

ASYMMETRIC GARCH MODELLING WITHOUT MOMENT CONDITIONS

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Supplementary Material

The Supplementary Material contains additional simulation results, additional empirical results, and proofs of all theorems in the main text. Appendix S1 provides plots of strict stationarity regions, additional finite-sample performance of the MLE in the stationary and explosive cases, results for stationarity and asymmetry tests, as well as empirical results on portfolio returns. Appendix S2 gives the proofs of Theorems 1–6. Appendix S3 provides useful properties of stable distribution and some technical lemmas with proofs. Appendix S4 gives the explicit expressions of the first- and second-order partial derivatives of $\ell_t(\theta)$ and the Fisher information matrix Υ in explosive cases.

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S1 Additional Simulations and Empirical Studies

S1.1 Stable densities and strict stationarity region

Recall that the proposed sAGARCH model (2.2) is strictly stationary *if and only if* the top Lyapunov exponent

$$\gamma_\alpha := E \log a(\eta_1) < 0, \quad \text{where } a(x) = \phi_+(x^+)^2 + \phi_-(x^-)^2 + \psi.$$

Fig. S.1(a) plots the densities $f_\alpha(x)$ for different values of α , and Fig. S.1(b)-(d) plots the surface of $\gamma_\alpha = 0$ in (2.3) in terms of (ϕ_+, ϕ_-, ψ) . The strict stationarity region of model (2.2) is the closed area bounded by the surface $\gamma_\alpha = 0$ and the planes $\phi_+ = 0$, $\phi_- = 0$, and $\psi = 0$. From the subfigures, we observe that: (i) As α increases, the strictly stationary region expands; (ii) within this region, ψ is strictly no more than 1, while ϕ_+ or ϕ_- may exceed 1 if the other two parameters are sufficiently small.

S1.2 Performance of the MLE in the stationary cases

In this subsection, we report the full simulation results for the stationary cases with $\alpha_0 = 1.5, 1.0$, and 0.5 . The number of observations is $n = 200, 500, 1000$, and 2000 , each with 1000 replications. We consider the following three stationary cases, with true parameters $\theta_0 = (0.2, 0.1, 0.2, 0.5, 1.5)'$, $(0.1, 0.1, 0.2, 0.3, 1)'$, and $(0.05, 0.02, 0.05, 0.1, 0.5)'$, respectively.

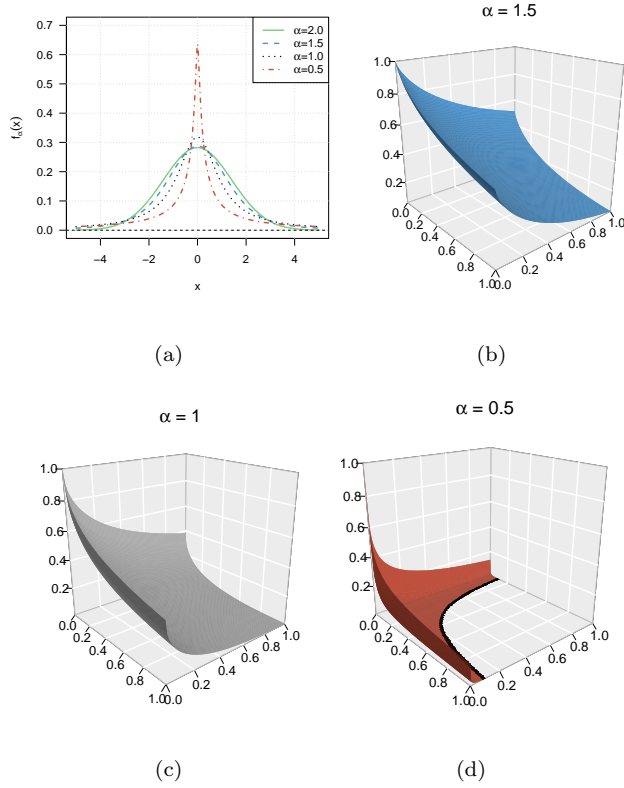


Figure S.1: (a) The densities $f_\alpha(x)$; (b)-(d) The surfaces determined by $\{(\phi_+, \phi_-, \psi) : \gamma_\alpha = 0\}$ in (2.3) for different fixed values of α . The vertical axis represents ψ , and the other two represent ϕ_+ and ϕ_- . In (d), the black curve represents the boundary of the surface in the $\psi = 0$ plane. The surfaces are plotted within a finite region $(0, 1)^3$ for the sake of clarity, while the whole surface can stretch very far, e.g. the extremely small values of ϕ_+, ψ and large value of ϕ_- .

Table S.1 reports the empirical biases (Bias), empirical standard deviations (ESD), asymptotic standard deviations (ASD) of the MLE $\widehat{\theta}_n$, together with two estimators $\widehat{\text{ASD}}^{\text{int}}$ and $\widehat{\text{ASD}}^{\text{res}}$ of ASDs from $\widehat{\Sigma}^{\text{int}}$ and $\widehat{\Sigma}^{\text{res}}$ in Section 3.2. The ASDs are calculated from Theorem 1 (ii). Table S.1 shows that: (i) For the MLE $\widehat{\vartheta}_n$, the biases are small in each case, and the values of the ESD and ASD are close to each other, particularly for large n . (ii) The two estimators of the ASD for $\{\widehat{\phi}_{n+}, \widehat{\phi}_{n-}, \widehat{\psi}_n\}$ perform similarly well. For $\widehat{\alpha}_n$, $\widehat{\text{ASD}}^{\text{int}}$ tends to outperform $\widehat{\text{ASD}}^{\text{res}}$ when $\alpha_0 < 1$, whereas $\widehat{\text{ASD}}^{\text{res}}$ outperforms $\widehat{\text{ASD}}^{\text{int}}$ when $\alpha_0 \in [1, 2)$. Their differences diminish as n increases. (iii) Although $\widehat{\omega}_n$ is theoretically consistent and asymptotically normal for the stationary case, its finite-sample performance is unsatisfactory, particularly when $\alpha_0 \leq 1$: both the ESDs and estimated ASDs are relatively large. Notably, the performance of $\widehat{\omega}_n$ improves as n increases. In fact, this issue has appeared in Francq and Zakoïan (2012), but no plausible explanation has been given so far. According to our model, this phenomenon is likely related to the extreme values of the observations. Intuitively, the intercept ω can be viewed as a *scale parameter* for y_t , as model (2.2) is equivalent to

$$\begin{cases} y_t/\sqrt{\omega} = (\sigma_t/\sqrt{\omega})\eta_t, \\ (\sigma_t/\sqrt{\omega})^2 = 1 + \phi_+(y_{t-1}^+/\sqrt{\omega})^2 + \phi_-(y_{t-1}^-/\sqrt{\omega})^2 + \psi(\sigma_{t-1}/\sqrt{\omega})^2. \end{cases}$$

This is also an sAGARCH model for the scaled series with intercept 1 and other parameters unchanged. As α decreases, the tail of the innovation becomes heavier, leading to extreme values of y_t . Consequently, the estimator of the scale parameter ω is likely influenced by the scale of y_t and may be overestimated in some cases. Additionally, random number generators often perform poorly for heavy-tailed distributions, as outliers are frequently generated and may deviate from the theoretical distribution. Further, numerical evaluations of stable densities and their partial derivatives can exhibit unreliable behavior near the boundary when α is close to 1, or when $\alpha < 1$ and x is large (see, e.g., Matsui and Takemura (2006)). These factors all contribute to the poor performance of $\hat{\omega}_n$, and also explain why the estimated ASDs of $\hat{\alpha}_n$ perform less well when $\alpha_0 = 1$. We further examine the medians of the MLE and the medians of estimated $\widehat{\text{ASD}}$ across 1000 replications in Table S.2. As a result, the median estimators of all parameters are much closer to the true values, and the medians of the estimated $\widehat{\text{ASD}}$ are closer to the theoretical ASD, even when n is small. It indicates that the poor performance of $\hat{\omega}_n$ is indeed driven by outliers.

Table S.1: Simulation results for the MLE $\hat{\theta}_n$ for sAGARCH(1,1) under stationary cases.

n	$\alpha_0 = 1.5$						$\alpha_0 = 1.0$						$\alpha_0 = 0.5$					
	$\omega_0=0.2 \phi_{0+}=0.1 \phi_{0-}=0.2 \psi_0=0.5 \alpha_0=1.5$						$\omega_0=0.1 \phi_{0+}=0.1 \phi_{0-}=0.2 \psi_0=0.3 \alpha_0=1.0$						$\omega_0=0.05 \phi_{0+}=0.02 \phi_{0-}=0.05 \psi_0=0.1 \alpha_0=0.5$					
	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	$\hat{\alpha}_n$	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	$\hat{\alpha}_n$	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	
200	Bias	0.0360	0.0038	0.0008	-0.0007	0.0253	0.0334	0.0039	0.0072	0.0085	0.0200	0.3249	0.0019	0.0064	0.0109	0.0107		
	ESD	0.1213	0.0610	0.0883	0.1086	0.1131	0.1082	0.0527	0.0985	0.0764	0.0710	1.4575	0.0170	0.0413	0.0410	0.0361		
	ASD	0.0890	0.0504	0.0813	0.0878	0.1085	0.0636	0.0489	0.0866	0.0657	0.0780	0.0665	0.0136	0.0315	0.0311	0.0342		
	$\widehat{\text{ASD}}_{\text{int}}$	0.1095	0.0519	0.0818	0.0948	0.1070	0.3908	0.0506	0.0897	0.0702	0.0561	3.5211	0.0150	0.0357	0.0353	0.0351		
	$\widehat{\text{ASD}}_{\text{res}}$	0.1099	0.0521	0.0821	0.0951	0.1076	0.3899	0.0510	0.0902	0.0705	0.0580	3.4737	0.0147	0.0350	0.0347	0.0355		
	Bias	0.0181	0.0001	-0.0026	0.0016	0.0113	0.0135	-0.0002	0.0003	0.0030	0.0084	0.0486	0.0005	0.0004	0.0045	0.0048		
500	ESD	0.0653	0.0330	0.0518	0.0585	0.0712	0.0649	0.0313	0.0519	0.0424	0.0403	0.1573	0.0094	0.0206	0.0210	0.0215		
	ASD	0.0563	0.0319	0.0514	0.0555	0.0687	0.0402	0.0310	0.0547	0.0416	0.0493	0.0420	0.0086	0.0199	0.0197	0.0216		
	$\widehat{\text{ASD}}_{\text{int}}$	0.0627	0.0320	0.0511	0.0576	0.0683	0.0540	0.0309	0.0548	0.0427	0.0292	3.3471	0.0088	0.0201	0.0205	0.0219		
	$\widehat{\text{ASD}}_{\text{res}}$	0.0628	0.0320	0.0512	0.0576	0.0684	0.0538	0.0309	0.0548	0.0427	0.0300	3.2913	0.0086	0.0197	0.0201	0.0219		
	Bias	0.0080	0.0008	0.0005	-0.0002	0.0052	0.0059	0.0008	0.0017	0.0004	0.0040	0.0209	0.0004	0.0005	0.0022	0.0020		
	ESD	0.0435	0.0239	0.0361	0.0409	0.0474	0.0367	0.0239	0.0397	0.0314	0.0285	0.0704	0.0062	0.0141	0.0144	0.0156		
1000	ASD	0.0398	0.0225	0.0364	0.0392	0.0485	0.0284	0.0219	0.0387	0.0294	0.0349	0.0297	0.0061	0.0141	0.0139	0.0153		
	$\widehat{\text{ASD}}_{\text{int}}$	0.0422	0.0226	0.0366	0.0397	0.0485	0.0321	0.0219	0.0387	0.0295	0.0205	0.2901	0.0062	0.0142	0.0141	0.0154		
	$\widehat{\text{ASD}}_{\text{res}}$	0.0423	0.0227	0.0367	0.0397	0.0485	0.0321	0.0219	0.0387	0.0294	0.0210	0.2821	0.0060	0.0139	0.0137	0.0154		
	Bias	0.0014	-0.0002	-0.0005	0.0005	0.0025	0.0032	-0.0011	0.0001	-0.0001	0.0011	0.0136	0.0002	0.0003	0.0010	0.0008		
	ESD	0.0284	0.0164	0.0253	0.0276	0.0328	0.0254	0.0199	0.0278	0.0233	0.0200	0.0420	0.0045	0.0097	0.0098	0.0108		
	ASD	0.0282	0.0159	0.0257	0.0278	0.0343	0.0201	0.0155	0.0274	0.0208	0.0247	0.0210	0.0043	0.0100	0.0098	0.0108		
2000	$\widehat{\text{ASD}}_{\text{int}}$	0.0288	0.0159	0.0256	0.0279	0.0343	0.0216	0.0152	0.0271	0.0206	0.0135	0.0371	0.0043	0.0099	0.0097	0.0108		
	$\widehat{\text{ASD}}_{\text{res}}$	0.0288	0.0159	0.0256	0.0279	0.0343	0.0215	0.0152	0.0270	0.0205	0.0137	0.0363	0.0042	0.0097	0.0095	0.0109		

Table S.2: Simulation results (Median Bias/ASD) for the MLE $\hat{\theta}_n$ for sAGARCH(1,1) under stationary cases.

n	θ_0	$\alpha_0 = 1.0$					$\alpha_0 = 0.5$				
		0.10	0.10	0.20	0.30	1.00	0.05	0.02	0.05	0.10	0.50
	MLE	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$
200	Bias	0.0064	-0.0030	-0.0107	-0.0023	0.0138	0.0114	-0.0024	-0.0019	0.0061	0.0082
	ASD	0.0636	0.0489	0.0866	0.0657	0.0780	0.0665	0.0136	0.0315	0.0311	0.0342
	$\widehat{\text{ASD}}^{\text{int}}$	0.0667	0.0480	0.0845	0.0688	0.0694	0.0667	0.0120	0.0308	0.0343	0.0350
	$\widehat{\text{ASD}}^{\text{res}}$	0.0663	0.0479	0.0848	0.0691	0.0733	0.0655	0.0118	0.0305	0.0337	0.0353
500	Bias	0.0009	-0.0036	-0.0037	0.0017	0.0078	0.0062	-0.0008	-0.0023	0.0035	0.0041
	ASD	0.0402	0.0310	0.0547	0.0416	0.0493	0.0420	0.0086	0.0199	0.0197	0.0216
	$\widehat{\text{ASD}}^{\text{int}}$	0.0411	0.0303	0.0537	0.0424	0.0496	0.0433	0.0082	0.0191	0.0203	0.0219
	$\widehat{\text{ASD}}^{\text{res}}$	0.0412	0.0303	0.0538	0.0424	0.0499	0.0426	0.0081	0.0187	0.0199	0.0219
1000	Bias	0.0020	0.0001	-0.0011	-0.0007	0.0030	0.0039	-0.0004	-0.0012	0.0012	0.0007
	ASD	0.0284	0.0219	0.0387	0.0294	0.0349	0.0297	0.0061	0.0141	0.0139	0.0153
	$\widehat{\text{ASD}}^{\text{int}}$	0.0290	0.0218	0.0382	0.0294	0.0350	0.0312	0.0059	0.0138	0.0140	0.0153
	$\widehat{\text{ASD}}^{\text{res}}$	0.0291	0.0218	0.0383	0.0293	0.0352	0.0307	0.0058	0.0135	0.0137	0.0154
2000	Bias	0.0015	-0.0027	-0.0007	-0.0004	-0.0023	0.0039	-0.0004	-0.0006	0.0005	0.0009
	ASD	0.0201	0.0155	0.0274	0.0208	0.0247	0.0210	0.0043	0.0100	0.0098	0.0108
	$\widehat{\text{ASD}}^{\text{int}}$	0.0210	0.0152	0.0271	0.0206	0.0247	0.0245	0.0042	0.0098	0.0097	0.0109
	$\widehat{\text{ASD}}^{\text{res}}$	0.0208	0.0152	0.0269	0.0206	0.0245	0.0239	0.0041	0.0096	0.0095	0.0109
4000	Bias	-0.0002	-0.0013	-0.0012	-0.0004	-0.0023	0.0001	0.0000	-0.0003	0.0006	0.0001
	ASD	0.0142	0.0110	0.0194	0.0147	0.0175	0.0149	0.0030	0.0070	0.0069	0.0077
	$\widehat{\text{ASD}}^{\text{int}}$	0.0146	0.0108	0.0191	0.0145	0.0173	0.0181	0.0030	0.0069	0.0067	0.0077
	$\widehat{\text{ASD}}^{\text{res}}$	0.0146	0.0108	0.0191	0.0145	0.0171	0.0177	0.0029	0.0068	0.0066	0.0077

S1.3 Performance of the MLE in the explosive cases

Next, we examine the finite-sample performance of $\widehat{\theta}_n$ in explosive cases. Similar to the stationary case, three true parameters are set to be $\theta_0 = (0.2, 0.1, 0.2, 0.8, 1.5)'$, $(0.1, 0.1, 0.2, 0.5, 1)'$, and $(0.05, 0.04, 0.08, 0.1, 0.5)'$, respectively. The true top Lyapunov exponents γ_{α_0} are calculated and reported in Table S.4.

Table S.3 reports the Bias, ESD, ASD of $\widehat{\theta}_n$, together with two estimators $\widehat{\text{ASD}}^{\text{int}}$ and $\widehat{\text{ASD}}^{\text{res}}$ of the ASD of $\widehat{\vartheta}_n$. The ASDs are calculated based on the asymptotic covariance matrix Υ in Theorem 2 (ii). We can see that $\widehat{\omega}_n$ is completely biased and the performance does not improve with increasing n (e.g. when $\alpha_0 = 1$). This point confirms the fact that ω_0 is not estimable in the explosive case. The other findings are in accordance with those in the stationary case. Fig. S.2 plots the histograms of $\sqrt{n}(\widehat{\vartheta}_n - \vartheta_0)'$ with $n = 1000$ and $\vartheta_0 = (0.1, 0.2, 0.8, 1.5)'$.

As for the Lyapunov exponent γ_{α_0} , we provide here two estimators in similar ways as Section 3.2:

$$\begin{aligned}\widehat{\gamma}_n^{\text{int}} &= \int_{\mathbb{R}} \log [\widehat{\phi}_{n+}(u^+)^2 + \widehat{\phi}_{n-}(u^-)^2 + \widehat{\psi}_n] f_{\widehat{\alpha}_n}(u) du, \\ \widehat{\gamma}_n^{\text{res}} &= \frac{1}{n} \sum_{t=1}^n \log [\widehat{\phi}_{n+}(\widehat{\eta}_t^+)^2 + \widehat{\phi}_{n-}(\widehat{\eta}_t^-)^2 + \widehat{\psi}_n].\end{aligned}$$

Table S.4 summarizes the true value of the top Lyapunov exponent for

Table S.3: Simulation results for the MLE $\hat{\theta}_n$ for sAGARCH(1,1) under explosive cases.

n	$\alpha_0 = 1.5$					$\alpha_0 = 1.0$					$\alpha_0 = 0.5$					
	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	$\hat{\omega}_n$	$\hat{\phi}_{n+}$	$\hat{\phi}_{n-}$	$\hat{\psi}_n$	$\hat{\alpha}_n$	
200	Bias	33.810	-0.0058	-0.0105	0.0207	0.0278	15.845	0.0022	0.0045	0.0176	0.0200	1.933×10^5	0.0046	0.0076	0.0145	0.0098
	ESD	495.58	0.0471	0.0722	0.0684	0.1146	162.99	0.0494	0.0901	0.0829	0.0833	5.892×10^6	0.0322	0.0581	0.0433	0.0364
500	ASD	/	0.0446	0.0711	0.0646	0.1085	/	0.0465	0.0820	0.0700	0.0780	/	0.0254	0.0482	0.0317	0.0342
	\widehat{ASD}_{int}	/	0.0412	0.0660	0.0601	0.1070	/	0.0463	0.0818	0.0680	0.0636	/	0.0276	0.0515	0.0328	0.0351
1000	\widehat{ASD}_{res}	/	0.0422	0.0667	0.0607	0.1081	/	0.0467	0.0827	0.0682	0.0658	/	0.0271	0.0507	0.0322	0.0353
	Bias	7.3643	-0.0016	-0.0009	0.0061	0.0083	7.9374	0.0020	0.0016	0.0071	0.0100	3.032×10^5	0.0015	0.0043	0.0056	0.0034
500	ESD	41.017	0.0299	0.0460	0.0509	0.0863	108.21	0.0300	0.0523	0.0470	0.0417	9.585×10^6	0.0177	0.0345	0.0232	0.0215
	ASD	/	0.0282	0.0449	0.0408	0.0687	/	0.0294	0.0518	0.0443	0.0493	/	0.0161	0.0305	0.0200	0.0216
1000	\widehat{ASD}_{int}	/	0.0275	0.0443	0.0399	0.0683	/	0.0296	0.0516	0.0438	0.0292	/	0.0165	0.0318	0.0205	0.0218
	\widehat{ASD}_{res}	/	0.0278	0.0445	0.0401	0.0684	/	0.0297	0.0520	0.0439	0.0300	/	0.0162	0.0312	0.0200	0.0219
1000	Bias	23.358	-0.0012	-0.0009	0.0044	0.0019	15.452	0.0001	0.0018	0.0048	0.0018	4.7857	0.0002	0.0008	0.0035	0.0018
	ESD	631.87	0.0203	0.0317	0.0293	0.0498	436.81	0.0199	0.0363	0.0318	0.0328	70.947	0.0118	0.0218	0.0159	0.0159
1000	ASD	/	0.0199	0.0318	0.0289	0.0485	/	0.0208	0.0367	0.0313	0.0349	/	0.0114	0.0215	0.0142	0.0153
	\widehat{ASD}_{int}	/	0.0196	0.0315	0.0286	0.0484	/	0.0207	0.0368	0.0313	0.0226	/	0.0114	0.0217	0.0144	0.0154
1000	\widehat{ASD}_{res}	/	0.0197	0.0316	0.0286	0.0486	/	0.0208	0.0369	0.0313	0.0230	/	0.0112	0.0213	0.0140	0.0154

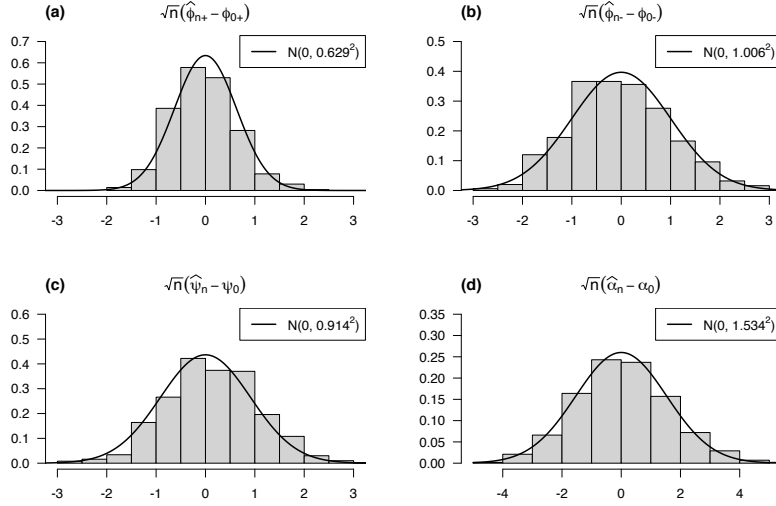


Figure S.2: The histograms of $\sqrt{n}(\hat{\vartheta}_n - \vartheta_0)'$ with $n = 1000$ and $\vartheta_0 = (0.1, 0.2, 0.8, 1.5)'$.

both stationary and explosive cases, together with its empirical mean (EM) and ESD of two estimators $\hat{\gamma}_n^{\text{int}}$ and $\hat{\gamma}_n^{\text{res}}$. From Table S.4, we find that the empirical mean of $\hat{\gamma}_n^{\text{res}}$ is closer to the true value than that of $\hat{\gamma}_n^{\text{int}}$, while its ESD is slightly larger than that of the latter. Moreover, deriving the asymptotics of $\hat{\gamma}_n^{\text{res}}$ is more straightforward. Thus, we use $\hat{\gamma}_n^{\text{res}}$ in the stationarity test in Section 4.1.

S1.4 Performance of the test statistics

In this subsection, we examine the finite-sample performance of the stationarity and asymmetry tests proposed in Section 4.

We use $n = 200, 500$, and 1000 , each with 1000 replications for the

Table S.4: True values of Lyapunov exponent γ_{α_0} with estimates $\hat{\gamma}_n^{\text{int}}$ and $\hat{\gamma}_n^{\text{res}}$.

α_0	n		$\gamma_{\alpha_0} < 0$		$\gamma_{\alpha_0} > 0$	
			$\hat{\gamma}_n^{\text{int}}$	$\hat{\gamma}_n^{\text{res}}$	$\hat{\gamma}_n^{\text{int}}$	$\hat{\gamma}_n^{\text{res}}$
1.5		TRUE	$\gamma_{\alpha_0} = -0.1796$ $(\vartheta_0 = (0.1, 0.2, 0.5, 1.5)')$		$\gamma_{\alpha_0} = 0.1663$ $(\vartheta_0 = (0.1, 0.2, 0.8, 1.5)')$	
	200	EM	-0.2079	-0.2064	0.1575	0.1578
		ESD	0.1230	0.1243	0.0455	0.0472
	500	EM	-0.1907	-0.1901	0.1648	0.1653
		ESD	0.0595	0.0611	0.0292	0.0310
	1000	EM	-0.1840	-0.1838	0.1669	0.1665
		ESD	0.0400	0.0407	0.0203	0.0215
	1.0		TRUE	$\gamma_{\alpha_0} = -0.1516$ $(\vartheta_0 = (0.1, 0.2, 0.3, 1.0)')$		$\gamma_{\alpha_0} = 0.1663$ $(\vartheta_0 = (0.1, 0.2, 0.5, 1.0)')$
200		EM	-0.1682	-0.1604	0.1578	0.1582
		ESD	0.1303	0.1349	0.1109	0.1148
500		EM	-0.1614	-0.1564	0.1620	0.1640
		ESD	0.0796	0.0832	0.0598	0.0641
1000		EM	-0.1617	-0.1563	0.1694	0.1700
		ESD	0.0582	0.0596	0.0458	0.0480
0.5			TRUE	$\gamma_{\alpha_0} = -0.0959$ $(\vartheta_0 = (0.02, 0.05, 0.1, 0.5)')$		$\gamma_{\alpha_0} = 0.1789$ $(\vartheta_0 = (0.04, 0.08, 0.1, 0.5)')$
	200	EM	-0.1305	-0.1279	0.1510	0.1535
		ESD	0.2666	0.2708	0.2772	0.2798
	500	EM	-0.1101	-0.1094	0.1743	0.1765
		ESD	0.1607	0.1641	0.1668	0.1706
	1000	EM	-0.0977	-0.0980	0.1734	0.1747
		ESD	0.1149	0.1182	0.1225	0.1256

sAGARCH model (2.2) with $\theta_0 = (0.1, \phi_{0+}, \phi_{0-}, 0.5, 1.5)'$. The choice of $\alpha_0 = 1.5$ is based on the empirical examples in Section 6. Consider two scenarios: (I) a symmetric case with $\phi_{0+} = \phi_{0-}$, ranging from 0.18 to 0.32; (II) an asymmetric case with $\phi_{0+} = 0.2$ and ϕ_{0-} varying from 0.24 to 0.38. Note that the true top Lyapunov exponent γ_{α_0} equals 0 for $\phi_{0+} = \phi_{0-} = 0.2492$ and for $\phi_{0+} = 0.2$ and $\phi_{0-} = 0.3052$. Table S.5 summarizes the finite-sample performances of the stationarity and asymmetry tests based on the test statistics T_n and T_n^S respectively under these two scenarios. Specifically, the stationarity test considers the hypothesis $H_0^\gamma : \gamma_{\alpha_0} < 0$, and the asymmetry test considers the hypothesis $H_0^S : \phi_{0+} = \phi_{0-}$. The relative frequencies of rejection are reported at the significance level $\underline{\alpha} = 5\%$.

From the upper panel of Table S.5, we can see that the empirical size of asymmetry test T_n^S under the null hypothesis is about 5% (see the numbers in the row of H_0^S), regardless of the value of ϕ_{0-} . The empirical size of the stationarity test under the null is close to the nominal level, shown in the bold-face column. The frequency of rejection of H_0^γ rapidly decreases to 0 when γ_0 goes below zero, and tends to 1 when ϕ_{0-} increases from 0.2492. The power improves with increasing n . On the other hand, the bottom panel of Table S.5 also reveals the satisfactory size of the stationarity test marked in bold-face, and the rejection frequency approaches 0 when $\gamma_{\alpha_0} < 0$

Table S.5: Finite-sample performances of the stationarity and asymmetry tests based on the statistics T_n and T_n^S respectively under two scenarios at the significance level $\alpha = 5\%$. The relative frequencies of rejection of hypotheses are written in percentage (%) and the sizes of the tests are displayed in bold.

Model with $\phi_{0+} = \phi_{0-}$		$\omega_0 = 0.1, \psi_0 = 0.5, \alpha_0 = 1.5$							
		ϕ_{0-}							
n	Null	0.18	0.2	0.22	0.2492	0.26	0.28	0.3	0.32
500	H_0^γ	0.0	0.0	0.3	4.2	8.9	22.8	45.3	69.2
	H_0^S	3.6	4.0	5.2	3.6	5.8	3.8	3.8	3.5
1000	H_0^γ	0.0	0.0	0.0	4.1	13.6	41.3	74.2	94.5
	H_0^S	4.9	4.0	4.8	4.7	5.0	4.5	4.7	4.6
2000	H_0^γ	0.0	0.0	0.0	4.5	17.3	67.7	95.8	99.9
	H_0^S	6.0	4.1	4.0	4.8	3.8	3.7	5.5	6.2
Model with $\phi_{0+} < \phi_{0-}$		$\omega_0 = 0.1, \phi_{0+} = 0.2, \psi_0 = 0.5, \alpha_0 = 1.5$							
		ϕ_{0-}							
n	Null	0.24	0.26	0.28	0.3052	0.32	0.34	0.36	0.38
500	H_0^γ	0.1	0.2	1.1	3.8	5.1	9.9	15.8	23.3
	H_0^S	7.6	12.9	17.5	27.6	35.9	44.2	53.3	61.6
1000	H_0^γ	0.0	0.2	1.0	4.6	6.8	16.4	27.0	44.5
	H_0^S	12.9	23.2	37.6	54.2	65.6	73.5	82.7	89.6
2000	H_0^γ	0.0	0.1	0.7	4.9	11.3	22.8	47.6	69.5
	H_0^S	22.7	41.3	64.3	83.8	93.1	96.8	98.5	99.3

and tends to 1 when $\gamma_{\alpha_0} > 0$ as expected. Under the alternative hypothesis, the power of T_n^S tends to 1 when ϕ_{0-} increases from 0.24 and when n increases. These results illustrate the satisfactory finite-sample performances of stationarity and asymmetry tests.

S1.5 Empirical Studies on Portfolio Returns

To illustrate the merits of model (2.2) for modelling aggregate behavior of random variables, we analyze three portfolio returns and compare the performance with that of the AGARCH model with t -innovation (t -AGARCH). They are the daily series of the S&P 500 Index (SPX, Mar 3, 2021 – Aug. 26, 2024), Dow Jones Industrial Average (DJI, May. 7, 2020 – Aug. 26, 2024), and NASDAQ Composite (IXIC, Dec. 4, 2019 – Aug. 26, 2024).

Fig. S.3 plots the three portfolio return series.

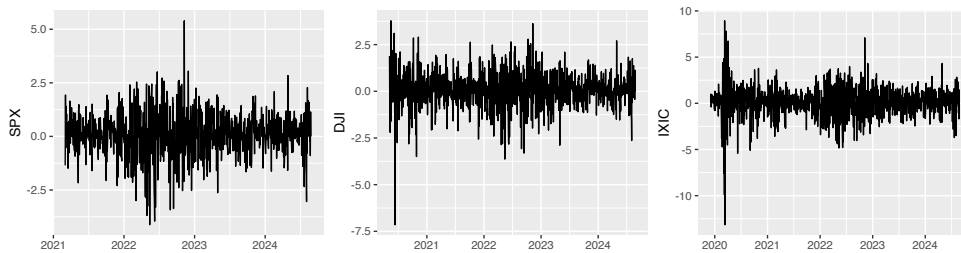


Figure S.3: The plots of three portfolio return series (%): SPX, DJI, and IXIC.

We fit the sAGARCH(1,1) model (2.2) to the data and the results are

Table S.6: The upper panel displays the fitting results of sAGARCH(1,1) model, diagnostic checking statistics T_n^D , the log-likelihood, and the value of the AIC for portfolio returns, respectively. The lower displays the fitting of t -AGARCH(1,1) model for comparison.

Model	S&P500	DJI	IXIC
n	846	1046	1146
$\hat{\omega}_n$	0.013	0.008	0.032
$\hat{\phi}_{n+}$	1.4e-4 (0.009)	1.4e-4 (0.006)	2.8e-4 (0.009)
$\hat{\phi}_{n-}$	0.105 (0.017)	0.050 (0.009)	0.110 (0.016)
$\hat{\psi}_n$	0.875 (0.022)	0.919 (0.015)	0.860 (0.019)
sAGARCH(1,1) $\hat{\alpha}_n$	1.954 (0.039)	1.854 (0.046)	1.939 (0.032)
T_n^D	2.577	2.676	3.726
p-value	0.020	0.015	0.000
log-lik	-1133.5	-1349.9	-1934.0
AIC	2277.1	2709.8	3878.0
$\hat{\omega}_n$	0.007	0.029	0.047
$\hat{\phi}_{n+}$	0.061	0.027	4.4e-4
$\hat{\phi}_{n-}$	0.171	0.120	0.187
t -AGARCH(1,1) $\hat{\psi}_n$	0.883	0.873	0.870
$\hat{\nu}_n$	27.577	11.737	9.140
log-lik	-1143.0	-1350.7	-1936.3
AIC	2295.9	2711.3	3882.5

summarized in Table S.6, together with the diagnostic checking statistics T_n^D . The estimated ASDs in parentheses are calculated by the universal estimator $\hat{\Upsilon}_*$ in Section 3.3. Moreover, we also fit a t -AGARCH(1,1) model to the data and report the estimation results for comparison, as well as the values of the log-likelihood and the AIC.

Table S.6 reveals the following findings:

- (i) We can see that $\hat{\phi}_{n-}$, $\hat{\psi}_n$ and $\hat{\alpha}_n$ are significant. Interestingly, for all portfolio returns, $\hat{\phi}_{n+}$ are very small and less than $\hat{\phi}_{n-}$, which accords with the phenomenon that negative returns tend to have a stronger impact on future volatilities than positive returns of the same magnitude.
- (ii) As evidenced by the values of the AIC, we can see that the sAGARCH(1,1) model has a better fit than the t -AGARCH(1,1) for all three portfolio returns. It indicates that stable distributions are useful for modelling aggregate behavior and characterizing the risk of portfolios. The t -distribution is sometimes insufficient for capturing the tail-thickness of the innovation.

We further compare the fitting performances of sAGARCH(1,1) and t -AGARCH(1,1) models by examining the empirical cumulative distribution function (ECDF) of residuals. Fig. S.4 plots the CDF of stable distribution with estimated $\hat{\alpha}_n$, or CDF of t -distribution with estimated degree of freedom $\hat{\nu}_n$, and the ECDF of the residual. From the figure, we can

see that for all three portfolio returns, the ECDF nearly coincides with the true one, which illustrates the goodness-of-fit of both sAGARCH(1,1) model and t -AGARCH(1,1) model on the whole. Nevertheless, the value of the AIC implies the sAGARCH(1,1) model has a better fit than the t -AGARCH(1,1).

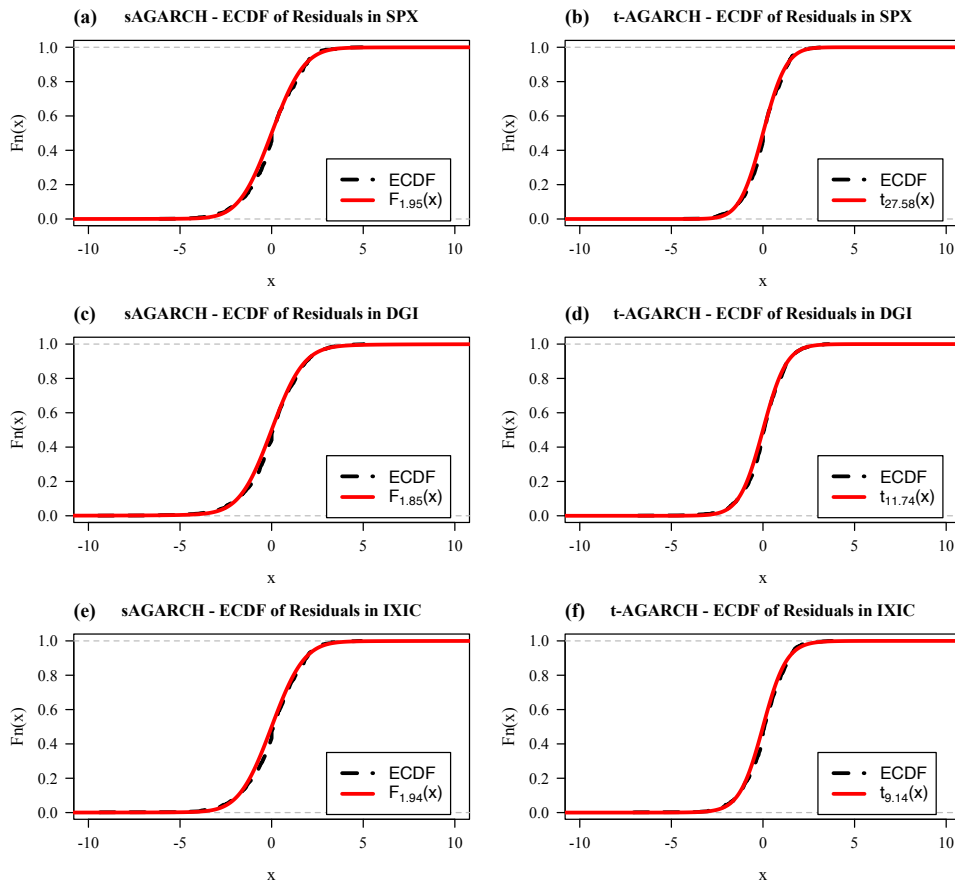


Figure S.4: The ECDFs of residuals of portfolio returns, i.e., SPX, DJI, and IXIC.

S2 Proofs of Theorems

Let $\underline{\omega} = \inf\{\omega \mid \theta \in \Theta\}$, $\underline{\phi}_+ = \inf\{\phi_+ \mid \theta \in \Theta\}$, $\underline{\phi}_- = \inf\{\phi_- \mid \theta \in \Theta\}$,
 $\underline{\phi} = \inf\{\phi_+, \phi_- \mid \theta \in \Theta\}$, $\underline{\psi} = \inf\{\psi \mid \theta \in \Theta\}$, $\underline{\alpha} = \inf\{\alpha : \theta \in \Theta\}$,
 $\bar{\omega} = \sup\{\omega \mid \theta \in \Theta\}$, $\bar{\phi}_+ = \sup\{\phi_+ \mid \theta \in \Theta\}$, $\bar{\phi}_- = \sup\{\phi_- \mid \theta \in \Theta\}$,
 $\bar{\phi} = \sup\{\phi_+, \phi_- \mid \theta \in \Theta\}$, $\bar{\psi} = \sup\{\psi \mid \theta \in \Theta\}$, $\bar{\alpha} = \sup\{\alpha : \theta \in \Theta\}$. Let
 $\underline{\vartheta} = (\underline{\phi}_+, \underline{\phi}_-, \underline{\psi}, \underline{\alpha})'$. The notation K is a generic constant that may vary
across lines. Define the $[0, \infty]$ -valued random sequence

$$v_t(\vartheta) = \sum_{j=1}^{\infty} \frac{\{\phi_+(\eta_{t-j}^+)^2 + \phi_-(\eta_{t-j}^-)^2\}}{a_0(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a_0(\eta_{t-k})},$$

with the convention $\prod_{k=1}^{j-1} \equiv 1$ if $j \leq 1$, where $a_0(x) = \phi_{0+}(x^+)^2 + \phi_{0-}(x^-)^2 + \psi_0$ and $\{\eta_t\}$ satisfies Assumption 1. Let $\Theta_0 = \{\theta \in \Theta : \psi < \exp(\gamma_{\alpha_0})\}$.

S2.1 Proof of Theorem 1

Let $L_n^*(\theta) = \{L_n(\theta) - L_n(\theta_0)\}/n$, $\theta \in \Theta$. Equivalently, $\hat{\theta}_n = \arg \max_{\theta \in \Theta} L_n^*(\theta)$.

(i). For the stationary case $\gamma_{\alpha_0} < 0$, we first show the strong consistency of $\hat{\theta}_n$. From (3.5), $\sigma_t(\theta_0)/\sigma_t(\theta)$ is a measurable function of strictly stationary and ergodic series $\{y_{t-j} : j \geq 1\}$, and y_t is measurable of i.i.d. sequence $\{\eta_t\}$. By the independence of η_t and $\sigma_t(\theta_0)/\sigma_t(\theta)$, $\{(\sigma_t(\theta_0)/\sigma_t(\theta), \eta_t)\}$ is also

strictly stationary and ergodic. By the strong law of large numbers,

$$\begin{aligned} L_n^*(\theta) &= \frac{1}{n} \sum_{t=1}^n \left\{ \log \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) \right] - \log f_{\alpha_0}(\eta_t) \right\} \\ &\xrightarrow{\text{a.s.}} E \left\{ \log \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) \right] - \log f_{\alpha_0}(\eta_t) \right\}. \end{aligned}$$

Define $\mathcal{F}_t = \sigma(y_t, y_{t-1}, \dots)$. Further, by the inequality $\log x \leq x - 1$ for

$x > 0$, we have

$$\begin{aligned} & E \left\{ \log \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) \right] - \log f_{\alpha_0}(\eta_t) \right\} \\ &= E \left\{ E \left[\log \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) \right] - \log f_{\alpha_0}(\eta_t) \middle| \mathcal{F}_{t-1} \right] \right\} \\ &= E \left\{ \int_{\mathbb{R}} f_{\alpha_0}(\eta) \left[\log \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) \right] - \log f_{\alpha_0}(\eta) \right] d\eta \right\} \quad (\text{S2.1}) \\ &\leq E \left\{ \int_{\mathbb{R}} f_{\alpha_0}(\eta) \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) / f_{\alpha_0}(\eta) - 1 \right] d\eta \right\} \\ &= E \left\{ \int_{\mathbb{R}} \left[\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) - f_{\alpha_0}(\eta) \right] d\eta \right\} \\ &= E \left\{ \int_{\mathbb{R}} f_\alpha(x) dx - \int_{\mathbb{R}} f_{\alpha_0}(x) dx \right\} = 1 - 1 = 0, \end{aligned}$$

and the equality holds *if and only if*

$$\frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) = f_{\alpha_0}(\eta_t), \quad \text{a.s.}, \forall t \in \mathbb{Z}. \quad (\text{S2.2})$$

By Lemma 1, we have $\alpha = \alpha_0$, and $\sigma_t(\theta_0)/\sigma_t(\theta) = 1$, a.s., $\forall t \in \mathbb{Z}$. Thus,

$$\begin{aligned} 0 &= \sigma_t^2(\theta) - \sigma_t^2(\theta_0) \\ &= (\omega - \omega_0) + (\phi_+ - \phi_{0+})(y_{t-1}^+)^2 + (\phi_- - \phi_{0-})(y_{t-1}^-)^2 + \psi \sigma_{t-1}^2(\theta) - \psi_0 \sigma_{t-1}^2(\theta_0) \\ &= (\omega - \omega_0) + (\phi_+ - \phi_{0+})(y_{t-1}^+)^2 + (\phi_- - \phi_{0-})(y_{t-1}^-)^2 + (\psi - \psi_0) \sigma_{t-1}^2(\theta_0), \quad \text{a.s.} \end{aligned}$$

Since $y_{t-1} \in \mathcal{F}_{t-1}$ and $\sigma_{t-1}^2 \in \mathcal{F}_{t-2}$, it implies that $\phi_+ - \phi_{0+} = 0$ and $\phi_- - \phi_{0-} = 0$, otherwise y_{t-1} would be measurable with respect to \mathcal{F}_{t-2} . Therefore, $\phi_+ = \phi_{0+}$, $\phi_- = \phi_{0-}$, and we have $0 = (\omega - \omega_0) + (\psi - \psi_0)\sigma_{t-1}^2(\theta_0)$, a.s. By the randomness of $\sigma_{t-1}^2(\theta_0)$, we obtain that $\psi - \psi_0 = 0$, $\omega - \omega_0 = 0$, i.e., $\psi = \psi_0$, $\omega = \omega_0$. Thus, the equality holds *if and only if* $\theta = \theta_0$.

The rest of the proof uses a standard argument invoking the compactness of Θ , following the framework of proof of Theorem 2.1 in Francq and Zakoïan (2004). For any $\theta \in \Theta$, and $\theta \neq \theta_0$, let $V_k(\theta)$ be an open ball with center θ and radius $1/k$. Note that $\sup_{\theta \in \Theta} |\log \sigma_t(\theta)| \leq \frac{1}{2}\{|\log \underline{\omega}| + [\sup_{\theta \in \Theta} \sigma_t^{2\iota}(\theta)]/\iota\} \leq K\{1 + \sup_{\theta \in \Theta} \sigma_t^{2\iota}(\theta)\}$, for some $\iota \in (0, \alpha_0/4)$. Similar to the proof of Lemma 3, by the C_r -inequality we have that $E\{\sup_{\theta \in \Theta} \sigma_t^{2\iota}(\theta)\} < \infty$. In view of the definition of $\ell_t(\theta)$ and Lemma 2,

$$E\left\{\sup_{\theta \in \Theta} |\ell_t(\theta)|\right\} \leq K\left\{1 + E\left\{\sup_{\theta \in \Theta} \sigma_t^{2\iota}(\theta)\right\} + E(y_t^{2\iota})\right\} < \infty.$$

By Lemma 3 and ergodic theorem, it yields that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta^* \in V_k(\theta) \cap \Theta} \tilde{L}_n(\theta^*) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta^* \in V_k(\theta) \cap \Theta} L_n(\theta^*) + \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta \in \Theta} |L_n(\theta) - \tilde{L}_n(\theta)| \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \sup_{\theta^* \in V_k(\theta) \cap \Theta} \ell_t(\theta^*) \\ & = E \sup_{\theta^* \in V_k(\theta) \cap \Theta} \ell_1(\theta^*), \quad \text{a.s.} \end{aligned}$$

As $\sup_{\theta^* \in V_k(\theta) \cap \Theta} \ell_1(\theta^*)$ is non-increasing with respect to k , $\sup_{\theta^* \in V_k(\theta) \cap \Theta} \ell_1(\theta^*) \rightarrow \ell_1(\theta)$ a.s. as $k \rightarrow \infty$, and $E \sup_{\theta^* \in V_1(\theta) \cap \Theta} \ell_1(\theta^*) < \infty$, by the Monotone Convergence Theorem, we have $E \sup_{\theta^* \in V_k(\theta) \cap \Theta} \ell_1(\theta^*) \rightarrow E \ell_1(\theta)$. By (S2.1), when $\theta \neq \theta_0$, $E \ell_1(\theta) < E \ell_1(\theta_0)$. Note that $E \ell_1(\theta)$ is continuous with respect to $\theta \in \Theta$. Thus, for any $\theta \neq \theta_0$, there exists an open set $V(\theta)$, such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta^* \in V(\theta) \cap \Theta} L_n(\theta^*) < E \ell_1(\theta_0), \quad \text{a.s.} \quad (\text{S2.3})$$

Consider an arbitrarily small open neighborhood $V(\theta_0)$ of θ_0 . By the definition of the MLE and Lemma 3,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta^* \in V(\theta_0) \cap \Theta} \tilde{L}_n(\theta^*) \geq \lim_{n \rightarrow \infty} \frac{1}{n} \tilde{L}_n(\theta_0) = \lim_{n \rightarrow \infty} \frac{1}{n} L_n(\theta_0) = E \ell_1(\theta_0), \quad \text{a.s.} \quad (\text{S2.4})$$

Let $V^c(\theta_0)$ be the complementary set of $V(\theta_0)$. Since $V^c(\theta_0) \cap \Theta$ is compact, there exists a finite subcover of $V^c(\theta_0) \cap \Theta$ of the form $\{V(\theta_i), \theta_i \in V^c(\theta_0), i = 1, \dots, k\}$, satisfying $V^c(\theta_0) \cap \Theta \subset \bigcup_{i=1}^k V(\theta_i)$. Thus,

$$\begin{aligned} \frac{1}{n} \sup_{\theta \in \Theta} \tilde{L}_n(\theta) &= \frac{1}{n} \max_{0 \leq i \leq k} \sup_{\theta \in V(\theta_i) \cap \Theta} \tilde{L}_n(\theta) \\ &= \max \left\{ \frac{1}{n} \sup_{\theta \in V(\theta_0) \cap \Theta} \tilde{L}_n(\theta), \max_{1 \leq i \leq k} \frac{1}{n} \sup_{\theta \in V(\theta_i) \cap \Theta} \tilde{L}_n(\theta) \right\}. \end{aligned}$$

Combining (S2.3)-(S2.4), it follows that a.s.

$$\frac{1}{n} \sup_{\theta \in \Theta} \tilde{L}_n(\theta) = \frac{1}{n} \sup_{\theta \in V(\theta_0) \cap \Theta} \tilde{L}_n(\theta), \quad \text{for } n \text{ large enough.}$$

Thus, a.s., $\widehat{\theta}_n \in V(\theta_0)$ for n large enough. Due to the arbitrariness of $V(\theta_0)$, then $\widehat{\theta}_n \xrightarrow{a.s.} \theta_0$. The proof is complete. \square

(ii). By Taylor's expansion, we have

$$0 = \frac{\partial \widetilde{L}_n(\widehat{\theta}_n)}{\partial \theta} = \frac{\partial \widetilde{L}_n(\theta_0)}{\partial \theta} + \frac{\partial^2 \widetilde{L}_n(\theta^*)}{\partial \theta \partial \theta'} (\widehat{\theta}_n - \theta_0),$$

where θ^* lies between $\widehat{\theta}_n$ and θ_0 , i.e., $\|\theta^* - \theta_0\| \leq \|\widehat{\theta}_n - \theta_0\|$. By Lemmas

7-9, it follows that

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial \widetilde{L}_n(\theta_0)}{\partial \theta} &= \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} + o_p(1) \xrightarrow{d} \mathcal{N}(0, \Sigma), \\ \frac{1}{n} \frac{\partial^2 \widetilde{L}_n(\theta^*)}{\partial \theta \partial \theta'} &= \frac{1}{n} \frac{\partial^2 L_n(\theta_0)}{\partial \theta \partial \theta'} + o_p(1) \xrightarrow{p} -\Sigma. \end{aligned}$$

Thus, it yields that

$$\begin{aligned} \sqrt{n}(\widehat{\theta}_n - \theta_0) &= \left(-\frac{1}{n} \frac{\partial^2 \widetilde{L}_n(\theta^*)}{\partial \theta \partial \theta'} \right)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \widetilde{L}_n(\theta_0)}{\partial \theta} \\ &= \{\Sigma + o_p(1)\}^{-1} \frac{1}{\sqrt{n}} \frac{\partial \widetilde{L}_n(\theta_0)}{\partial \theta} \xrightarrow{d} \mathcal{N}(0, \Sigma^{-1}). \quad \blacksquare \end{aligned}$$

S2.2 Proof of Theorem 2

For ease of notation, the tilde symbol “ \sim ” over $L_n(\theta)$ is omitted under explosive cases, and the following abbreviations are used:

$$L_n(\theta) := \widetilde{L}_n(\theta), \quad \ell_t(\theta) := \widetilde{\ell}_t(\theta), \quad \sigma_t^2(\theta) := \widetilde{\sigma}_t^2(\theta).$$

It is because we do not need the stationarity of $\sigma_t^2(\theta)$ for explosive cases. As a matter of fact, the process $\sigma_t^2(\theta)$ will tend to infinity a.s. in these cases,

regardless of the initial values. We thus need to construct a stationary and ergodic process to approximate $\sigma_t^2(\theta)/\sigma_t^2(\theta_0)$.

(i). When $\gamma_{\alpha_0} > 0$, we decompose $L_n^*(\theta)$ into three parts, i.e.,

$$L_n^*(\theta) = O_n(\theta) + R_{1n}(\theta) + R_{2n}(\theta),$$

where

$$\begin{aligned} O_n(\vartheta) &= \frac{1}{n} \sum_{t=1}^n \left\{ \log \left[\frac{1}{\sqrt{v_t(\vartheta)}} f_\alpha \left(\eta_t \frac{1}{\sqrt{v_t(\vartheta)}} \right) \right] - \log f_{\alpha_0}(\eta_t) \right\}; \\ R_{1n}(\theta) &= \frac{1}{2n} \sum_{t=1}^n \left\{ \log \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} - \log \frac{1}{v_t(\vartheta)} \right\}; \\ R_{2n}(\theta) &= \frac{1}{n} \sum_{t=1}^n \left\{ \log f_\alpha \left(\eta_t \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right) - \log f_\alpha \left(\eta_t \frac{1}{\sqrt{v_t(\vartheta)}} \right) \right\}. \end{aligned}$$

By Lemma 4, if $\theta \notin \Theta_0$, then $L_n^*(\theta) \xrightarrow{a.s.} -\infty$. Thus it suffices to consider the case $\theta \in \Theta_0^*$, where Θ_0^* is an arbitrary compact subset of Θ_0 . By the strong law of large numbers,

$$O_n(\vartheta) \xrightarrow{a.s.} E \left\{ \log \left[\frac{1}{\sqrt{v_t(\vartheta)}} f_\alpha \left(\eta_t \frac{1}{\sqrt{v_t(\vartheta)}} \right) \right] - \log f_{\alpha_0}(\eta_t) \right\} \leq 0,$$

where the equality holds *if and only if* $v_t(\vartheta) = 1$, a.s., and $\alpha_0 = \alpha$. By Lemma 5, $\vartheta = \vartheta_0$.

For the term $R_{1n}(\theta)$, by the Lagrange mean value theorem, $v_t(\vartheta) \geq$

$v_t(\underline{\vartheta}) > 0$, a.s., it follows that

$$\begin{aligned} \sup_{\theta \in \Theta_0^*} |R_{1n}(\theta)| &\leq \frac{1}{2n} \sum_{t=1}^n \sup_{\theta \in \Theta_0^*} \left\{ \frac{1}{m_t} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\underline{\vartheta}) \right| \right\} \\ &\leq \frac{1}{2n} \sum_{t=1}^n \sup_{\theta \in \Theta_0^*} \left\{ \frac{v_t(\underline{\vartheta})}{m_t} \right\} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} - 1 \right|, \end{aligned}$$

where $m_t = p_t \sigma_t^2(\theta)/\sigma_t^2(\theta_0) + (1 - p_t)v_t(\underline{\vartheta}) > 0$, $p_t \in (0, 1)$, $1 \leq t \leq n$. By

Lemma 4, there exists $c_0 > 0$ such that $e^{c_0 t} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\underline{\vartheta}) \right| \xrightarrow{a.s.} 0$

as $t \rightarrow \infty$. By the Markov inequality and Lemma 6 that $1/v_t(\underline{\vartheta})$ is strictly

stationary and ergodic with finite moments of any orders, it follows that

$$\sum_{t=1}^{\infty} P\left(\frac{1}{v_t(\underline{\vartheta})} e^{-c_0 t} > \epsilon\right) \leq \sum_{t=1}^{\infty} \epsilon^{-1} E\left(\frac{1}{v_t(\underline{\vartheta})}\right) e^{-c_0 t} \leq \epsilon^{-1} E\left(\frac{1}{v_t(\underline{\vartheta})}\right) \sum_{t=1}^{\infty} e^{-c_0 t} < \infty.$$

By the Borel-Cantelli lemma, we have $e^{-c_0 t}/v_t(\underline{\vartheta}) \xrightarrow{a.s.} 0$. Thus, as $t \rightarrow \infty$,

$$\begin{aligned} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} - 1 \right| &\leq \sup_{\theta \in \Theta_0^*} \left\{ \frac{1}{v_t(\underline{\vartheta})} \right\} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\underline{\vartheta}) \right| \\ &\leq \left(e^{-c_0 t}/v_t(\underline{\vartheta}) \right) \left(e^{c_0 t} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\underline{\vartheta}) \right| \right) \xrightarrow{a.s.} 0, \end{aligned}$$

and it follows that

$$\sup_{\theta \in \Theta_0^*} \left\{ \frac{v_t(\underline{\vartheta})}{m_t} \right\} = \sup_{\theta \in \Theta_0^*} \frac{1}{p_t \frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} + (1 - p_t)} = \sup_{\theta \in \Theta_0^*} \frac{1}{p_t \left[\frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} - 1 \right] + 1} \xrightarrow{a.s.} 1.$$

Thus, by the Stolz-Cesàro theorem, it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\theta \in \Theta_0^*} |R_{1n}(\theta)| &\leq \lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{t=1}^n \sup_{\theta \in \Theta_0^*} \left\{ \frac{v_t(\underline{\vartheta})}{m_t} \right\} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} - 1 \right| \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \sup_{\theta \in \Theta_0^*} \left\{ \frac{v_t(\underline{\vartheta})}{m_t} \right\} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)/\sigma_t^2(\theta_0)}{v_t(\underline{\vartheta})} - 1 \right| \\ &= 0, \quad \text{a.s.} \end{aligned}$$

For the term $R_{2n}(\theta)$, similarly it follows that

$$\begin{aligned}
& \sup_{\theta \in \Theta_0^*} |R_{2n}(\theta)| \\
& \leq K \left(\sup_{x \in \mathbb{R}} \sup_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \left| \frac{\partial \log f_\alpha(x)}{\partial x} \right| \right) \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta_0^*} \left| \eta_t \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \left\{ \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right\} \frac{1}{2\sqrt{v_t(\vartheta)}} \right| \\
& \leq K \left(\sup_{x \in \mathbb{R}} \sup_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \left| \frac{\partial \log f_\alpha(x)}{\partial x} \right| \right) \frac{1}{n} \sum_{t=1}^n \frac{|\eta_t| V_t}{2\sqrt{v_t(\vartheta)}} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right|,
\end{aligned}$$

where V_t is defined in (S3.16) in Lemma 6. Using the fact that $1/v_t(\vartheta)$ and V_t are strictly stationary and ergodic with finite moments of any order by Lemma 6 and $E|\eta_t|^{\alpha_0/2} < \infty$, it follows that by Markov's inequality and the Cauchy-Schwarz inequality

$$\begin{aligned}
& \sum_{t=1}^{\infty} P\left(\frac{|\eta_t| V_t}{\sqrt{v_t(\vartheta)}} e^{-c_0 t} > \epsilon\right) \\
& \leq \sum_{t=1}^{\infty} \epsilon^{-\alpha_0/4} E\left[|\eta_t|^{\alpha_0/4} \left(\frac{V_t}{\sqrt{v_t(\vartheta)}}\right)^{\alpha_0/4}\right] e^{-\frac{c_0 \alpha_0}{4} t} \\
& \leq \epsilon^{-\alpha_0/4} (E|\eta_t|^{\alpha_0/2})^{1/2} \left[E\left(\frac{V_t}{\sqrt{v_t(\vartheta)}}\right)^{\alpha_0/2}\right]^{1/2} \sum_{t=1}^{\infty} e^{-\frac{c_0 \alpha_0}{4} t} \\
& \leq \epsilon^{-\alpha_0/4} (E|\eta_t|^{\alpha_0/2})^{1/2} \left[E(V_t)^{\alpha_0} E\left(\frac{1}{\sqrt{v_t(\vartheta)}}\right)^{\alpha_0}\right]^{1/4} \sum_{t=1}^{\infty} e^{-\frac{c_0 \alpha_0}{4} t} < \infty.
\end{aligned}$$

Using the Borel-Cantelli lemma, we have $(|\eta_t| V_t / \sqrt{v_t(\vartheta)}) e^{-c_0 t} \xrightarrow{a.s.} 0$. By

the Stolz-Cesàro theorem, it follows that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n \frac{|\eta_t| V_t}{\sqrt{v_t(\vartheta)}} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right| \\
&= \lim_{t \rightarrow \infty} \frac{|\eta_t| V_t}{\sqrt{v_t(\vartheta)}} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right| \\
&= \lim_{t \rightarrow \infty} \left(\frac{|\eta_t| V_t}{\sqrt{v_t(\vartheta)}} e^{-c_0 t} \right) e^{c_0 t} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right| = 0, \quad \text{a.s.}
\end{aligned}$$

By Proposition 3, we can obtain that

$$\sup_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \left| \frac{\partial \log f_\alpha(x)}{\partial x} \right| \leq K \min(1, |x|^{-1}), \quad (\text{S2.5})$$

where $[\underline{\alpha}, \bar{\alpha}] \subset (0, 2)$. Hence, it yields that

$$\sup_{\theta \in \Theta_0^*} |R_{2n}(\theta)| \xrightarrow{\text{a.s.}} 0.$$

The conclusion of the proof uses similar arguments in Theorem 1(i) invoking the compactness of Θ . More specifically, Lemma 4 entails that when $\theta \notin \Theta_0$, $L_n^*(\theta) \xrightarrow{\text{a.s.}} -\infty$. Thus, it remains to prove the case $\theta \in \Theta_0^*$, where Θ_0^* is an arbitrary compact subset of Θ_0 . For any $\theta \in \Theta_0^*$, and $\theta \neq \theta_0$, let $V_k(\theta)$ be an open ball with center θ and radius $1/k$. Recall that $\theta = (\omega, \vartheta)'$. Let $O_n(\vartheta) := \frac{1}{n} \sum_{t=1}^n q_t(\vartheta)$, and denote Ξ_0^* to be a restricted

subspace of Θ_0^* with respect to ϑ . By the ergodic theorem,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{\theta^* \in V_k(\vartheta) \cap \Theta_0^*} (\theta^*) &= \limsup_{n \rightarrow \infty} \sup_{\theta^* \in V_k(\vartheta) \cap \Theta_0^*} [O_n(\vartheta^*) + R_{1n}(\theta^*) + R_{2n}(\theta^*)] \\ &= \limsup_{n \rightarrow \infty} \sup_{\vartheta^* \in V_k(\vartheta) \cap \Xi_0^*} O_n(\vartheta^*) \\ &= E \sup_{\vartheta^* \in V_k(\vartheta) \cap \Xi_0^*} q_t(\vartheta^*), \quad \text{a.s.} \end{aligned}$$

Note that $\sup_{\vartheta^* \in V_k(\vartheta) \cap \Xi_0^*} q_t(\vartheta^*)$ is non-increasing w.r.t. k , and $\sup_{\vartheta^* \in V_k(\vartheta) \cap \Xi_0^*} q_t(\vartheta^*) \xrightarrow{\text{a.s.}} q_t(\vartheta)$ as $k \rightarrow \infty$. Moreover,

$$\begin{aligned} \sup_{\vartheta \in \Xi_0^*} |q_t(\vartheta)| &\leq \sup_{\vartheta \in \Xi_0^*} \log(1 + \sqrt{v_t(\vartheta)}) + \sup_{\vartheta \in \Xi_0^*} \log\left(1 + \frac{1}{\sqrt{v_t(\vartheta)}}\right) \\ &\quad + \sup_{\vartheta \in \Xi_0^*} K \left\{ 1 + \log\left(1 + \left| \eta_t \frac{1}{\sqrt{v_t(\vartheta)}} \right| \right) \right\} + K \{1 + \log(1 + |\eta_t|)\}. \end{aligned}$$

By $\sup_{\vartheta \in \Xi_0^*} v_t(\vartheta) \leq K \sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \{\bar{\psi}_0/a_0(\eta_{t-k})\}$ and Lemma 2.2 in Francq and Zakoïan (2019), we have $E(\sup_{\vartheta \in \Xi_0^*} v_t(\vartheta))^\iota < \infty$, $\iota \in (0, 1/4)$. In view of Lemma 6 and independence of η_t and $v_t(\vartheta)$, it entails that $E \sup_{\vartheta \in \Xi_0^*} |q_t(\vartheta)| < \infty$. Thus, $E \sup_{\vartheta^* \in V_1(\vartheta) \cap \Xi_0^*} q_t(\vartheta^*) < \infty$. By the Monotone Convergence Theorem, $E \sup_{\vartheta^* \in V_k(\vartheta) \cap \Xi_0^*} q_t(\vartheta^*) \rightarrow E q_t(\vartheta)$. From the preceding arguments, $E q_t(\vartheta) < 0$ if $\vartheta \neq \vartheta_0$. Note that $E q_t(\vartheta)$ is continuous with respect to $\vartheta \in \Xi_0^*$. Thus, for any $\vartheta \neq \vartheta_0$, there exists an open neighborhood $V(\vartheta)$, such that

$$\limsup_{n \rightarrow \infty} \sup_{\vartheta^* \in V(\vartheta) \cap \Xi_0^*} O_n(\vartheta^*) < 0, \quad \text{a.s.}$$

For an arbitrarily small open neighborhood $V(\vartheta_0)$ of ϑ_0 , by the definition of the MLE and Lemma 3, it follows that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sup_{\theta^* \in V(\theta_0) \cap \Theta_0^*} L_n^*(\theta^*) &= \liminf_{n \rightarrow \infty} \sup_{\vartheta^* \in V(\vartheta_0) \cap \Xi_0^*} O_n(\vartheta^*) \\ &\geq \lim_{n \rightarrow \infty} O_n(\vartheta_0) = Eq_t(\vartheta_0) = 0, \quad \text{a.s.} \end{aligned}$$

Since $V^c(\vartheta_0) \cap \Xi_0^*$ is compact, there exists a finite subcover of the form $\{V(\vartheta_i) : \vartheta_i \in V^c(\vartheta_0), i = 1, \dots, k\}$, satisfying $V^c(\vartheta_0) \cap \Xi_0^* \subset \bigcup_{i=1}^k V(\vartheta_i)$.

Thus, a.s.,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{\theta \in \Theta_0^*} L_n^*(\theta) &= \lim_{n \rightarrow \infty} \sup_{\vartheta \in \Xi_0^*} O_n(\vartheta) \\ &= \lim_{n \rightarrow \infty} \max \left\{ \sup_{\vartheta \in V(\vartheta_0) \cap \Xi_0^*} O_n(\vartheta), \max_{1 \leq i \leq k} \sup_{\vartheta \in V(\vartheta_i) \cap \Xi_0^*} O_n(\vartheta) \right\}. \end{aligned}$$

Combining the above two inequalities, it follows that a.s.

$$\sup_{\vartheta \in \Xi_0^*} O_n(\vartheta) = \sup_{\vartheta \in V(\vartheta_0) \cap \Xi_0^*} O_n(\vartheta), \quad \text{for } n \text{ large enough.}$$

Thus, a.s., we have $\widehat{\vartheta}_n \in V(\vartheta_0)$ for n large enough. Since $V(\vartheta_0)$ is arbitrary, then $\widehat{\vartheta}_n \xrightarrow{\text{a.s.}} \vartheta_0$. The proof is complete. \square

(ii). Let $\vartheta_0 = (\phi_{0+}, \phi_{0-}, \psi_0, \alpha_0)'$ and $\widehat{\vartheta}_n = (\widehat{\phi}_{n+}, \widehat{\phi}_{n-}, \widehat{\psi}_n, \widehat{\alpha}_n)'$. By the definition of $\widehat{\theta}_n$ and Taylor's expansion, it yields that

$$\begin{pmatrix} \frac{1}{\sqrt{n}} \frac{\partial L_n(\widehat{\theta}_n)}{\partial \omega} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} + \mathcal{J}_n \sqrt{n} (\widehat{\theta}_n - \theta_0), \quad (\text{S2.6})$$

where \mathcal{J}_n is a 5×5 matrix with elements

$$\mathcal{J}_n(i, j) = \frac{1}{n} \frac{\partial^2 L_n(\theta_i^*)}{\partial \theta_i \partial \theta_j},$$

where $\theta_i^* = (\omega_i^*, \phi_{i+}^*, \phi_{i-}^*, \psi_i^*, \alpha_i^*)'$ lies between $\widehat{\theta}_n$ and θ_0 .

Since $\theta_0 \in \Theta_0$, we have $\theta_i^* \in \Theta_0$ for n large enough. Due to the compactness of Θ in Assumption 2, $0 < \underline{\omega} < \omega < \bar{\omega} < \infty$. In view of Lemma 12(ii), we have, for $i = 2, \dots, 5$,

$$|\mathcal{J}_n(i, 1) \sqrt{n} (\widehat{\omega}_n - \omega_0)| \leq \sum_{t=1}^{\infty} \sup_{\theta \in \Theta_0} \left\| \frac{\partial^2 \ell_t(\theta)}{\partial \omega \partial \theta} \right\| \frac{1}{\sqrt{n}} (\bar{\omega} - \underline{\omega}) \xrightarrow{a.s.} 0.$$

Further, Taylor's expansion entails that

$$\mathcal{J}_n(2, 2) = \frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \ell_t(\omega_2^*, \vartheta_0)}{\partial \phi_+^2} + \frac{1}{n} \sum_{t=1}^n \frac{\partial^3 \ell_t(\omega_2^*, \vartheta^*)}{\partial \phi_+^2 \partial \vartheta'} (\vartheta_2^* - \vartheta_0),$$

where ϑ^* is between ϑ_2^* and ϑ_0 . By Lemma 12(iii)-(iv) and the consistency of ϑ_2^* , it yields that $\mathcal{J}_n(2, 2) \rightarrow \Upsilon(1, 1)$ a.s.. Similarly, we have $\mathcal{J}_n(i+1, j+1) \rightarrow \Upsilon(i, j)$ a.s. for $i, j = 1, \dots, 4$. By Lemma 11 and (S2.6), it follows that

$$\sqrt{n}(\widehat{\vartheta}_n - \vartheta_0) = \{\Upsilon + o_p(1)\}^{-1} \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \vartheta} \xrightarrow{d} \mathcal{N}(0, \Upsilon^{-1}). \quad \blacksquare$$

S2.3 Proof of Theorem 3

(i) When $\gamma_{\alpha_0} < 0$, without loss of generality, take the entry $\Sigma_{\phi_+ \phi_+} := \Sigma_{\sigma} \Sigma_{\eta}$

as example, where $\Sigma_{\sigma} = \frac{1}{4} E \left\{ \frac{1}{\sigma_i^4(\theta_0)} \frac{\partial \sigma_i^2(\theta_0)}{\partial \phi_+} \frac{\partial \sigma_i^2(\theta_0)}{\partial \phi_+} \right\}$, and $\Sigma_{\eta} = E \left[\left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta)}{\partial x} \eta \right\}^2 \right]$.

We aim to show that the following three estimators

$$\begin{aligned}\widehat{\Sigma}_\sigma &:= \frac{1}{4n} \sum_{t=1}^n \frac{1}{\widehat{\sigma}_t^4(\widehat{\theta}_n)} \frac{\partial \widehat{\sigma}_t^2(\widehat{\theta}_n)}{\partial \phi_+} \frac{\partial \widehat{\sigma}_t^2(\widehat{\theta}_n)}{\partial \phi_+}, \quad \widehat{\Sigma}_\eta^{\text{res}} := \frac{1}{n} \sum_{t=1}^n \left\{ 1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right\}^2, \\ \widehat{\Sigma}_\eta^{\text{int}} &:= \int_{\mathbb{R}} \left\{ 1 + \frac{\partial \log f_{\widehat{\alpha}_n}(u)}{\partial x} u \right\}^2 f_{\widehat{\alpha}_n}(u) du,\end{aligned}$$

are strongly consistent estimators of Σ_σ and Σ_η , respectively.

For $\widehat{\Sigma}_\sigma$, by the Lagrange mean value theorem,

$$\begin{aligned}\frac{1}{4n} \sum_{t=1}^n \frac{1}{\sigma_t^4(\widehat{\theta}_n)} \frac{\partial \sigma_t^2}{\partial \phi_+} \frac{\partial \sigma_t^2}{\partial \phi_+}(\widehat{\theta}_n) &= \frac{1}{4n} \sum_{t=1}^n \left\{ \frac{1}{\sigma_t^2(\widehat{\theta}_n)} \sum_{j=1}^{\infty} \widehat{\psi}_n^{j-1}(y_{t-j}^+)^2 \right\}^2 \\ &= \frac{1}{4n} \sum_{t=1}^n \left\{ \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2}{\partial \phi_+}(\theta) \right\}^2 + S_n,\end{aligned}$$

where S_n is bounded by

$$\begin{aligned}|S_n| &\leq \frac{K}{n} \sum_{t=1}^n \left\{ \frac{1}{\sigma_t^2(\theta^*)} \sum_{j=1}^{\infty} (\psi^*)^{j-1} (y_{t-j}^+)^2 \right\}^2 \left\| \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\theta^*)}{\partial \bar{\theta}} \right\| \|\widehat{\theta}_n - \bar{\theta}_0\| \\ &\quad + \frac{K}{n} \sum_{t=1}^n \left\{ \frac{1}{\sigma_t^2(\theta^*)} \sum_{j=1}^{\infty} (\psi^*)^{j-1} (y_{t-j}^+)^2 \right\} \\ &\quad \left\{ \frac{1}{\sigma_t^2(\theta^*)} \sum_{j=1}^{\infty} (j-1) (\psi^*)^{j-2} (y_{t-j}^+)^2 \right\} |\widehat{\psi}_n - \psi_0| \\ &= o(1), \quad \text{a.s.},\end{aligned}$$

for θ^* such that $\|\theta^* - \theta_0\| \leq \|\widehat{\theta}_n - \theta_0\|$, where $\bar{\theta} = (\omega, \phi_+, \phi_-)'$, using the ergodic theorem, the strong consistency of $\widehat{\psi}_n$ from Theorem 1(i), and $\sup_{\theta \in \Theta, \psi < 1} \left\| \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \bar{\theta}} \right\|_p < \infty$ for all $p \geq 1$ with compact Θ from Lemma 6

and Lemma 11. Combining Lemma 10, we obtain that

$$\frac{1}{4n} \sum_{t=1}^n \frac{1}{\tilde{\sigma}_t^4(\hat{\theta}_n)} \frac{\partial \tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial \phi_+} \frac{\partial \tilde{\sigma}_t^2(\hat{\theta}_n)}{\partial \phi_+} \xrightarrow{a.s.} \frac{1}{4} E \left\{ \frac{1}{\sigma_t^4(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \phi_+} \frac{\partial \sigma_t^2(\theta_0)}{\partial \phi_+} \right\}. \quad (\text{S2.7})$$

For $\widehat{\Sigma}_\eta^{\text{res}}$, let $q_t(\theta) = y_t/\sigma_t(\theta)$, then $\hat{\eta}_t = q_t(\hat{\theta}_n) = y_t/\sigma_t(\hat{\theta}_n)$, and $\eta_t = q_t(\theta_0)$. Note that $\partial q_t(\theta)/\partial \theta = \frac{1}{2} q_t(\theta) \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \theta}$. By the mean value theorem, we have

$$\frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\hat{\alpha}_n}(\hat{\eta}_t)}{\partial x} \hat{\eta}_t \right)^2 = \frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right)^2 + S'_n,$$

with

$$\begin{aligned} S'_n &:= -\frac{1}{n} \sum_{t=1}^n \left\{ 1 + q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} \right\} \\ &\quad \times \left\{ q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} + q_t^2(\theta^*) \frac{\partial^2 \log f_{\alpha^*}(q_t(\theta^*))}{\partial x^2} \right\} \\ &\quad \times \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\tilde{\theta}^*)}{\partial \theta'} (\hat{\theta}_n - \tilde{\theta}_0) \\ &\quad + \frac{2}{n} \sum_{t=1}^n \left\{ 1 + q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} \right\} q_t(\theta^*) \frac{\partial^2 \log f_{\alpha^*}(q_t(\theta^*))}{\partial x \partial \alpha} (\hat{\alpha}_n - \alpha_0), \end{aligned}$$

where $\theta^* = (\omega^*, \vartheta^*)'$ is between $\hat{\theta}_n$ and θ_0 . By Proposition 3, the ergodic theorem, $\sup_{\theta \in \Theta, \psi < 1} \left\| \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \theta} \right\|_p < \infty$ for all $p \geq 1$ with compact Θ , and

the strong consistency of $\widehat{\vartheta}_n$ from Theorem 1(i), it follows that

$$\begin{aligned}
& |S'_n| \\
& \leq \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| 1 + x \frac{\partial \log f_\alpha(x)}{\partial x} \right| \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| x \frac{\partial \log f_\alpha(x)}{\partial x} + x^2 \frac{\partial^2 \log f_\alpha(x)}{\partial x^2} \right| \\
& \quad \times \frac{1}{n} \sum_{t=1}^n \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\tilde{\theta}^*)}{\partial \theta'} (\widehat{\theta}_n - \tilde{\theta}_0) \\
& \quad + 2 \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| 1 + x \frac{\partial \log f_\alpha(x)}{\partial x} \right| \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| x \frac{\partial^2 \log f_\alpha(x)}{\partial x \partial \alpha} \right| |\widehat{\alpha}_n - \alpha_0| \\
& = o(1), \quad \text{a.s.}
\end{aligned}$$

Thus, by the strong law of large numbers, we have

$$\frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 \xrightarrow{\text{a.s.}} E \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right)^2. \quad (\text{S2.8})$$

Finally, for $\widehat{\Sigma}_\eta^{\text{int}}$, the mean value theorem yields that

$$\int_{\mathbb{R}} \left\{ 1 + \frac{\partial \log f_{\widehat{\alpha}_n}(u)}{\partial x} u \right\}^2 f_{\widehat{\alpha}_n}(u) du = \int_{\mathbb{R}} \left\{ 1 + \frac{\partial \log f_{\alpha_0}(u)}{\partial x} u \right\}^2 f_{\alpha_0}(u) du + S''_n,$$

with

$$\begin{aligned}
S''_n & := \int_{\mathbb{R}} \left\{ 2 \left(1 + \frac{\partial \log f_{\alpha^*}(u)}{\partial x} u \right) \frac{\partial^2 \log f_{\alpha^*}(u)}{\partial x \partial \alpha} u f_{\alpha^*}(u) \right. \\
& \quad \left. + \left(1 + \frac{\partial \log f_{\alpha^*}(u)}{\partial x} u \right)^2 \frac{\partial \log f_{\alpha^*}(u)}{\partial \alpha} f_{\alpha^*}(u) \right\} du (\widehat{\alpha}_n - \alpha_0) \\
& = E_{\alpha^*} \left\{ 2 \left(1 + \frac{\partial \log f_{\alpha^*}(\eta)}{\partial x} \eta \right) \frac{\partial^2 \log f_{\alpha^*}(\eta)}{\partial x \partial \alpha} \eta \right. \\
& \quad \left. + \left(1 + \frac{\partial \log f_{\alpha^*}(\eta)}{\partial x} \eta \right)^2 \frac{\partial \log f_{\alpha^*}(\eta)}{\partial \alpha} \right\} (\widehat{\alpha}_n - \alpha_0),
\end{aligned}$$

where α^* is between α_0 and $\widehat{\alpha}_n$, and $\eta \sim f_\alpha(x)$. By Propositions 1 – 3 and

(2.5)-(2.10) in Andrews, Calder and Davis (2009), it can be shown that this

expectation is uniformly bounded in the neighborhood of α_0 . By Theorem 1(i), it yields that $|S_n''| = o(1)$, a.s. The convergence results for other entries can be obtained similarly. Combining S2.7–S2.8, it yields that $\widehat{\Sigma}^{\text{int}} \xrightarrow{a.s.} \Sigma$, and $\widehat{\Sigma}^{\text{res}} \xrightarrow{a.s.} \Sigma$, as $n \rightarrow \infty$.

(ii) When $\gamma_{\alpha_0} > 0$, the consistency of $\widehat{\Sigma}_\eta^{\text{int}}$ is already shown in (i), as it only involves the strong consistency result of $\widehat{\alpha}_n$, and the consistency of $\widehat{\Sigma}_\eta^{\text{res}}$ will be shown in S2.10 of Section S2.4. It remains to prove the consistency of estimators of $E(d_t d_t')$ and $E(d_t)$. Without loss of generality, take the entry $E\{(d_t^\psi)^2\} = \nu_2(1+\nu_1)/\{\psi_0^2(1-\nu_2)(1-\nu_1)\}$ as example, where $\nu_i = E[\{\psi_0/a_0(\eta_t)\}^i]$ for $i = 1, 2$, and $a_0(x) = \phi_{0+}(x^+)^2 + \phi_{0-}(x^-)^2 + \psi_0$. We aim to show that

$$\begin{aligned}\widehat{\nu}_i^{\text{int}} &= \int_{\mathbb{R}} \left\{ \frac{\widehat{\psi}_n}{\widehat{\phi}_{n+}(u^+)^2 + \widehat{\phi}_{n-}(u^-)^2 + \widehat{\psi}_n} \right\}^i f_{\widehat{\alpha}_n}(u) du, \\ \widehat{\nu}_i^{\text{res}} &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{\widehat{\psi}_n}{\widehat{\phi}_{n+}(\widehat{\eta}_t^+)^2 + \widehat{\phi}_{n-}(\widehat{\eta}_t^-)^2 + \widehat{\psi}_n} \right\}^i,\end{aligned}$$

are consistent estimators of ν_i , for $i = 1, 2$, respectively.

For $\widehat{\nu}_i^{\text{int}}$ and $i = 1$, the mean value theorem yields that

$$\begin{aligned}& \int_{\mathbb{R}} \left\{ \frac{\widehat{\psi}_n}{\widehat{\phi}_{n+}(u^+)^2 + \widehat{\phi}_{n-}(u^-)^2 + \widehat{\psi}_n} \right\} f_{\widehat{\alpha}_n}(u) du \\ &= \int_{\mathbb{R}} \left\{ \frac{\psi_0}{\phi_{0+}(u^+)^2 + \phi_{0-}(u^-)^2 + \psi_0} \right\} f_{\alpha_0}(u) du + S_n,\end{aligned}$$

where

$$\begin{aligned}
S_n &:= \int_{\mathbb{R}} \left\{ \frac{-\psi^*(u^+)^2}{(\phi_+^*(u^+)^2 + \phi_-^*(u^-)^2 + \psi^*)^2} \right\} f_{\alpha^*}(u) du (\widehat{\phi}_{n+} - \phi_{0+}) \\
&+ \int_{\mathbb{R}} \left\{ \frac{-\psi^*(u^-)^2}{(\phi_+^*(u^+)^2 + \phi_-^*(u^-)^2 + \psi^*)^2} \right\} f_{\alpha^*}(u) du (\widehat{\phi}_{n-} - \phi_{0-}) \\
&+ \int_{\mathbb{R}} \left\{ \frac{\phi_+^*(u^+)^2 + \phi_-^*(u^-)^2}{(\phi_+^*(u^+)^2 + \phi_-^*(u^-)^2 + \psi^*)^2} \right\} f_{\alpha^*}(u) du (\widehat{\psi}_n - \psi_0) \\
&+ \int_{\mathbb{R}} \left\{ \frac{\psi^*}{\phi_+^*(u^+)^2 + \phi_-^*(u^-)^2 + \psi^*} \right\} \frac{\partial \log f_{\alpha^*}(u)}{\partial \alpha} f_{\alpha^*}(u) du (\widehat{\alpha}_n - \alpha_0),
\end{aligned}$$

where θ^* is between $\widehat{\theta}_n$ and θ_0 . By Propositions 1 – 3 and the compactness of Θ , it can be shown that these integrals are uniformly bounded in the neighborhood of θ_0 . Combining Theorem 2(i), it yields that $|S_n| = o(1)$, a.s. The case of $i = 2$ can be proved similarly.

As for $\widehat{\nu}_i^{\text{res}}$, by the mean value theorem, we have

$$\frac{1}{n} \sum_{t=1}^n \frac{\widehat{\psi}_n}{\widehat{\phi}_{n+}(\widehat{\eta}_t^+)^2 + \widehat{\phi}_{n-}(\widehat{\eta}_t^-)^2 + \widehat{\psi}_n} = \frac{1}{n} \sum_{t=1}^n \frac{\psi_0}{\phi_{0+}(\eta_t^+)^2 + \phi_{0-}(\eta_t^-)^2 + \psi_0} + S'_n,$$

with

$$\begin{aligned}
& S'_n \\
& := \frac{1}{n} \sum_{t=1}^n \frac{\psi^* [\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2]}{[\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2 + \psi^*]^2} \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\theta^*)}{\partial \omega} (\widehat{\omega}_n - \omega_0) \\
& + \frac{1}{n} \sum_{t=1}^n \left\{ \frac{-\psi^* \{ (q_t^+(\theta^*))^2 - [\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2] \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\theta^*)}{\partial \phi_+} \}}{[\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2 + \psi^*]^2} \right\} (\widehat{\phi}_{n+} - \phi_{0+}) \\
& + \frac{1}{n} \sum_{t=1}^n \left\{ \frac{-\psi^* \{ (q_t^-(\theta^*))^2 - [\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2] \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\theta^*)}{\partial \phi_-} \}}{[\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2 + \psi^*]^2} \right\} (\widehat{\phi}_{n-} - \phi_{0-}) \\
& + \frac{1}{n} \sum_{t=1}^n \left\{ \frac{[\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2] \left\{ 1 + \psi^* \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\theta^*)}{\partial \psi} \right\}}{[\phi_+^*(q_t^+(\theta^*))^2 + \phi_-^*(q_t^-(\theta^*))^2 + \psi^*]^2} \right\} (\widehat{\psi}_n - \psi_0).
\end{aligned}$$

where $\theta^* = (\omega^*, \vartheta^*)'$ is between $\widehat{\theta}_n$ and θ_0 . By Proposition 2.1 in Francq and Zakoïan (2013), the strong consistency of $\widehat{\vartheta}_n$ from 2(i), and the finite moments of any order for $d_t(\vartheta)$ from Lemma 11, it holds that $|S_n| = o(1)$, a.s. The proof for the case $i = 2$ is analogous and is omitted. Combining the above results, it follows that $\widehat{\Upsilon}^{\text{int}} \xrightarrow{a.s.} \Upsilon$, and $\widehat{\Upsilon}^{\text{res}} \xrightarrow{a.s.} \Upsilon$, as $n \rightarrow \infty$.

■

S2.4 Proof of Theorem 4

(i). When $\gamma_{\alpha_0} < 0$, by the Taylor expansions of

$$\frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 \quad \text{and} \quad \frac{1}{n} \sum_{t=1}^n \frac{1}{\widetilde{\sigma}_t^4(\widehat{\theta}_n)} \frac{\partial \widetilde{\sigma}_t^2(\widehat{\theta}_n)}{\partial \theta_i} \frac{\partial \widetilde{\sigma}_t^2(\widehat{\theta}_n)}{\partial \theta_j} \quad (\text{S2.9})$$

around θ_0 , where $\widehat{\eta}_t = q_t(\widehat{\theta}_n) = y_t/\sigma_t(\widehat{\theta}_n)$, the ergodic theorem, and Theorem 1(i), the proof of (i) can be completed by standard arguments, similar

to the proof of Theorem 3(i), and it is thus omitted. \square

(ii). When $\gamma_{\alpha_0} > 0$, we will consider the two terms in (S2.9) respectively, and the other terms in $\widehat{\Upsilon}_*$ can be dealt with analogously. As for the first term, by the mean value theorem, we have

$$\frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 = \frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right)^2 + S_n,$$

with

$$\begin{aligned} S_n := & -\frac{1}{n} \sum_{t=1}^n \left\{ 1 + q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} \right\} \\ & \times \left\{ q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} + q_t^2(\theta^*) \frac{\partial^2 \log f_{\alpha^*}(q_t(\theta^*))}{\partial x^2} \right\} \frac{1}{\sigma_t^2(\theta^*)} \frac{\partial \sigma_t^2(\tilde{\theta}^*)}{\partial \theta'} (\widehat{\theta}_n - \tilde{\theta}_0) \\ & + \frac{2}{n} \sum_{t=1}^n \left\{ 1 + q_t(\theta^*) \frac{\partial \log f_{\alpha^*}(q_t(\theta^*))}{\partial x} \right\} q_t(\theta^*) \frac{\partial^2 \log f_{\alpha^*}(q_t(\theta^*))}{\partial x \partial \alpha} (\widehat{\alpha}_n - \alpha_0), \end{aligned}$$

where $\theta^* = (\omega^*, \vartheta^*)'$ is between $\widehat{\theta}_n$ and θ_0 . Denote $d_t(\vartheta) = (d_t^{\phi^+}(\vartheta), d_t^{\phi^-}(\vartheta), d_t^{\psi}(\vartheta))'$.

By Proposition 2.1 in Francq and Zakoïan (2013), it yields that for some

$$\rho \in (\psi^* \exp(-\gamma_{\alpha_0}), 1),$$

$$\begin{aligned} & |S_n| \\ \leq & \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| 1 + x \frac{\partial \log f_{\alpha}(x)}{\partial x} \right| \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| x \frac{\partial \log f_{\alpha}(x)}{\partial x} + x^2 \frac{\partial^2 \log f_{\alpha}(x)}{\partial x^2} \right| \\ & \times \frac{K}{n} \sum_{t=1}^n (\rho^t |\widehat{\omega}_n - \omega_0| + d_t^{\phi^+}(\vartheta) |\widehat{\phi}_{n+} - \phi_{0+}| + d_t^{\phi^-}(\vartheta) |\widehat{\phi}_{n-} - \phi_{0-}| + d_t^{\psi}(\vartheta) |\widehat{\psi}_n - \psi_0|) \\ & + 2 \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| 1 + x \frac{\partial \log f_{\alpha}(x)}{\partial x} \right| \sup_{x \in \mathbb{R}} \sup_{|\alpha - \alpha_0| < \epsilon} \left| x \frac{\partial^2 \log f_{\alpha}(x)}{\partial x \partial \alpha} \right| |\widehat{\alpha}_n - \alpha_0| \\ = & o(1), \quad \text{a.s.}, \end{aligned}$$

where the last equality follows from Proposition 3, the strong consistency of $\widehat{\vartheta}_n$, and the finite moments of any order for $d_t(\vartheta)$. Thus, by the strong law of large numbers, we have

$$\frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t\right)^2 \xrightarrow{a.s.} E \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t\right)^2. \quad (\text{S2.10})$$

Similarly, the convergence result can be obtained for the terms $n^{-1} \sum_{t=1}^n$

$$\frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial \alpha} \widehat{\eta}_t \text{ and } n^{-1} \sum_{t=1}^n \left\{ \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial \alpha} \right\}^2.$$

Now we turn to the second term in (S2.9). For brevity, we only consider the case $i = j = 1$, and the other terms can be handled similarly. By the Lagrange mean value theorem,

$$\begin{aligned} & \frac{1}{n} \sum_{t=1}^n \frac{1}{\sigma_t^4(\widehat{\theta}_n)} \frac{\partial \sigma_t^2}{\partial \phi_+} \frac{\partial \sigma_t^2}{\partial \phi_+}(\widehat{\theta}_n) \\ &= \frac{1}{n} \sum_{t=1}^n \left\{ \frac{1}{\sigma_t^2(\widehat{\theta}_n)} \sum_{j=1}^t \widehat{\psi}_n^{j-1} (y_{t-j}^+)^2 \right\}^2 \\ &= \frac{1}{n} \sum_{t=1}^n \{d_t^{\phi_+}(\vartheta_0)\}^2 + \frac{1}{n} \sum_{t=1}^n \left[\left\{ \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2}{\partial \phi_+}(\theta) \right\}^2 - \{d_t^{\phi_+}(\vartheta_0)\}^2 \right] + S'_n, \end{aligned} \quad (\text{S2.11})$$

where S'_n is bounded by

$$\begin{aligned} & |S'_n| \\ &\leq \frac{K}{n} \sum_{t=1}^n \left\{ \sigma_t^{-2}(\theta^*) \sum_{j=1}^t (\psi^*)^{j-1} (y_{t-j}^+)^2 \right\}^2 (\rho^t |\widehat{\omega}_n - \omega_0| + \|d_t(\vartheta^*)\| \|\widehat{\vartheta}_n - \vartheta_0\|) \\ &+ \frac{K}{n} \sum_{t=1}^n \left\{ \sigma_t^{-2}(\theta^*) \sum_{j=1}^t (\psi^*)^{j-1} (y_{t-j}^+)^2 \right\} \left\{ \sigma_t^{-2}(\theta^*) \sum_{j=1}^t (j-1) (\psi^*)^{j-2} (y_{t-j}^+)^2 \right\} |\widehat{\psi}_n - \psi_0| \\ &= o(1), \quad \text{a.s.} \end{aligned}$$

for θ^* such that $\|\theta^* - \theta_0\| \leq \|\widehat{\theta}_n - \theta_0\|$ and some $\rho \in (\psi^* \exp(-\gamma_{\alpha_0}), 1)$, by similar arguments as above. Moreover, the second term in the right-hand side of (S2.11) converges to 0 a.s. in view of (S3.22), while the first term converges to $E\{(d_1^{\phi_+}(\vartheta_0))^2\}$. Thus, we obtain that

$$\frac{1}{n} \sum_{t=1}^n \frac{1}{\sigma_t^4(\widehat{\theta}_n)} \frac{\partial \sigma_t^2}{\partial \theta_i} \frac{\partial \sigma_t^2}{\partial \theta_j}(\widehat{\theta}_n) \xrightarrow{a.s.} E(d_t d_t').$$

Together with (S2.10), it yields that $\widehat{\Sigma}_{\vartheta\vartheta} \xrightarrow{a.s.} \Upsilon$.

Finally, we consider the term $\widehat{\Sigma}_{\vartheta\omega}$ and $\widehat{\Sigma}_{\omega\omega}$. Note that $\{\sum_{j=1}^t \psi^{j-1}(y_{t-j}^+)^2\} / \sigma_t^2(\theta) \leq 1/\phi_+$. As for the first entry of $\widehat{\Sigma}_{\vartheta\omega}$,

$$\begin{aligned} n\widehat{\Sigma}_{\phi_+\omega} &= \frac{1}{4} \left\{ \sum_{t=1}^n \frac{1}{\sigma_t^4(\widehat{\theta}_n)} \frac{\partial \sigma_t^2}{\partial \phi_+} \frac{\partial \sigma_t^2}{\partial \omega}(\widehat{\theta}_n) \right\} \left\{ \frac{1}{n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 \right\} \\ &\leq \left\{ \frac{1}{4n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 \right\} \left\{ \sum_{t=1}^n \frac{1}{\widehat{\phi}_{n+}} \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\widehat{\theta}_n)} \frac{1}{\sigma_t^2(\theta_0)} \sum_{j=1}^t \widehat{\psi}_n^{j-1} \right\} \\ &\leq \left\{ \frac{K}{4n} \sum_{t=1}^n \left(1 + \frac{\partial \log f_{\widehat{\alpha}_n}(\widehat{\eta}_t)}{\partial x} \widehat{\eta}_t \right)^2 \right\} \sum_{t=1}^n \rho^t \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\widehat{\theta}_n)}, \end{aligned}$$

for some $\rho \in (0, 1)$ when n is large enough. Together with (S2.10), it follows that $n\widehat{\Sigma}_{\phi_+\omega} = O(1)$ a.s.. Similarly, it can be shown that $n\widehat{\Sigma}_{\phi_-\omega} = O(1)$ a.s., $n\widehat{\Sigma}_{\psi\omega} = O(1)$ a.s., and $n\widehat{\Sigma}_{\alpha\omega} = O(1)$ a.s. As for $\widehat{\Sigma}_{\omega\omega}$, we have $n\widehat{\Sigma}_{\omega\omega} \geq K > 0$. Thus it yields that

$$\widehat{\Sigma}_{\vartheta\omega} \widehat{\Sigma}_{\omega\omega}^{-1} \widehat{\Sigma}'_{\vartheta\omega} = o(1) \quad \text{a.s.}$$

The proof of (ii) is thus completed. ■

S2.5 Proof of Theorem 5

The complete version of Theorem 5 is as follows, including the explicit forms of a_1, a_2, ν_1, c_1 :

Theorem 5. *Let $u_t = \log a_0(\eta_t) - \gamma_{\alpha_0}$ and $\sigma_u^2 = Eu_t^2 < \infty$. Suppose that Assumptions 1–3 hold, it follows that*

$$\sqrt{n}(\hat{\gamma}_n - \gamma_{\alpha_0}) \xrightarrow{d} \mathcal{N}(0, \sigma_\gamma^2) \quad \text{as } n \rightarrow \infty,$$

where

$$\sigma_\gamma^2 = \begin{cases} \sigma_u^2 + \{a_1' \Sigma^{-1} a_2 - 4(1 - \nu_1)^2 / c_1\}, & \text{if } \gamma_{\alpha_0} < 0, \\ \sigma_u^2, & \text{if } \gamma_{\alpha_0} > 0, \end{cases}$$

with

$$\begin{aligned} a_1 &= (0, -\tilde{\nu}_{1+}, -\tilde{\nu}_{1-}, -\nu_1/\psi_0, 2c_2(1 - \nu_1 + c^*)/c_1 - 2\tilde{c})', \\ a_2 &= (0, -\tilde{\nu}_{1+}, -\tilde{\nu}_{1-}, -\nu_1/\psi_0, 2c_2(1 - \nu_1)/c_1)', \\ c^* &= E\left[u_t \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t\right)\right], \tilde{c} = E\left(u_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}\right), \\ c_1 &= E\left[\left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t\right)^2\right], c_2 = E\left(\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \eta_t\right), \\ \tilde{\nu}_{1+} &= E\left\{\frac{(\eta_1^+)^2}{a_0(\eta_1)}\right\}, \tilde{\nu}_{1-} = E\left\{\frac{(\eta_1^-)^2}{a_0(\eta_1)}\right\}, \nu_1 = E\left\{\frac{\psi_0}{a_0(\eta_1)}\right\}. \end{aligned}$$

PROOF. Define $\gamma_n(\theta) = \frac{1}{n} \sum_{t=1}^n \log[\phi_+\{q_t^+(\theta)\}^2 + \phi_-\{q_t^-(\theta)\}^2 + \psi]$, where $q_t(\theta) = y_t/\tilde{\sigma}_t(\theta)$, and $\tilde{\sigma}_t(\theta)$ is defined in (3.4). Note that $\gamma_n(\theta)$ does not involve the stability parameter α , so it is equivalent to $\gamma_n(\tilde{\theta})$.

In the stationary case that $\gamma_{\alpha_0} < 0$, by standard arguments we have

$$\widehat{\gamma}_n = \gamma_n(\tilde{\theta}_0) + \frac{\partial \gamma_n(\tilde{\theta}_0)}{\partial \tilde{\theta}'} (\widehat{\theta}_n - \theta_0)_{(4 \times 1)} + o_p(n^{-1/2}), \quad (\text{S2.12})$$

with (4×1) representing the first four entries of the vector $(\widehat{\theta}_n - \theta_0)$, and

$$\begin{aligned} \frac{\partial \gamma_n(\tilde{\theta}_0)}{\partial \tilde{\theta}} &= \frac{-1}{n} \sum_{t=1}^n \frac{1}{a_0(\eta_t)} \left\{ \left[\phi_{0+}(\eta_t^+)^2 + \phi_{0-}(\eta_t^-)^2 \right] \frac{1}{\tilde{\sigma}_t^2(\theta_0)} \frac{\partial \tilde{\sigma}_t^2(\theta_0)}{\partial \theta} - \begin{pmatrix} 0 \\ (\eta_t^+)^2 \\ (\eta_t^-)^2 \\ 1 \end{pmatrix} \right\} \\ &= -\Psi + o_p(1), \end{aligned}$$

where $\Psi = (1 - \nu_1)\Omega - a$ and $\Omega = E \left\{ \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta} \right\}$, with $a = (0, \tilde{\nu}_{1+}, \tilde{\nu}_{1-}, \nu_1/\psi_0)'$

and

$$\tilde{\nu}_{1+} = E \left\{ \frac{(\eta_1^+)^2}{a_0(\eta_1)} \right\}, \quad \tilde{\nu}_{1-} = E \left\{ \frac{(\eta_1^-)^2}{a_0(\eta_1)} \right\}, \quad \nu_1 = E \left\{ \frac{\psi_0}{a_0(\eta_1)} \right\}.$$

Moreover, the MLE $\widehat{\theta}_n$ satisfies

$$\sqrt{n}(\widehat{\theta}_n - \theta_0)_{(4 \times 1)} = (\Sigma^{-1})_{(4 \times 5)} \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} + o_p(1),$$

where $(\Sigma^{-1})_{(4 \times 5)}$ represents the submatrix formed by the first 4 rows and

first 5 columns of Σ^{-1} . Therefore, it yields that

$$\sqrt{n}(\widehat{\gamma}_n - \gamma_{\alpha_0}) = \frac{1}{\sqrt{n}} \sum_{t=1}^n u_t - \Psi'(\Sigma^{-1})_{(4 \times 5)} \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} + o_p(1).$$

Note that

$$\text{Cov} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n u_t, \Psi'(\Sigma^{-1})_{(4 \times 5)} \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} \right) = \left(-\frac{1}{2} c^* \Omega', \tilde{c} \right) (\Sigma^{-1})_{(5 \times 4)} \Psi,$$

where

$$c^* = E \left[u_t \left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right) \right], \quad \tilde{c} = E \left(u_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \right).$$

By the Slutsky lemma and the central limit theorem for martingale differences, it entails that

$$\sqrt{n}(\hat{\gamma}_n - \gamma_{\alpha_0}) \xrightarrow{d} \mathcal{N} \left(0, \sigma_u^2 + (c^* \Omega', -2\tilde{c}) \Sigma^{-1} (\Psi', 0)' + (\Psi', 0) \Sigma^{-1} (\Psi', 0)' \right).$$

Denote $\bar{\theta}_0 = (\omega_0, \phi_{0+}, \phi_{0-}, 0)'$. Then $\bar{\theta}'_0 \partial \sigma_t^2(\theta_0) / \partial \bar{\theta} = \sigma_t^2(\theta_0)$ a.s. and

$$E \left\{ \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \bar{\theta}} \left(1 - \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \bar{\theta}'} \bar{\theta}_0 \right) \right\} = 0,$$

which is $\Sigma(\bar{\theta}'_0, 0)' = (c_1/4, -c_2)'$, where

$$c_1 = E \left[\left(1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right)^2 \right], \quad c_2 = E \left(\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \eta_t \right).$$

Moreover, $(\bar{\theta}'_0, 0) \Sigma (\bar{\theta}'_0, 0)' = c_1/4$. Note that $(1 - \nu_1) = \bar{\theta}'_0 a$, it follows that

$$\begin{aligned} (c^* \Omega', -2\tilde{c}) \Sigma^{-1} (\Psi', 0)' &= \left(\mathbf{0}, \frac{2c_2 c^*}{c_1} - 2\tilde{c} \right) \Sigma^{-1} \left(-a, \frac{2c_2(1 - \nu_1)}{c_1} \right)', \\ (\Psi', 0) \Sigma^{-1} (\Psi', 0)' &= \left(-a, \frac{2c_2(1 - \nu_1)}{c_1} \right) \Sigma^{-1} \left(-a, \frac{2c_2(1 - \nu_1)}{c_1} \right)' - \frac{4}{c_1} (1 - \nu_1)^2. \end{aligned}$$

Thus, the proof in the case of $\gamma_{\alpha_0} < 0$ is completed. \square

For the explosive case $\gamma_{\alpha_0} > 0$, let θ_n^* be a sequence such that $\|\theta_n^* - \theta_0\| \leq \|\hat{\theta}_n - \theta_0\|$. By Proposition 2.1 in Francq and Zakoian (2013), we have

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{1}{\sigma_t^2(\theta_n^*)} \frac{\partial \sigma_t^2(\theta_n^*)}{\partial \omega} = o(1) \quad \text{a.s. as } n \rightarrow \infty.$$

Similarly, it is not hard to get $\sqrt{n} \frac{\partial^2 \gamma_n(\theta_n^*)}{\partial \omega \partial \theta} = o(1)$ a.s., and $\sqrt{n}(\widehat{\theta} - \theta_0)' \frac{\partial^2 \gamma_n(\theta_n^*)}{\partial \theta \partial \theta'} (\widehat{\theta} - \theta_0) = o_p(1)$, which shows that (S2.12) still holds. By similar arguments, we can complete the proof in the case $\gamma_{\alpha_0} > 0$, with

$$\Omega = E \begin{pmatrix} 0 \\ d_t^{\phi+}(\theta_0) \\ d_t^{\phi-}(\theta_0) \\ d_t^{\psi}(\theta_0) \end{pmatrix} = \frac{1}{1 - \nu_1} \begin{pmatrix} 0 \\ \tilde{\nu}_{1+} \\ \tilde{\nu}_{1-} \\ \nu_1/\psi_0 \end{pmatrix} \quad \text{and} \quad \Psi = \mathbf{0}. \quad \blacksquare$$

S2.6 Proof of Theorem 6

Recall that $\tilde{\eta}_t = y_t/\sigma_t(\tilde{\theta}_n)$, $t = 1, \dots, n$, and $\tilde{\theta}_n = (\widehat{\omega}_n, \widehat{\phi}_{n+}, \widehat{\phi}_{n-}, \widehat{\psi}_n)'$ is the restricted MLE of $\tilde{\theta}_0$ under H_0 that $\alpha_0 = \alpha_*$. We have

$$\begin{aligned} \widehat{V}_n(r) &= \frac{1}{\sqrt{n}} \sum_{t=1}^n [I(F_{\alpha_*}(\tilde{\eta}_t) \leq r) - r] \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[I\left(\eta_t \frac{\tilde{\sigma}_t(\tilde{\theta}_0)}{\tilde{\sigma}_t(\tilde{\theta}_n)} \leq F_{\alpha_*}^{-1}(r)\right) - r \right] \\ &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \left[I\left(U_t \leq F_{\alpha_*} \left\{ F_{\alpha_*}^{-1}(r) \frac{\tilde{\sigma}_t(\tilde{\theta}_n)}{\tilde{\sigma}_t(\tilde{\theta}_0)} \right\}\right) - r \right] \\ &= V_n(r) + \frac{1}{n} \sum_{t=1}^n d_{nt}(r) + o_p(1), \end{aligned}$$

where $d_{nt}(r) = \sqrt{n} \left\{ F_{\alpha_*} \left(F_{\alpha_*}^{-1}(r) \frac{\tilde{\sigma}_t(\tilde{\theta}_n)}{\tilde{\sigma}_t(\tilde{\theta}_0)} \right) - r \right\}$, and $o_p(1)$ holds uniformly in $r \in [0, 1]$. The third term in the last equality can be proved using similar arguments in the proof of Theorem A.2 in Bai (1996). As for the second

term, we have

$$\begin{aligned}
\frac{1}{n} \sum_{t=1}^n d_{nt}(r) &= \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ F_{\alpha_*} \left(F_{\alpha_*}^{-1}(r) \frac{\tilde{\sigma}_t(\tilde{\theta}_n)}{\tilde{\sigma}_t(\tilde{\theta}_0)} \right) - F_{\alpha_*}(F_{\alpha_*}^{-1}(r)) \right\} \\
&= f_{\alpha_*}(F_{\alpha_*}^{-1}(r)) \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ F_{\alpha_*}^{-1}(r) \frac{\tilde{\sigma}_t(\tilde{\theta}_n)}{\tilde{\sigma}_t(\tilde{\theta}_0)} - F_{\alpha_*}^{-1}(r) \right\} + o_p(1) \\
&= \frac{1}{2} f_{\alpha_*}(F_{\alpha_*}^{-1}(r)) F_{\alpha_*}^{-1}(r) \frac{1}{n} \sum_{t=1}^n \left\{ \frac{1}{\tilde{\sigma}_t^2(\tilde{\theta}_0)} \frac{\partial \tilde{\sigma}_t^2(\tilde{\theta}_0)}{\partial \tilde{\theta}'} \right\} \sqrt{n} (\tilde{\theta}_n - \tilde{\theta}_0) + o_p(1),
\end{aligned}$$

where $o_p(1)$ holds uniformly in $r \in [0, 1]$.

In the stationary case, i.e., $\gamma_{\alpha_*} < 0$, we have

$$\frac{1}{n} \sum_{t=1}^n d_{nt}(r) = \frac{1}{2} f_{\alpha_*}(F_{\alpha_*}^{-1}(r)) F_{\alpha_*}^{-1}(r) E \left\{ \frac{1}{\sigma_t^2(\tilde{\theta}_0)} \frac{\partial \sigma_t^2(\tilde{\theta}_0)}{\partial \tilde{\theta}'} \right\} \sqrt{n} (\tilde{\theta}_n - \tilde{\theta}_0) + o_p(1).$$

Hence, it entails that

$$\widehat{V}_n(r) = V_n(r) + \frac{1}{2} f_{\alpha_*}(F_{\alpha_*}^{-1}(r)) F_{\alpha_*}^{-1}(r) E \left\{ \frac{1}{\sigma_t^2(\tilde{\theta}_0)} \frac{\partial \sigma_t^2(\tilde{\theta}_0)}{\partial \tilde{\theta}'} \right\} \sqrt{n} (\tilde{\theta}_n - \tilde{\theta}_0) + o_p(1).$$

and then the conclusion follows by the same arguments as for Corollary 1 in Bai (2003).

In the explosive case when $\gamma_{\alpha_*} > 0$, there exist constants K_y and $\rho^* \in (\psi_0 \exp(-\gamma_{\alpha_0}), 1)$ such that, for t large enough,

$$\frac{1}{\tilde{\sigma}_t^2(\tilde{\theta}_0)} \frac{\partial \tilde{\sigma}_t^2(\tilde{\theta}_0)}{\partial \omega} = \frac{1}{\tilde{\sigma}_t^2(\tilde{\theta}_0)} \sum_{j=1}^t \psi_0^{j-1} \leq \frac{K \rho^{*t}}{y_{t-1}^2 (\rho^*/\psi_0)^t} \leq K_y \rho^{*t}, \quad \text{a.s.}$$

where the inequality follows by the fact from Proposition 2.1 in Francq and

Zakoïan (2013) that, for any $\rho > \exp(-\gamma_{\alpha_0})$, $\rho^t y_t^2 \xrightarrow{\text{a.s.}} \infty$, a.s. as $t \rightarrow \infty$.

Then $\sum_{t=1}^{\infty} K_y \rho^{*t} = K_y \sum_{t=1}^{\infty} \rho^{*t} < \infty$, a.s., which implies that

$$\frac{1}{n} \sum_{t=1}^n \left\{ \frac{1}{\tilde{\sigma}_t^2(\tilde{\theta}_0)} \frac{\partial \tilde{\sigma}_t^2(\tilde{\theta}_0)}{\partial \omega} \right\} \sqrt{n} |\hat{\omega}_n - \omega_0| \leq \frac{K}{\sqrt{n}} K_y (\bar{\omega} - \underline{\omega}) \xrightarrow{a.s.} 0.$$

Thus, by the boundedness of $f_{\alpha_*}(F_{\alpha_*}^{-1}(r))F_{\alpha_*}^{-1}(r)$, the Stolz theorem, and (S3.22), we can show that

$$\frac{1}{n} \sum_{t=1}^n d_{nt}(r) = \frac{1}{2} f_{\alpha_*}(F_{\alpha_*}^{-1}(r))F_{\alpha_*}^{-1}(r)E(d_t)' \sqrt{n}(\tilde{\vartheta}_n - \tilde{\vartheta}_0) + o_p(1).$$

It follows that

$$\hat{V}_n(r) = V_n(r) + \frac{1}{2} f_{\alpha_*}(F_{\alpha_*}^{-1}(r))F_{\alpha_*}^{-1}(r)E(d_t)' \sqrt{n}(\tilde{\vartheta}_n - \tilde{\vartheta}_0) + o_p(1),$$

where $o_p(1)$ holds uniformly in $r \in [0, 1]$. Note that the derivative of $f_{\alpha_*}(F_{\alpha_*}^{-1}(r))F_{\alpha_*}^{-1}(r)$ with respect to r is bounded variation function on $[0, 1]$.

The conclusion follows by the similar arguments as for Corollary 1 in Bai (2003). ■

S3 Propositions and Technical Lemmas

The following propositions state some important properties of stable distribution used in the proof. Propositions 1 and 2 are basic properties of stable distribution (see, e.g., Chapter 1 in Nolan (2020)). Proposition 3 describes the asymptotic behavior of derivatives of log-stable densities (see, e.g., Calder (1998) or (2.5)-(2.10) in Andrews, Calder and Davis (2009)).

The proof utilizes the series expansions (see Theorem 3.5 in Nolan (2020)) and duality (see Theorem 3.6 in Nolan (2020)) of stable densities.

Proposition 1 (Tail approximation). *For the density of $\mathbf{S}(\alpha, \beta, \gamma, \delta)$, where $0 < \alpha < 2, -1 < \beta \leq 1$, when $x \rightarrow \infty$, it follows that*

$$f(x \mid \alpha, \beta, \gamma, \delta) \sim \alpha \gamma^\alpha c_\alpha (1 + \beta) |x|^{-(\alpha+1)},$$

where “ \sim ” represents limit equivalence, $c_\alpha = \sin(\pi\alpha/2)\Gamma(\alpha)/\pi$, and $\Gamma(\cdot)$ is the gamma function. It entails that when $0 < \alpha < 2$, the tails of stable distributions are asymptotically power laws with heavy tails.

Proposition 2 (Moments). *For a non-Gaussian stable random variable $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta)$, $0 < \alpha < 2$, when $0 < p < \alpha$, $E|X|^p < \infty$; when $p \geq \alpha$, $E|X|^p = +\infty$. Moreover, $\log |X|$ has moments of any order, i.e., $E|\log |X||^q < \infty, \forall q > 0$.*

Proposition 3 (Order of partial derivatives of log-stable densities). *For non-Gaussian stable densities (2.1),*

$$\begin{aligned} \frac{\partial \log f_\alpha(x)}{\partial x} &\asymp -x^{-1}, & \frac{\partial \log f_\alpha(x)}{\partial \alpha} &\asymp -\log x; \\ \frac{\partial^2 \log f_\alpha(x)}{\partial x^2} &\asymp x^{-2}, & \frac{\partial^2 \log f_\alpha(x)}{\partial x \partial \alpha} &\asymp -x^{-1}, \end{aligned}$$

as $x \rightarrow \infty$, where “ \asymp ” represents “of the same order”. To be more precise,

for symmetric non-Gaussian stable densities, as $x \rightarrow \infty$,

- (i). $\frac{\partial \log f_\alpha(x)}{\partial x} = -(\alpha + 1)x^{-1} + O(x^{-\alpha-1});$
- (ii). $\frac{\partial \log f_\alpha(x)}{\partial \alpha} = \frac{\dot{g}_1}{g_1} I_{(0 < \alpha \leq 1)} + \frac{\dot{A}_1}{A_1} I_{(1 < \alpha < 2)} - \log x + O(x^{-\alpha} \log x);$
- (iii). $\frac{\partial^2 \log f_\alpha(x)}{\partial x^2} = (\alpha + 1)x^{-2} + O(x^{-\alpha-2});$
- (iv). $\frac{\partial^2 \log f_\alpha(x)}{\partial x \partial \alpha} = -x^{-1} + O(x^{-\alpha-1} \log x),$

where

$$g_1 = \Gamma(\alpha + 1) \sin(\pi\alpha/2), \quad \dot{g}_1 = \frac{d\{\Gamma(\alpha + 1) \sin(\pi\alpha/2)\}}{d\alpha},$$

$$A_1 = \frac{1}{\sin^\alpha(\pi/(2\alpha))}, \quad \dot{A}_1 = \frac{d\{1/\sin^\alpha(\pi/(2\alpha))\}}{d\alpha}.$$

Remark 1. For (ii) and (iv), we made some corrections based on the results in p.14-15 of Calder (1998). Properties of other partial derivatives can be found in p.14-15 of Calder (1998) or (2.5)-(2.10) in Andrews, Calder and Davis (2009).

Lemma 1. Suppose random variables ε and η are independent, $\varepsilon > 0$ a.s., and the density function of η is $f_{\alpha_0}(x)$. If ε and η satisfies

$$\varepsilon f_\alpha(\varepsilon\eta) = f_{\alpha_0}(\eta), \quad a.s., \quad (\text{S3.13})$$

where $f_\alpha(x)$ is the density function of standardized symmetric stable random variable (see (2.1)) with stability parameter $\alpha \in (0, 2)$. Then it holds that $\alpha = \alpha_0$, and $\varepsilon = 1$ a.s.

PROOF. For any $x \in (0, \infty)$, (S3.13) implies that

$$\varepsilon f_\alpha(\varepsilon\eta)I_{(\eta \in [x, \infty))} = f_{\alpha_0}(\eta)I_{(\eta \in [x, \infty))}, \quad \text{a.s.}$$

Note that $f_\alpha(x)$ is bounded, by taking conditional expectation it yields that

$$\varepsilon E\{f_\alpha(\varepsilon\eta)I_{(\eta \in [x, \infty))}|\varepsilon\} = E\{f_{\alpha_0}(\eta)I_{(\eta \in [x, \infty))}|\varepsilon\}, \quad \text{a.s.}$$

As ε and η are independent and using the density function of η , we have

$$\varepsilon \int_x^\infty f_{\alpha_0}(u)f_\alpha(\varepsilon u)du = \int_x^\infty f_{\alpha_0}^2(u)du, \quad \text{a.s.}$$

Thus, there exists a set A with $P(A) = 0$ such that for any fixed $\tilde{\omega} \in \Omega \setminus A$,

$$\varepsilon(\tilde{\omega}) \int_x^\infty f_{\alpha_0}(u)f_\alpha(\varepsilon(\tilde{\omega})u)du = \int_x^\infty f_{\alpha_0}^2(u)du, \quad \tilde{\omega} \in \Omega \setminus A, \quad (\text{S3.14})$$

and $\varepsilon(\tilde{\omega})$ is positive and finite, i.e., $0 < \varepsilon(\tilde{\omega}) < \infty$.

By Proposition 1 (when $\beta = 0$, $\gamma = 1$), as $x \rightarrow \infty$,

$$f_\alpha(x) \sim \frac{\Gamma(\alpha + 1) \sin(\pi\alpha/2)x^{-(\alpha+1)}}{\pi},$$

i.e.,

$$\lim_{x \rightarrow \infty} \frac{f_\alpha(x)}{\Gamma(\alpha + 1) \sin(\pi\alpha/2)x^{-(\alpha+1)}/\pi} = 1.$$

For any given $\tilde{\omega} \in \Omega \setminus A$, from (S3.14), we have

$$\begin{aligned} \frac{[\Gamma(\alpha_0 + 1) \sin(\pi\alpha_0/2)][\Gamma(\alpha + 1) \sin(\pi\alpha/2)]}{\pi^2[\varepsilon(\tilde{\omega})]^\alpha(\alpha + \alpha_0 + 1)x^{\alpha+\alpha_0+1}} &\sim [\varepsilon(\tilde{\omega})] \int_x^\infty f_{\alpha_0}(u)f_\alpha([\varepsilon(\tilde{\omega})]u)du \\ &= \int_x^\infty f_{\alpha_0}^2(u)du \sim \frac{[\Gamma(\alpha_0 + 1) \sin(\pi\alpha_0/2)]^2}{\pi^2(2\alpha_0 + 1)x^{2\alpha_0+1}}. \end{aligned}$$

It follows that

$$\frac{\Gamma(\alpha_0 + 1) \sin(\pi\alpha_0/2)}{\Gamma(\alpha + 1) \sin(\pi\alpha/2)} \frac{(\alpha + \alpha_0 + 1)[\varepsilon(\tilde{\omega})]^\alpha}{2\alpha_0 + 1} \lim_{x \rightarrow \infty} x^{\alpha - \alpha_0} = 1.$$

Thus, we obtain that $\alpha = \alpha_0$, and in turn $[\varepsilon(\tilde{\omega})]^\alpha = 1$, i.e., $\varepsilon(\tilde{\omega}) = 1$, $\tilde{\omega} \in \Omega \setminus A$. Since $P(A) = 0$, we have $\varepsilon = 1$ a.s., which completes the proof.

■

Lemma 2. *Suppose Assumption 1 holds and $\gamma_{\alpha_0} < 0$, then there exists an $\iota \in (0, \alpha_0/4)$ such that $E(\sigma_t^{2\iota}) < \infty$, and $E(y_t^{2\iota}) < \infty$.*

PROOF. From model (2.2) we have

$$\sigma_t^2 = \omega_0 + [\phi_{0+} (\eta_{t-1}^+)^2 + \phi_{0-} (\eta_{t-1}^-)^2 + \psi_0] \sigma_{t-1}^2, \quad \forall t \in \mathbb{Z}.$$

By Theorem 2.4 in Bougerol and Picard (1992b), the process $\{\sigma_t^2 : t \in \mathbb{Z}\}$ has a unique strictly stationary solution, which is

$$\sigma_t^2 = \omega_0 + \sum_{j=1}^{\infty} \omega_0 \left\{ \prod_{k=1}^j [\phi_{0+} (\eta_{t-k}^+)^2 + \phi_{0-} (\eta_{t-k}^-)^2 + \psi_0] \right\}, \quad \text{a.s.,} \quad \forall t \in \mathbb{Z}.$$

Note that $E(|\eta_t|^{\alpha_0/2}) < \infty$, by the C_r -inequality and $\alpha_0/4 \in (0, 1)$, we have $E([\phi_{0+} (\eta_t^+)^2 + \phi_{0-} (\eta_t^-)^2 + \psi_0]^{\alpha_0/4}) < \infty$. By Lemma 2.2 in Francq and Zakoïan (2019), there exists an $\iota \in (0, \alpha_0/4)$ such that

$$0 \leq \rho := E([\phi_{0+} (\eta_t^+)^2 + \phi_{0-} (\eta_t^-)^2 + \psi_0]^\iota) < 1.$$

Using the C_r -inequality again and i.i.d. series $\{\eta_t\}$, it follows that

$$\begin{aligned} E(\sigma_t^{2\iota}) &\leq \omega_0^\iota + \sum_{j=1}^{\infty} \omega_0^\iota E \left\{ \prod_{k=1}^j [\phi_{0+} (\eta_{t-k}^+)^2 + \phi_{0-} (\eta_{t-k}^-)^2 + \psi_0]^\iota \right\} \\ &= \omega_0^\iota + \omega_0^\iota \sum_{j=1}^{\infty} \rho^j = \frac{\omega_0^\iota}{1-\rho} < \infty. \end{aligned}$$

Since η_t and σ_t are independent, it yields that $E(y_t^{2\iota}) = E(\sigma_t^{2\iota})E(\eta_t^{2\iota}) < \infty$.

■

Lemma 3. *Suppose Assumptions 1-2 hold and $\gamma_{\alpha_0} < 0$, if Θ satisfies $\forall \theta \in \Theta$, $\psi < 1$, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\theta \in \Theta} |L_n(\theta) - \tilde{L}_n(\theta)| = 0, \quad a.s.$$

PROOF. Since Θ is compact, and $\forall \theta \in \Theta$, $\psi < 1$, there exists a constant ρ such that $\sup_{\theta \in \Theta} \psi \leq \rho < 1$. By iteration, we have

$$\sigma_t^2(\theta) = (\omega + \phi_+(y_{t-1}^+)^2 + \phi_-(y_{t-1}^-)^2) + \cdots + \psi^{t-1}(\omega + \phi_+(y_0^+)^2 + \phi_-(y_0^-)^2) + \psi^t \sigma_0^2(\theta),$$

$$\tilde{\sigma}_t^2(\theta) = (\omega + \phi_+(y_{t-1}^+)^2 + \phi_-(y_{t-1}^-)^2) + \cdots + \psi^{t-1} \omega, \quad \forall t \geq 1.$$

Then

$$\begin{aligned} \sup_{\theta \in \Theta} |\sigma_t^2(\theta) - \tilde{\sigma}_t^2(\theta)| &= \sup_{\theta \in \Theta} |\psi^{t-1} \phi_+(y_0^+)^2 + \psi^{t-1} \phi_-(y_0^-)^2 + \psi^t \sigma_0^2(\theta)| \\ &\leq K \rho^t \{ (y_0^+)^2 + (y_0^-)^2 + \sup_{\theta \in \Theta} \sigma_0^2(\theta) \}, \quad \forall t \geq 1. \end{aligned}$$

Note that

$$\begin{aligned} \sup_{\theta \in \Theta} \sigma_0^2(\theta) &\leq \sup_{\theta \in \Theta} \sum_{j=0}^{\infty} \rho^j \{ \omega + \phi_+(y_{-j}^+)^2 + \phi_-(y_{-j}^-)^2 \} \\ &\leq \sum_{j=0}^{\infty} \rho^j \{ \bar{\omega} + \bar{\phi}_+(y_{-j}^+)^2 + \bar{\phi}_-(y_{-j}^-)^2 \}. \end{aligned}$$

Lemma 2 gives that $Ey_t^{2\iota} < \infty$, $\iota \in (0, \alpha_0/4)$. By the C_r -inequality and strict stationarity of $\{y_t\}$, we have

$$\begin{aligned} &E \left(\sum_{j=0}^{\infty} \rho^j \{ \bar{\omega} + \bar{\phi}_+(y_{-j}^+)^2 + \bar{\phi}_-(y_{-j}^-)^2 \} \right)^\iota \\ &\leq \sum_{j=0}^{\infty} E \left(\rho^{tj} \{ \bar{\omega} + \bar{\phi}_+(y_{-j}^+)^2 + \bar{\phi}_-(y_{-j}^-)^2 \}^\iota \right) \\ &\leq \sum_{j=0}^{\infty} \rho^{tj} \{ \bar{\omega}^\iota + \bar{\phi}_+^\iota E(y_{-j}^+)^{2\iota} + \bar{\phi}_-^\iota E(y_{-j}^-)^{2\iota} \} \\ &\leq \left\{ \sum_{j=0}^{\infty} \rho^{tj} \right\} \{ \bar{\omega}^\iota + \bar{\phi}_+^\iota E(y_0^+)^{2\iota} + \bar{\phi}_-^\iota E(y_0^-)^{2\iota} \} \\ &< \infty. \end{aligned}$$

Thus $\sum_{j=0}^{\infty} \rho^j \{ \bar{\omega} + \bar{\phi}_+(y_{-j}^+)^2 + \bar{\phi}_-(y_{-j}^-)^2 \} < \infty$, a.s. and $\sup_{\theta \in \Theta} \sigma_0^2(\theta) < \infty$,

a.s. It follows that

$$\sup_{\theta \in \Theta} |\sigma_t^2(\theta) - \tilde{\sigma}_t^2(\theta)| \leq K_y \rho^t, \quad \text{a.s.} \quad \forall t \geq 1, \quad (\text{S3.15})$$

where $K_y := K \{ (y_0^+)^2 + (y_0^-)^2 + \sup_{\theta \in \Theta} \sigma_0^2(\theta) \}$ is a random variable independent of index $t \geq 1$ and parameter θ . Using inequalities $\log x \leq x - 1$

and $|\log x - \log y| \leq |x - y|/\min(x, y)$, $x, y > 0$, we have

$$\begin{aligned} & \frac{1}{n} \sup_{\theta \in \Theta} \left| L_n(\theta) - \tilde{L}_n(\theta) \right| \\ & \leq \frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \Theta} \left\{ \frac{1}{2} \left| \log \frac{\tilde{\sigma}_t^2(\theta)}{\sigma_t^2(\theta)} \right| + \left| \log f_\alpha \left(\frac{y_t}{\sigma_t(\theta)} \right) - \log f_\alpha \left(\frac{y_t}{\tilde{\sigma}_t(\theta)} \right) \right| \right\} \\ & \leq \frac{K}{\underline{\omega}} \frac{K_y}{n} \sum_{t=1}^n \rho^t + \frac{K}{\underline{\omega}} \left(\sup_{x \in \mathbb{R}} \sup_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} \left| \frac{\partial \log f_\alpha(x)}{\partial x} \right| \right) \frac{K_y}{n} \sum_{t=1}^n \rho^t |y_t|. \end{aligned}$$

First, $n^{-1} \sum_{t=1}^n \rho^t \rightarrow 0$. By Markov's inequality and Lemma 2,

$$\sum_{t=1}^{\infty} P(\rho^t |y_t| > \epsilon) \leq \sum_{t=1}^{\infty} \frac{E(\rho^t |y_t|)^{2t}}{\epsilon^{2t}} = \frac{E(|y_0|^{2t})}{\epsilon^{2t}} \sum_{t=1}^{\infty} (\rho^{2t})^t < \infty, \quad \forall \epsilon > 0.$$

By the Borel-Cantelli lemma, $\rho^t |y_t| \xrightarrow{a.s.} 0$. Using Cesàro's lemma, we have

$\sum_{t=1}^n \rho^t |y_t| / n \xrightarrow{a.s.} 0$. Since $K_y < \infty$, a.s., combining (S2.5) yields that

$$\frac{1}{n} \sup_{\theta \in \Theta} \left| L_n(\theta) - \tilde{L}_n(\theta) \right| \xrightarrow{a.s.} 0. \quad \blacksquare$$

Lemma 4. *When $\gamma_{\alpha_0} > 0$, for any $\theta \in \Theta_0$, the process $v_t(\vartheta)$ is strictly stationary and ergodic. Further, for any compact $\Theta_0^* \subset \Theta_0$, there exists a constant $c_0 > 0$ such that*

$$e^{c_0 t} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right| \xrightarrow{a.s.} 0.$$

Finally, for any $\theta \notin \Theta_0$ it holds that $\sigma_t^2(\theta)/\sigma_t^2(\theta_0) \rightarrow \infty$ a.s.

PROOF. Notice that $\sigma_t^2(\theta) = \sum_{j=1}^t \psi^{j-1} (\omega + \phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2)$, and

$$\frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} = \sum_{j=1}^t \psi^{j-1} \left\{ \prod_{k=1}^j \frac{\sigma_{t-k}^2(\theta_0)}{\sigma_{t-k+1}^2(\theta_0)} \right\} \frac{\omega + \phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2}{\sigma_{t-j}^2(\theta_0)} := a_t + b_t,$$

where

$$a_t = \sum_{j=1}^t \frac{\phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2}{\sigma_{t-j+1}^2(\theta_0)} \left\{ \prod_{k=1}^{j-1} \psi \frac{\sigma_{t-k}^2(\theta_0)}{\sigma_{t-k+1}^2(\theta_0)} \right\} := \sum_{j=1}^t a_{tj},$$

$$b_t = \sum_{j=1}^t \psi^{j-1} \frac{\omega}{\sigma_t^2(\theta_0)}.$$

As $\Theta_0 = \{\theta \in \Theta : 0 < \psi < \exp(\gamma_{\alpha_0})\}$, when $\gamma_{\alpha_0} > 0$, by Cauchy's root test, it can be shown that $v_t(\vartheta)$ is finite a.s. for any $\theta \in \Theta_0$. Since $v_t(\vartheta)$ is a measurable function of i.i.d. sequence $\{\eta_j, j < t\}$, then $v_t(\vartheta)$ is strictly stationary and ergodic.

Let $\bar{\psi}_0 = \sup\{\psi \mid \theta \in \Theta_0^*\}$. By Proposition 2.1 in Francq and Zakoian (2013), for any $\rho > \exp(-\gamma_{\alpha_0})$, $\rho^t \sigma_t^2 \xrightarrow{a.s.} \infty$ as $t \rightarrow \infty$. Choose $\rho_0 = (\gamma_{\alpha_0} - \log \bar{\psi}_0)/2$, then $\rho_0 > 0$, and $\exp(\rho_0) \bar{\psi}_0 < \exp(\gamma_{\alpha_0})$. Thus, we have $e^{\rho_0 t} \sup_{\theta \in \Theta_0^*} b_t \leq K \bar{\omega} \{e^{\rho_0} \bar{\psi}_0\}^t / \sigma_t^2(\theta_0) \xrightarrow{a.s.} 0$.

On the other hand, notice that

$$\frac{\sigma_{t-k}^2(\theta_0)}{\sigma_{t-k+1}^2(\theta_0)} = \frac{\sigma_{t-k}^2(\theta_0)}{\omega_0 + a_0(\eta_{t-k})\sigma_{t-k}^2(\theta_0)} \leq \frac{1}{a_0(\eta_{t-k})},$$

$$0 \leq \frac{1}{a_0(\eta_{t-k})} - \frac{\sigma_{t-k}^2(\theta_0)}{\sigma_{t-k+1}^2(\theta_0)} \leq \frac{\omega_0}{a_0^2(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)}.$$

A simple calculation yields that

$$0 \leq \prod_{k=1}^j \frac{1}{a_0(\eta_{t-k})} - \prod_{k=1}^j \frac{\sigma_{t-k}^2(\theta_0)}{\sigma_{t-k+1}^2(\theta_0)} \leq \left[\prod_{i=1}^j \frac{1}{a_0(\eta_{t-i})} \right] \sum_{k=1}^j \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)}.$$

Then,

$$0 \leq v_t(\vartheta) - a_t \leq \sum_{j=1}^t (v_{tj} - a_{tj}) + \sum_{j=t+1}^{\infty} v_{tj},$$

where $v_t(\vartheta) := \sum_{j=1}^{\infty} v_{tj}$. For the second term, clearly,

$$\begin{aligned} e^{\rho_0 t} \sup_{\theta \in \Theta_0^*} \sum_{j=t+1}^{\infty} v_{tj} &\leq e^{\rho_0 t} \sum_{j=t+1}^{\infty} \frac{\{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2\}}{a_0(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\ &= e^{\rho_0 t} \sum_{i=1}^{\infty} \frac{\{\bar{\phi}_+(\eta_{-i}^+)^2 + \bar{\phi}_-(\eta_{-i}^-)^2\}}{a_0(\eta_{-i})} \prod_{k=1}^{t+i-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\ &= e^{\rho_0 t} \left\{ \prod_{k=0}^{t-1} \frac{\bar{\psi}_0}{a_0(\eta_k)} \right\} \left\{ \sum_{i=1}^{\infty} \frac{\{\bar{\phi}_+(\eta_{-i}^+)^2 + \bar{\phi}_-(\eta_{-i}^-)^2\}}{a_0(\eta_{-i})} \prod_{k=1}^{i-1} \frac{\bar{\psi}_0}{a_0(\eta_{-k})} \right\}. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} \left\{ -\rho_0 - \log \bar{\psi}_0 + 1/t \sum_{k=0}^{t-1} \log a_0(\eta_k) \right\} = -\rho_0 - \log \bar{\psi}_0 + \gamma_{\alpha_0} > 0$,

a.s., we have that $e^{\rho_0 t} \left\{ \prod_{k=0}^{t-1} \bar{\psi}_0 / a_0(\eta_k) \right\} \xrightarrow{a.s.} 0$. By Cauchy's root test, it

follows that

$$\sum_{i=1}^{\infty} \frac{\bar{\phi}_+(\eta_{-i}^+)^2 + \bar{\phi}_-(\eta_{-i}^-)^2}{a_0(\eta_{-i})} \prod_{k=1}^{i-1} \frac{\bar{\psi}_0}{a_0(\eta_{-k})} < \infty \quad \text{a.s.}$$

Thus, $e^{\rho_0 t} \sup_{\theta \in \Theta_0^*} \sum_{j=t+1}^{\infty} v_{tj} \xrightarrow{a.s.} 0$ as $t \rightarrow \infty$.

It remains to show that $e^{c_0 t} \sup_{\theta \in \Theta_0^*} \sum_{j=1}^t (v_{tj} - a_{tj}) \xrightarrow{a.s.} 0$. Note that

$$\begin{aligned} 0 &\leq \sup_{\theta \in \Theta_0^*} \sum_{j=1}^t (v_{tj} - a_{tj}) \\ &\leq \sum_{j=1}^t \frac{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} \left[\prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \sum_{k=1}^{j-1} \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)} \\ &\quad + \sum_{j=1}^t \frac{(\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2) \omega_0}{a_0^2(\eta_{t-j}) \sigma_{t-j}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\ &:= I_1 + I_2. \end{aligned}$$

For I_1 , exchanging summation yields that

$$\begin{aligned}
I_1 &= \sum_{k=1}^{t-1} \sum_{j=k+1}^t \frac{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} \left[\prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)} \\
&= \left\{ \sum_{k=1}^{\lfloor t/2 \rfloor} + \sum_{k=\lfloor t/2 \rfloor+1}^{t-1} \right\} \left[\sum_{j=k+1}^t \frac{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} \prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)} \\
&:= I_{1,1} + I_{1,2}.
\end{aligned}$$

where $\lfloor x \rfloor$ is the floor function of x . For $I_{1,1}$, note that $\sigma_t^2(\theta_0) \geq a_0(\eta_{t-1})\sigma_{t-1}^2(\theta_0)$,

then

$$a_0(\eta_{t-k})\sigma_{t-k}^2(\theta_0) \geq \sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0) \prod_{i=k}^{t-\lfloor t/2 \rfloor} a_0(\eta_{t-i}), \quad \text{when } k < \lfloor t/2 \rfloor.$$

Choose $\rho_1 = -2(\log \sqrt{\rho})/3$, where $\rho \in (\exp(-\gamma_{\alpha_0}), 1)$, then $\rho^t \sigma_t^2 \xrightarrow{a.s.} \infty$ as

$t \rightarrow \infty$, and

$$\begin{aligned}
e^{\rho_1 t} I_{1,1} &\leq e^{\rho_1 t} \sum_{k=1}^{\lfloor t/2 \rfloor} \left[\sum_{j=k+1}^{\infty} \frac{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} \prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)} \\
&= e^{\rho_1 t} v_t(\bar{\vartheta}) \sum_{k=1}^{\lfloor t/2 \rfloor} \frac{\omega_0}{a_0(\eta_{t-k})} \frac{1}{\sigma_{t-k}^2(\theta_0)} \\
&\leq \omega_0 e^{\rho_1 t} v_t(\bar{\vartheta}) \left\{ \sum_{k=1}^{\lfloor t/2 \rfloor} \prod_{i=k}^{t-\lfloor t/2 \rfloor} \frac{1}{a_0(\eta_{t-i})} \right\} \frac{1}{\sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0)} \\
&= \omega_0 e^{-\frac{\rho_1}{2} t} v_t(\bar{\vartheta}) \left\{ \sum_{k=1}^{\lfloor t/2 \rfloor} \prod_{i=\lfloor t/2 \rfloor}^{t-k} \frac{1}{a_0(\eta_i)} \right\} \frac{\rho^{t-\lfloor t/2 \rfloor}}{\rho^{t/2}} \frac{1}{\rho^{t-\lfloor t/2 \rfloor} \sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0)}.
\end{aligned}$$

By Cauchy's root test and the C_r -inequality, it holds that $\sum_{k=1}^{\lfloor t/2 \rfloor} \prod_{i=\lfloor t/2 \rfloor}^{t-k} \frac{1}{a_0(\eta_i)}$

and $v_t(\bar{\vartheta})$ converge a.s., and have fractional moments. Thus, by the fact

$\rho^{t-\lfloor t/2 \rfloor} / \rho^{t/2} \leq 1$, the Markov inequality, and the Borel-Cantelli lemma, it

follows that $e^{\rho_1 t} I_{1,1} \xrightarrow{a.s.} 0$.

For $I_{1,2}$, since $\sigma_{t-k}^2(\theta_0) \geq \omega_0$, we have

$$\begin{aligned}
I_{1,2} &\leq \sum_{k=\lfloor t/2 \rfloor + 1}^{t-1} \left[\sum_{j=k+1}^t \frac{\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} \prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \frac{1}{a_0(\eta_{t-k})} \\
&\leq \frac{1}{\bar{\psi}_0} \left(\frac{\bar{\phi}_+}{\bar{\phi}_{0+}} + \frac{\bar{\phi}_-}{\bar{\phi}_{0-}} \right) \sum_{k=\lfloor t/2 \rfloor + 1}^{t-1} \sum_{j=k+1}^t \left[\prod_{i=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \\
&\leq K \left\{ \prod_{i=1}^{\lfloor t/2 \rfloor} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right\} \sum_{k=\lfloor t/2 \rfloor + 1}^{t-1} \sum_{j=k+1}^t \left[\prod_{i=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \\
&\leq K \frac{t}{2} \left\{ \prod_{i=1}^{\lfloor t/2 \rfloor} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right\} \left[\sum_{j=\lfloor t/2 \rfloor + 2}^t \prod_{i=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right].
\end{aligned}$$

By Cauchy's root test and the C_r -inequality, $\sum_{j=\lfloor t/2 \rfloor + 2}^t \prod_{i=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})}$ converges a.s. and has fractional moments. Moreover, choose $\rho_2 := (\gamma_{\alpha_0} - \log \bar{\psi}_0)/4$, then

$$\begin{aligned}
e^{\rho_2 t} I_{1,2} &\leq e^{\rho_2 t} K \frac{t}{2} \left\{ \prod_{i=1}^{\lfloor t/2 \rfloor} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right\} \left[\sum_{j=\lfloor t/2 \rfloor + 2}^t \prod_{i=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right] \\
&\leq K \exp \left\{ \frac{3}{2} \rho_2 t + \lfloor t/2 \rfloor \log \bar{\psi}_0 - \sum_{i=1}^{\lfloor t/2 \rfloor} \log a_0(\eta_{t-i}) + \log(t/2) \right\} \\
&\quad \times e^{-\frac{\rho_2}{2} t} \left[\sum_{j=\lfloor t/2 \rfloor + 2}^t \prod_{i=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-i})} \right].
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \left\{ -3\rho_2 - \log \bar{\psi}_0 + \frac{1}{\lfloor t/2 \rfloor} \sum_{i=1}^{\lfloor t/2 \rfloor} \log a_0(\eta_{t-i}) - \frac{\log(t/2)}{\lfloor t/2 \rfloor} \right\} \\
&= -3\rho_2 - \log \bar{\psi}_0 + \gamma_{\alpha_0} > 0, \quad \text{a.s.},
\end{aligned}$$

by the strong law of large numbers, Markov's inequality, and the Borel-Cantelli lemma, it yields that $e^{\rho_1 t} I_{1,2} \xrightarrow{a.s.} 0$.

For I_2 , similarly, we have

$$\begin{aligned}
I_2 &\leq \sum_{j=1}^t \frac{(\bar{\phi}_+(\eta_{t-j}^+)^2 + \bar{\phi}_-(\eta_{t-j}^-)^2) \omega_0}{a_0^2(\eta_{t-j}) \sigma_{t-j}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\
&\leq \left(\frac{\bar{\phi}_+}{\phi_{0+}} + \frac{\bar{\phi}_-}{\phi_{0-}} \right) \sum_{j=1}^t \frac{\omega_0}{a_0(\eta_{t-j}) \sigma_{t-j}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\
&\leq K \left\{ \sum_{j=1}^{\lfloor t/2 \rfloor} + \sum_{j=\lfloor t/2 \rfloor+1}^{t-1} \right\} \frac{\omega_0}{a_0(\eta_{t-j}) \sigma_{t-j}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\
&:= I_{2,1} + I_{2,2}.
\end{aligned}$$

For $I_{2,1}$, by $\sigma_t^2(\theta_0) \geq a_0(\eta_{t-1}) \sigma_{t-1}^2(\theta_0)$, it follows that

$$\begin{aligned}
e^{\rho_1 t} I_{2,1} &\leq e^{\rho_1 t} K \sum_{j=1}^{\lfloor t/2 \rfloor} \frac{\omega_0}{a_0(\eta_{t-j}) \sigma_{t-j}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\
&\leq e^{\rho_1 t} \omega_0 K \sum_{j=1}^{\lfloor t/2 \rfloor} \prod_{i=j}^{t-\lfloor t/2 \rfloor} \frac{1}{a_0(\eta_{t-i})} \frac{1}{\sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0)} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\
&\leq e^{\rho_1 t} \omega_0 K \left\{ \sum_{j=1}^{\lfloor t/2 \rfloor} \prod_{i=1}^{t-\lfloor t/2 \rfloor} \frac{\bar{\psi}_0^{j-1}}{a_0(\eta_{t-i})} \right\} \frac{1}{\sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0)} \\
&\leq e^{-\frac{\rho_1}{2} t} \omega_0 K \left\{ \sum_{j=1}^{\lfloor t/2 \rfloor} \prod_{i=1}^{t-\lfloor t/2 \rfloor} \frac{\bar{\psi}_0^{j-1}}{a_0(\eta_{t-i})} \right\} \frac{\rho^{t-\lfloor t/2 \rfloor}}{\rho^{t/2}} \frac{1}{\rho^{t-\lfloor t/2 \rfloor} \sigma_{t-\lfloor t/2 \rfloor}^2(\theta_0)} \xrightarrow{a.s.} 0.
\end{aligned}$$

For $I_{2,2}$, since $\sigma_{t-k}^2(\theta_0) \geq \omega_0$, we have

$$\begin{aligned} e^{\rho_2 t} I_{2,2} &\leq e^{\rho_2 t} K \sum_{j=\lfloor t/2 \rfloor + 1}^{t-1} \frac{1}{a_0(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\ &\leq e^{\rho_2 t} K \frac{1}{\psi_0} \sum_{j=\lfloor t/2 \rfloor + 1}^{t-1} \prod_{k=1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \\ &\leq e^{\frac{3}{2}\rho_2 t} K \frac{1}{\psi_0} \prod_{k=1}^{\lfloor t/2 \rfloor} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \left\{ e^{-\frac{\rho_2}{2} t} \sum_{j=\lfloor t/2 \rfloor + 1}^{t-1} \prod_{k=\lfloor t/2 \rfloor + 1}^{j-1} \frac{\bar{\psi}_0}{a_0(\eta_{t-k})} \right\} \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

by the fact

$$\lim_{t \rightarrow \infty} \left\{ -3\rho_2 - \log \bar{\psi}_0 + \frac{1}{\lfloor t/2 \rfloor} \sum_{i=1}^{\lfloor t/2 \rfloor} \log a_0(\eta_{t-i}) \right\} = -3\rho_2 - \log \bar{\psi}_0 + \gamma_{\alpha_0} > 0, \quad \text{a.s.}$$

To sum up, choose $c_0 := \min(\rho_0, \rho_1, \rho_2) = (\gamma_{\alpha_0} - \max(\log \bar{\psi}_0, 0))/4 > 0$, it follows that

$$e^{c_0 t} \sup_{\theta \in \Theta_0^*} \left| \frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} - v_t(\vartheta) \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as } t \rightarrow \infty.$$

Finally, for $\theta \notin \Theta_0$, $\forall t_0 < t$, $\rho > e^{-\gamma_{\alpha_0}}$, as $t \rightarrow \infty$,

$$\frac{\sigma_t^2(\theta)}{\sigma_t^2(\theta_0)} \geq \sum_{j=1}^{t_0} a_{tj} = \sum_{j=1}^{t_0} v_{tj} + o(\rho^t t_0^2) \quad \text{a.s.}$$

By Cauchy's root test for the case $\psi > e^{\gamma_{\alpha_0}}$, and the Chung-Fuchs Theorem for the case $\psi = e^{\gamma_{\alpha_0}}$, we can obtain that $\sum_{j=1}^{t_0} v_{tj} \xrightarrow{\text{a.s.}} \infty$, as $t_0 \rightarrow \infty$. The proof is complete. ■

Lemma 5. *If $\theta \in \Theta_0$, then $v_t(\vartheta) = 1$, a.s. if and only if $(\phi_+, \phi_-, \psi) = (\phi_{0+}, \phi_{0-}, \psi_0)$.*

PROOF. From the definition of $v_t(\vartheta)$, it follows that

$$v_t(\vartheta)a_0(\eta_{t-1}) = \psi v_{t-1}(\vartheta) + \phi_+(\eta_{t-1}^+)^2 + \phi_-(\eta_{t-1}^-)^2.$$

Then,

$$\{v_t(\vartheta) - 1\}a_0(\eta_{t-1}) = \psi v_{t-1}(\vartheta) - \psi_0 + (\phi_+ - \phi_{0+})(\eta_{t-1}^+)^2 + (\phi_- - \phi_{0-})(\eta_{t-1}^-)^2.$$

By the strict stationarity of $v_t(\vartheta)$, it follows that $v_t(\vartheta) = 1$ a.s. *if and only*

if

$$\psi - \psi_0 + (\phi_+ - \phi_{0+})(\eta_{t-1}^+)^2 + (\phi_- - \phi_{0-})(\eta_{t-1}^-)^2 = 0.$$

By the randomness of η_t , then $(\phi_+, \phi_-, \psi)' = (\phi_{0+}, \phi_{0-}, \psi_0)'$. ■

Let $\check{\Theta}$ be the compact set of ϑ 's such that $(\omega, \vartheta)' \in \Theta$.

Lemma 6. *For any $m \in \mathbb{Z}_+$,*

$$E \sup_{\theta \in \Theta} \left(\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \right)^m < \infty \quad \text{and} \quad E \sup_{\vartheta \in \check{\Theta}} \left(\frac{1}{v_t(\vartheta)} \right)^m < \infty.$$

PROOF. Let $\epsilon > 0$ such that $p(\epsilon) := P(|\eta_t| \leq \epsilon) \in [0, 1)$. If $|\eta_{t-1}| > \epsilon$, we

have

$$\begin{aligned} \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} &\leq \frac{\omega_0 + a_0(\eta_{t-1})\sigma_{t-1}^2(\theta_0)}{\omega + \phi_+\sigma_{t-1}^2(\theta)(\eta_{t-1}^+)^2 + \phi_-\sigma_{t-1}^2(\theta)(\eta_{t-1}^-)^2 + \psi\sigma_{t-1}^2(\theta)} \\ &\leq \frac{\omega_0}{\underline{\omega}} + \frac{\bar{\phi}}{\underline{\phi}} + \frac{\psi_0}{\underline{\phi}\epsilon^2} := K(\epsilon). \end{aligned}$$

Similarly, if $|\eta_{t-1}| \leq \epsilon$ and $|\eta_{t-2}| > \epsilon$, then

$$\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \leq \frac{\omega_0}{\underline{\omega}} + \frac{a_0(\epsilon)}{\underline{\psi}} K(\epsilon).$$

By iteration, we can obtain that

$$\sup_{\theta \in \Theta} \frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \leq V_t,$$

where

$$V_t := \sum_{i=1}^{\infty} I_{|\eta_{t-1}| \leq \epsilon} \cdots I_{|\eta_{t-i+1}| \leq \epsilon} I_{|\eta_{t-i}| > \epsilon} \left\{ \frac{\omega_0}{\underline{\omega}} \sum_{j=0}^{i-2} \left(\frac{a_0(\epsilon)}{\underline{\psi}} \right)^j + \left(\frac{a_0(\epsilon)}{\underline{\psi}} \right)^{i-1} K(\epsilon) \right\}. \quad (\text{S3.16})$$

For any $m \in \mathbb{Z}_+$, it follows that

$$\begin{aligned} E \sup_{\theta \in \Theta} \left(\frac{\sigma_t^2(\theta_0)}{\sigma_t^2(\theta)} \right)^m &\leq \{1 - p(\epsilon)\} \sum_{i=1}^{\infty} p(\epsilon)^{i-1} \left\{ \frac{\omega_0}{\underline{\omega}} \sum_{j=0}^{i-2} \left(\frac{a_0(\epsilon)}{\underline{\psi}} \right)^j + \left(\frac{a_0(\epsilon)}{\underline{\psi}} \right)^{i-1} K(\epsilon) \right\}^m \\ &\leq \{K(\epsilon) + K\}^m \{1 - p(\epsilon)\} \sum_{i=1}^{\infty} p(\epsilon)^{i-1} \left(\frac{a_0(\epsilon)}{\underline{\psi}} \right)^{m(i-1)}. \end{aligned}$$

Since $\lim_{\epsilon \downarrow 0} p(\epsilon) = 0$ and $\lim_{\epsilon \downarrow 0} a_0(\epsilon) = \psi_0$, we have $p(\epsilon)(a_0(\epsilon)/\underline{\psi})^m < 1$ for ϵ small enough. Thus, the first result holds. The proof of the second one is similar to that of Lemma A.3 in Francq and Zakoian (2012), and is thus omitted. ■

Lemma 7. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} < 0$, then*

$$\frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} \xrightarrow{d} \mathcal{N}(0, \Sigma), \quad \text{as } n \rightarrow \infty.$$

PROOF. Let $\tilde{\theta} = (\omega, \phi_+, \phi_-, \psi)'$. By the expression of $\partial \ell_t(\theta)/\partial \theta$, it follows

that

$$\frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}} = -\frac{1}{2\sigma_t^2} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}, \quad \frac{\partial \ell_t(\theta_0)}{\partial \alpha} = \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}. \quad (\text{S3.17})$$

By Proposition 3, $\partial \log f_{\alpha_0}(x)/\partial x$, $x \partial \log f_{\alpha_0}(x)/\partial x$ are bounded and $\partial \log f_{\alpha_0}(x)/\partial \alpha \asymp -\log|x|$ as $|x| \rightarrow \infty$. Thus, by the fact that $E|\eta_t|^{\alpha_0/2} < \infty$, we have $E\left|\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x}\right| < \infty$, $E\left|\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}\right| < \infty$, and $E\left|\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t\right| < \infty$. Further, since $f_{\alpha_0}(x)$ and $\partial f_{\alpha_0}(x)/\partial \alpha$ are even, and $\partial f_{\alpha_0}(x)/\partial x$ is odd on \mathbb{R} , and $f_{\alpha_0}(x) \asymp |x|^{-(1+\alpha_0)}$ as $|x| \rightarrow \infty$, a simple algebraic calculation yields that

$$E\left\{\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x}\right\} = E\left\{\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}\right\} = 0, \quad E\left\{\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t\right\} = -1. \quad (\text{S3.18})$$

By Lemma 2, there exists $0 < \iota < \alpha_0/4$ such that $E(y_t^{2\iota} + \sigma_t^{2\iota}) < \infty$. Similar to the proof of (4.14) in Francq and Zakoian (2004), we have $E\left\|\frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}}\right\| < \infty$, which implies $E\left\|\frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}}\right\| < \infty$. Similarly, it can be shown that $E\left\|\frac{\partial \ell_t(\theta_0)}{\partial \theta} \frac{\partial \ell_t(\theta_0)}{\partial \theta'}\right\| < \infty$. By (S3.18) and the independence of η_t and σ_t^2 , $\{\partial \ell_t(\theta_0)/\partial \theta\}$ is a martingale difference sequence. Thus,

$$\frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \theta} \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

by the martingale central limit theorem in Brown (1971), where

$$\Sigma = E\left\{\frac{\partial \ell_t(\theta_0)}{\partial \theta} \frac{\partial \ell_t(\theta_0)}{\partial \theta'}\right\} = (\Sigma_{\theta_i \theta_j})_{5 \times 5}, \quad \theta_i, \theta_j \in \{\omega, \phi_+, \phi_-, \psi, \alpha\},$$

with

$$\begin{aligned}\Sigma_{\tilde{\theta}\tilde{\theta}'} &= \frac{1}{4}E\left\{\frac{1}{\sigma_t^4(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\tilde{\theta}}\frac{\partial\sigma_t^2(\theta_0)}{\partial\tilde{\theta}'}\right\}E\left[\left\{1+\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial x}\eta_t\right\}^2\right], \\ \Sigma_{\tilde{\theta}\alpha} &= -\frac{1}{2}E\left\{\frac{1}{\sigma_t^2(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\tilde{\theta}}\right\}E\left\{\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial x}\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial\alpha}\eta_t\right\}, \\ \Sigma_{\alpha\alpha} &= E\left[\left\{\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial\alpha}\right\}^2\right].\end{aligned}\quad \blacksquare$$

Lemma 8. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} < 0$, then Σ is positive definite.*

PROOF. First, it is easy to see that Σ is a positive semi-definite matrix.

For any $\mathbf{u} = (u_1, \dots, u_5)' \in \mathbb{R}^5$, suppose $\mathbf{u}'\Sigma\mathbf{u} = 0$. That is,

$$E\left\{\mathbf{u}'\frac{\partial\ell_t(\theta_0)}{\partial\theta}\frac{\partial\ell_t(\theta_0)}{\partial\theta'}\mathbf{u}\right\} = 0.$$

Then, $\mathbf{u}'\partial\ell_t(\theta_0)/\partial\theta = 0$ a.s., which is equivalent to

$$u_1\partial\ell_t(\theta_0)/\partial\omega + u_2\partial\ell_t(\theta_0)/\partial\phi_+ + u_3\partial\ell_t(\theta_0)/\partial\phi_- + u_4\partial\ell_t(\theta_0)/\partial\psi + u_5\partial\ell_t(\theta_0)/\partial\alpha = 0 \quad \text{a.s.}$$

Thus,

$$(u_1, \dots, u_4)\frac{1}{2\sigma_t^2(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\tilde{\theta}}\left\{1+\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial x}\eta_t\right\} - u_5\frac{\partial\log f_{\alpha_0}(\eta_t)}{\partial\alpha} = 0, \quad \text{a.s..}$$

Due to the independence of η_t and y_{t-1} , there must be $u_5 = 0$, and in turn

$$\begin{aligned}0 &= (u_1, \dots, u_4)\frac{\partial\sigma_t^2(\theta_0)}{\partial\tilde{\theta}} = (u_1, \dots, u_4)\begin{pmatrix} 1 \\ (y_{t-1}^+)^2 \\ (y_{t-1}^-)^2 \\ \sigma_{t-1}^2(\theta_0) \end{pmatrix} + (u_1, \dots, u_4)\frac{\partial\sigma_{t-1}^2(\theta_0)}{\partial\tilde{\theta}} \\ &= u_1 + u_2(y_{t-1}^+)^2 + u_3(y_{t-1}^-)^2 + u_4\sigma_{t-1}^2, \quad \text{a.s..}\end{aligned}$$

The stationarity of $\{\partial\sigma_t^2(\theta_0)/\partial\tilde{\theta}\}$ implies that $u_2 = u_3 = 0$, otherwise y_{t-1}^2 would be measurable with respect to the σ -field generated by $\{\eta_i, i < t-1\}$. By the randomness of σ_{t-1}^2 , we have $u_4 = 0$ and in turn $u_1 = 0$. Thus, $\mathbf{u} = \mathbf{0}$ and then Σ is positive definite. \blacksquare

Lemma 9. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} < 0$, then, for any $\epsilon > 0$,*

- (i). $E \left\{ \sup_{\|\theta - \theta_0\| < \epsilon} \left\| \frac{1}{n} \frac{\partial^2 L_n(\theta)}{\partial\theta\partial\theta'} - \frac{1}{n} \frac{\partial^2 L_n(\theta_0)}{\partial\theta\partial\theta'} \right\| \right\} = O(\epsilon).$
- (ii). $\frac{1}{n} \frac{\partial^2 L_n(\theta_0)}{\partial\theta\partial\theta'} \rightarrow -\Sigma, \quad a.s..$

PROOF. (i). By the explicit expressions of $\partial^2 \ell_t(\theta)/\partial\theta\partial\theta'$ in the Appendix S3, Proposition 3, and the mean value theorem, the proof can be completed by simple algebraic calculations and is thus omitted. \square

(ii). By the strong law of large numbers for strictly stationary and ergodic sequences, we have

$$\frac{1}{n} \frac{\partial^2 L_n(\theta_0)}{\partial\theta\partial\theta'} \xrightarrow{a.s.} E \left(\frac{\partial^2 \ell_t(\theta_0)}{\partial\theta\partial\theta'} \right).$$

Since $f_{\alpha_0}(x) \sim |x|^{-(1+\alpha_0)}$ as $|x| \rightarrow \infty$, a simple calculation gives that

$$\begin{aligned} E \left\{ \eta_t^2 \frac{\partial^2 \log f_{\alpha_0}(\eta_t)}{\partial x^2} + \eta_t^2 \left(\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \right)^2 \right\} &= 2, \\ E \left\{ \eta_t \frac{\partial^2 \log f_{\alpha_0}(\eta_t)}{\partial x \partial \alpha} + \eta_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \right\} &= 0, \\ E \left\{ \frac{\partial^2 \log f_{\alpha_0}(\eta_t)}{\partial \alpha^2} + \left(\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \right)^2 \right\} &= 0. \end{aligned} \tag{S3.19}$$

In view of (S3.18), Lemma 7, and the explicit expressions of $\partial^2 \ell_t(\theta)/\partial\theta\partial\theta'$,

it follows that

$$E\left(\frac{\partial^2 \ell_t(\theta_0)}{\partial \theta \partial \theta'}\right) = -E\left(\frac{\partial \ell_t(\theta_0)}{\partial \theta} \frac{\partial \ell_t(\theta_0)}{\partial \theta'}\right) = -\Sigma.$$

Thus (ii) holds. ■

Lemma 10. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} < 0$, then, as $n \rightarrow \infty$,*

$$\begin{aligned} \text{(i).} \quad & \left\| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ \frac{\partial \ell_t(\theta_0)}{\partial \theta} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \theta} \right\} \right\| \xrightarrow{p} 0. \\ \text{(ii).} \quad & \sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{t=1}^n \left\{ \frac{\partial^2 \ell_t(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \tilde{\ell}_t(\theta)}{\partial \theta \partial \theta'} \right\} \right\| \xrightarrow{a.s.} 0, \end{aligned}$$

PROOF. First, taking the first-order partial derivatives of $\tilde{\sigma}_t^2(\theta)$ with respect to θ gives

$$\begin{aligned} \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \omega} &= \sum_{k=0}^{t-1} \psi^k; \\ \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \phi_+} &= \sum_{k=0}^{t-2} \psi^k (y_{t-k-1}^+)^2; \\ \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \phi_-} &= \sum_{k=0}^{t-2} \psi^k (y_{t-k-1}^-)^2; \\ \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \psi} &= \sum_{k=1}^{t-2} k \psi^{k-1} (\omega + \phi_+ (y_{t-k-1}^+)^2 + \phi_- (y_{t-k-1}^-)^2) + (t-1) \psi^{t-2} \omega. \end{aligned}$$

The second-order partial derivatives of $\tilde{\sigma}_t^2(\theta)$ and $\sigma_t^2(\theta)$ with respect to θ can be derived similarly. Since Θ is compact and $\forall \theta \in \Theta, \psi < 1$, similar to

Lemma 3, we have, $\forall t \geq 1$,

$$\sup_{\theta \in \Theta} \left\| \frac{\partial \sigma_t^2(\theta)}{\partial \theta} - \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \theta} \right\| < K_y \rho^t, \quad \sup_{\theta \in \Theta} \left\| \frac{\partial^2 \sigma_t^2(\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \tilde{\sigma}_t^2(\theta)}{\partial \theta \partial \theta'} \right\| < K_y \rho^t, \quad a.s.,$$

(S3.20)

where K_y is some random variable independent of index $t \geq 1$ and parameter θ . Moreover, from (S3.15) we have

$$\left| \frac{1}{\sigma_t^2(\theta_0)} - \frac{1}{\tilde{\sigma}_t^2(\theta_0)} \right| = \left| \frac{\tilde{\sigma}_t^2(\theta_0) - \sigma_t^2(\theta_0)}{\sigma_t^2(\theta_0)\tilde{\sigma}_t^2(\theta_0)} \right| \leq \frac{K_y \rho^t}{\sigma_t^2(\theta_0)}, \quad \frac{\sigma_t^2(\theta_0)}{\tilde{\sigma}_t^2(\theta_0)} \leq 1 + K_y \rho^t.$$

(S3.21)

Since

$$\begin{aligned} \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}} &= -\frac{1}{2\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \left\{ 1 + \eta_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \right\}, & \frac{\partial \ell_t(\theta_0)}{\partial \alpha} &= \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}, \\ \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \tilde{\theta}} &= -\frac{1}{2\tilde{\sigma}_t^2(\theta_0)} \frac{\partial \tilde{\sigma}_t^2(\theta_0)}{\partial \tilde{\theta}} \left\{ 1 + \tilde{q}_t(\theta_0) \frac{\partial \log f_{\alpha_0}(\tilde{q}_t(\theta_0))}{\partial x} \right\}, & \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \alpha} &= \frac{\partial \log f_{\alpha_0}(\tilde{q}_t(\theta_0))}{\partial \alpha}, \end{aligned}$$

where $\tilde{q}_t(\theta_0) = y_t/\tilde{\sigma}_t(\theta_0)$, it follows that

$$\begin{aligned} \left| \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \tilde{\theta}} \right| &= \left| \frac{1}{2} \left(\frac{1}{\sigma_t^2(\theta_0)} - \frac{1}{\tilde{\sigma}_t^2(\theta_0)} \right) \frac{\partial \tilde{\sigma}_t^2(\theta_0)}{\partial \tilde{\theta}} \left\{ 1 + \tilde{q}_t(\theta_0) \frac{\partial \log f_{\alpha_0}(\tilde{q}_t(\theta_0))}{\partial x} \right\} \right. \\ &\quad + \frac{1}{2} \frac{1}{\sigma_t^2(\theta_0)} \left(\frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} - \frac{\partial \tilde{\sigma}_t^2(\theta_0)}{\partial \tilde{\theta}} \right) \left\{ 1 + \tilde{q}_t(\theta_0) \frac{\partial \log f_{\alpha_0}(\tilde{q}_t(\theta_0))}{\partial x} \right\} \\ &\quad \left. + \frac{1}{2} \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \left\{ \eta_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} - \tilde{q}_t(\theta_0) \frac{\partial \log f_{\alpha_0}(\tilde{q}_t(\theta_0))}{\partial x} \right\} \right| \\ &\leq K_y \rho^t \sup_{x \in \mathbb{R}} \left| x \frac{\partial \log f_{\alpha_0}(x)}{\partial x} \right| \left| 1 + \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \right| \\ &\quad + K_y \rho^t \sup_{x \in \mathbb{R}} \left| \frac{\partial \log f_{\alpha_0}(x)}{\partial x} \right| \left| \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \eta_t \right| \end{aligned}$$

by (S3.15), (S3.20), (S3.21), and the Lagrange mean value theorem. Thus,

$$\begin{aligned} & \left| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}_i} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \tilde{\theta}_i} \right\} \right| \\ & \leq K_y \sup_{x \in \mathbb{R}} \left| x \frac{\partial \log f_{\alpha_0}(x)}{\partial x} \right| \frac{1}{\sqrt{n}} \sum_{t=1}^n \rho^t \left\{ \left| 1 + \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \right| + \left| \frac{1}{\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\theta}} \eta_t \right| \right\}. \end{aligned}$$

By Markov's inequality, the independence of η_t and $\sigma_t^2(\theta_0)$, $E|\eta_t|^{\alpha_0/2} < \infty$,

and Proposition 3, it follows that

$$\left| \frac{1}{\sqrt{n}} \sum_{t=1}^n \left\{ \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\theta}_i} - \frac{\partial \tilde{\ell}_t(\theta_0)}{\partial \tilde{\theta}_i} \right\} \right| \xrightarrow{p} 0.$$

Thus, (i) holds. (ii) can be similarly proved by (S3.20), Hölder's inequality,

Markov's inequality, and Borel-Cantelli Lemma, and is thus omitted. \blacksquare

Let $\tilde{\vartheta} = (\phi_+, \phi_-, \psi)'$, $\tilde{\vartheta}_0 = (\phi_{0+}, \phi_{0-}, \psi_0)'$, $a(\eta_t) = \phi_+(\eta_t^+)^2 + \phi_-(\eta_t^-)^2 + \psi$. We now introduce three new $[0, \infty]$ -valued processes:

$$\begin{aligned} d_t^{\phi_+}(\tilde{\vartheta}) &= \sum_{j=1}^{\infty} \frac{(\eta_{t-j}^+)^2}{a(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a(\eta_{t-k})}, & d_t^{\phi_-}(\tilde{\vartheta}) &= \sum_{j=1}^{\infty} \frac{(\eta_{t-j}^-)^2}{a(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a(\eta_{t-k})}, \\ d_t^{\psi}(\tilde{\vartheta}) &= \sum_{j=2}^{\infty} \frac{(j-1) \{ \phi_+(\eta_{t-j}^+)^2 + \phi_-(\eta_{t-j}^-)^2 \}}{\psi a(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a(\eta_{t-k})}. \end{aligned}$$

Further, let $q_t(\theta) = y_t/\sigma_t(\theta)$.

Lemma 11. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} > 0$, then*

$$\frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \tilde{\vartheta}} \xrightarrow{d} \mathcal{N}(0, \Upsilon).$$

Moreover, Υ is positive definite.

PROOF. Since $E\{\log[\psi/a(\eta_1)]\} < 0$, the processes $d_t^{\phi+}$, $d_t^{\phi-}$, and d_t^ψ are strictly stationary and ergodic by the Cauchy root test. Further, we have

$$\begin{aligned}\frac{\partial\sigma_t^2(\theta)}{\partial(\phi_+, \phi_-)} &= \sum_{j=1}^t \psi^{j-1} ((y_{t-j}^+)^2, (y_{t-j}^-)^2), \\ \frac{\partial\sigma_t^2(\theta)}{\partial\psi} &= \sum_{j=2}^t (j-1)\psi^{j-2} \{\omega + \phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2\}.\end{aligned}$$

A simple calculation yields that

$$\begin{aligned}\frac{1}{\sigma_t^2(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial(\phi_+, \phi_-)} &= \sum_{j=1}^t \psi^{j-1} \left\{ \prod_{k=1}^j \frac{\sigma_{t-k}^2(\theta)}{\sigma_{t-k+1}^2(\theta)} \right\} \frac{((y_{t-j}^+)^2, (y_{t-j}^-)^2)}{\sigma_{t-j}^2(\theta)} \leq (d_t^{\phi+}(\tilde{\vartheta}), d_t^{\phi-}(\tilde{\vartheta})), \\ \frac{1}{\sigma_t^2(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\psi} &= \sum_{j=2}^t (j-1)\psi^{j-2} \left\{ \prod_{k=1}^j \frac{\sigma_{t-k}^2(\theta)}{\sigma_{t-k+1}^2(\theta)} \right\} \frac{\{\omega + \phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2\}}{\sigma_{t-j}^2(\theta)} \leq d_t^\psi(\tilde{\vartheta}),\end{aligned}$$

where the first inequality holds element-wisely. Moreover, for any fixed

$t_0 < t$,

$$0 \leq d_t^{\phi+}(\tilde{\vartheta}) - \frac{1}{\sigma_t^2(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\phi_+} \leq s_{t_0} + r_{t_0},$$

where

$$\begin{aligned}s_{t_0} &= \sum_{j=1}^{t_0} \left\{ \frac{(\eta_{t-j}^+)^2}{a(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a(\eta_{t-k})} - \frac{(y_{t-j}^+)^2}{\sigma_{t-j+1}^2(\theta)} \prod_{k=1}^{j-1} \frac{\psi\sigma_{t-k}^2(\theta)}{\sigma_{t-k+1}^2(\theta)} \right\}, \\ r_{t_0} &= \sum_{j=t_0+1}^{\infty} \frac{(\eta_{t-j}^+)^2}{a(\eta_{t-j})} \prod_{k=1}^{j-1} \frac{\psi}{a(\eta_{t-k})}.\end{aligned}$$

For all $p \geq 1$, since $\|\psi/a(\eta_1)\|_p < 1$ and $\|(\eta_1^+)^2/a(\eta_1)\|_p < 1/\phi_+$, we have

$\|r_{t_0}\|_p \rightarrow 0$ as $t_0 \rightarrow \infty$. In addition, $\|\psi\sigma_{t-1}^2(\theta)/\sigma_t^2(\theta)\|_p < 1$, and by the

Dominated Convergence Theorem,

$$\left\| \frac{\psi}{a(\eta_{t-1})} - \frac{\psi \sigma_{t-1}^2(\theta)}{\sigma_t^2(\theta)} \right\|_p = \left\| \frac{\psi \omega}{a(\eta_{t-1}) \sigma_t^2(\theta)} \right\|_p \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Thus, $\|s_{t_0}\|_p \rightarrow 0$ as $t \rightarrow \infty$. The same arguments hold true when $d_t^{\phi^+}$ is replaced by $d_t^{\phi^-}$ and d_t^ψ . Therefore, $d_t^{\phi^+}(\tilde{\vartheta})$, $d_t^{\phi^-}(\tilde{\vartheta})$, and $d_t^\psi(\tilde{\vartheta})$ have finite moments of any order, and

$$\left\| \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \tilde{\vartheta}} - (d_t^{\phi^+}(\tilde{\vartheta}), d_t^{\phi^-}(\tilde{\vartheta}), d_t^\psi(\tilde{\vartheta}))' \right\|_p \rightarrow 0. \quad (\text{S3.22})$$

Let $d_t = (d_t^{\phi^+}(\tilde{\vartheta}_0), d_t^{\phi^-}(\tilde{\vartheta}_0), d_t^\psi(\tilde{\vartheta}_0))'$, and by the explicit expression of $\partial \ell_t(\theta)/\partial \theta$ and the ergodic theorem, it follows that

$$\begin{aligned} \text{Var} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\vartheta}} \right\} &= \frac{1}{n} \sum_{t=1}^n E \left[-\frac{1}{2\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \tilde{\vartheta}} \left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}^2 \right] \\ &= \frac{1}{4n} E \left[\left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}^2 \right] \sum_{t=1}^n E(d_t d_t') + o(1) \\ &\rightarrow \frac{1}{4} E(d_t d_t') E \left[\left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}^2 \right], \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Cov} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial \ell_t(\theta_0)}{\partial \tilde{\vartheta}}, \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial \ell_t(\theta_0)}{\partial \alpha} \right) &\rightarrow -\frac{1}{2} E(d_t) E \left\{ \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \eta_t \right\}, \\ \text{Var} \left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^n \frac{\partial \ell_t(\theta_0)}{\partial \alpha} \right\} &\rightarrow E \left[\left\{ \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha} \right\}^2 \right]. \end{aligned}$$

By the martingale central limit theorem in Brown (1971), we have

$$\frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial \tilde{\vartheta}} \xrightarrow{d} \mathcal{N}(0, \Upsilon),$$

where

$$\begin{aligned}\Upsilon_{\tilde{\partial}\tilde{\partial}'} &= \frac{1}{4}E(d_t d_t')E\left[\left\{1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x}\eta_t\right\}^2\right], \\ \Upsilon_{\tilde{\partial}\alpha} &= -\frac{1}{2}E(d_t)E\left\{\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x}\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}\eta_t\right\}, \quad \Upsilon_{\alpha\alpha} = E\left[\left\{\frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial \alpha}\right\}^2\right].\end{aligned}$$

The explicit form of $E(d_t d_t')$ and $E(d_t)$ are given in the Appendix S3 below.

Finally, we show that Υ is positive definite. First, it is easy to see that Υ is positive semi-definite. Assume there exists $\mathbf{u} = (u_1, u_2, u_3, u_4)' \in \mathbb{R}^4$, such that $u'\Upsilon u = 0$ holds. From the form of Υ and arguments in Lemma 8, we must have $u_4 = 0$. Thus it holds that $(u_1, u_2, u_3)d_t = 0$, a.s., which is

$$\sum_{j=1}^{\infty} \left\{ u_1 \frac{(\eta_{t-j}^+)^2}{a_0(\eta_{t-j})} + u_2 \frac{(\eta_{t-j}^-)^2}{a_0(\eta_{t-j})} + u_3 \frac{(j-1)\{\phi_{0+}(\eta_{t-j}^+)^2 + \phi_{0-}(\eta_{t-j}^-)^2\}}{\psi_0 a_0(\eta_{t-j})} \right\} \prod_{k=1}^{j-1} \frac{\psi_0}{a_0(\eta_{t-k})} = 0.$$

It is equivalent to

$$\frac{a_0(\eta_{t-1})}{\psi_0} \left\{ u_1 \frac{(\eta_{t-1}^+)^2}{a_0(\eta_{t-1})} + u_2 \frac{(\eta_{t-1}^-)^2}{a_0(\eta_{t-1})} \right\} = f_{t-2}, \quad \text{a.s.},$$

which is

$$\frac{u_1}{\psi_0}(\eta_{t-1}^+)^2 + \frac{u_2}{\psi_0}(\eta_{t-1}^-)^2 = f_{t-2}, \quad \text{a.s.},$$

where f_{t-2} is measurable with respect to $\sigma(\{\eta_j, j \leq t-2\})$. Due to the independence of η_{t-1} and f_{t-2} , and the definition of η_{t-1} , it entails that $u_1 = u_2 = 0$. Thus, $u_3 = 0$. Hence, $\mathbf{u} = 0$, which completes the proof that Υ is positive definite. ■

Lemma 12. *If Assumptions 1–3 hold and $\gamma_{\alpha_0} > 0$, then,*

- (i). $\sum_{t=1}^{\infty} \sup_{\theta \in \Theta_0} \left| \frac{\partial \ell_t(\theta)}{\partial \omega} \right| < \infty$, *a.s.*;
- (ii). $\sum_{t=1}^{\infty} \sup_{\theta \in \Theta_0} \left\| \frac{\partial^2 \ell_t(\theta)}{\partial \omega \partial \theta} \right\| < \infty$, *a.s.*;
- (iii). $\sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| \frac{1}{n} \sum_{t=1}^n \frac{\partial^2 \ell_t(\omega, \vartheta_0)}{\partial \theta_{i+1} \partial \theta_{j+1}} - \Upsilon_{ij} \right| = o(1)$, *a.s. for all $i, j \in \{1, 2, 3, 4\}$;*
- (iv). $\frac{1}{n} \sum_{t=1}^n \sup_{\theta \in \mathcal{V}(\theta_0)} \left| \frac{\partial^3 \ell_t(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_k} \right| = O(1)$, *a.s., for all $i, j, k \in \{2, 3, 4, 5\}$.*

PROOF. (i). For all $\theta \in \Theta_0$, there exists a random variable K_y and $\rho^* \in (\psi \exp(-\gamma_{\alpha_0}), 1)$, such that, for t large enough,

$$\left| \frac{\partial \ell_t(\theta)}{\partial \omega} \right| = \frac{1}{2\sigma_t^2(\theta)} \left| 1 + q_t \frac{\partial \log f_{\alpha}(q_t)}{\partial x} \right| \left| \sum_{j=1}^t \psi^{j-1} \right| \leq K_y \rho^{*t} \sup_{x \in \mathbb{R}} \left| 1 + x \frac{\partial \log f_{\alpha}(x)}{\partial x} \right|, \quad \text{a.s.}$$

where the inequality follows by the fact from Proposition 2.1 in Francq and Zakoïan (2013) that, for any $\rho > \exp(-\gamma_{\alpha_0})$, $\rho^t y_t^2 \xrightarrow{\text{a.s.}} \infty$, a.s. as $t \rightarrow \infty$. Since $x \partial \log f_{\alpha}(x) / \partial x$ is bounded, we have $\sum_{t=1}^{\infty} K_y \rho^{*t} \sup_{x \in \mathbb{R}} |1 + x \partial \log f_{\alpha}(x) / \partial x| < \infty$, a.s., uniformly in $\theta \in \Theta_0$, which completes the proof of (i). \square

(ii). The proof is similar to that of (i), and is thus omitted. \square

(iii). The proof can be completed element-wise. For brevity, we only consider the case $i = 2$ and $j = 4$. The other terms can be dealt with

similarly. Recall that

$$\begin{aligned}\Upsilon(2, 4) &= \frac{1}{4} E(d_t^{\phi_+} d_t^\psi) E \left[\left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}^2 \right] \\ &= \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n \frac{1}{4} d_t^{\phi_+}(\tilde{\vartheta}_0) d_t^\psi(\tilde{\vartheta}_0) \left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \eta_t \right\}^2.\end{aligned}$$

Further, note that

$$\begin{aligned}\frac{\partial^2 \ell_t(\omega, \vartheta_0)}{\partial \phi_+ \partial \psi} &= \frac{1}{\sigma_t^4} \frac{\partial \sigma_t^2}{\partial \phi_+} \frac{\partial \sigma_t^2}{\partial \psi} \left\{ \frac{1}{2} + \frac{3}{4} q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} + \frac{1}{4} q_t^2 \frac{\partial^2 \log f_{\alpha_0}(q_t)}{\partial x^2} \right\} \\ &\quad - \frac{1}{2} \frac{1}{\sigma_t^2} \frac{\partial^2 \sigma_t^2}{\partial \phi_+ \partial \psi} \left\{ 1 + q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} \right\} \\ &= \left\{ \frac{1}{\sigma_t^2} \sum_{j=1}^t \psi_0^{j-1} (y_{t-j}^+)^2 \right\} \left\{ \frac{1}{\sigma_t^2} \sum_{j=2}^t (j-1) \psi_0^{j-2} (\omega + \phi_{0+} (y_{t-j}^+)^2 + \phi_{0-} (y_{t-j}^-)^2) \right\} \\ &\quad \times \left\{ \frac{1}{2} + \frac{3}{4} q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} + \frac{1}{4} q_t^2 \frac{\partial^2 \log f_{\alpha_0}(q_t)}{\partial x^2} \right\} \\ &\quad - \frac{1}{2} \left\{ \frac{1}{\sigma_t^2} \sum_{j=2}^t (j-1) \psi_0^{j-2} (y_{t-j}^+)^2 \right\} \left\{ 1 + q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} \right\},\end{aligned}$$

where $\sigma_t^2 = \sigma_t^2(\omega, \vartheta_0)$ and $q_t = q_t(\omega, \vartheta_0)$ for ease of simplicity here. Thus, it

suffices to show that, as $t \rightarrow \infty$,

$$\left| \frac{1}{\sigma_t^2(\theta_0)} \sum_{j=1}^t \psi_0^{j-1} (y_{t-j}^+)^2 - d_t^{\phi_+}(\tilde{\vartheta}_0) \right| \xrightarrow{a.s.} 0; \quad (\text{S3.23})$$

$$\sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| \frac{1}{\sigma_t^2(\theta_0)} \sum_{j=2}^t (j-1) \psi_0^{j-2} \{ \omega + \phi_{0+} (y_{t-j}^+)^2 + \phi_{0-} (y_{t-j}^-)^2 \} - d_t^\psi(\tilde{\vartheta}_0) \right| \xrightarrow{a.s.} 0; \quad (\text{S3.24})$$

$$\left| \frac{1}{\sigma_t^2(\theta_0)} \sum_{j=2}^t (j-1) \psi_0^{j-2} (y_{t-j}^+)^2 - \frac{1}{\phi_{0+}} d_t^\psi(\tilde{\vartheta}_0) \right| \xrightarrow{a.s.} 0; \quad (\text{S3.25})$$

$$\sup_{\omega \in [\underline{\omega}, \bar{\omega}]} e^{c_0 t} \left| \frac{\sigma_t^2(\omega, \vartheta_0)}{\sigma_t^2(\theta_0)} - 1 \right| \xrightarrow{a.s.} 0, \quad \text{where } c_0 = (\gamma_{\alpha_0} - \max(\log \psi_0, 0))/2 > 0; \quad (\text{S3.26})$$

$$\sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} - \eta_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \right| \xrightarrow{a.s.} 0; \quad (\text{S3.27})$$

$$\sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| q_t^2 \frac{\partial^2 \log f_{\alpha_0}(q_t)}{\partial x^2} - \eta_t^2 \frac{\partial^2 \log f_{\alpha_0}(\eta_t)}{\partial x^2} \right| \xrightarrow{a.s.} 0. \quad (\text{S3.28})$$

First, (S3.23)-(S3.25) can be proved by the similar arguments used to establish the convergence in Lemma 4. Second, since $\sigma_t^2(\theta_0) - \sigma_t^2(\omega, \vartheta_0) = \sum_{j=1}^t \psi_0^{j-1}(\omega - \omega_0)$ and $1/\sigma_t^2(\theta_0) = o(\rho^t)$ a.s. for some $\rho \in (\exp(-\gamma_{\alpha_0}), \psi_0^{-1})$, (S3.26) can be proved. Third, for (S3.27), we have

$$\begin{aligned} & \sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| q_t \frac{\partial \log f_{\alpha_0}(q_t)}{\partial x} - \eta_t \frac{\partial \log f_{\alpha_0}(\eta_t)}{\partial x} \right| \\ & \leq \left\{ \sup_{x \in \mathbb{R}} \left| \frac{\partial \log f_{\alpha_0}(x)}{\partial x} \right| + \sup_{x \in \mathbb{R}} \left| x \frac{\partial^2 \log f_{\alpha_0}(x)}{\partial x^2} \right| \right\} \sup_{\omega \in [\underline{\omega}, \bar{\omega}]} |q_t(\omega, \vartheta_0) - \eta_t| \\ & \leq \left\{ \sup_{x \in \mathbb{R}} \left| \frac{\partial \log f_{\alpha_0}(x)}{\partial x} \right| + \sup_{x \in \mathbb{R}} \left| x \frac{\partial^2 \log f_{\alpha_0}(x)}{\partial x^2} \right| \right\} |\eta_t| \sup_{\omega \in [\underline{\omega}, \bar{\omega}]} \left| \frac{\sigma_t(\theta_0)}{\sigma_t(\omega, \vartheta_0)} - 1 \right| \\ & \leq K \left\{ e^{-c_0 t} |\eta_t| \frac{\sigma_t(\theta_0)}{\sigma_t(\underline{\omega}, \vartheta_0)} \right\} \left\{ \sup_{\omega \in [\underline{\omega}, \bar{\omega}]} e^{c_0 t} \left| \frac{\sigma_t(\omega, \vartheta_0)}{\sigma_t(\theta_0)} - 1 \right| \right\} \\ & \xrightarrow{a.s.} 0. \end{aligned}$$

by the Lagrange mean value theorem and (S3.26). Similar arguments can be used in the proof of (S3.28) by the fact that $x^2 \partial^2 \log f_{\alpha_0}(x) / \partial x^2$ is bounded.

Thus, (iii) holds. \square

(iv). Similar to (iii), we only consider the case $i = j = 2$ and $k = 4$. A

simple calculation yields that

$$\begin{aligned}
& \frac{\partial^3 \ell_t(\theta)}{\partial \phi_+^2 \partial \psi} \\
&= - \left\{ \frac{1}{\sigma_t^2(\theta)} \sum_{j=1}^t \psi^{j-1} (y_{t-j}^+)^2 \right\}^2 \left\{ \frac{1}{\sigma_t^2(\theta)} \sum_{j=2}^t (j-1) \psi^{j-2} [\omega + \phi_+(y_{t-j}^+)^2 + \phi_-(y_{t-j}^-)^2] \right\} \\
&\quad \times \left\{ 1 + \frac{15}{8} q_t \frac{\partial \log f_\alpha(q_t)}{\partial x} + \frac{9}{8} q_t^2 \frac{\partial^2 \log f_\alpha(q_t)}{\partial x^2} + \frac{1}{8} q_t^3 \frac{\partial^3 \log f_\alpha(q_t)}{\partial x^3} \right\} \\
&\quad + \left\{ \frac{1}{\sigma_t^2(\theta)} \sum_{j=1}^t \psi^{j-1} (y_{t-j}^+)^2 \right\} \left\{ \frac{1}{\sigma_t^2(\theta)} \sum_{j=2}^t (j-1) \psi^{j-2} (y_{t-j}^+)^2 \right\} \\
&\quad \times \left\{ 1 + \frac{3}{2} q_t \frac{\partial \log f_\alpha(q_t)}{\partial x} + \frac{1}{2} q_t^2 \frac{\partial^2 \log f_\alpha(q_t)}{\partial x^2} \right\} \\
&= \left(d_t^{\phi_+}(\tilde{\vartheta}) \right)^2 d_t^\psi(\tilde{\vartheta}) \left\{ 1 + \frac{15}{8} q_t \frac{\partial \log f_\alpha(q_t)}{\partial x} + \frac{9}{8} q_t^2 \frac{\partial^2 \log f_\alpha(q_t)}{\partial x^2} + \frac{1}{8} q_t^3 \frac{\partial^3 \log f_\alpha(q_t)}{\partial x^3} \right\} \\
&\quad + d_t^{\phi_+}(\tilde{\vartheta}) \frac{1}{\phi_+} d_t^\psi(\tilde{\vartheta}) \left\{ 1 + \frac{3}{2} q_t \frac{\partial \log f_\alpha(q_t)}{\partial x} + \frac{1}{2} q_t^2 \frac{\partial^2 \log f_\alpha(q_t)}{\partial x^2} \right\} + o(1) \quad \text{a.s.}
\end{aligned}$$

where the term $o(1)$ is obtained by similar arguments used in the proof of (S3.23)-(S3.25). Similar to the facts (2.5) in Andrews, Calder and Davis (2009), using the series expansion of stable densities $f_\alpha(x)$, we can obtain that

$$\sup_{|\alpha - \alpha_0| < \epsilon} \left| \frac{\partial^3 \log f_\alpha(x)}{\partial x^3} \right| = O(|x|^{-3}), \quad \text{as } |x| \rightarrow \infty.$$

Since $d_t^{\phi_+}(\tilde{\vartheta})$ and $d_t^\psi(\tilde{\vartheta})$ have finite moments of any order, by the ergodic theorem and the Cauchy-Schwarz inequality, the proof of (iv) is complete.

■

S4 Some Explicit Expressions

S4.1 Partial derivatives of $\ell_t(\theta)$

We display the explicit expressions of partial derivatives of $\ell_t(\theta)$. For ease of notation, let

$$q_t(\theta) = y_t/\sigma_t(\theta) = y_t/\{\omega + \phi_+(y_{t-1}^+)^2 + \phi_-(y_{t-1}^-)^2 + \psi\sigma_{t-1}^2(\theta)\}^{1/2}.$$

The first-order partial derivatives of $\sigma_t^2(\theta)$ with respect to $\tilde{\theta} = (\omega, \phi_+, \phi_-, \psi)'$ are

$$\begin{aligned} \frac{\partial \sigma_t^2(\theta)}{\partial \omega} &= 1 + \psi \frac{\partial \sigma_{t-1}^2(\theta)}{\partial \omega}; \\ \frac{\partial \sigma_t^2(\theta)}{\partial \phi_+} &= (y_{t-1}^+)^2 + \psi \frac{\partial \sigma_{t-1}^2(\theta)}{\partial \phi_+}; \\ \frac{\partial \sigma_t^2(\theta)}{\partial \phi_-} &= (y_{t-1}^-)^2 + \psi \frac{\partial \sigma_{t-1}^2(\theta)}{\partial \phi_-}; \\ \frac{\partial \sigma_t^2(\theta)}{\partial \psi} &= \sigma_{t-1}^2(\theta) + \psi \frac{\partial \sigma_{t-1}^2(\theta)}{\partial \psi}. \end{aligned}$$

Recall the definition of $\ell_t(\theta)$: $\ell_t(\theta) = -\log \sigma_t(\theta) + \log f_\alpha(y_t/\sigma_t(\theta))$, where $f_\alpha(x)$ is defined in (2.1) and $\sigma_t^2(\theta) = \omega + \phi_+(y_{t-1}^+)^2 + \phi_-(y_{t-1}^-)^2 + \psi\sigma_{t-1}^2(\theta)$. Then, the first-order partial derivatives of $\ell_t(\theta)$ with respect to θ are given by

$$\frac{\partial \ell_t(\theta)}{\partial \tilde{\theta}} = -\frac{1}{2\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \tilde{\theta}} \left\{ 1 + q_t(\theta) \frac{\partial \log f_\alpha(q_t(\theta))}{\partial x} \right\}; \quad \frac{\partial \ell_t(\theta)}{\partial \alpha} = \frac{\partial \log f_\alpha(q_t(\theta))}{\partial \alpha},$$

where $\frac{\partial \log f_\alpha(q_t(\theta))}{\partial x} := \frac{\partial \log f_\alpha(x)}{\partial x} \Big|_{x=q_t(\theta)}$.

The second-order partial derivatives of $\ell_t(\theta)$ with respect to θ are given

by

$$\begin{aligned}\frac{\partial^2 \ell_t}{\partial \tilde{\theta} \partial \tilde{\theta}'} &= \frac{1}{\sigma_t^4(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \tilde{\theta}} \frac{\partial \sigma_t^2(\theta)}{\partial \tilde{\theta}'} \left\{ \frac{1}{2} + \frac{3}{4} q_t(\theta) \frac{\partial \log f_\alpha(q_t(\theta))}{\partial x} + \frac{1}{4} q_t^2(\theta) \frac{\partial^2 \log f_\alpha(q_t(\theta))}{\partial x^2} \right\} \\ &\quad - \frac{1}{2} \frac{1}{\sigma_t^2(\theta)} \frac{\partial^2 \sigma_t^2(\theta)}{\partial \tilde{\theta} \partial \tilde{\theta}'} \left\{ 1 + q_t(\theta) \frac{\partial \log f_\alpha(q_t(\theta))}{\partial x} \right\}; \\ \frac{\partial^2 \ell_t(\theta)}{\partial \tilde{\theta} \partial \alpha} &= -\frac{1}{2} \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \tilde{\theta}} q_t(\theta) \frac{\partial^2 \log f_\alpha(q_t(\theta))}{\partial x \partial \alpha}; \\ \frac{\partial^2 \ell_t(\theta)}{\partial \alpha^2} &= \frac{\partial^2 \log f_\alpha(q_t(\theta))}{\partial \alpha^2}.\end{aligned}$$

S4.2 Fisher matrix Υ in explosive cases

Next, we give the exact value of Fisher matrix Υ in the explosive case when

$\alpha_0 = 1$. Recall that $d_t = (d_t^{\phi+}, d_t^{\phi-}, d_t^\psi)'$. Let $a_{0+}(\eta_t) = \phi_{0+}(\eta_t^+)^2 + \psi_0$,

$a_{0-}(\eta_t) = \phi_{0-}(\eta_t^-)^2 + \psi_0$, and

$$\nu_i = E \left[\left\{ \frac{\psi_0}{a_0(\eta_t)} \right\}^i \right], \quad \nu_{i+} = E \left[\left\{ \frac{\psi_0}{a_{0+}(\eta_t)} \right\}^i \right], \quad \nu_{i-} = E \left[\left\{ \frac{\psi_0}{a_{0-}(\eta_t)} \right\}^i \right],$$

for $i = 1, 2$. Then

$$\begin{aligned}E\{(d_t^{\phi+})^2\} &= \frac{(1 - 2\nu_{1+} + \nu_{2+})(1 - \nu_1) + 2(\nu_{1+} - \nu_{2+})(1 - \nu_{1+})}{\phi_{0+}^2(1 - \nu_1)(1 - \nu_2)}, \\ E(d_t^{\phi+} d_t^{\phi-}) &= \frac{(\nu_{1+} - \nu_{2+})(1 - \nu_{1-}) + (\nu_{1-} - \nu_{2-})(1 - \nu_{1+})}{\phi_{0+} \phi_{0-} (1 - \nu_1)(1 - \nu_2)}, \\ E(d_t^{\phi+} d_t^\psi) &= \frac{\nu_2(1 - \nu_{1+}) + \nu_{1+} - \nu_{2+}}{\psi_0 \phi_{0+} (1 - \nu_2)(1 - \nu_1)}, \\ E\{(d_t^\psi)^2\} &= \frac{\nu_2(1 + \nu_1)}{\psi_0^2(1 - \nu_2)(1 - \nu_1)},\end{aligned}\tag{S4.29}$$

and $E\{(d_t^{\phi^-})^2\}$ (resp. $E(d_t^{\phi^-} d_t^\psi)$) is obtained by replacing ϕ_{0+} by ϕ_{0-} and ν_{i+} by ν_{i-} in $E\{(d_t^{\phi^+})^2\}$ (resp. $E(d_t^{\phi^+} d_t^\psi)$). Moreover,

$$E(d_t) = \left(\frac{1 - \nu_{1+}}{\phi_{0+}(1 - \nu_1)}, \frac{1 - \nu_{1-}}{\phi_{0-}(1 - \nu_1)}, \frac{\nu_1}{\psi_0(1 - \nu_1)} \right)'. \quad (\text{S4.30})$$

Particularly, when $\alpha_0 = 1$, i.e., η follows the standard Cauchy distribution, the following values (also given in Li, et.al. (2023)) are available:

$$\begin{aligned} E \left[\left\{ \frac{\partial \log f_{\alpha_0}(\eta)}{\partial x} \right\}^2 \right] &= E \left[\left\{ 1 + \frac{\partial \log f_{\alpha_0}(\eta)}{\partial x} \eta \right\}^2 \right] = \frac{1}{2}, \\ E \left\{ \frac{\partial \log f_{\alpha_0}(\eta)}{\partial x} \frac{\partial \log f_{\alpha_0}(\eta)}{\partial \alpha} \eta \right\} &= \frac{\mathcal{C} - 1 + \log 2}{2}, \\ E \left[\left\{ \frac{\partial \log f_{\alpha_0}(\eta)}{\partial \alpha} \right\}^2 \right] &= \frac{(\mathcal{C} - 1 + \log 2)^2}{2} + \frac{\pi^2}{12}, \end{aligned}$$

where $\mathcal{C} = 0.577\ 215\ 664 \dots$ is the Euler-Mascheroni constant. Based on these, the exact value of Fisher matrix Υ can be calculated. Specifically,

$$\begin{aligned} \nu_1 &= \frac{a}{2(a+1)} + \frac{b}{2(b+1)}, & \nu_{1+} &= \frac{a}{2(a+1)} + \frac{1}{2}, & \nu_{1-} &= \frac{b}{2(b+1)} + \frac{1}{2}, \\ \nu_2 &= \frac{a(2a+1)}{4(a+1)^2} + \frac{b(2b+1)}{4(b+1)^2}, & \nu_{2+} &= \frac{a(2a+1)}{4(a+1)^2} + \frac{1}{2}, & \nu_{2-} &= \frac{b(2b+1)}{4(b+1)^2} + \frac{1}{2}, \end{aligned}$$

where $a = \sqrt{\psi_0/\phi_{0+}}$ and $b = \sqrt{\psi_0/\phi_{0-}}$. Thus, the exact values of $E(d_t d_t')$

and $E(d_t)$ can be calculated, and Υ is given by

$$\Upsilon = \frac{1}{8} \begin{pmatrix} E\{(d_t^{\phi^+})^2\} & E(d_t^{\phi^+} d_t^{\phi^-}) & E(d_t^{\phi^+} d_t^\psi) & -2\tau E(d_t^{\phi^+}) \\ E(d_t^{\phi^+} d_t^{\phi^-}) & E\{(d_t^{\phi^-})^2\} & E(d_t^{\phi^-} d_t^\psi) & -2\tau E(d_t^{\phi^-}) \\ E(d_t^{\phi^+} d_t^\psi) & E(d_t^{\phi^-} d_t^\psi) & E\{(d_t^\psi)^2\} & -2\tau E(d_t^\psi) \\ -2\tau E(d_t^{\phi^+}) & -2\tau E(d_t^{\phi^-}) & -2\tau E(d_t^\psi) & 4\tau^2 + \frac{2\pi^2}{3} \end{pmatrix},$$

where $\tau = \mathcal{C} - 1 + \log 2$.

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