots, N$

$$
\sqrt{T}\big(\widehat{\Lambda}_i-\Lambda_i^{\dagger}\big)\to \mathbf{N}_c\Big(\mathbf{0},\sigma_{\varepsilon}^{\dagger 2}\Sigma_E^{\dagger -1}\Big).
$$

Note that $\mathcal{J} = T\begin{pmatrix}\mathcal{I}_{\Sigma_E,\Sigma_E} & \mathcal{I}_{\Sigma_E,\Phi}\\ \mathcal{I}_{\Phi,\Sigma_E} & \mathcal{I}_{\Phi,\Phi}\end{pmatrix}$, then we have

$$
\begin{aligned}
&\Big(\mathrm{vec}^{\tau}\big(\widehat{\Sigma}_E-\Sigma_E^{\dagger}\big),\mathrm{vec}^{\tau}\big(\widehat{\Phi}-\Phi^{\dagger}\big)\Big)^{\tau}\\
=\ & -\frac{1}{2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\Omega_E(I_T\otimes\Lambda^{\dagger\tau})\big(\Omega_Y^{\dagger -1}-\Omega_Y^{\dagger -1}S\Omega_Y^{\dagger -1}\big)(I_T\otimes\Lambda^{\dagger})\Omega_E^{\dagger}\big]\\
\approx\ & -\frac{1}{2N^2}\sigma_{\varepsilon}^{\dagger -4}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[(I_T\otimes\Lambda^{\tau})\big(\Omega_Y-S\big)(I_T\otimes\Lambda)\big]\\
=\ & -\frac{1}{2N^2}\sigma_{\varepsilon}^{\dagger -4}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big\{\big(L^{\dagger}\otimes\Lambda^{\dagger\tau}\big)\big[\Sigma_F^{\dagger}\otimes I_N-\mathrm{vec}(F)\mathrm{vec}(F)^{\tau}\big]\big(L^{\dagger\tau}\otimes\Lambda^{\dagger}\big)\big\}\\
& -\frac{1}{2N^2}\sigma_{\varepsilon}^{\dagger -4}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big\{(I_T\otimes\Lambda^{\dagger\tau})\big[\sigma_{\varepsilon}^{\dagger 2}I_{TN}-\mathrm{vec}(\varepsilon)\mathrm{vec}(\varepsilon)^{\tau}\big](I_T\otimes\Lambda^{\dagger})\big\}\\
& +\frac{1}{2N}\sigma_{\varepsilon}^{\dagger -2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\mathrm{vec}(E^{\tau})\mathrm{vec}(F)^{\tau}\big(L^{\dagger\tau}\otimes\Lambda^{\dagger}\big)\big]\\
& +\frac{1}{2N}\sigma_{\varepsilon}^{\dagger -2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\big(L^{\dagger}\otimes\Lambda^{\dagger\tau}\big)\mathrm{vec}(F)\mathrm{vec}(E^{\tau})^{\tau}\big]\\
& +\frac{1}{2N^2}\sigma_{\varepsilon}^{\dagger -4}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\big(L^{\dagger}\otimes\Lambda^{\tau}\big)\mathrm{vec}(F)\mathrm{vec}(\varepsilon)^{\tau}(I_T\otimes\Lambda^{\dagger})\big]\\
& +\frac{1}{2N^2}\sigma_{\varepsilon}^{\dagger -4}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[(I_T\otimes\Lambda^{\dagger\tau})\mathrm{vec}(\varepsilon)\mathrm{vec}(F)^{\tau}\big(L^{\dagger\tau}\otimes\Lambda^{\dagger}\big)\big]\\
& +\frac{1}{2N}\sigma_{\varepsilon}^{\dagger -2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\mathrm{vec}(E^{\tau})\mathrm{vec}(\varepsilon)^{\tau}\big(L^{\dagger\tau}\otimes\Lambda^{\dagger}\big)\big]\\
& +\frac{1}{2N}\sigma_{\varepsilon}^{\dagger -2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[(I_T\otimes\Lambda^{\dagger\tau})\mathrm{vec}(\varepsilon)\mathrm{vec}(E^{\tau})^{\tau}\big]\\
& -\frac{1}{2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\Omega_E^{\dagger}-\mathrm{vec}(E^{\tau})\mathrm{vec}(E^{\tau})^{\tau}\big]\\
\approx\ & -\frac{1}{2}\mathcal{J}^{-1}\big(\Upsilon_1,\Upsilon_2\big)^{\tau}\big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathrm{vec}\big[\Omega_E^{\dagger}-\mathrm{vec}(E^{\tau})\mathrm{vec}(E^{\tau})^{\tau}\big]+\mathbf{o}_P(T^{-1/2})\mathbf{1}_{2c^2}\\
=\ & \frac{1}{2}\mathcal{J}^{-1}\Big(\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma)}{\boldsymbol{\partial}\mathrm{vec}^{\tau}(\Sigma_E)},\frac{\boldsymbol{\partial}\mathrm{vec}(\Gamma)}{\boldsymbol{\partial}\mathrm{vec}^{\tau}(\Phi)}\Big)^{\tau}\Big|_{\Sigma_E^{\dagger},\Phi^{\dagger}}\mathbf{E}+\mathbf{o}_P(T^{-1/2})\mathbf{1}_{2c^2},
\end{aligned}
$$

where $\mathbf{E} = \Big(\mathrm{vec}^{\tau}\big(E_1E_1^{\tau}-\Sigma_E^{\dagger}\big),\sum_{t=2}^{T-1}\mathrm{vec}^{\tau}\big(E_tE_t^{\tau}-\Sigma_E^{\dagger}\big),\mathrm{vec}^{\tau}\big(E_TE_T^{\tau}-\Sigma_E^{\dagger}\big),2\sum_{t=1}^{T-1}\mathrm{vec}^{\tau}\big(E_tE_{t+1}^{\tau}-\Sigma_E^{\dagger}\Phi^{\dagger\tau}\big)\Big)^{\tau}$.

From the above, we can see that whether $\left(\frac{\partial \mathrm{vec}(\Gamma)}{\partial \mathrm{vec}^\tau(\Sigma_E)}\right)^\tau\Big|_{\Sigma_E^\dagger,\Phi^\dagger}\mathbf{E}$ or $\left(\frac{\partial \mathrm{vec}(\Gamma)}{\partial \mathrm{vec}^\tau(\Phi)}\right)^\tau\Big|_{\Sigma_E^\dagger,\Phi^\dagger}\mathbf{E}$ is a weighted sum of $\mathrm{vec}\Big(\big(E_1E_1^\tau + E_1E_2^\tau\big) - \big(\Sigma_E^\dagger + \Sigma_E^\dagger\Phi^{\dagger\tau}\big)\Big)$, $\sum_{t=2}^{T-1}\mathrm{vec}\Big(\big(E_tE_t^\tau + E_tE_{t+1}^\tau\big) - \big(\Sigma_E^\dagger + \Sigma_E^\dagger\Phi^{\dagger\tau}\big)\Big)$, and $\mathrm{vec}\Big(\big(E_TE_T^\tau + E_{T-1}E_T^\tau\big) - \big(\Sigma_E^\dagger + \Sigma_E^\dagger\Phi^{\dagger\tau}\big)\Big)$. By Lemma 5 for $\{E_tE_t^\tau + E_tE_{t+1}^\tau\}$, thus we have

$$\sqrt{T}\Big(\mathrm{vec}\big(\widehat{\Sigma}_E - \Sigma_E^\dagger\big)^\tau, \mathrm{vec}\big(\widehat{\Phi} - \Phi^\dagger\big)^\tau\Big)^\tau \to \mathbf{N}_T\Big(\mathbf{0}, \mathbf{J}\mathbf{\Pi}\mathbf{J}^\tau\Big),$$

where $\mathbf{J} = \frac{1}{2}\mathcal{J}^{-1}\left(\frac{\partial \mathrm{vec}(\Gamma)}{\partial \mathrm{vec}^\tau(\Sigma_E)}, \frac{\partial \mathrm{vec}(\Gamma)}{\partial \mathrm{vec}^\tau(\Phi)}\right)^\tau\Bigg|_{\Sigma_E^\dagger,\Phi^\dagger}$ and $\mathbf{\Pi} = \mathrm{Cov}\big(\mathbf{E}\big)$.

Next, in (A.12), we will show the order of the second term

$$\mathbb{I}^{-1}(\Theta^\dagger)\Big(\widehat{\mathbb{I}}(\Theta^\dagger) - \mathbb{I}(\Theta^\dagger)\Big)\big(\widehat{\Theta} - \Theta^\dagger\big) = \mathbf{O}_P\big(N^{1/2}T^{-3/2} + T^{-1}\big)\mathbf{1}_{1+Tr+Nc+r^2+2c^2}. \tag{A.13}$$

Here, since the other elements is similar to the diagonal elements, we only show the order of the diagonal elements of $\widehat{\mathbb{I}}(\Theta^\dagger) - \mathbb{I}(\Theta^\dagger)$. The order of $\widehat{\mathbb{I}}(\Theta^\dagger) - \mathbb{I}(\Theta^\dagger)$ is first to be showed by considering

$$\begin{aligned}
d^2\mathbb{L} - \mathbb{E}\big(\boldsymbol{d}^2\mathbb{L}\big) &= tr\Big(\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-1}\boldsymbol{d}\Omega_Y\Omega_Y^{\dagger-1}\boldsymbol{d}\Omega_Y\Big)\\
&= tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Big[\Omega_Y^{\dagger-1}\boldsymbol{d}\sigma_\varepsilon^2 + \Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\\
&\quad + \Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big) + \Omega_Y^{\dagger-1}\Big(\big(L^\dagger\boldsymbol{d}\Sigma_F L^{\dagger\tau}\big)\otimes I_N\Big)\\
&\quad + \Omega_Y^{\dagger-1}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Omega_E^\dagger\big(I_T\otimes\Lambda^{\dagger\tau}\big) + \Omega_Y^{\dagger-1}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^\dagger\big(I_T\otimes\boldsymbol{d}\Lambda^\tau\big)\\
&\quad + \Omega_Y^{\dagger-1}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^\dagger\boldsymbol{d}\Omega_E^{-1}\Omega_E^\dagger\big(I_T\otimes\Lambda^{\dagger\tau}\big)\Big]^2\Big\}.
\end{aligned}\tag{A.14}$$

We then have by $tr\Big(\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-2}\big(\boldsymbol{d}\sigma_\varepsilon^2\big)^2\Big)$ in (A.14) and Lemma 6,

$$\widehat{\mathcal{I}}_{\sigma_\varepsilon^{\dagger2},\sigma_\varepsilon^{\dagger2}} - \mathcal{I}_{\sigma_\varepsilon^{\dagger2},\sigma_\varepsilon^{\dagger2}} = \frac{1}{T}tr\Big(\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-2}\Big) = \mathbf{O}_P\big(N^{1/2}T^{-1/2}\big).$$

Since in (A.14)

$$\begin{aligned}
& tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Big[\Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big) + \Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big)\Big]^2\Big\}\\
={}& tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Big\}\\
&+tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big)\Big\}\\
&+tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(L^\dagger\Sigma_F^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Big\}\\
&+tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^\dagger L^{\dagger\tau}\big)\otimes I_N\Big)\Big\}\\
={}& \frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\Omega_Y^{\dagger-1}\big(S - \Omega_Y^\dagger\big)\Big(\big(L^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Big(\big(L^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Big\}\\
&+\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\big(S - \Omega_Y^\dagger\big)\Big(\big(L^\dagger\boldsymbol{d}L^\tau\big)\otimes I_N\Big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}LL^{\dagger\tau}\big)\otimes I_N\Big)\Big\}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}LL^{\dagger\tau}\big)\otimes I_N\Big)\Big(\big(\boldsymbol{d}LL^{\dagger\tau}\big)\otimes I_N\Big)\Big\} \\
& +tr\Big\{\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}\Big(\big(\boldsymbol{d}L\Sigma_F^{\dagger}\big)\otimes I_N\Big)\big(\boldsymbol{d}L^{\tau}\otimes I_N\big)\Big\} \\
= & \frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big(I_N\otimes\big(dL^{\tau}L^{\dagger}\boldsymbol{d}L^{\tau}\big)\Big)\Big\} \\
& +\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big(\big(\boldsymbol{d}L^{\tau}\boldsymbol{d}L\big)\otimes I_N\Big)\Big\} \\
& +\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{K_{NT}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big(I_N\otimes\big(\boldsymbol{d}LL^{\dagger\tau}\boldsymbol{d}L\big)\Big)\Big\} \\
& +tr\Big\{K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big(I_N\otimes\big(\boldsymbol{d}L\Sigma_F^{\dagger}dL^{\tau}\big)\Big)\Big\} \\
= & \frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\mathrm{trb}\Big(K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big)\big(dL^{\tau}L^{\dagger}\boldsymbol{d}L^{\tau}\big)\Big\} \\
& +\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\mathrm{trb}\Big(K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big)\big(\boldsymbol{d}L^{\tau}\boldsymbol{d}L\big)\Big\} \\
& +\frac{1}{T^2}\sigma^{\dagger-4}tr\Big\{\mathrm{trb}\Big(K_{NT}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big)\big(\boldsymbol{d}LL^{\dagger\tau}\boldsymbol{d}L\big)\Big\} \\
& +tr\Big\{\mathrm{trb}\Big(K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big)\big(\boldsymbol{d}L\Sigma_F^{\dagger}dL^{\tau}\big)\Big\}.
\end{aligned}
$$

Then, by (A.6) and (A.7) in Lemma 6 and $tr\big(A^{\tau}BCD\big)=\big[\mathrm{vec}(A)\big]^{\tau}\big(D^{\tau}\otimes B\big)\mathrm{vec}(C)$ for any matrices A,B,C and D, we have

$$
\begin{aligned}
\widehat{\mathcal{I}}_{L^{\dagger},L^{\dagger}}-\mathcal{I}_{L^{\dagger},L^{\dagger}} = & \frac{1}{T^3}\sigma^{\dagger-4}K_{Tr}\Big\{L^{\dagger\tau}\otimes\mathrm{trb}\Big(K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big)\Big\} \\
& +\frac{1}{T^3}\sigma^{\dagger-4}\Big\{I_T\otimes\mathrm{trb}\Big(K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\big(L^{\dagger}\otimes I_N\big)K_{rN}\Big)\Big\} \\
& +\frac{1}{T^3}\sigma^{\dagger-4}\Big\{L^{\dagger}\otimes\mathrm{trb}\Big(K_{NT}\big(L^{\dagger\tau}\otimes I_N\big)\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big)\Big\}K_{rT} \\
& +\frac{1}{T}\Big\{\mathrm{trb}\Big(K_{NT}\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}K_{TN}\Big)\otimes\Sigma_F^{\dagger}\Big\} \\
= & \frac{N}{T}\Big\{I_T\otimes\Big(\Sigma_F-\frac{1}{N}\sum_{i=1}^{N}F_iF_i^{\tau}\Big)\Big\}+\frac{N}{T}\Big\{\Big(\sigma_{\varepsilon}^2I_T-\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i\cdot}\varepsilon_{i\cdot}^{\tau}\Big)\otimes\Sigma_F\Big\} \\
& +\mathbf{O}_P\big(N^{-1}T^{-1}\big)\mathbf{1}_{Tr}+\mathbf{O}_P\big(T^{-2}\big)\mathbf{1}_{Tr} \\
= & \mathbf{O}_P\big(N^{1/2}T^{-1}\big)\mathbf{1}_{Tr}.
\end{aligned}
$$

Since in (A.14)

$$
\begin{aligned}
& tr\Big\{\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Big[\Omega_Y^{\dagger-1}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Omega_E^{\dagger}\big(I_T\otimes\Lambda^{\dagger\tau}\big)+\Omega_Y^{\dagger-1}\big(I_T\otimes\Lambda^{\dagger}\big)\Omega_E^{\dagger}\big(I_T\otimes\boldsymbol{d}\Lambda^{\tau}\big)\Big]^2\Big\} \\
= & \frac{1}{N^2}\sigma^{\dagger-4}tr\Big\{\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(I_T\otimes\boldsymbol{d}\Lambda\big)\big(I_T\otimes\Lambda^{\dagger\tau}\big)\Big\} \\
& +\frac{1}{N^2}\sigma^{\dagger-4}tr\Big\{\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\big(I_T\otimes\Lambda^{\dagger}\big)\big(I_T\otimes\boldsymbol{d}\Lambda^{\tau}\big)\big(I_T\otimes\Lambda^{\dagger}\big)\big(I_T\otimes\boldsymbol{d}\Lambda^{\tau}\big)\Big\} \\
& +\frac{1}{N^2}\sigma^{\dagger-4}tr\Big\{\big(S-\Omega_Y^{\dagger}\big)\big(I_T\otimes\Lambda^{\dagger}\big)\big(I_T\otimes\boldsymbol{d}\Lambda^{\tau}\big)\Omega_Y^{\dagger-1}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\big(I_T\otimes\Lambda^{\dagger\tau}\big)\Big\} \\
& +tr\Big\{\Omega_Y^{\dagger-1}\big(S-\Omega_Y^{\dagger}\big)\Omega_Y^{\dagger-1}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Omega_E^{\dagger}\big(I_T\otimes\boldsymbol{d}\Lambda\big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{N^2}\sigma^{\dagger -4} tr\Big\{\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\Big(\big(I_T\otimes\Lambda^\tau\big)\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda\big)\Big)_{t_1,t_2}\boldsymbol{d}\Lambda^\tau\big(\Omega_Y^{\dagger -1}\big)_{t_2,t_1}\boldsymbol{d}\Lambda\Big\}\\
&+\frac{1}{N^2}\sigma^{\dagger -4} tr\Big\{\mathrm{trb}\Big(\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\Big)\boldsymbol{d}\Lambda\Lambda^{\dagger\tau}\boldsymbol{d}\Lambda\Big\}\\
&+\frac{1}{N^2}\sigma^{\dagger -4} tr\Big\{\mathrm{trb}\Big(\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Big)\Lambda^\dagger\boldsymbol{d}\Lambda^\tau\Lambda^\dagger\boldsymbol{d}\Lambda^\tau\Big\}\\
&+tr\Big\{\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\Big(\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\Big)_{t_1,t_2}\boldsymbol{d}\Lambda\big(\Omega_E^\dagger\big)_{t_2,t_1}\boldsymbol{d}\Lambda^\tau\Big\}.
\end{aligned}
$$

We similarly obtain that

$$
\begin{aligned}
\widehat{\mathcal{I}}_{\Lambda^\dagger,\Lambda^\dagger}-\mathcal{I}_{\Lambda^\dagger,\Lambda^\dagger} = & \frac{1}{TN^2}\sigma^{\dagger -4}\Big\{\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\big(\Omega_Y^{\dagger -1}\big)^\tau_{t_2,t_1}\otimes\Big(\big(I_T\otimes\Lambda^\tau\big)\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda\big)\Big)_{t_1,t_2}\Big\}\\
&+\frac{1}{TN^2}\sigma^{\dagger -4}\Big\{\Lambda^\dagger\otimes\mathrm{trb}\Big(\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\Big)\Big\}K_{cN}\\
&+\frac{1}{TN^2}\sigma^{\dagger -4}K_{cN}\Big\{\Lambda^{\dagger\tau}\otimes\mathrm{trb}\Big(\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda^\dagger\big)\Big)\Big\}\\
&+\frac{1}{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\Big(\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\Big)^\tau_{t_1,t_2}\otimes\big(\Omega_E^\dagger\big)_{t_2,t_1}\\
\approx & \frac{N}{T}\Big(I_N\otimes\big(\frac{1}{N}\sum_{t=1}^{T}E_tE_t^\tau-\Sigma_E^\dagger\big)\Big)+\Big(\sigma^{\dagger 2}I_N-\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{\cdot t}\varepsilon_{\cdot t}^\tau\Big)\otimes\Sigma_E^\dagger\\
= & \mathbf{O}_P\big(N^{1/2}T^{-1}\big)\mathbf{1}_{Nc}+\mathbf{O}_P\big(T^{-1/2}\big)\mathbf{1}_{Nc}.
\end{aligned}
$$

Since in (A.14)

$$
\begin{aligned}
& tr\Big\{\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\Big(\big(L^\dagger\boldsymbol{d}\Sigma_F L^{\dagger\tau}\big)\otimes I_N\Big)\Omega_Y^{\dagger -1}\Big(\big(L^\dagger\boldsymbol{d}\Sigma_F L^{\dagger\tau}\big)\otimes I_N\Big)\Big\}\\
\approx & tr\Big(\big(L^{\dagger\tau}\otimes I_N\big)\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\big(L^\dagger\otimes I_N\big)\big(\boldsymbol{d}\Sigma_F\otimes I_N\big)\big(\Sigma_F^{\dagger -1}\otimes I_N\big)\big(\boldsymbol{d}\Sigma_F\otimes I_N\big)\Big)\\
= & tr\Big\{\mathrm{trb}\Big(K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\big(L^\dagger\otimes I_N\big)K_{rN}\Big)\boldsymbol{d}\Sigma_F\Sigma_F^{\dagger -1}\boldsymbol{d}\Sigma_F\Big\}.
\end{aligned}
$$

By (A.7) in Lemma 6, we have

$$
\begin{aligned}
\widehat{\mathcal{I}}_{\Sigma_F^\dagger,\Sigma_F^\dagger}-\mathcal{I}_{\Sigma_F^\dagger,\Sigma_F^\dagger} = & \frac{1}{T}\Big\{\Sigma_F^{\dagger -1}\otimes\mathrm{trb}\Big(K_{Nr}\big(L^{\dagger\tau}\otimes I_N\big)\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Omega_Y^{\dagger -1}\big(L^\dagger\otimes I_N\big)K_{rN}\Big)\Big\}\\
\approx & \frac{N}{T}\Big\{\Sigma_F^{\dagger -1}\otimes\Big(\Sigma_F^{\dagger -1}\big(\Sigma_F^\dagger-\frac{1}{N}\sum_{i=1}^{N}F_iF_i^\tau\big)\Sigma_F^{\dagger -1}\Big)\Big\}+\mathbf{O}_P\big(NT^{-2}\big)\mathbf{1}_{r^2}\\
&+\frac{N}{T}\Big\{\Sigma_F^{\dagger -1}\otimes\Big(\Sigma_F^{\dagger -1}\frac{1}{N}\sum_{i=1}^{N}F_i\varepsilon_{i\cdot}^\tau\Big)\Big\}+\mathbf{O}_P\big(N^{-1}T^{-1}\big)\mathbf{1}_{r^2}\\
= & \mathbf{O}_P\big(N^{1/2}T^{-1}\big)\mathbf{1}_{r^2}.
\end{aligned}
$$

Since in (A.14)

$$
tr\Big\{\Omega_Y^{\dagger -1}\big(S-\Omega_Y^\dagger\big)\Big[\Omega_Y^{\dagger -1}\big(I_T\otimes\Lambda^\dagger\big)\Omega_E^\dagger\boldsymbol{d}\Omega_E^{-1}\Omega_E^\dagger\big(I_T\otimes\Lambda^{\dagger\tau}\big)\Big]^2\Big\}
$$

$$\approx \frac{1}{N^2}\sigma^{\dagger-4} tr\Big\{\boldsymbol{d}\Omega_E^{-1}\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda^\dagger\big)\boldsymbol{d}\Omega_E^{-1}\Omega_E^\dagger\Big\}.$$

By $\text{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big)=\Upsilon_1\text{vec}\big(\boldsymbol{d}\Sigma_E\big)$ and $\text{vec}\big(\boldsymbol{d}\Omega_E^{-1}\big)=\Upsilon_2\text{vec}\big(\boldsymbol{d}\Phi\big)$, we have

$$\begin{aligned}\widehat{\mathcal{I}}_{\Sigma_E^\dagger,\Sigma_E^\dagger}-\mathcal{I}_{\Sigma_E^\dagger,\Sigma_E^\dagger} &= \frac{1}{N^2T}\Upsilon_1^\tau\Big\{\Omega_E^\dagger\otimes\Big(\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda^\dagger\big)\Big)\Big\}\Upsilon_1\\ &\approx \frac{1}{T}\Upsilon_1^\tau\Big\{\Omega_E^\dagger\otimes\Big(\text{vec}(E^\tau)\text{vec}(E^\tau)^\tau-\Omega_E^\dagger\Big)\Big\}\Upsilon_1+\mathbf{O}_P\big(N^{-1}T^{-1}\big)\mathbf{1}_{c^2}\\ &= \mathbf{O}_P\big(N^{1/2}T^{-1}\big)\mathbf{1}_{c^2},\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathcal{I}}_{\Phi^\dagger,\Phi^\dagger}-\mathcal{I}_{\Phi^\dagger,\Phi^\dagger} &= \frac{1}{N^2T}\Upsilon_2^\tau\Big\{\Omega_E^\dagger\otimes\Big(\big(I_T\otimes\Lambda^{\dagger\tau}\big)\big(S-\Omega_Y^\dagger\big)\big(I_T\otimes\Lambda^\dagger\big)\Big)\Big\}\Upsilon_2\\ &\approx \frac{1}{T}\Upsilon_2^\tau\Big\{\Omega_E^\dagger\otimes\Big(\text{vec}(E^\tau)\text{vec}(E^\tau)^\tau-\Omega_E^\dagger\Big)\Big\}\Upsilon_2+\mathbf{O}_P\big(N^{-1}T^{-1}\big)\mathbf{1}_{c^2}\\ &= \mathbf{O}_P\big(N^{1/2}T^{-1}\big)\mathbf{1}_{c^2}.\end{aligned}$$

Then the order of the second term is determined.

Finally, in (A.12), similar to the order of $\frac{\boldsymbol{d}^3\mathbb{Q}}{\boldsymbol{d}x^3}(u|\widehat{\Theta},\Theta^\dagger)$, we can show the order of the third term

$$\mathbb{I}^{-1}(\Theta^\dagger)\frac{\boldsymbol{d}^2\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x^2}(u|\widehat{\Theta},\Theta^\dagger)=\mathbf{O}_P\big(N^{1/2}T^{-3/2}+NT^{-2}\big)\mathbf{1}_{1+Tr+Nc+r^2+2c^2}. \tag{A.15}$$

Put (A.13) and (A.15) together, we can obtain the asymptotic normality of MLE for $\widehat{\Theta}$.

□

Proof of Theorem 2.5. Recall the second order Taylor expansion of the score function at $x=1$ around $x=0$

$$\mathbf{0}=\boldsymbol{\nabla}\mathbb{Q}(1|\widehat{\Theta},\Theta^\dagger)=\boldsymbol{\nabla}\mathbb{Q}(0|\widehat{\Theta},\Theta^\dagger)+\frac{\boldsymbol{d}\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x}(0|\widehat{\Theta},\Theta^\dagger)+\frac{1}{2}\frac{\boldsymbol{d}^2\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x^2}(u|\widehat{\Theta},\Theta^\dagger),$$

for $u\in(0,1)$. Then we have

$$\begin{aligned}\widehat{\Theta}-\Theta^\dagger &= -\mathbb{I}^{-1}(\Theta^\dagger)\boldsymbol{\nabla}\mathbb{Q}(0|\widehat{\Theta},\Theta^\dagger)-\mathbb{I}^{-1}(\Theta^\dagger)\Big(\widehat{\mathbb{I}}(\Theta^\dagger)-\mathbb{I}(\Theta^\dagger)\Big)\big(\widehat{\Theta}-\Theta^\dagger\big)\\ &\quad -\frac{1}{2}\mathbb{I}^{-1}(\Theta^\dagger)\frac{\boldsymbol{d}^2\boldsymbol{\nabla}\mathbb{Q}}{\boldsymbol{d}x^2}(u|\widehat{\Theta},\Theta^\dagger),\end{aligned}$$

where $\widehat{\mathbb{I}}(\Theta^\dagger)$ is the Hessian matrix.

By the proof of Theorem 2.4, Theorem 2.5 can be proved.

□

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