Supplementary Materials for "Bi-optimal Quantile-based Test Planning for Accelerated Degradation Tests Based on a Wiener Process"

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1 Technical Derivations and Auxiliary Results

1.1 Fisher Information Matrix of the Accelerated Degradation Model Based on a Wiener Process

The single degradation path $\boldsymbol{Y}_{ik}$ follows an m-variate normal distribution, i.e.,

$$\boldsymbol{Y}_{ik} \sim \mathcal{N}_m(g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k}, g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})\boldsymbol{Q}_{i,k}),\ i=1,\ldots,n_k,\ k=1,\ldots,l,$$

where $\boldsymbol{t}_{i,k} = (t_{i,1,k},\ldots,t_{i,m-1,k},t_{m,k})'$ and $\boldsymbol{Q}_{i,k} = [\min\{t_{i,j_1,k}, t_{i,j_2,k}\}]_{1\le j_1,j_2\le m}$. The log-likelihood function is given by

$$\mathcal{L}(\boldsymbol{\theta}) = -\frac{m}{2}\Big(\ln(2\pi)+\ln(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))\Big) - \frac{1}{2}\ln|\boldsymbol{Q}_{i,k}| - \frac{(\boldsymbol{y}_{i,k}-g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})'\boldsymbol{Q}_{i,k}^{-1}(\boldsymbol{y}_{i,k}-g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})}{2g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}.$$

The first and second derivatives of $\mathcal{L}(\boldsymbol{\theta})$ are

$$\begin{aligned}
\frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}} &= \frac{\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k}}{g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial\boldsymbol{X}_k'\boldsymbol{\beta}}\boldsymbol{X}_k,\\
\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} &= \left\{\frac{\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k}}{g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\frac{\partial^2 g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial\left(\boldsymbol{X}_k'\boldsymbol{\beta}\right)^2} - \frac{\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k}}{g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\left(\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial\boldsymbol{X}_k'\boldsymbol{\beta}}\right)^2\right\}\boldsymbol{X}_k\boldsymbol{X}_k',\\
\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\gamma}'} &= \frac{g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k} - \boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{y}_{i,k}}{(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2}\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial\boldsymbol{X}_k'\boldsymbol{\beta}}\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial\boldsymbol{Z}_k'\boldsymbol{\gamma}}\boldsymbol{X}_k\boldsymbol{Z}_k',\\
\frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\gamma}} &= \left\{\frac{(\boldsymbol{y}_{i,k}-g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})'\boldsymbol{Q}_{i,k}^{-1}(\boldsymbol{y}_{i,k}-g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})}{2(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2} - \frac{m}{2g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\right\}\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial\boldsymbol{Z}_k'\boldsymbol{\gamma}}\boldsymbol{Z}_k,\\
\frac{\partial^2\mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\beta}'} &= \frac{g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k} - \boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{y}_{i,k}}{(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2}\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial\boldsymbol{X}_k'\boldsymbol{\beta}}\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial\boldsymbol{Z}_k'\boldsymbol{\gamma}}\boldsymbol{Z}_k\boldsymbol{X}_k',
\end{aligned}$$

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$$\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\gamma}'} = \left\{ \left(\frac{(\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})'\boldsymbol{Q}_{i,k}^{-1}(\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})}{2(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2} - \frac{m}{2g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})} \right) \frac{\partial^2 g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial(\boldsymbol{Z}_k'\boldsymbol{\gamma})^2} \right.$$
$$\left. + \left(\frac{m}{2(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2} - \frac{(\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})'\boldsymbol{Q}_{i,k}^{-1}(\boldsymbol{y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})}{(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^3} \right) \left(\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial \boldsymbol{Z}_k'\boldsymbol{\gamma}} \right)^2 \right\} \boldsymbol{Z}_k\boldsymbol{Z}_k'.$$

Since $\boldsymbol{t}_{i,k}'\boldsymbol{Q}_{i,k}^{-1}\boldsymbol{t}_{i,k} = t_{m,k}$ and

$$(\boldsymbol{Y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})'\boldsymbol{Q}_{i,k}^{-1}(\boldsymbol{Y}_{i,k} - g_1(\boldsymbol{X}_k'\boldsymbol{\beta})\boldsymbol{t}_{i,k})/g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}) \sim \chi_m^2,$$

the elements of $(N_1+N_2)\times(N_1+N_2)$ FIM, $\mathcal{I}_{i,k}(\boldsymbol{\theta})$, for $\boldsymbol{Y}_{ik}$ are simplified as

$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}'}\right) = \frac{t_{m,k}}{g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\left(\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial \boldsymbol{X}_k'\boldsymbol{\beta}}\right)^2 \boldsymbol{X}_k\boldsymbol{X}_k', \quad \mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\beta}\partial \boldsymbol{\gamma}'}\right) = \boldsymbol{0}_{N_1\times N_2},$$
$$\mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\beta}'}\right) = \boldsymbol{0}_{N_2\times N_1}, \quad \mathrm{E}\left(-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}\partial \boldsymbol{\gamma}'}\right) = \frac{m}{2(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2}\left(\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial \boldsymbol{Z}_k'\boldsymbol{\gamma}}\right)^2 \boldsymbol{Z}_k\boldsymbol{Z}_k',$$

where $\boldsymbol{0}_{N_1\times N_2}$ denotes a $N_1 \times N_2$ matrix of zeros. Consequently, the overall FIM, $\mathcal{I}_n(\boldsymbol{\theta})$, is a block diagonal matrix, i.e.,

$$\mathcal{I}_n(\boldsymbol{\theta}) = \sum_{k=1}^{l}\sum_{i=1}^{n_k}\mathcal{I}_{i,k}(\boldsymbol{\theta}) = n((t_m\boldsymbol{B}) \oplus (m\boldsymbol{G}/2)),$$

where "$\oplus$" denotes the direct sum,

$$\boldsymbol{B} = \sum_{k=1}^{l}\frac{p_k\psi_k}{g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}\left(\frac{\partial g_1(\boldsymbol{X}_k'\boldsymbol{\beta})}{\partial \boldsymbol{X}_k'\boldsymbol{\beta}}\right)^2 \boldsymbol{X}_k\boldsymbol{X}_k' \text{ and } \boldsymbol{G} = \sum_{k=1}^{l}\frac{p_k}{(g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma}))^2}\left(\frac{\partial g_2(\boldsymbol{Z}_k'\boldsymbol{\gamma})}{\partial \boldsymbol{Z}_k'\boldsymbol{\gamma}}\right)^2 \boldsymbol{Z}_k\boldsymbol{Z}_k'.$$

1.2 Proof of Theorem 2.1

Proof. Let

$$G(\boldsymbol{\xi}) = -n^{N_0}t_m^{N_1}m^{N_2} + \mu_0(C_{op}t_m + C_{mea}mn + C_{it}n - 1) + \mu_1(l-n) - \mu_2 t_m + \mu_3(1-m), \tag{S.1}$$

where $\mu_0, \mu_1, \mu_2, \mu_3 \geq 0$. The Karush-Kuhn-Tucker (KKT) conditions of (S.1) are

$$\frac{\partial G(\boldsymbol{\xi})}{\partial n} = 0 \Leftrightarrow -N_0 n^{N_0-1}t_m^{N_1}m^{N_2} + \mu_0(C_{it} + C_{mea}m) - \mu_1 = 0, \tag{S.2}$$
$$\frac{\partial G(\boldsymbol{\xi})}{\partial t_m} = 0 \Leftrightarrow -N_1 n^{N_0}t_m^{N_1-1}m^{N_2} + \mu_0 C_{op} - \mu_2 = 0, \tag{S.3}$$
$$\frac{\partial G(\boldsymbol{\xi})}{\partial m} = 0 \Leftrightarrow -N_2 n^{N_0}t_m^{N_1}m^{N_2-1} + \mu_0 C_{mea}n - \mu_3 = 0, \tag{S.4}$$
$$\mu_0(C_{op}t_m + C_{mea}mn + C_{it}n - 1) = \mu_1(l-n) = -\mu_2 t_m = \mu_3(1-m) = 0, \tag{S.5}$$

$\mu_0, \mu_1, \mu_2, \mu_3 \geq 0.$

When $t_m = 0$ or $\mu_0 = 0$, there is no solution for n, t_m, and m in (S.2)–(S.4). Hence, we have $\mu_0 > 0$ and $\mu_2 = 0$ by the complementary slackness in (S.5). According to the distinctions between n, m, μ_1, μ_3, there are four cases as follows.

(a) $n = l, m > 1, \mu_1 \geq 0$, and $\mu_3 = 0$:

Substituting $n = l$ and $\mu_2 = \mu_3 = 0$ into (S.3) and (S.4) gives $\tilde{m}(t_m) = N_2 C_{op} t_m / (N_1 l C_{mea})$. Since $\mu_0 > 0$, substituting $\tilde{m}(t_m)$, and $n = l$ into (S.5), we have

$$\widetilde{t_m} = \frac{N_1(1 - lC_{it})}{(N_1 + N_2)C_{op}} > 0 \Leftrightarrow C_{it} < 1/l \text{ and}$$
$$\tilde{m}(\widetilde{t_m}) = \frac{N_2(1 - lC_{it})}{(N_1 + N_2)lC_{mea}} > 1 \Leftrightarrow C_{it} + \frac{(N_1 + N_2)C_{mea}}{N_2} < \frac{1}{l}.$$

Again, substituting $\widetilde{t_m}$, $\tilde{m}(\widetilde{t_m})$, $\mu_2 = 0$, and $n = l$ into (S.3), we get

$$\widetilde{\mu_0}(\widetilde{t_m}, \tilde{m}(\widetilde{t_m})) = \frac{N_1 l^{N_0}}{C_{op}} \left(\frac{N_1(1 - lC_{it})}{(N_1 + N_2)C_{op}} \right)^{N_1 - 1} \left(\frac{N_2(1 - lC_{it})}{(N_1 + N_2)lC_{mea}} \right)^{N_2} \propto (1 - lC_{it})^{N_1 + N_2 - 1} > 0$$
$$\Leftrightarrow C_{it} < 1/l \text{ or } \{C_{it} \geq 1/l \text{ and } (N_1 + N_2 - 1)/2 \in \mathbb{N} \cup \{0\}\}.$$

In addition, substituting $\widetilde{t_m}$, $\tilde{m}(\widetilde{t_m})$, $\widetilde{\mu_0}(\widetilde{t_m}, \tilde{m}(\widetilde{t_m}))$, and $n = l$ into (S.2), we have

$$\widetilde{\mu_1} = \frac{\widetilde{\mu_0}(\widetilde{t_m}, \tilde{m}(\widetilde{t_m}))\,(N_1 l C_{it} + N_2 - N_0(1 - lC_{it}))}{l(N_1 + N_2)} \geq 0 \Leftrightarrow C_{it} \geq \frac{N_0 - N_2}{l(N_0 + N_1)}.$$

(b) $n > l, m > 1, \mu_1 = 0$, and $\mu_3 = 0$:

Substituting $\mu_1 = \mu_2 = \mu_3 = 0$ into (S.2), (S.3), and (S.4) gives

$$\tilde{m} = \frac{N_2 C_{it}}{(N_0 - N_2)C_{mea}} \quad \text{and} \quad \tilde{n}(t_m) = \frac{(N_0 - N_2)C_{op} t_m}{N_1 C_{it}}.$$

Again, substituting $\tilde{m}$ and $\tilde{n}(t_m)$ into (S.5), we obtain $\widetilde{t_m} = N_1/((N_0 + N_1)C_{op}) > 0$. Hence, we have

$$\tilde{m} > 1 \Leftrightarrow N_0 > N_2 \text{ and } \frac{C_{it}}{C_{mea}} > \frac{N_0 - N_2}{N_2},$$
$$\tilde{n}(\widetilde{t_m}) = \frac{N_0 - N_2}{(N_0 + N_1)C_{it}} > l \Leftrightarrow N_0 > N_2 \text{ and } C_{it} < \frac{N_0 - N_2}{l(N_0 + N_1)}.$$

Substituting $\tilde{n}(\widetilde{t_m})$, $\widetilde{t_m}$, $\tilde{m}$, and $\mu_2 = 0$ into (S.3), we have

$$\widetilde{\mu_0}(\tilde{n}(\widetilde{t_m}), \widetilde{t_m}, \tilde{m}) \propto (N_0 - N_2)^{N_0 - N_2} > 0 \Leftrightarrow N_0 > N_2 \text{ or } \{N_0 \leq N_2 \text{ and } (N_2 - N_0)/2 \in \mathbb{N} \cup \{0\}\},$$

where $\lim_{x \to 0^+} x^x = 1$.

(c) $n > l, m = 1, \mu_1 = 0$, and $\mu_3 \geq 0$:

Substituting $m = 1$ and $\mu_1 = 0$ into (S.2) and (S.3) gives $\tilde{n}(t_m) = N_0 C_{op} t_m / (N_1(C_{it} + C_{mea}))$. Since $\mu_0 > 0$, substituting $\tilde{n}(t_m)$ and $m = 1$ into (S.5), we have

$$\widetilde{t_m} = \frac{N_1}{(N_0 + N_1)C_{op}} > 0 \text{ and } \tilde{n}(\widetilde{t_m}) = \frac{N_0}{(N_0 + N_1)(C_{it} + C_{mea})} > l \Leftrightarrow C_{it} + C_{mea} < \frac{N_0}{l(N_0 + N_1)}.$$

Again, substituting $\widetilde{t_m}$, $\tilde{n}(\widetilde{t_m})$, $\mu_2 = 0$ and $m = 1$ into (S.3), we get

$$\widetilde{\mu_0} = \frac{N_1}{C_{op}} \left(\frac{N_0}{(N_0 + N_1)(C_{it} + C_{mea})} \right)^{N_0} \left(\frac{N_1}{(N_0 + N_1)C_{op}} \right)^{N_1 - 1} > 0.$$

In addition, substituting $\widetilde{t_m}$, $\tilde{n}(\widetilde{t_m})$, $\widetilde{\mu_0}$, and $m = 1$ into (S.4), we have

$$\widetilde{\mu_3} = \frac{\widetilde{\mu_0}}{N_0 + N_1} \left(\frac{N_0 C_{mea}}{C_{it} + C_{mea}} - N_2 \right) \geq 0 \Leftrightarrow N_0 > N_2 \text{ and } \frac{C_{it}}{C_{mea}} \leq \frac{N_0 - N_2}{N_2}.$$

(d) $n = l, m = 1, \mu_1 \geq 0$, and $\mu_3 \geq 0$:

Since $\mu_0 > 0$, substituting $n = l$ and $m = 1$ into (S.5), we have

$$\widetilde{t_m} = (1 - lC_{it} - lC_{mea})/C_{op} > 0 \Leftrightarrow C_{it} + C_{mea} < 1/l.$$

Substituting $\widetilde{t_m}$, $n = l$, $m = 1$, and $\mu_2 = 0$ into (S.3) gives

$$\begin{aligned} \widetilde{\mu_0}(\widetilde{t_m}) &= \frac{N_1 l^{N_0}}{C_{op}} \left(\frac{1 - lC_{it} - lC_{mea}}{C_{op}} \right)^{N_1 - 1} > 0 \\ &\Leftrightarrow C_{it} + C_{mea} < 1/l \text{ or } \{C_{it} + C_{mea} \geq 1/l \text{ and } (N_1 - 1)/2 \in \mathbb{N} \cup \{0\}\}. \end{aligned}$$

Again, substituting $\widetilde{t_m}$, $\widetilde{\mu_0}(\widetilde{t_m})$, $n = l$, and $m = 1$ into (S.2) and (S.4) gives

$$\begin{aligned} \widetilde{\mu_1}(\widetilde{t_m}) &= \left(\frac{N_1 l(C_{it} + C_{mea})}{C_{op}} - \frac{N_0(1 - lC_{it} - lC_{mea})}{C_{op}} \right) l^{N_0 - 1} \widetilde{t_m}^{N_1 - 1} \geq 0 \\ &\Leftrightarrow C_{it} + C_{mea} \geq \frac{N_0}{l(N_0 + N_1)} \text{ and} \\ \widetilde{\mu_3}(\widetilde{t_m}) &= \left(\frac{N_1 l C_{mea}}{C_{op}} - \frac{N_2(1 - lC_{it} - lC_{mea})}{C_{op}} \right) l^{N_0} \widetilde{t_m}^{N_1 - 1} \geq 0 \\ &\Leftrightarrow C_{it} + \frac{N_1 + N_2}{N_2} C_{mea} \geq \frac{1}{l}. \end{aligned}$$

From cases (a)–(d), the KKT conditions can respectively be summarized as follows.

($\bar{\text{a}}$) Since $\{C_{it} \geq 1/l$ and $(N_1 + N_2 - 1)/2 \in \mathbb{N} \cup \{0\}\}$ for $\widetilde{\mu_0}(\widetilde{t_m}, \tilde{m}(\widetilde{t_m})) > 0$ contradicts $C_{it} < 1/l$ for $\widetilde{t_m} > 0$, only two conditions $C_{it} + (N_1 + N_2)C_{mea}/N_2 < 1/l$ and $C_{it} \geq (N_0 - N_2)/(l(N_0 + N_1))$ if and only if $\boldsymbol{\xi}_D = \left(l, \widetilde{t_m}, \tilde{m}(\widetilde{t_m})\right)$.

($\bar{\text{b}}$) Since $\{N_0 \leq N_2$ and $(N_2 - N_0)/2 \in \mathbb{N} \cup \{0\}\}$ for $\widetilde{\mu_0}(\tilde{n}(\widetilde{t_m}), \widetilde{t_m}, \tilde{m}) > 0$ contradicts the positive experimental cost $C_{it} < (N_0 - N_2)/(l(N_0 + N_1))$, only three conditions $N_0 > N_2$, $C_{it}/C_{mea} > (N_0 - N_2)/N_2$ and $C_{it} < (N_0 - N_2)/(l(N_0 + N_1))$ if and only if $\boldsymbol{\xi}_D = \left(\tilde{n}(\widetilde{t_m}), \widetilde{t_m}, \tilde{m}\right)$.

($\bar{\text{c}}$) $N_0 > N_2$, $C_{it}/C_{mea} \leq (N_0 - N_2)/N_2$ and $C_{it} + C_{mea} < N_0/(l(N_0 + N_1))$ if and only if $\boldsymbol{\xi}_D = \left(\tilde{n}(\widetilde{t_m}), \widetilde{t_m}, 1\right)$.

($\bar{\text{d}}$) Since $\{C_{it} + C_{mea} \geq 1/l$ and $(N_1 - 1)/2 \in \mathbb{N} \cup \{0\}\}$ for $\widetilde{\mu_0}(\widetilde{t_m}) > 0$ contradicts $C_{it} + C_{mea} < 1/l$ for $\widetilde{t_m} > 0$, only three conditions $C_{it} + C_{mea} < 1/l$, $C_{it} + C_{mea} \geq N_0/(l(N_0 + N_1))$ and $C_{it} + (N_1 + N_2)C_{mea}/N_2 \geq 1/l$ if and only if $\boldsymbol{\xi}_D = \left(l, \widetilde{t_m}, 1\right)$.

The results can be divided into two cases $N_0 > N_2$ and $N_0 \leq N_2$ immediately. □

1.3 Proof of Corollary 2.1

Proof. The results are easy to show according to the definition of occurrence probability. □

1.4 Proof of Corollary 2.2

Proof. (ii) By Theorem 2.1(i)-(2), we have

$$C_{op} t_{m,D} = \frac{N_1}{N_0 + N_1}, \; C_{mea} m_D n_D = \frac{N_2}{N_0 + N_1}, \quad \text{and} \quad C_{it} n_D = \frac{N_0 - N_2}{N_0 + N_1},$$

as desired. The remaining cases (i) and (iii) are easy to verify. □

1.5 Proof of Theorem 3.1

Proof. For the $D(q^*)$-optimal test plan $\boldsymbol{\xi}_{D(q^*)} (= \boldsymbol{\xi}_{V_{t_{q^*}}} = \boldsymbol{\xi}_D)$, the intersection of necessary and sufficient conditions in both V_{t_q} and D-optimality criteria should be satisfied at each interior or boundary case.

(i) For $n_{D(q^*)} = l$, since $C_{it} + C_{mea} < C_{it} + (N_1 + N_2)C_{mea}/N_2$, by Theorem 2.1(i)-(1) (or (ii)-(1)), the condition $C_{it} + C_{mea} < 1/l$ holds directly. Using the optimal total termination time in Theorem 2.1(i)-(1) (or (ii)-(1)) and Theorem 2.2(i), we have

$$t_{m,D} = t_{m,V_{t_q}} \Leftrightarrow \frac{N_1(1 - lC_{it})}{(N_1 + N_2)C_{op}} = \frac{(1 - lC_{it})(\sqrt{lC_{op}C_{mea}\tilde{\alpha}(q)} - C_{op})}{C_{op}(lC_{mea}\tilde{\alpha}(q) - C_{op})},$$

indicating the DQ-equation (11). From $m_D = m_{V_{t_q}}$, we obtain the same results. Substituting the DQ-equation into the necessary and sufficient conditions in Theorem 2.2(i) gives

$$\frac{(1-2lC_{it})^2 C_{op}}{l^3 C_{mea} C_{it}^2} \leq \frac{N_2^2 C_{op}}{N_1^2 l C_{mea}} \Leftrightarrow C_{it} \geq \frac{N_1}{l(2N_1+N_2)}, \tag{S.6}$$

$$\frac{N_2^2 C_{op}}{N_1^2 l C_{mea}} > \frac{l C_{mea} C_{op}}{(1-lC_{it}-lC_{mea})^2} \Leftrightarrow C_{it} + (N_1+N_2)C_{mea}/N_2 < 1/l. \tag{S.7}$$

The inequality (S.6) is a new lower bound of C_{it} for $2C_{it} + C_{mea} < 1/l$. From (S.6), applying $C_{it} \geq N_1/(l(2N_1+N_2))$ for $2C_{it}+C_{mea} < 1/l$, we have $C_{mea} < N_2/(l(2N_1+N_2))$. Therefore, it is easy to show that $C_{it} + (N_1+N_2)C_{mea}/N_2 < 2C_{it} + C_{mea}$, implying that the condition $C_{it} + (N_1+N_2)C_{mea}/N_2 < 1/l$ holds. Combining with the inequality (S.7) for $1/l \leq 2C_{it}+C_{mea}$, the condition $C_{it}+(N_1+N_2)C_{mea}/N_2 < 1/l$ is a common condition for both optimality criteria. For $N_0 > N_2$, comparing the two lower bounds of C_{it} gives

$$\frac{N_1}{l(2N_1+N_2)} \leq \frac{N_0-N_2}{l(N_0+N_1)} \Leftrightarrow N_0 \geq N_1 + N_2. \tag{S.8}$$

Consequently, the results can be divided into two cases with the common condition $C_{it}+(N_1+N_2)C_{mea}/N_2 < 1/l$ and the DQ-equation by (S.8). Supplementary Figures 1–3 (for $l = 1$), with the "pale-green" area, are plotted for $N_0 \geq N_1 + N_2$ and $N_0 < N_1 + N_2$, respectively.

(ii) By solving $m_D = m_{V_{t_q}}$, we have

$$\begin{aligned} c_0 &= \sqrt[3]{k_1(\tilde{\alpha}(q)) + k_2(\tilde{\alpha}(q))} + \sqrt[3]{k_1(\tilde{\alpha}(q)) - k_2(\tilde{\alpha}(q))} \\ \Leftrightarrow \; c_0^3 &= 2k_1(\tilde{\alpha}(q)) + 3\sqrt[3]{k_1(\tilde{\alpha}(q))^2 - k_2(\tilde{\alpha}(q))^2}c_0, \end{aligned} \tag{S.9}$$

where

$$\begin{aligned} c_0 &= \frac{3N_2}{(N_0-N_2)} + 2, \\ k_1(\tilde{\alpha}(q)) &= \frac{27C_{mea}\tilde{\alpha}(q)}{2C_{it}C_{op}} - 8, \\ k_2(\tilde{\alpha}(q)) &= \frac{3}{2}\sqrt{\frac{3C_{mea}\tilde{\alpha}(q)}{C_{it}C_{op}}\left(\frac{27C_{mea}\tilde{\alpha}(q)}{C_{it}C_{op}} - 32\right)}. \end{aligned}$$

Substituting $k_1(\tilde{\alpha}(q))$ and $k_2(\tilde{\alpha}(q))$ into (S.9), a cubic equation of $\tilde{\alpha}(q)$ is given by

$$19683C_{mea}^3\tilde{\alpha}(q)^3 - 2187C_{it}C_{op}C_{mea}^2(c_0^3+16)\tilde{\alpha}(q)^2 + 81C_{it}^2C_{op}^2C_{mea}(c_0^3+16)^2\tilde{\alpha}(q)$$

$$- (c_0^3 - 8)^2(c_0^3 + 64)C_{it}^3 C_{op}^3 = 0. \tag{S.10}$$

Since the discriminant (defined in Supplementary Lemma 1 of Cheng and Peng, 2024) of (S.10) is proportional to $c_0^6 C_{it}^6 C_{op}^6 C_{mea}^6$ and positive, there is only one positive real root

$$\tilde{\alpha}(q) = \frac{(2N_0 - N_2)N_2^2 C_{it} C_{op}}{(N_0 - N_2)^3 C_{mea}} \tag{S.11}$$

by using Supplementary Lemma 2(ii) in Cheng and Peng (2024). Substituting m_D in Theorem 2.1(i)-(2) into $t_{m,D} = t_{m,V_{t_q}}$, we have

$$\frac{N_1}{(N_0 + N_1)C_{op}} = \frac{\sqrt{\frac{N_2 C_{it} C_{op}}{(N_0 - N_2)C_{mea}}\left(\frac{N_2 C_{it} C_{op}}{(N_0 - N_2)C_{mea}} + \tilde{\alpha}(q)\right)} - \frac{N_2 C_{it} C_{op}}{(N_0 - N_2)C_{mea}}}{C_{op}\tilde{\alpha}(q)}.$$

Simplifying the above equation, we get

$$\tilde{\alpha}(q) = \frac{(N_0^2 - N_1^2)N_2 C_{it} C_{op}}{N_1^2(N_0 - N_2)C_{mea}}. \tag{S.12}$$

Moreover, under the condition $N_0 > N_2$ for (S.12), we have $\tilde{\alpha}(q) > 0 \Leftrightarrow N_0 > N_1$. Similarly, substituting m_D in Theorem 2.1(i)-(2) into $n_D = n_{V_{t_q}}$, we obtain the same equation (S.12). To achieve $D(q^*)$-optimality, the equalities (S.11) and (S.12) should be the same, i.e.,

$$\frac{(2N_0 - N_2)N_2^2 C_{it} C_{op}}{(N_0 - N_2)^3 C_{mea}} = \frac{(N_0^2 - N_1^2)N_2 C_{it} C_{op}}{N_1^2(N_0 - N_2)C_{mea}} \Leftrightarrow N_0 = 0, \text{ or } N_0 + N_1 = N_2, \text{ or } N_0 = N_1 + N_2.$$

Since $N_0 > N_2$ and $N_0, N_1, N_2 \in \mathbb{N}$, we have $N_0 = N_1 + N_2$, implying the DQ-equation (12). Substituting $N_0 = N_1 + N_2$ into the necessary and sufficient conditions in Theorem 2.1(i)-(2) and Theorem 2.2(ii), we have

$$\frac{(2C_{it} + C_{mea})C_{mea}C_{op}}{C_{it}^2} < \frac{(2N_1 + N_2)N_2^2 C_{it} C_{op}}{N_1^3 C_{mea}} \Leftrightarrow \frac{C_{mea}}{C_{it}} < \frac{N_2}{N_1},$$
$$\frac{(1 - 2lC_{it})^2 C_{op}}{l^3 C_{it}^2 C_{mea}} > \frac{(2N_1 + N_2)N_2^2 C_{it} C_{op}}{N_1^3 C_{mea}} \Leftrightarrow C_{it} < \frac{N_1}{l(2N_1 + N_2)}.$$

Furthermore, it is easy to verify that $2C_{it} + C_{mea} < (2 + N_2/N_1)C_{it} < 1/l$. The feasible region for $l = 1$ can be referred to Supplementary Figure 1 (the "khaki" area).

(iii) For $m_{D(q^*)} = 1$ and $N_0 > N_2$, solving $n_D = n_{V_{t_q}}$, we get

$$\frac{N_0}{(N_0 + N_1)(C_{it} + C_{mea})} = \frac{C_{op} + \alpha_\Xi(\boldsymbol{\theta}) - \sqrt{C_{op}(C_{op} + \alpha_\Xi(\boldsymbol{\theta}))}}{\alpha_\Xi(\boldsymbol{\theta})(C_{it} + C_{mea})} \Leftrightarrow \tilde{\alpha}(q) = \frac{(N_0^2 - N_1^2)C_{op}}{N_1^2}.$$

Since $\tilde{\alpha}(q) > 0$, we have $N_0 > N_1$. From $t_{m,D} = t_{m,V_{t_q}}$, we obtain the same results. Substituting the DQ-equation (13) into the necessary and sufficient conditions in Theorem 2.2(iii) gives

$$\frac{(2l(C_{it}+C_{mea})-1)\,C_{op}}{(1-lC_{it}-l^2C_{mea})^2} < \frac{(N_0^2-N_1^2)C_{op}}{N_1^2} \leq \frac{(2C_{it}+C_{mea})C_{mea}C_{op}}{C_{it}^2}. \tag{S.13}$$

Applying the condition $C_{it} + lC_{mea} < N_0/(l(N_0+N_1))$ for the first inequality of (S.13), we have

$$\begin{aligned}&\frac{(2l(C_{it}+C_{mea})-1)\,C_{op}}{(1-lC_{it}-lC_{mea})^2} - \frac{(N_0^2-N_1^2)C_{op}}{N_1^2}\\ &= -\frac{C_{op}(N_0-l(C_{it}+C_{mea})(N_0+N_1))((1-lC_{it}-lC_{mea})N_0+l(C_{it}+C_{mea})N_1)}{N_1^2(1-lC_{it}-lC_{mea})^2} < 0,\end{aligned}$$

and vice versa. For the second inequality of (S.13), we obtain the new lower bound of C_{mea}/C_{it}, i.e.,

$$\frac{(2C_{it}+C_{mea})C_{mea}C_{op}}{C_{it}^2} \geq \frac{(N_0^2-N_1^2)C_{op}}{N_1^2} \Leftrightarrow \frac{C_{mea}}{C_{it}} \geq \frac{N_0-N_1}{N_1}.$$

The new inequality is equivalent to $2C_{it} + C_{mea} \leq (C_{it}+C_{mea})(N_0+N_1)/N_0$. This indicates that the condition $2C_{it} + C_{mea} < 1/l$ holds automatically. Consequently, comparing the two lower bounds of C_{mea}/C_{it} gives the key condition: for $N_0 > N_1$ and $N_0 > N_2$,

$$\frac{N_0-N_1}{N_1} > \frac{N_2}{N_0-N_2} \Leftrightarrow N_0 > N_1 + N_2. \tag{S.14}$$

The results can be divided into two cases with the common condition $C_{it} + C_{mea} < N_0/(l(N_0+N_1))$ and the DQ-equation by using (S.14). The feasible region for $l = 1$ can be referred to Supplementary Figure 2 (the "pink" area).

(iv) For $n_{D(q^*)} = l, m_{D(q^*)} = 1$, all need to do is to find intersection of the feasible regions for both V_{t_q} and D-optimal test plans. According to Theorem 2.2(iv), the results can be divided into two cases as follows.

  (a) Since $C_{it}+C_{mea} < 2C_{it}+C_{mea}$, the condition $C_{it}+C_{mea} < 1/l$ holds by Theorem 2.2(iv)-(a). The common conditions are $2C_{it} + C_{mea} < 1/l \leq 2(C_{it}+C_{mea})$ and (14). For $N_0 < N_1 + N_2$, we have three disjoint sets:

    (1) $N_0 < N_1 + N_2$, $N_2 < N_0$ and $N_1 < N_0$:
    Since $N_1 < N_0$, we have $2(C_{it}+C_{mea}) > (N_0+N_1)(C_{it}+C_{mea})/N_0$. It means that the condition $2(C_{it}+C_{mea}) > 1/l$ holds.

(2) $N_0 \le N_1,\ N_2 < N_1$:

Since $N_0 \le N_1$, we have $2(C_{it} + C_{mea}) \le (N_0 + N_1)(C_{it} + C_{mea})/N_0$. It means that the condition $C_{it} + C_{mea} \ge N_0/(l(N_0 + N_1))$ is satisfied.

(3) $N_0 \le N_2,\ N_1 < N_2$:

For $N_1 < N_2$ and $C_{it} + (N_1 + N_2)C_{mea}/N_2 \ge 1/l$, we have $2(C_{it} + C_{mea}) > (C_{it} + (N_1 + N_2)C_{mea}/N_2) + C_{it} \ge 1/l + C_{it} > 1/l$.

For the last case (4), recalling the intersection point P $= ((N_0 - N_2)/(l(N_0 + N_1)), N_2/(l(N_0 + N_1)))$ in Figure 1(a) of Theorem 2.1, it can be verified that the point P is located on $2C_{it} + C_{mea} = 1/l$ when $N_0 = N_1 + N_2$. This means that the point P satisfies $2C_{it} + C_{mea} < 1/l$ for $N_0 < N_1 + N_2$ and $2C_{it} + C_{mea} > 1/l$ for $N_0 > N_1 + N_2$. When $N_0 \ge N_1 + N_2$, we have $2(C_{it} + C_{mea}) > 2C_{it} + C_{mea} \ge 1/l$. Using $C_{it} \le (N_0 - N_2)/(l(N_0 + N_1))$ and $C_{mea} \ge N_2/(l(N_0 + N_1))$ for the intersection point P, we have

$$C_{it} \le \frac{N_0 - N_2}{l(N_0 + N_1)} = \frac{N_0 - N_2}{N_2} \frac{N_2}{l(N_0 + N_1)} \le \frac{N_0 - N_2}{N_2} C_{mea},$$

which is equivalent to $(N_0 + N_1)(C_{it} + C_{mea})/N_0 \le C_{it} + (N_1 + N_2)C_{mea}/N_2$. It means that the condition $C_{it} + (N_1 + N_2)C_{mea}/N_2 \ge 1/l$ holds. Supplementary Figures 1–3 (for $l = 1$), with the "light-blue" area labeled as (a), are plotted for the case $2C_{it} + C_{mea} < 1/l$.

(b) From Theorem 2.1(i)-(4) and (ii)-(2), the results can be divided into two cases with the common condition $1/l \le C_{it} + (N_1 + N_2)C_{mea}/N_2$. Supplementary Figures 1–3 (for $l = 1$), with the "light-blue" area labeled as (b), are plotted for the case $C_{it} + C_{mea} < 1/l \le 2C_{it} + C_{mea}$ and $1/l \le C_{it} + (N_1 + N_2)C_{mea}/N_2$. (1) This is the case $N_0 > N_1 + N_2$ and $C_{it} + C_{mea} \ge N_0/(l(N_0 + N_1))$. (2) For $N_0 \le N_1 + N_2$ and $N_0 > N_2$, the intersection point for $2C_{it} + C_{mea} = 1/l$ and $C_{it} + (N_1 + N_2)C_{mea}/N_2 = 1/l$ is Q $= (N_1/(l(2N_1 + N_2)), N_2/(l(2N_1 + N_2)))$. Using $C_{it} \le N_1/(l(2N_1 + N_2))$ and $C_{mea} \ge N_2/(l(2N_1 + N_2))$ for the intersection point Q, we have

$$C_{mea} \ge \frac{N_2}{l(2N_1 + N_2)} \ge \frac{N_0 - N_1}{N_1} \frac{N_1}{l(2N_1 + N_2)} \ge \frac{N_0 - N_1}{N_1} C_{it},$$

which is equivalent to $(N_0 + N_1)(C_{it} + C_{mea})/N_0 \ge 2C_{it} + C_{mea}$. This means that the condition $C_{it} + C_{mea} \ge N_0/(l(N_0 + N_1))$ is satisfied.

This completes the proof. □

1.6 Proof of Proposition 1

Proof. To ensure $\tilde{\alpha}(q) < \infty$, the denominator of (18) is not equal to zero (i.e., $q \neq \Phi\left(\rho^{-1} - 2\rho\right)$), which means that there is a vertical asymptote at $q = \Phi(\rho^{-1} - 2\rho)$. In addition, the first derivative of $\tilde{\alpha}(q)$ is proportional to $-h_q/(1-h_q)^3$ with $h_q = \left(\rho^{-2} - \Phi^{-1}(q)\rho^{-1}\right)/2$ and is zero at $q = \Phi\left(\rho^{-1}\right)$. Since the sign of $-h_q/(1-h_q)^3$ is the same as $h_q(h_q - 1)$, we have

$$\frac{\mathrm{d}\tilde{\alpha}(q)}{\mathrm{d}q} < 0 \Leftrightarrow 0 < h_q < 1 \Leftrightarrow \Phi\left(\rho^{-1} - 2\rho\right) < q < \Phi\left(\rho^{-1}\right) \text{ and}$$
$$\frac{\mathrm{d}\tilde{\alpha}(q)}{\mathrm{d}q} > 0 \Leftrightarrow h_q < 0 \text{ or } h_q > 1 \Leftrightarrow q < \Phi\left(\rho^{-1} - 2\rho\right) \text{ or } q > \Phi\left(\rho^{-1}\right).$$

Hence, it is easy to check that $\lim_{q\to 0} \tilde{\alpha}(q) = \lim_{q\to 1} \tilde{\alpha}(q) = 2\eta^2/\sigma^2 > 0 = \tilde{\alpha}(\Phi(\rho^{-1}))$, indicating that there is an absolute minimum at $q = \Phi(\rho^{-1})$ with $\tilde{\alpha}(\Phi(\rho^{-1})) = 0$. Moreover, the second derivative of $\tilde{\alpha}(q)$ is given by

$$\frac{\mathrm{d}^2\tilde{\alpha}(q)}{\mathrm{d}q^2} \propto \rho^2\left(\Phi^{-1}(q)\right)^3 + 2\rho(\rho^2 - 1)\left(\Phi^{-1}(q)\right)^2 + (1 - 4\rho^2)\Phi^{-1}(q) + 2\rho(\rho^2 + 1), \quad \text{(S.15)}$$

which is a cubic polynomial of $\Phi^{-1}(q)$. Since the discriminant of the cubic equation (defined in Supplementary Lemma 1 of Cheng and Peng, 2024) for (S.15) is $\Delta = 4\rho^6(16\rho^6 + 51\rho^4 + 12\rho^2 + 2) > 0$, there is one negative real root z_0 defined in Proposition 1(i) by Vieta's formula. Thus, there is an inflection point at $q = \Phi(z_0)$, i.e., $\mathrm{d}^2\tilde{\alpha}(q)/\mathrm{d}q^2 > 0$ for $\Phi(z_0) < q < 1$ and $\mathrm{d}^2\tilde{\alpha}(q)/\mathrm{d}q^2 < 0$ for $0 < q < \Phi(z_0)$. Now, we claim that the inflection point is on the left-hand side of the vertical asymptote, i.e., $z_0 < \rho^{-1} - 2\rho$. It can be verified that $z_0 < \rho^{-1} - 2\rho$ is equivalent to $\sqrt[3]{(-z_1 - \sqrt{27\rho^4\Delta})/2} + \sqrt[3]{(-z_1 + \sqrt{27\rho^4\Delta})/2} > \rho(4\rho^2 - 1)$, where z_1 is defined in Proposition 1(i). By using Cauchy-Schwarz inequality, we have

$$\sqrt[3]{(-z_1 - \sqrt{27\rho^4\Delta})/2} + \sqrt[3]{(-z_1 + \sqrt{27\rho^4\Delta})/2} - \rho(4\rho^2 - 1) \geq 2(\rho + 2\rho^3) - \rho(4\rho^2 - 1) > 0.$$

The proof is complete. □

1.7 *Derivation of (20)*

Let the IQ function $\tilde{\alpha}(q)$ be defined on the interested interval (19), i.e.,

$$\tilde{\alpha} : \left(\Phi\left(\rho^{-1} - 2\rho\right), \Phi\left(\rho^{-1}\right)\right) \to (0, \infty) : \tilde{\alpha}(q) = \frac{2\eta^2}{\sigma^2}\left(\frac{1 - \rho\Phi^{-1}(q)}{2\rho^2 - 1 + \rho\Phi^{-1}(q)}\right)^2.$$

The inverse function of $\tilde{\alpha}(q)$ can be solved as follows. Let $u = 2\eta^2/(\sigma^2 c)$, then

$$\tilde{\alpha}(q) = c \Leftrightarrow q = \Phi\left(\rho^{-1} - \frac{2\rho}{1 \pm \sqrt{u}}\right). \quad \text{(S.16)}$$

Substituting (S.16) into the interested interval (19), we have

$$\Phi\left(\rho^{-1}-2\rho\right)<\Phi\left(\rho^{-1}-\frac{2\rho}{1\pm\sqrt{u}}\right)<\Phi\left(\rho^{-1}\right)\Leftrightarrow 1>\frac{1}{1\pm\sqrt{u}}>0.$$

Since $\frac{1}{1-\sqrt{u}}>1$ for $u\le 1$ or $\frac{1}{1-\sqrt{u}}<0$ for $u>1$, the inverse function of $\tilde{\alpha}(q)$ is expressed as (20).

1.8 *Proof of Theorem 4.1*

Proof. For the Wiener process, we have $N_0=N_1+N_2$. Substituting $N_0=N_1+N_2$ into Theorem 3.1, the necessary and sufficient conditions for $\boldsymbol{\xi}_{D(q^*)}$ can be simplified. By using (20), the corresponding bi-optimal quantile for interior and boundary cases can be obtained directly. □

References

[1] Cheng, Y. S. and Peng, C. Y. (2024), Optimal test planning for heterogeneous Wiener processes. *Naval Research Logistics*, **71**, 509–520.

[2] Peng, C. Y. and Cheng, Y. S. (2021), Profile optimum planning for degradation analysis. *Naval Research Logistics*, **68**, 951–962.

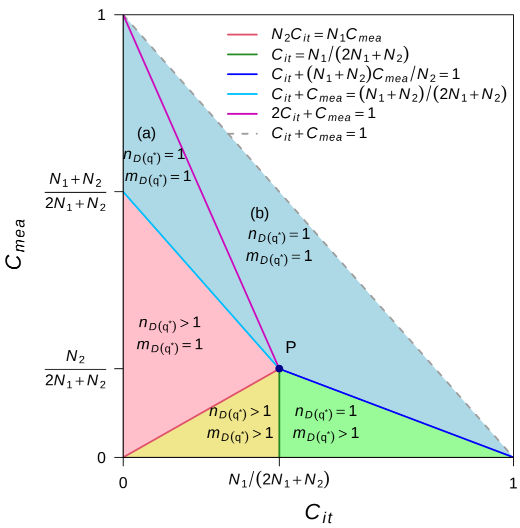
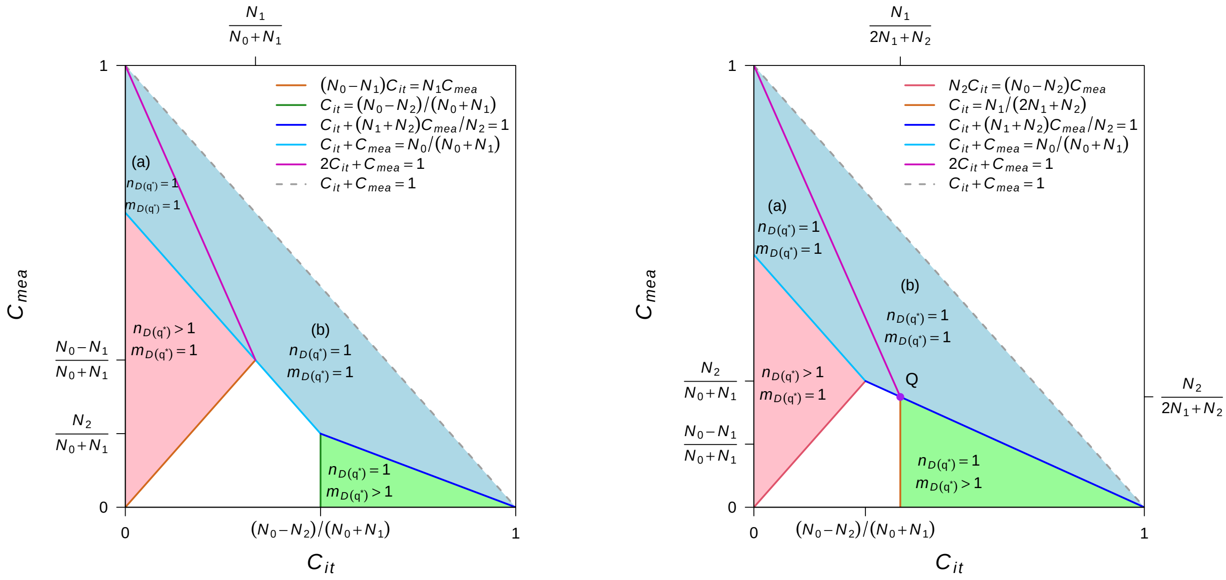


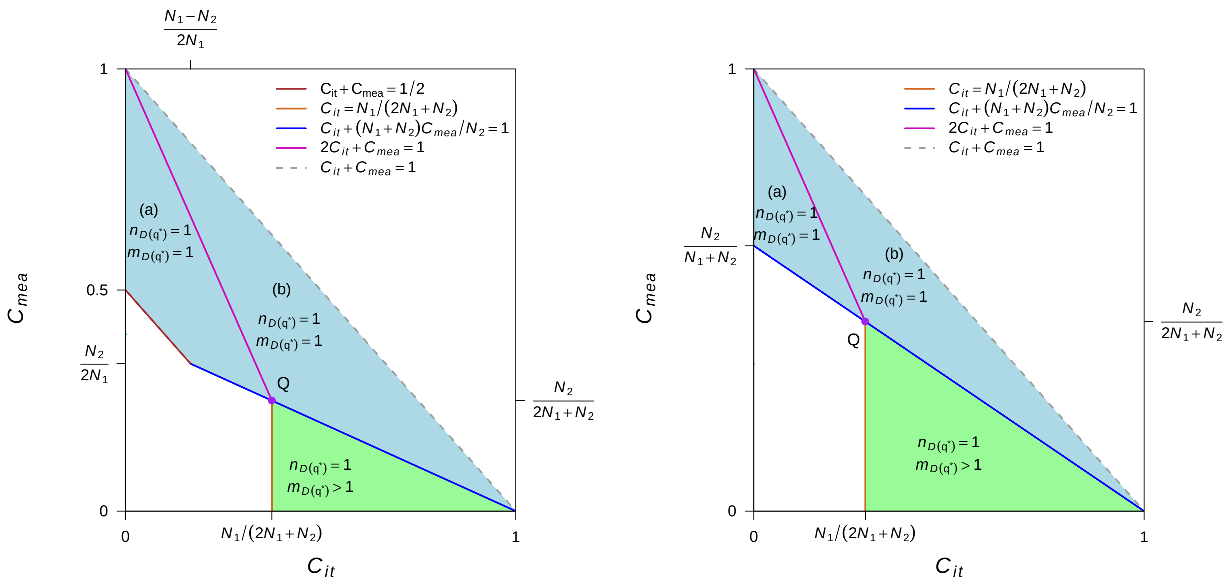
Figure 1: $N_0=N_1+N_2$ $(l=1)$



(a) $N_0 > N_1 + N_2$

(b) $\max\{N_1, N_2\} < N_0 \leq N_1 + N_2$

Figure 2: $\max\{N_1, N_2\} < N_0 \neq N_1 + N_2$ $(l = 1)$



(a) $N_0 \leq N_1$, $N_2 < N_1$

(b) $N_0 \leq N_2$, $N_1 < N_2$

Figure 3: $N_0 \leq \max\{N_1, N_2\}$ $(l = 1)$