ON CORRELATION ROBUSTNESS OF SUPREMUM-BASED p-VALUE COMBINATION TESTS

Xiaohui Chen, Fangfang Wang, Hong Zhang, and Zheyang Wu

Department of Mathematical Sciences, Worcester Polytechnic Institute

Global Biometrics & Data Management, Pfizer Inc., New York

Supplementary Material

The Supplementary Material provides detailed explanations and discussions of related comments in the manuscript (Sections S1 and S2), as well as auxiliary lemmas (Section S3), proofs of theoretical results (Section S4), and additional numerical studies (Section S5), including a statistical power study.

S1 Problem Description and Formulation

S1.1 Supremum-based p-value Combination Tests

The supremum-based p-value combination tests are a class of methods that combine input p-values from individual hypothesis tests into a single test statistic, as defined by (2.8) in the manuscript, to assess the global null hypothesis. This approach encompasses a broad range of test statistics.

The minP test is considered to belong to this family because we can define a statistic equivalent to the minP statistic $P_{(1)}$: $T_f(\boldsymbol{P}) = \sup_{1 \leq i \leq n} f(P_{(i)})$, where $f(\cdot)$ is a strictly decreasing function satisfying Assumption 1. This equivalence follows from the one-to-one correspondence of these events: for any $p \in (0,1)$, with $t = f(p)$,

$$\{P_{(1)} < p\} = \{f(P_{(1)}) > f(p)\} = \left\{\sup_i f\big(P_{(i)}\big) > t\right\} = \{T_f(\boldsymbol{P}) > t\}.$$

Therefore, in testing, $P_{(1)}$ and $T_f(\boldsymbol{P})$ yield the same decision rule at any chosen significance level.

Similarly, the Simes test is also a member of this family because $T_{Simes}(\boldsymbol{P}) = \min_{i=1,\ldots,n}\left\{P_{(i)}\frac{n}{i}\right\}$ is equivalent to $T_f(\boldsymbol{P}) = \sup_{1\le i\le n} f_i(P_{(i)})$, where $f_i(x) = f\big(x\frac{n}{i}\big)$ and $f(\cdot)$ is a strictly decreasing function with $\lim_{x\to 0} f(x) = +\infty$. Thus, f_i satisfies Assumption 1. This equivalence follows because, for any $p \in (0,1)$ with $t = f(p)$,

$$\left\{\min_i\big\{P_{(i)}\tfrac{n}{i}\big\} < p\right\} = \left\{\sup_i f\big(P_{(i)}\tfrac{n}{i}\big) > f(p)\right\} = \left\{\sup_i f_i\big(P_{(i)}\big) > t\right\}.$$

The HC statistic $T_{HC}(\boldsymbol{P}) = \max_{i=1,\ldots,n} \frac{\sqrt{n}(\frac{i}{n}-P_{(i)})}{\sqrt{P_{(i)}\big(1-P_{(i)}\big)}}$ takes the form of (2.8) with $f_i(P_{(i)}) = f(\frac{i}{n}, P_{(i)})$ and $f(\frac{i}{n}, x) = \frac{\sqrt{n}(\frac{i}{n}-x)}{\sqrt{x(1-x)}}$, which compares the magnitude of $P_{(i)}$ with its expected value $\frac{i}{n}$ under the null (Donoho and Jin, 2004) (although, strictly speaking, $\mathbb{E}(P_{(i)}|H_0) = \frac{i}{n+1}$ (Zhang et al., 2020)).

The one-sided ϕ-divergence statistics defined by (2.7) are also special cases of the statistic family in (2.8), noting that $f_i(x) = f_s(\frac{i}{n}, x)$ is strictly decreasing for a given s, i, and n.

In general, when $f_i(P_{(i)}) = f\big(\frac{i}{n}, P_{(i)}\big)$ is defined as a decreasing function of $P_{(i)}$ that measures the deviance from its null expectation approximated by $\frac{i}{n}$, the statistic is said to belong to the general goodness-of-fit (GGOF) family (Zhang et al., 2020).

S2 Asymptotic Correlation Robustness under Gaussianity

S2.1 On the tail bounds for the minP statistic

Corollary 2.1 of Li and Shao (2002) has shown that for $\boldsymbol{X}$ satisfying the Gaussian model in (1) with $\sigma_{ij} \in [-1, 1]$,

$$\begin{aligned} &|\mathbb{P}\left(T_{maxN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right) - \mathbb{P}\left(T_{maxN}(\boldsymbol{X}) > t|\mathbf{I}\right)| \\ \leq &\frac{1}{4}\sum_{1\leq i<j\leq n}\Big\{|\sigma_{ij}|\exp(-t^2/(1+|\sigma_{ij}|)\Big\}, \qquad \forall t > 0. \end{aligned} \tag{S2.1}$$

Equation (3.14) in the manuscript provides a sharper bound than (S2.1), provided that the X_i are not perfectly correlated and t is large. To demonstrate this, we obtain the following result using (3.14):

$$\begin{aligned} &|\mathbb{P}\left(T_{maxN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right) - \mathbb{P}\left(T_{maxN}(\boldsymbol{X}) > t|\mathbf{I}\right)| \\ \leq &A_n(t) + B_n(t) + C_n(t), \end{aligned} \tag{S2.2}$$

where

$$\begin{aligned} A_n(t) &= \sum_{1\leq i<j\leq n} A_{ij}(t), \text{ with } A_{ij}(t) = \frac{(1+\sigma_{ij})^2}{2\pi\sqrt{1-\sigma_{ij}^2}}\frac{\exp(-t^2/(1+\sigma_{ij}))}{t^2}; \\ B_n(t) &= \frac{n(n-1)\bar{\Phi}^2(t)}{2}; \\ C_n(t) &= \sum_{i=3}^{n}\binom{n}{i}(-\bar{\Phi}(t))^i. \end{aligned}$$

As $t \to \infty$, $A_{ij}(t) = O(t^{-2}\exp(-t^2/(1+\sigma_{ij})))$, $B_n(t) = O(t^{-2}e^{-t^2})$, and $C_n(t) = o(B_n(t))$. Therefore, (S2.2) improves upon (S2.1) by a factor of t^{-2} if $|\sigma_{ij}| < 1$ for $i \neq j$.

S2.2 The Simes test versus the minP test

Equations (3.13) and (3.16) in the manuscript provide an insightful comparison between the minP and Simes tests under arbitrary correlations. Specifically, we have the following:

$$\lim_{p\to 0} \frac{\mathbb{P}\left(T_{Simes}(\boldsymbol{P}) < p|\boldsymbol{\Sigma}\right)}{\mathbb{P}\left(T_{minP}(\boldsymbol{P}) < p/n|\boldsymbol{\Sigma}\right)} = 1.$$

This indicates that for a small p, the p-quantile of T_{Simes} (which is roughly p) is approximately n times the p-quantile of T_{minP} under the null hypothesis and arbitrary correlations. Moreover, the p-value of the minP test is always asymptotically greater than or equal to that of the Simes test. To see this, let $p_{(1)}$ be the observed statistic of T_{minP}. The observed statistic of T_{Simes} is given by $\min_i \left\{p_{(i)}\frac{n}{i}\right\}$, which is less than or equal to $np_{(1)}$. The p-value of the minP test is

$$\mathbb{P}(T_{minP} < p_{(1)}|\boldsymbol{\Sigma}) \sim \mathbb{P}(T_{Simes} < np_{(1)}|\boldsymbol{\Sigma}) \geq \mathbb{P}\left(T_{Simes} < \min_i \left\{p_{(i)}\frac{n}{i}\right\}|\boldsymbol{\Sigma}\right), \tag{S2.3}$$

where the right-hand side is the p-value of the Simes test. Therefore, for the same data, the Simes test is more likely to yield a significant result than the minP test. In other words, for a sufficiently small α, if $\mathbb{P}(T_{minP} < p_{(1)}) = \alpha$, then $\mathbb{P}\left(T_{Simes} < \min_i \left\{p_{(i)}\frac{n}{i}\right\}|\boldsymbol{\Sigma}\right) \leq \alpha$, indicating that the Simes test (which rejects H_0 if $p_{(i)} < i\alpha/n$ for at least one i) effectively controls the probability of type I error (i.e., it is a level-α test). Although both tests are level-α tests, the statistical power of the Simes test is asymptotically greater than or equal to that of the minP test under arbitrary correlations. It is worth noting that this conclusion holds for any arbitrary correlation structure (see Corollaries 6 and 7 for results in the non-Gaussian case), whereas Sarkar (1998) requires the test statistics vector $\boldsymbol{X}$ to be multivariate totally positive of order two (see also Simes (1986); Sarkar and Chang (1997)).

S3 Auxiliary Lemmas and Proofs

In this section, we present auxiliary lemmas that are essential for proving the results in the manuscript. Lemmas A1 – A4 introduce important techniques from the existing literature. Lemma A5 provides an asymptotic calculation of tail probabilities for bivariate normal random variables. Lemma A6 identifies the conditions under which the joint bivariate tail probability becomes negligible compared to the marginal tail probability. Lemma A7 provides non-asymptotic upper and lower bounds for the tail of a multivariate normal distribution. Finally, Lemma A8 presents necessary and sufficient conditions for checking whether the supremum-based p-value combination test statistics satisfy Assumption 1 in the manuscript.

Lemma A1 (Bonferroni Inequalities (Bonferroni, 1936; Liu and Xie, 2020))**.** *For every* $1 \leq p \leq \lfloor n/2 \rfloor$,

$$\sum_{k=1}^{2p}(-1)^{k-1}S_k \leq \mathbb{P}(\cup_{i=1}^{n}A_i) \leq \sum_{k=1}^{2p-1}(-1)^{k-1}S_k,$$

where $S_k = \sum_{1\leq i_1 \leq \ldots \leq i_k \leq n}\mathbb{P}(A_{i_1}\cap \ldots \cap A_{i_k})$.

When $p = 1$, *the inequalities reduce to*

$$\sum_{i=1}^{n}\mathbb{P}\left(A_i\right) - \sum_{1\leq i<j\leq n}\mathbb{P}\left(A_i \cap A_j\right) \leq \mathbb{P}\left(\cup_{i=1}^{n}A_i\right) \leq \sum_{i=1}^{n}\mathbb{P}\left(A_i\right). \tag{S3.1}$$

Lemma A2 (Sarkar's Equation (Lemma 2.1 and Remark 2.1 in Sarkar (1998)))**.** *Let* $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ *be the ordered components of an arbitrary random vector* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\top$. *Denote* $X_{(1)}^{(-i)}, \ldots, X_{(n-1)}^{(-i)}$ *the ordered components of* $(n-1)$ *dimensional random vector* $\boldsymbol{X}^{(-i)}$ *by eliminating* X_i *from* $\boldsymbol{X}$.

1. For any fixed $a_1 \leq a_2 \leq ... \leq a_n$*,*

$$\mathbb{P}\left(X_{(1)} \geq a_1, ..., X_{(n)} \geq a_n\right) = 1 - \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(X_i < a_n)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(X_i \leq a_{j+1})}{j+1} - \frac{I(X_i \leq a_j)}{j}\right\} \times \mathbb{P}\left(X_{(j)}^{(-i)} \geq a_{j+1}, .., X_{(n-1)}^{(-i)} \geq a_n | X_i\right)\right], \tag{S3.2}$$

2. For any fixed $b_1 \leq b_2 \leq ... \leq b_n$*,*

$$\mathbb{P}\left(X_{(1)} \leq b_1, ..., X_{(n)} \leq b_n\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(X_i < b_1)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(X_i \geq b_{n-j})}{j+1} - \frac{I(X_i \geq b_{n-j+1})}{j}\right\} \times \mathbb{P}\left(X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j} | X_i\right)\right]. \tag{S3.3}$$

If X_i*'s are i.i.d. with a common CDF* $F(x)$*, then Equations* (S3.2) *and* (S3.3) *reduce to, respectively,*

$$\mathbb{P}\left(X_{(1)} \geq a_1, ..., X_{(n)} \geq a_n\right) = 1 - \mathbb{P}(X_1 < a_n)$$
$$+ n\sum_{j=1}^{n-1}\left[\frac{\mathbb{P}(X_i \leq a_{j+1})}{j+1} - \frac{\mathbb{P}(X_i \leq a_j)}{j}\right] \times \mathbb{P}\left(X_{(j):n-1} \geq a_{j+1}, .., X_{(n-1):n-1} \geq a_n\right), \tag{S3.4}$$

and

$$\mathbb{P}\left(X_{(1)} \leq b_1, ..., X_{(n)} \leq b_n\right) = \mathbb{P}(X_1 < b_1)$$
$$+ n\sum_{j=1}^{n-1}\left[\frac{\mathbb{P}(X_i \geq b_{n-j})}{j+1} - \frac{\mathbb{P}(X_i \geq b_{n-j+1})}{j}\right] \times \mathbb{P}\left(X_{(1):n-1} \leq b_1, .., X_{(n-j):n-1} \leq b_{n-j}\right), \tag{S3.5}$$

where $X_{(1):n-1}, ..., X_{(n-1):n-1}$ *are the order statistics of* $(n-1)$ *dimensional random vector* $\boldsymbol{X} = (X_1, X_2, ..., X_{n-1})^\top$*.*

Lemma A3 (Lemma 2 in the supplementary material of Long et al. (2023))**.** *As* $y \to 0^+$*,* *the inverse Gaussian function is in the following range,*

$$\sqrt{2\log(1/y)+1}-1 \le \bar{\Phi}^{-1}(y) \le \sqrt{2\log(1/y)}. \tag{S3.6}$$

Lemma A4 (Multivariate Mill's Inequality (Savage, 1961))**.** *Let $\boldsymbol{X}$ follow a multivariate normal distribution $N(\mathbf{0},\Sigma)$, with $\boldsymbol{\Sigma}=(\sigma_{ij})_{1\le i,j\le n}$, $\sigma_{ii}=1$, and $|\sigma_{ij}|<1$ for $i\neq j$. The probability density function (pdf) of $\boldsymbol{X}$ is denoted by $f(\boldsymbol{x})$. Let $\boldsymbol{\Sigma}^{-1}=(\sigma'_{ij})_{1\le i,j\le n}$, $\Delta_i=\sum_{j=1}^n t_j\sigma'_{ji}$, and $\boldsymbol{t}$ be a row vector satisfying $\boldsymbol{t}\Sigma^{-1}>0$ entrywise. Then, the multivariate tail probability is bounded by*

$$\frac{1-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n\sigma'_{ij}(1+\delta_{ij})/(\Delta_i\Delta_j)}{\prod_{i=1}^n\Delta_i} \le \frac{\mathbb{P}(\boldsymbol{X}>\boldsymbol{t})}{f(\boldsymbol{t})} \le \frac{1}{\prod_{i=1}^n\Delta_i}, \tag{S3.7}$$

where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$.

Furthermore, the univariate Mill's inequality (Mills, 1926; Borjesson and Sundberg, 1979) is that for any $t_i>0$,

$$\frac{\phi(t_i)t_i}{1+t_i^2} \le \mathbb{P}(X_i>t_i) \le \frac{\phi(t_i)}{t_i}, \tag{S3.8}$$

and

$$\frac{2\phi(t_i)t_i}{1+t_i^2} \le \mathbb{P}(|X_i|>t_i) \le \frac{2\phi(t_i)}{t_i}. \tag{S3.9}$$

Lemma A5 (Bivariate normal tail probabilities)**.** *Suppose $(X_1,X_2)^\top$ follows a standard bivariate normal distribution with the density function $f(x_1,x_2)=(2\pi\sqrt{1-\sigma^2})^{-1}\exp(-\frac{1}{2(1-\sigma^2)}(x_1^2-2\sigma x_1x_2+x_2^2))$ and $\sigma\in(-1,1)$. If $|\sigma|<t_1/t_2<|\frac{1}{\sigma}|$, then as $t_1,t_2\to\infty$,*

$$\mathbb{P}(X_1>t_1,X_2>t_2)=\mathbb{P}(X_1<-t_1,X_2<-t_2)\sim f(t_1,t_2)\frac{(1-\sigma^2)^2}{(t_1-\sigma t_2)(t_2-\sigma t_1)};$$
$$\mathbb{P}(X_1<-t_1,X_2>t_2)=\mathbb{P}(X_1>t_1,X_2<-t_2)\sim f(t_1,-t_2)\frac{(1-\sigma^2)^2}{(t_1+\sigma t_2)(t_2+\sigma t_1)}.$$

Proof. First, notice that, by the symmetry of the distribution of the bivariate vector $(X_1,X_2)^\top$, we have $\mathbb{P}(X_1>t_1,X_2>t_2)=\mathbb{P}(X_1<-t_1,X_2<-t_2)$ and

$\mathbb{P}\left(X_1 < -t_1, X_2 > t_2\right) = \mathbb{P}\left(X_1 > t_1, X_2 < -t_2\right)$.

Second, the magnitudes of the tail probabilities follow Lemma A4. Specifically, the bivariate vector $(X_1, X_2)^\top$ corresponds to $\boldsymbol{\Sigma} = \left(\begin{smallmatrix} 1 & \sigma \\ \sigma & 1 \end{smallmatrix}\right)$ and $\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} \frac{1}{1-\sigma^2} & \frac{\sigma}{\sigma^2-1} \\ \frac{\sigma}{\sigma^2-1} & \frac{1}{1-\sigma^2} \end{pmatrix}$. Let $\boldsymbol{t} = (t_1, t_2)$. Since $|\sigma| < t_1/t_2 < |\frac{1}{\sigma}|$, for sufficiently large t_1, t_2, we have $t_1 - \sigma t_2 > 0$, and $t_2 - \sigma t_1 > 0$. Therefore, $\boldsymbol{t}\Sigma^{-1} = \left(\frac{t_1-\sigma t_2}{1-\sigma^2}, \frac{t_2-\sigma t_1}{1-\sigma^2}\right) = (\Delta_1, \Delta_2)$ with

$$\Delta_1 = \frac{t_1 - \sigma t_2}{1-\sigma^2} \to \infty, \quad \Delta_2 = \frac{t_2 - \sigma t_1}{1-\sigma^2} \to \infty,$$

as $t_1, t_2 \to \infty$.

Accordingly, Equation (S3.7) reduces to

$$\frac{1}{\Delta_1\Delta_2}\left(1 - \frac{(2-\sigma)/(1-\sigma^2)}{\Delta_1\Delta_2}\right) \leq \frac{\mathbb{P}(\boldsymbol{X} > \boldsymbol{t})}{f(\boldsymbol{t})} = \frac{\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right)}{f(t_1, t_2)} \leq \frac{1}{\Delta_1\Delta_2}. \tag{S3.10}$$

As $t_1, t_2 \to \infty$, we have $\frac{1}{\Delta_1\Delta_2} = \frac{(1-\sigma^2)^2}{(t_1-\sigma t_2)(t_2-\sigma t_1)} \to 0$. Therefore,

$$\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right) \sim f(t_1, t_2)\frac{(1-\sigma^2)^2}{(t_1 - \sigma t_2)(t_2 - \sigma t_1)}. \tag{S3.11}$$

Similarly, we can get $\mathbb{P}\left(X_1 < -t_1, X_2 > t_2\right) = \mathbb{P}\left(-X_1 > t_1, X_2 > t_2\right)$.

Notice that $(-X_1, X_2)^\top$ is bivariate normal with $\boldsymbol{\Sigma} = \left(\begin{smallmatrix} 1 & -\sigma \\ -\sigma & 1 \end{smallmatrix}\right)$ and $\boldsymbol{\Sigma}^{-1} = \begin{pmatrix} \frac{1}{1-\sigma^2} & \frac{\sigma}{\sigma^2-1} \\ \frac{\sigma}{\sigma^2-1} & \frac{1}{1-\sigma^2} \end{pmatrix}$.

We have $\boldsymbol{t}\Sigma^{-1} = \left(\frac{t_1+\sigma t_2}{1-\sigma^2}, \frac{t_2+\sigma t_1}{1-\sigma^2}\right) = (\Delta_1, \Delta_2)$. Under the condition $|\sigma| < t_1/t_2 < |\frac{1}{\sigma}|$, we have

$$\Delta_1 = \frac{t_1 + \sigma t_2}{1-\sigma^2} \to \infty, \quad \Delta_2 = \frac{t_2 + \sigma t_1}{1-\sigma^2} \to \infty,$$

as $t_1, t_2 \to \infty$.

Similar to the calculation in (S3.10), we have

$$\frac{\mathbb{P}(\boldsymbol{X} > \boldsymbol{t})}{f_{-X_1,X_2}(\boldsymbol{t})} = \frac{\mathbb{P}(-X_1 > t_1, X_2 > t_2)}{f_{-X_1,X_2}(t_1, t_2)} \sim \frac{1}{\Delta_1\Delta_2} = \frac{(1-\sigma^2)^2}{(t_1 + \sigma t_2)(t_2 + \sigma t_1)}.$$

Since $f_{-X_1,X_2}(t_1, t_2) = f_{X_1,X_2}(-t_1, t_2) = f(-t_1, t_2)$, and $f(-t_1, t_2) = f(t_1, -t_2)$ by symmetry, we get

$$\mathbb{P}\left(X_1 < -t_1, X_2 > t_2\right) = \mathbb{P}\left(-X_1 > t_1, X_2 > t_2\right) \sim f(t_1, -t_2)\frac{(1-\sigma^2)^2}{(t_1+\sigma t_2)(t_2+\sigma t_1)}.$$

□

Lemma A6. *Suppose $(X_1, X_2)^\top$ follows a standard bivariate normal distribution with the correlation coefficient $\sigma \in (-1, 1)$. For $t_1, t_2 \to \infty$ and $|\sigma| < \lim_{t_1,t_2\to\infty} t_1/t_2 < |1/\sigma|$, we have*

$$\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right) = o\left(\min\left\{\mathbb{P}\left(X_1 > t_1\right), \mathbb{P}\left(X_2 > t_2\right)\right\}\right). \tag{S3.12}$$

$$\mathbb{P}\left(|X_1| > t_1, |X_2| > t_2\right) = o\left(\min\left\{\mathbb{P}\left(|X_1| > t_1\right), \mathbb{P}\left(|X_2| > t_2\right)\right\}\right). \tag{S3.13}$$

Proof. By Lemma A5,

$$\begin{aligned}\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right) &\sim f(t_1, t_2)\frac{(1-\sigma^2)^2}{\left(t_1-\sigma t_2\right)\left(t_1-\sigma t_2\right)}\\ &= \frac{(1-\sigma^2)^2 \exp\left(-\frac{1}{2(1-\sigma^2)}\left(t_1^2 - 2\sigma t_1 t_2 + t_2^2\right)\right)}{2\pi(t_1-\sigma t_2)(t_2-\sigma t_1)\sqrt{1-\sigma^2}}\\ &= \frac{(1-\sigma^2)^{\frac{3}{2}}}{2\pi(t_1-\sigma t_2)(t_2-\sigma t_1)}\exp\left(-\frac{1}{2(1-\sigma^2)}\left(t_1^2 - 2\sigma t_1 t_2 + t_2^2\right)\right).\end{aligned}$$

Applying the univariate Mill's inequality in (S3.8), we have

$$\mathbb{P}\left(X_1 > t_1\right) \geq \frac{t_1}{1+t_1^2}\frac{1}{\sqrt{2\pi}}\exp(-\frac{t_1^2}{2}).$$

Thus, as $t_1, t_2 \to \infty$,

$$\begin{aligned}&\frac{\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right)}{\mathbb{P}\left(X_1 > t_1\right)}\\ \leq&\frac{(1-\sigma^2)^{\frac{3}{2}}\left(1+t_1^2\right)}{\sqrt{2\pi}t_1(t_1-\sigma t_2)(t_2-\sigma t_1)}\exp\left(-\frac{1}{2(1-\sigma^2)}\left(\sigma t_1 - t_2\right)^2\right) \to 0.\end{aligned}$$

Similarly, $\lim_{t_1,t_2\to\infty} \frac{\mathbb{P}(X_1>t_1,X_2>t_2)}{\mathbb{P}(X_2>t_2)} = 0$.

By Lemma A5,

$$
\begin{aligned}
\mathbb{P}(|X_1| > t_1, |X_2| > t_2) =& \mathbb{P}\left(X_1 > t_1, X_2 > t_2\right) + \mathbb{P}\left(X_1 < -t_1, X_2 < -t_2\right) \\
&+ \mathbb{P}\left(X_1 < -t_1, X_2 > t_2\right) + \mathbb{P}\left(X_1 > t_1, X_2 < -t_2\right) \\
=& 2\mathbb{P}\left(X_1 > t_1, X_2 > t_2\right) + 2\mathbb{P}\left(X_1 < -t_1, X_2 > t_2\right) \\
=& \frac{(1-\sigma^2)^{\frac{3}{2}}}{\pi(t_1 - \sigma t_2)(t_2 - \sigma t_1)} \exp\left(-\frac{1}{2(1-\sigma^2)}\left(t_1^2 - 2\sigma t_1 t_2 + t_2^2\right)\right) \\
&+ \frac{(1-\sigma^2)^{\frac{3}{2}}}{\pi(t_1 + \sigma t_2)(t_2 + \sigma t_1)} \exp\left(-\frac{1}{2(1-\sigma^2)}\left(t_1^2 + 2\sigma t_1 t_2 + t_2^2\right)\right) \\
=& C\frac{(1-\sigma^2)^{\frac{3}{2}}}{\pi(t_1 - |\sigma| t_2)(t_2 - |\sigma| t_1)} \exp\left(-\frac{1}{2(1-|\sigma|^2)}\left(t_1^2 - 2|\sigma| t_1 t_2 + t_2^2\right)\right).
\end{aligned}
$$

Also, by the univariate Mill's inequality in (S3.8), we have

$$
\begin{aligned}
\mathbb{P}\left(|X_1| > t_1\right) =& \mathbb{P}\left(X_1 > t_1\right) + \mathbb{P}\left(X_1 < -t_1\right) \\
\geq& \frac{t_1}{1+t_1^2}\frac{\sqrt{2}}{\sqrt{\pi}}\exp(-\frac{t_1^2}{2}).
\end{aligned}
$$

Thus, as $t_1, t_2 \to \infty$,

$$
\begin{aligned}
&\frac{\mathbb{P}(|X_1| > t_1, |X_2| > t_2)}{\mathbb{P}\left(|X_1| > t_1\right)} \\
\leq& \frac{C(1-\sigma^2)^{\frac{3}{2}}\left(1+t_1^2\right)}{t_1(t_1 - |\sigma| t_2)(t_2 - |\sigma| t_1)} \exp\left(-\frac{1}{2(1-|\sigma|^2)}\left(|\sigma| t_1 - t_2\right)^2\right) \to 0.
\end{aligned}
$$

Similarly, $\lim_{t_1,t_2\to\infty} \frac{\mathbb{P}(|X_1|>t_1,|X_2|>t_2)}{\mathbb{P}(|X_2|>t_2)} = 0$.

$\square$

Lemma A7. *Suppose that $\boldsymbol{X} = (X_1, ..., X_n)^\top$ is Gaussian as in Definition 1. Let*

$$
f_{ij}(u,v) = (2\pi\sqrt{1-\sigma_{ij}^2})^{-1}\exp(-\frac{1}{2(1-\sigma_{ij}^2)}(u^2 - 2\sigma_{ij}uv + v^2)),
$$

and $t_i > 0, 1 \leq i \leq n$.

1. *Define the events $A_i = \{X_i > t_i\}, 1 \leq i \leq n$. If $t_i - \sigma_{ij}t_j > 0$ and $t_j - \sigma_{ij}t_j > 0$, we*

have

$$\sum_{1\le i\le n}\frac{\phi(t_i)t_i}{1+t_i^2}-\sum_{1\le i<j\le n}\left\{f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}\right\}$$
$$\le \mathbb{P}\left(\bigcup_{i=1}^n A_i|\boldsymbol{\Sigma}\right)\le \sum_{1\le i\le n}\frac{\phi(t_i)}{t_i}; \tag{S3.14}$$

2. *Define the events* $A_i=\{|X_i|>t_i\}, 1\le i\le n$. *If* $t_i\pm\sigma_{ij}t_j>0$ *and* $t_j\pm\sigma_{ij}t_j>0$, *we have*

$$\sum_{1\le i\le n}\frac{2\phi(t_i)t_i}{1+t_i^2}-\sum_{1\le i<j\le n}\left\{2f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}+2f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i+\sigma_{ij}t_j)(t_j+\sigma_{ij}t_i)}\right\}$$
$$\le \mathbb{P}\left(\bigcup_{i=1}^n A_i|\boldsymbol{\Sigma}\right)\le \sum_{1\le i\le n}\frac{2\phi(t_i)}{t_i}. \tag{S3.15}$$

Proof. To ease the notation, we write $\mathbb{P}(\cdot|\boldsymbol{\Sigma})$ as $\mathbb{P}(\cdot)$ throughout the proof.

1. According to Bonferroni inequality in Lemma A1,

$$\sum_{i=1}^n\mathbb{P}(A_i)-\sum_{1\le i<j\le n}\mathbb{P}(A_i\cap A_j)\le\mathbb{P}(\cup_{i=1}^n A_i)\le\sum_{i=1}^n\mathbb{P}(A_i). \tag{S3.16}$$

By (S3.10), if $t_i-\sigma_{ij}t_j>0$ and $t_j-\sigma_{ij}t_j>0$,

$$\mathbb{P}(A_i\cap A_j)\le f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}.$$

Applying the univariate Mill's inequality in (S3.8), we have

$$\sum_{1\le i\le n}\frac{\phi(t_i)t_i}{1+t_i^2}-\sum_{1\le i<j\le n}\left\{f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}\right\}\le\mathbb{P}(\cup_{i=1}^n A_i)\le\sum_{1\le i\le n}\frac{\phi(t_i)}{t_i}.$$

2. Similar to the above deduction, by (S3.10), if $t_i\pm\sigma_{ij}t_j>0$ and $t_j\pm\sigma_{ij}t_j>0$,

$$\begin{aligned}\mathbb{P}(A_i\cap A_j)&=2\mathbb{P}(X_i>t_i,X_j>t_j)+2\mathbb{P}(X_i>t_i,X_j<-t_j)\\&\le 2f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}+2f_{ij}(t_i,-t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i+\sigma_{ij}t_j)(t_j+\sigma_{ij}t_i)}.\end{aligned}$$

Together with (S3.9), we have

$$\sum_{1\leq i\leq n}\frac{2\phi(t_i)t_i}{1+t_i^2}-\sum_{1\leq i<j\leq n}\left\{2f_{ij}(t_i,t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i-\sigma_{ij}t_j)(t_j-\sigma_{ij}t_i)}+2f_{ij}(t_i,-t_j)\frac{(1-\sigma_{ij}^2)^2}{(t_i+\sigma_{ij}t_j)(t_j+\sigma_{ij}t_i)}\right\}$$
$$\leq\mathbb{P}\left(\cup_{i=1}^n A_i\right)\leq\sum_{1\leq i\leq n}\frac{2\phi(t_i)}{t_i}.$$

□

Lemma A8. *Consider $f_i(\cdot)$ satisfying Assumption 1, $i=1,2,\ldots,n$. For $i<j$, $f_i(x)<f_j(x)$, $x\in[0,1]$ if and only if $f_i^{-1}(y)<f_j^{-1}(y)$, $\forall y$.*

Proof. Sufficiency. Suppose that $f_i^{-1}(y_0)\geq f_j^{-1}(y_0)$ for some y_0. Let $x_{i0}=f_i^{-1}(y_0)$ and $x_{j0}=f_j^{-1}(y_0)$. Then $f_j(x_{i0})\leq y_0\leq f_i(x_{j0})$. Note also that $f_i(x_{j0})<f_j(x_{j0})=y_0=f_i(x_{i0})<f_j(x_{i0})$. This is a contradiction! Thus, $f_i^{-1}(y)<f_j^{-1}(y)$, $\forall y$.

Necessity. Because f_i^{-1} is also strictly decreasing, it follows from the proof of sufficiency that $f_i(x)<f_j(x)$, $x\in[0,1]$ if $f_i^{-1}(y)<f_j^{-1}(y)$, $\forall y$.

□

S4 Proofs of Main Results

S4.1 Proof of Lemma 1

Proof. To ease the notation, we write $\mathbb{P}(\cdot|\mathbf{I})$ as $\mathbb{P}(\cdot)$, $T_{minP}(\boldsymbol{P})$ as T_{minP}, $T_{maxN}(\boldsymbol{X})$ as T_{maxN}, and $T_{maxhN}(\boldsymbol{X})$ as T_{maxhN} throughout the proof.

For $p \to 0$,

$$\begin{aligned}
\mathbb{P}(T_{minP} < p) =& 1 - \mathbb{P}(T_{minP} \geq p) \\
=& 1 - \mathbb{P}(P_i \geq p)^n \\
=& 1 - (1-p)^n \\
=& np - \sum_{i=2}^{n} \binom{n}{i} (-p)^i \\
=& np + O\left(p^2\right).
\end{aligned}$$

For $t \to \infty$, since $\bar{\Phi}(t) \to 0$,

$$\begin{aligned}
\mathbb{P}(T_{maxN} > t) =& 1 - \mathbb{P}(T_{maxN} \leq t) \\
=& 1 - \mathbb{P}(X_i \leq t)^n \\
=& 1 - (1 - \bar{\Phi}(t))^n \\
=& n\bar{\Phi}(t) + O(\bar{\Phi}^2(t)).
\end{aligned}$$

By (S3.8), $\lim_{t\to\infty} \frac{\bar{\Phi}(t)}{\phi(t)/t} = 1$. As a result, $\mathbb{P}(T_{maxN} > t) \sim n\phi(t)/t$.

Similarly,

$$\begin{aligned}
\mathbb{P}(T_{maxhN} > t) =& 1 - \mathbb{P}(T_{maxhN} \leq t) \\
=& 1 - \mathbb{P}(|X_i| \leq t)^n \\
=& 1 - (1 - 2\bar{\Phi}(t))^n \\
=& 2n\bar{\Phi}(t) + O(\bar{\Phi}^2(t)) \sim \frac{2n\phi(t)}{t}.
\end{aligned}$$

$\square$

S4.2 Proof of Theorem 1

Proof. To simplify the notation, we write $\mathbb{P}(\cdot|\boldsymbol{\Sigma})$ as $\mathbb{P}(\cdot)$, $T_{maxN}(\boldsymbol{X})$ as T_{maxN}, and $T_{maxhN}(\boldsymbol{X})$ as T_{maxhN}.

We first prove the case of one-sided p-values. The tail probability can be expressed as the union of individual events

$$\mathbb{P}\left(T_{maxN} > t\right) = \mathbb{P}(\bigcup_{i=1}^{n} A_i),$$

where $A_i = \{X_i > t\}$, $1 \le i \le n$. By Bonferroni Inequality (S3.16),

$$\begin{aligned} \mathbb{P}(T_{maxN} > t) &= \sum_{i=1}^{n} \mathbb{P}(A_i) + O(\sum_{i<j} \mathbb{P}(A_i \cap A_j)) \\ &= n\mathbb{P}(X_1 > t) + O(\max_{i<j} \mathbb{P}(A_i \cap A_j)). \end{aligned} \tag{S4.17}$$

Let $f_{ij}(x_i, x_j)$ be the joint density function of X_i and X_j. We have

$f_{ij}(t,t) = \frac{1}{2\pi\sqrt{1-\sigma_{ij}^2}} \exp\left(-\frac{t^2}{1+\sigma_{ij}}\right)$.

By Lemma A5,

$$\begin{aligned} \mathbb{P}(A_i \cap A_j) &= \mathbb{P}\left(X_i > t, X_j > t\right) \\ &\sim \frac{\exp(-\frac{t^2}{1+\sigma_{ij}})}{2\pi\sqrt{1-\sigma_{ij}^2}} \cdot \frac{(1-\sigma_{ij}^2)^2}{(1-\sigma_{ij})^2 t^2} = C\frac{\exp(-\frac{t^2}{1+\sigma_{ij}})}{t^2}. \end{aligned}$$

Thus, $\mathbb{P}(T_{maxN} > t) = n\mathbb{P}(X_1 > t) + O(t^{-2}\exp(-t^2/(1+\max\sigma_{ij})))$.

Similarly, letting $A_i = \{|X_i| > t\}$, $1 \le i \le n$, we have

$$\mathbb{P}\left(T_{maxhN} > t\right) = \mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) + O(\sum_{i<j} \mathbb{P}(A_i \cap A_j)).$$

By Lemma A5,

$$\begin{aligned}\mathbb{P}(A_i \cap A_j) =& \mathbb{P}\left(X_1 > t, X_2 > t\right) + \mathbb{P}\left(X_1 < -t, X_2 < -t\right) \\ &+ \mathbb{P}\left(X_1 < -t, X_2 > t\right) + \mathbb{P}\left(X_1 > t, X_2 < -t\right) \\ =& C_1 \frac{\exp(-t^2/(1+\sigma_{ij}))}{t^2} + C_2 \frac{\exp(-t^2/(1-\sigma_{ij}))}{t^2} \\ =& O(t^{-2}\exp(-t^2/(1+|\sigma_{ij}|))).\end{aligned}$$

Therefore, $\mathbb{P}\left(T_{maxhN} > t\right) = n\mathbb{P}(|X_1| > t) + O(t^{-2}\exp(-t^2/(1+\max_{i\neq j}|\sigma_{ij}|)))$.

Moreover, for one-sided p-values, let $t = \bar{\Phi}^{-1}(p)$. Then

$$\mathbb{P}(T_{minP}(\boldsymbol{P}) < p) = \mathbb{P}(T_{maxN} > t) = n\mathbb{P}(A_1) + O\left(\sum_{i<j}\mathbb{P}(A_i \cap A_j)\right),$$

where $A_i = \{X_i > t\}$, $i = 1, \ldots, n$. Note that $\mathbb{P}(X_i > t) = p$ and $\mathbb{P}(A_i \cap A_j) = o(p)$ due to (S3.12), as $p \to 0$. We have

$$\mathbb{P}\left(T_{minP} < p\right) = np + o(p). \tag{S4.18}$$

Similarly, for two-sided p-values, let $t = \bar{\Phi}^{-1}(p/2)$ and $A_i = \{|X_i| > t\}$, $i = 1, \ldots, n$. $\mathbb{P}(|X_i| > t) = p$ for all $i = 1, ..., n$. Then

$$\mathbb{P}(T_{minP} < p) = \mathbb{P}(T_{maxhN} > t) = n\mathbb{P}(A_1) + O\left(\sum_{i<j}\mathbb{P}(A_i \cap A_j)\right). \tag{S4.19}$$

It follows from (S3.13) that, as $p \to 0$,

$$\mathbb{P}\left(T_{minP} < p\right) = np + o(p). \tag{S4.20}$$

□

S4.3 Proof of Corollary 2

Proof. Note that $\mathbb{P}\left(T_{maxN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right) = \mathbb{P}\left(\max_i X_i > t|\boldsymbol{\Sigma}\right) = \mathbb{P}\left(\bigcup_{i=1}^n\{X_i > t\}|\boldsymbol{\Sigma}\right)$ and

$\mathbb{P}\left(T_{maxhN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right) = \mathbb{P}\left(\bigcup_{i=1}^{n}\{|X_i| > t\}|\boldsymbol{\Sigma}\right)$. So, Corollary 2 is a direct application of Lemma A7 with $t_i = t$ for all i.

□

S4.4 Proof of Theorem 2

Proof. To simplify the notation, we write $T_{Simes}(\boldsymbol{P})$ as T_{Simes} and $\mathbb{P}\left(T_{Simes} < p|\boldsymbol{\Sigma}\right)$ as $\mathbb{P}\left(T_{Simes} < p\right)$.

First, the input p-values are one-sided, as defined in (3.11). Accordingly, the ordered p-values are $P_{(i)} = \Phi^{-1}(X_{(n-i+1)})$, where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics of $X_1, ..., X_n$. The tail probability of the Simes statistic becomes

$$\begin{aligned}
\mathbb{P}\left(T_{Simes} < p\right) &= \mathbb{P}\left(\min \frac{P_{(i)}n}{i} < p\right) \\
&= 1 - \mathbb{P}\left(P_{(1)} \geq \frac{p}{n}, P_{(2)} \geq \frac{2p}{n}, \ldots, P_{(n)} \geq p\right) \\
&= 1 - \mathbb{P}\left(X_{(n)} \leq \bar{\Phi}^{-1}(\frac{p}{n}), ..., X_{(2)} \leq \bar{\Phi}^{-1}(\frac{(n-2)p}{n}), X_{(1)} \leq \bar{\Phi}^{-1}(p)\right) \\
&= 1 - \mathbb{P}\left(X_{(n)} \leq b_n, ..., X_{(2)} \leq b_2, X_{(1)} \leq b_1\right),
\end{aligned}$$

where $b_i = \bar{\Phi}^{-1}(\frac{(n-i+1)p}{n}), 1 \leq i \leq n$.

Because b_i is increasing in i, applying (S3.3) in Lemma A2 yields the following:

$$\begin{aligned}
&\mathbb{P}\left(X_{(n)} \leq b_n, ..., X_{(2)} \leq b_2, X_{(1)} \leq b_1\right) \\
=&\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(X_i < b_1) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(X_i \geq b_{n-j})}{j+1} - \frac{I(X_i \geq b_{n-j+1})}{j}\right\} \times \mathbb{P}\left(X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}|X_i\right)\right]
\end{aligned} \tag{S4.21}$$

$$=1 - p + \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(X_i \geq b_{n-j})}{j+1} - \frac{I(X_i \geq b_{n-j+1})}{j}\right\} \times \mathbb{P}\left(X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}|X_i\right)\right],$$

where recall that $X_{(1)}^{(-i)}, \dots, X_{(n-1)}^{(-i)}$ are the order statistics of the variables $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$. The expectation term in the double-summation is

$$\begin{aligned}
&\mathbb{E}\left[\left\{\frac{I(X_i \geq b_{n-j})}{j+1} - \frac{I(X_i \geq b_{n-j+1})}{j}\right\}\mathbb{P}\left(X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j} | X_i\right)\right] \qquad \text{(S4.22)}\\
=&\frac{1}{j+1}\mathbb{P}\left(X_i \geq b_{n-j}, X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}\right)\\
&-\frac{1}{j}\mathbb{P}\left(X_i \geq b_{n-j+1}, X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}\right).
\end{aligned}$$

For $j = 1, \dots, n-1$, let $E_{i_1,i_2,\dots,i_{n-j}}$ be the event $\{X_{(1)} = X_{i_1}, X_{(2)} = X_{i_2}, ..., X_{(n-j)} = X_{i_{n-j}}\}$, where $i_1, i_2, ..., i_{n-j}$ are distinct indices from $\{1, 2, ..., i-1, i+1, ..., n\}$. Write $\mathbb{P}_E(\cdot)$ as the conditional probability $\mathbb{P}(\cdot \mid E_{i_1,i_2,\dots,i_{n-j}})$. Then we have

$$\begin{aligned}
&\mathbb{P}\left(X_i \geq b_{n-j}, X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}\right) \qquad \text{(S4.23)}\\
=&\sum_{i_1,i_2,\dots,i_{n-j}\in\{1,2,\dots,i-1,i+1,\dots,n\}} \mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, .., X_{i_{n-j}} \leq b_{n-j}\right)\mathbb{P}\left(E_{i_1,i_2,\dots,i_{n-j}}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, .., X_{i_{n-j}} \leq b_{n-j}\right)\\
=&\mathbb{P}_E\left(X_i \geq b_{n-j}\right) - \mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} > b_1\right)\\
&- \mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, X_{i_2} > b_2\right)\\
&- ... - \mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, .., X_{i_{n-j-1}} \leq b_{n-j-1}, X_{i_{n-j}} > b_{n-j}\right).
\end{aligned}$$

Because b_i is increasing in i,

$$\begin{aligned}
\mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} > b_1\right) \leq&\mathbb{P}_E\left(X_i \geq b_1, X_{i_1} > b_1\right)\\
=&\frac{\mathbb{P}\left(X_i \geq b_1, X_{i_1} > b_1, E_{i_1,i_2,\dots,i_{n-j}}\right)}{\mathbb{P}\left(E_{i_1,i_2,\dots,i_{n-j}}\right)} \qquad \text{(S4.24)}\\
\leq&\frac{\mathbb{P}\left(X_i \geq b_1, X_{i_1} > b_1\right)}{\mathbb{P}\left(E_{i_1,i_2,\dots,i_{n-j}}\right)}. \qquad \text{(S4.25)}
\end{aligned}$$

Similarly, for $k = 2, 3, \ldots, n-j$, we have

$$\mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, ..., X_{i_{k-1}} < b_{k-1}, X_{i_k} \geq b_k\right) \leq \frac{\mathbb{P}\left(X_i \geq b_k, X_{i_k} \geq b_k\right)}{\mathbb{P}\left(E_{i_1,i_2,...,i_{n-j}}\right)}.$$

Note also that $P\left(X_i \geq b_k, X_{i_k} \geq b_k\right) = o(\bar{\Phi}\left(b_k\right))$ for $k = 1, 2, \ldots, n-j$, due to Lemma A6. As a result, the summand in (S4.23) becomes

$$\begin{aligned} &\mathbb{P}_E\left(X_i \geq b_{n-j}, X_{i_1} \leq b_1, .., X_{i_{n-j}} \leq b_{n-j}\right) \mathbb{P}\left(E_{i_1,i_2,...,i_{n-j}}\right) \\ =&\mathbb{P}\left(X_i \geq b_{n-j}, E_{i_1,i_2,...,i_{n-j}}\right) + o(\bar{\Phi}\left(b_1\right)), \end{aligned}$$

and thus,

$$\begin{aligned} \mathbb{P}\left(X_i \geq b_{n-j}, X^{(-i)}_{(1)} \leq b_1, .., X^{(-i)}_{(n-j)} \leq b_{n-j}\right) =&\mathbb{P}\left(X_i \geq b_{n-j}\right) + o(\bar{\Phi}\left(b_1\right)) \\ =&\frac{j+1}{n}p + o(p). \end{aligned}$$

Similarly,

$$\begin{aligned} P\left(X_i \geq b_{n-j+1}, X^{(-i)}_{(1)} \leq b_1, .., X^{(-i)}_{(n-j)} \leq b_{n-j}\right) =& \mathbb{P}\left(X_i \geq b_{n-j+1}\right) + o(\bar{\Phi}\left(b_1\right)) \\ =& \frac{j}{n}p + o(p). \end{aligned}$$

Clearly, (S4.22) becomes

$$\mathbb{E}\left[\left\{\frac{I(X_i \geq b_{n-j})}{j+1} - \frac{I(X_i \geq b_{n-j+1})}{j}\right\} \mathbb{P}\left(X^{(-i)}_{(1)} \leq b_1, ..., X^{(-i)}_{(n-j)} \leq b_{n-j} | X_i\right)\right] =o(p). \tag{S4.26}$$

By (S4.21) and (S4.26), for any fixed n,

$$\mathbb{P}\left(X_{(n)} \leq \bar{\Phi}^{-1}(\frac{p}{n}), ..., X_{(2)} \leq \bar{\Phi}^{-1}(\frac{2p}{n}), X_{(1)} \leq \bar{\Phi}^{-1}(p)\right) = 1 - p + o(p), \tag{S4.27}$$

and thus the tail probability of the Simes test is

$$\mathbb{P}\left(T_{Simes} < p\right) = p + o(p), \tag{S4.28}$$

which is independent of the covariance matrix $\mathbf{\Sigma}$. Therefore, the equation in Theorem 2 follows. Note that we can also directly compare (S4.28) with the independence case when $\mathbf{\Sigma} = \mathbf{I}$, T_{Simes} follows a Uniform(0,1), i.e., $\mathbb{P}\left(T_{Simes} < p|\mathbf{I}\right) = p$ (Simes, 1986).

For the two-sided input p-values, $P_{(i)} = 2\bar{\Phi}(|X|_{(n-i+1)})$. Then,

$$\mathbb{P}\left(T_{Simes} < p\right) = 1 - \mathbb{P}\left(|X|_{(n)} < \bar{\Phi}(\frac{p}{2n}), ..., |X|_{(1)} < \bar{\Phi}(\frac{p}{2n})\right).$$

Let $b_i = \bar{\Phi}^{-1}(\frac{(n-i+1)p}{2n})$. Similar to the proof of (S4.27), we have

$$\begin{aligned}
&\mathbb{P}\left(T_{Simes} < p\right) \\
=&1 - \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_i| < b_1) \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(|X_i| \geq b_{n-j})}{j+1} - \frac{I(|X_i| \geq b_{n-j+1})}{j}\right\} \times \mathbb{P}\left(|X|_{(1)}^{(-i)} \leq b_1, .., |X|_{(n-j)}^{(-i)} \leq b_{n-j}|X_i\right)\right] \\
=&1 - \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_i| < \bar{\Phi}^{-1}(\frac{p}{2})) + o(p) \\
=&p + o(p), \qquad \text{(S4.29)}
\end{aligned}$$

which does not depend on the covariance matrix $\mathbf{\Sigma}$. □

S4.5 Proof of Theorem 3

Proof. Throughout the proof, we will write $T_{Simes}(\boldsymbol{P})$ as T_{Simes} for ease of notation.

Without loss of generality, we assume that the m perfectly dependent p-values are $P_{n-m+1} = P_{n-m+2} = ... = P_n$. Let $P_i^{\#} = P_i$, for $i = 1, 2, ..., n-m+1$. Under the assumption, $P_i^{\#}$, $i = 1, 2, ..., n-m+1$, are i.i.d. $U(0, 1)$.

For $l = 1, 2, \ldots, n-m+1$, we define A_l as the event that $P_{n-m+1}^{\#}$ is the lth ordered p-value $P_{(l)}^{\#}$, i.e.,

$$A_l = \{P_{n-m+1}^{\#} = P_{(l)}^{\#}\}.$$

The tail probability of the Simes test can be decomposed in the following way:

$$\mathbb{P}(T_{Simes} < p) = \sum_{l=1}^{n-m+1} \mathbb{P}(T_{Simes} < p | A_l)\mathbb{P}(A_l), \tag{S4.30}$$

where $\mathbb{P}(A_l) = (n-m+1)^{-1}, 1 \le l \le n-m+1$.

Note that

$$\begin{aligned}\mathbb{P}(T_{Simes} < p | A_l) &= \mathbb{P}(\min_{k=1,2,\dots,n}\{P_{(k)} \cdot n/k\} < p | A_l)\\ &= \mathbb{P}\left(\min\{P_{(1)}\frac{n}{1}, ..., P_{(m)}\frac{n}{m}, P_{(m+1)}\frac{n}{m+1}, ..., P_{(n)}\frac{n}{n}\} < p | A_l\right).\end{aligned}$$

When $l = 1$, $P_{(1)} = ... = P_{(m)}$ and thus $\min\{P_{(1)}\frac{n}{1}, ..., P_{(m)}\frac{n}{m}\} = P_{(m)}\frac{n}{m}$. We have

$$\begin{aligned}\mathbb{P}(T_{Simes} < p | A_1) &= \mathbb{P}\left(\min\{P_{(m)}\frac{n}{m}, P_{(m+1)}\frac{n}{m+1}, ..., P_{(n)}\frac{n}{n}\} < p\right)\\ &= 1 - \mathbb{P}\left(\min\{P_{(m)}\frac{n}{m}, P_{(m+1)}\frac{n}{m+1}, ..., P_{(n)}\frac{n}{n}\} \ge p\right)\\ &= 1 - \mathbb{P}\left(P_{(m)}\frac{n}{m} \ge p, P_{(m+1)}\frac{n}{m+1} \ge p, ..., P_{(n)}\frac{n}{n} \ge p\right)\\ &= 1 - \mathbb{P}\left(P_{(m)} \ge \frac{mp}{n}, P_{(m+1)} \ge \frac{(m+1)p}{n}, ..., P_{(n)} \ge p\right)\\ &= 1 - \mathbb{P}\left(P^{\#}_{(1)} \ge \frac{mp}{n}, P^{\#}_{(2)} \ge \frac{(m+1)p}{n}, ..., P^{\#}_{(n-m+1)} \ge p\right).\end{aligned}$$

Recall that $P^{\#}_{(k)}$, $k = 1, ..., n-m+1$, are the order statistics of $\{P^{\#}_i, i = 1, 2, ..., n-m+1\}$ which are i.i.d. $U(0,1)$. By (S3.4) in Lemma A2, for $p^{\#}_k = \frac{m+k-1}{n}p, 1 \le k \le n-m+1$,

$$\begin{aligned}&\mathbb{P}(P^{\#}_{(1)} \ge p^{\#}_1, ..., P^{\#}_{(n-m+1)} \ge p^{\#}_{n-m+1})\\ =&1 - p + (n-m+1)\sum_{j=1}^{n-m}\left[\frac{p^{\#}_{j+1}}{j+1} - \frac{p^{\#}_j}{j}\right]\mathbb{P}\left(P^{\#}_{(j):n-m} \ge p^{\#}_{j+1}, ..., P^{\#}_{(n-m):n-m} \ge p^{\#}_{n-m+1}\right)\end{aligned}$$

where $P^{\#}_{(n-m-q+1):n-m}, ..., P^{\#}_{(n-m):n-m}$ are the order statistics of the $(n-m)$-dimensional random vector obtained by eliminating $P^{\#}_i$ from $\mathbf{P}^{\#}$. Following the joint distribution of ordered

U(0, 1) random variables,

$$\begin{aligned}&\mathbb{P}\left(P^{\#}_{(j):n-m} \geq p^{\#}_{j+1}, ..., P^{\#}_{(n-m):n-m} \geq p^{\#}_{n-m+1}\right)\\ =&\int_{\frac{m+j}{n}p}^{1} ... \int_{p}^{1} \frac{(n-m)!}{(j-1)!}(p_j)^{j-1} dp_j...dp_{n-m}\\ \sim&\frac{(n-m)!}{j!}, \text{ as } p \to 0,\end{aligned}$$

we have

$$\begin{aligned}&\mathbb{P}\left(T_{Simes} < p|A_1\right) \\ \sim&p - (n-m+1)\sum_{j=1}^{n-m}\left[\frac{\frac{m+j}{n}p}{j+1} - \frac{\frac{m+j-1}{n}p}{j}\right]\frac{(n-m)!}{j!}.\end{aligned} \tag{S4.31}$$

Similarly, for $l \neq 1$, the conditional probability $\mathbb{P}(T_{Simes} < p|A_l)$ can be expressed as the joint probability of the order statistics $P^{\#}_{(k)}, 1 \leq k \leq n-m+1$. In particular, for $l = n-m+1$, the conditional tail probability is

$$\begin{aligned}&\mathbb{P}(T_{Simes} < p|A_{n-m+1})\\ =&\mathbb{P}\left(\min\{P_{(1)}\frac{n}{1}, P_{(2)}\frac{n}{2}, ..., P_{(n-m)}\frac{n}{n-m}, ..., P_{(n)}\frac{n}{n}\} < p\right)\\ =&1 - \mathbb{P}\left(P^{\#}_{(1)} \geq \frac{p}{n}, P^{\#}_{(2)} \geq \frac{2p}{n}, ..., P^{\#}_{(n-m)} \geq \frac{(n-m)p}{n}, P^{\#}_{(n-m+1)} \geq p\right).\end{aligned}$$

Using a similar argument with $p^{\#}_k = \frac{k}{n}p, 1 \leq k \leq n+m$, and $p^{\#}_{n-m+1} = p$, we obtain

$$\begin{aligned}&\mathbb{P}(T_{Simes} < p|A_{n-m+1})\\ =&1 - \left(1 - p + (n-m+1)\left(\frac{p}{n-m+1} - \frac{\frac{(n-m)p}{n}}{n-m}\right)\mathbb{P}(P^{\#}_{(n-m):n-m} \geq p)\right)\\ \sim&p - (n-m+1)\left(\frac{p}{n-m+1} - \frac{p}{n}\right).\end{aligned} \tag{S4.32}$$

For $l = 2, \ldots, n-m$, the conditional tail probability becomes

$$\mathbb{P}(T_{Simes} < p|A_l)$$

$$
\begin{aligned}
=&\mathbb{P}\left(\min\{P_{(1)}\frac{n}{1},...,P_{(l-1)}\frac{n}{l-1},P_{(l)}\frac{n}{l},...,P_{(l+m-1)}\frac{n}{l+m-1},...,P_{(n)}\frac{n}{n}\}<p\right)\\
=&1-\mathbb{P}\left(P^{\#}_{(1)}>\frac{p}{n},...,P^{\#}_{(l-1)}>\frac{(l-1)p}{n},P^{\#}_{(l)}>\frac{(l+m-1)p}{n},...,P^{\#}_{(n-m+1)}>p\right)\\
\sim&p-(n-m+1)\left[\sum_{j=l}^{n-m}\left(\frac{\frac{(j+m)p}{n}}{j+1}-\frac{\frac{(j+m-1)p}{n}}{j}\right)\frac{(n-m)!}{j!}\right.\\
&\qquad\left.+\left(\frac{\frac{(l+m-1)p}{n}}{l}-\frac{p}{n}\right)\frac{(n-m)!}{(l-1)!}\right]. \qquad (S4.33)
\end{aligned}
$$

In view of (S4.30) – (S4.33), we have

$$
\mathbb{P}(T_{Simes}<p)\sim p-I_{sum},
$$

where

$$
\begin{aligned}
I_{sum}=&\sum_{l=1}^{n-m}\left[\sum_{j=l}^{n-m}\left(\frac{\frac{(j+m)p}{n}}{j+1}-\frac{\frac{(j+m-1)p}{n}}{j}\right)\frac{(n-m)!}{j!}\right]\\
&+\sum_{l=2}^{n-m+1}\left(\frac{(m+l-1)p}{ln}-\frac{p}{n}\right)\frac{(n-m)!}{(l-1)!}\\
\doteq&I_{sum1}+I_{sum2}.
\end{aligned}
$$

Let $C_{sum}=n^{-1}p(n-m)!$. Note that by $\frac{j+m}{j+1}-\frac{j+m-1}{j}=(m-1)\left(\frac{1}{j+1}-\frac{1}{j}\right)$,

$$
\begin{aligned}
I_{sum1}=&C_{sum}\sum_{l=1}^{n-m}\left[\sum_{j=l}^{n-m}\left(\frac{j+m}{j+1}-\frac{j+m-1}{j}\right)\frac{1}{j!}\right]\\
=&C_{sum}\,(m-1)\sum_{l=1}^{n-m}\sum_{j=l}^{n-m}\left(\frac{1}{j+1}-\frac{1}{j}\right)\frac{1}{j!}\\
=&-C_{sum}\,(m-1)\sum_{j=1}^{n-m}\frac{1}{(j+1)!},
\end{aligned}
$$

and

$$
\begin{aligned}
I_{sum2}=&\sum_{i=1}^{n-m}\left(\frac{(m+l)p}{(l+1)n}-\frac{p}{n}\right)\frac{(n-m)!}{i!}\\
=&C_{sum}(m-1)\sum_{i=1}^{n-m}\frac{1}{l+1}\frac{1}{i!}
\end{aligned}
$$

$$=C_{sum}\left(m-1\right)\sum_{l=1}^{n-m}\frac{1}{(l+1)!}.$$

We have $I_{sum}=I_{sum1}+I_{sim2}=0$ and thus $\mathbb{P}(T_{Simes}<p)\sim p$. □

S4.6 Proof of Theorem 4

Proof. First consider the one-sided input p-values. Note that $f_i(x)\triangleq f(i/n,x)$ is strictly decreasing in $x\in[0,1]$. The tail probability of the test statistic is

$$\begin{aligned}\mathbb{P}(T_f(\boldsymbol{P})>t)=&1-\mathbb{P}(\max_i f_i(P_{(i)})<t)\\ =&1-\mathbb{P}(P_{(i)}>p_i, i=1,2,\dots,n)\\ =&1-\mathbb{P}(X_{(i)}<b_i, i=1,2,\dots,n),\end{aligned}\tag{S4.34}$$

where $p_i=f_i^{-1}(t)$ and $b_i=\bar{\Phi}^{-1}\left(f_{n-i+1}^{-1}(t)\right)$. Because b_i is non-decreasing in i, by Lemma A2, we have

$$\begin{aligned}&\mathbb{P}\left(X_{(n)}\le b_n,...,X_{(2)}\le b_2,X_{(1)}\le b_1\right)\\ =&\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(X_i<b_1)\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(X_i\ge b_{n-j})}{j+1}-\frac{I(X_i\ge b_{n-j+1})}{j}\right\}\times\mathbb{P}\left(X_{(1)}^{(-i)}\le b_1,..,X_{(n-j)}^{(-i)}\le b_{n-j}|X_i\right)\right]\\ =&1-p_n+\sum_{i=1}^{n}\sum_{j=1}^{n-1}\left[\frac{1}{j+1}\mathbb{P}\left(X_i\ge b_{n-j},X_{(1)}^{(-i)}\le b_1,..,X_{(n-j)}^{(-i)}\le b_{n-j}\right)\right.\\ &\qquad\left.-\frac{1}{j}\mathbb{P}\left(X_i\ge b_{n-j+1},X_{(1)}^{(-i)}\le b_1,..,X_{(n-j)}^{(-i)}\le b_{n-j}\right)\right].\end{aligned}$$

Similar to the proof of Theorem 2, we have

$$\begin{aligned}\mathbb{P}\left(X_i\ge b_{n-j},X_{(1)}^{(-i)}\le b_1,..,X_{(n-j)}^{(-i)}\le b_{n-j}\right)=&\mathbb{P}\left(X_i\ge b_{n-j}\right)+o(\bar{\Phi}\left(b_1\right))\\ =&p_{j+1}+o(p_n),\end{aligned}\tag{S4.35}$$

and

$$\mathbb{P}\left(X_i \geq b_{n-j+1}, X_{(1)}^{(-i)} \leq b_1, .., X_{(n-j)}^{(-i)} \leq b_{n-j}\right) = p_j + o(p_n). \tag{S4.36}$$

Plugging (S4.35) and (S4.36) into (S4.21) – (S4.23), we obtain

$$\begin{aligned}
&\mathbb{P}\left(X_{(n)} < b_n, ..., X_{(1)} < b_1\right) \\
=&1 - p_n + \sum_{i=1}^{n}\sum_{j=1}^{n-1}\left(\frac{p_{j+1}}{j+1} - \frac{p_j}{j} + o(p_n)\right) \\
=&1 - p_n + n\sum_{j=1}^{n-1}\left(\frac{p_{j+1}}{j+1} - \frac{p_j}{j}\right) + o(p_n) \\
=&1 - np_1 + o(p_n).
\end{aligned}$$

Therefore, $\mathbb{P}(T_f(\boldsymbol{P}) > t) = nf_1^{-1}(t) + o(f_n^{-1}(t))$.

For the two-sided input p-values in (3.11), let $b_i = \bar{\Phi}^{-1}\left(\frac{f_{n-i+1}^{-1}(t)}{2}\right)$. The tail probability of $T_f(\boldsymbol{P})$ becomes

$$\begin{aligned}
&\mathbb{P}(T_f(\boldsymbol{P}) > t) \\
=&1 - \mathbb{P}\left(|X|_{(i)} < b_i, i = 1, 2, \ldots, n\right) \\
=&1 - \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_i| < b_1) \\
&- \sum_{i=1}^{n}\sum_{j=1}^{n-1}\mathbb{E}\left[\left\{\frac{I(|X_i| \geq b_{n-j})}{j+1} - \frac{I(|X_i| \geq b_{n-j+1})}{j}\right\} \times \mathbb{P}\left(|X|_{(1)}^{(-i)} \leq b_1, .., |X|_{(n-j)}^{(-i)} \leq b_{n-j} | X_i\right)\right] \\
=&1 - \left(1 - p_n + \sum_{i=1}^{n}\sum_{j=1}^{n-1}\left(\frac{p_{j+1}}{j+1} - \frac{p_j}{j}\right) + o(p_n)\right) \\
=&nf_1^{-1}(t) + o(f_n^{-1}(t)).
\end{aligned}$$

Therefore, proof is completed.

$\square$

S4.7 Proof of Corollary 4

Proof. We just need to show that the conditions in Theorem 4 are satisfied.

For the HC statistic, $f_i(x) = \frac{\sqrt{n}(\frac{i}{n}-x)}{\sqrt{x(1-x)}}$, $x \in [0,1]$ with

$$f_i^{-1}(t) = \frac{t^2 - t\sqrt{t^2 + 4i(1-\frac{i}{n})} + 2i}{2(t^2+n)}, \quad t \in \mathbb{R},$$

for $i = 1, 2, \ldots, n$. Note $f_i'(x) = \sqrt{n}(x(1-x))^{-3/2}[(i/n - 1/2)x - i/(2n)]$, $x \in [0,1]$. If $i/n \leq 1/2$, $f_i'(x) < 0$ for $x \in (0,1)$; and if $i/n > 1/2$, $f_i'(x) < \sqrt{n}(x(1-x))^{-3/2}[i/(2n) - 1/2] \leq 0$, for $x \in (0,1)$. Moreover, $\lim_{x\to 0} f_i(x) = +\infty$. Thus, Assumption 1 is fulfilled and $f_1^{-1}(t) \leq f_2^{-1}(t) \leq \ldots \leq f_n^{-1}(t)$ for any t, due to Lemma A8.

Note that

$$\begin{aligned}
f_1^{-1}(t) =& \frac{t^2 - t\sqrt{t^2 + 4(1-\frac{1}{n})} + 2}{2(t^2+n)} \\
=& \frac{1 - \sqrt{1 + 4(1-\frac{1}{n})t^{-2}} + 2t^{-2}}{2(1+nt^{-2})} \\
=& \frac{1}{2}\left[1 - \left(1 + 2(1-\frac{1}{n})t^{-2} + o(t^{-2})\right) + 2t^{-2}\right]\left[1 - nt^{-2} + o(t^{-2})\right] \\
=& \frac{1}{nt^2} + o(t^{-2}), \quad t \to \infty,
\end{aligned}$$

and $f_n^{-1}(t) = \frac{n}{t^2+n}$. It follows from Theorem 4 that, as $t \to \infty$,

$$\mathbb{P}(T_{HC}(\boldsymbol{P}) > t) = \frac{1}{t^2} + o(\frac{1}{t^2}).$$

□

S4.8 Proof of Corollary 5

Proof. Write $f_s(i/n, P_{(i)})$ as $f_i(P_{(i)})$. It suffices to verify the conditions in Theorem 4.

Note that

$$f_i(x) = \sqrt{2n}(-1)^j \sqrt{\frac{1}{s(1-s)} g_i(x)}, \quad s > 1, x \in [0,1],$$

where $g_i(x) = 1 - (i/n)^s x^{1-s} - (1 - i/n)^s (1-x)^{1-s}$, and $j = 0$ if $x \leq i/n$ and $j = 1$ if $x > i/n$. Because $g_i'(x) = (s-1)\left[(i/n)^s x^{-s} - (1-i/n)^s (1-x)^{-s}\right]$ is strictly decreasing in $x \in (0,1)$ and $g_i'(i/n) = 0$, we have for $0 \leq x \leq i/n$, $g_i(x)$ is strictly increasing in x and thus $f_i(x) = \sqrt{2n[s(1-s)]^{-1} g_i(x)}$ is strictly decreasing in x; while for $i/n < x \leq 1$, $g_i(x)$ is strictly decreasing in x and thus $f_i(x) = -\sqrt{2n[s(1-s)]^{-1} g_i(x)}$ is also strictly decreasing in x. Moreover, $\lim_{x \to 0} f_i(x) = +\infty$. So, Assumption 1 is satisfied for $s > 1$.

Next we need to show that $f_i(x)$ is increasing in i for any fixed $x \in [0,1]$. Define $G(x,y) = 1 - y^s x^{1-s} - (1-y)^s (1-x)^{1-s}$, $x \in [0,1], y \in [0,1]$. The partial derivative of $G(x,y)$ with respect to y is $G_y(x,y) = -s\left[y^{s-1} x^{1-s} - (1-y)^{s-1}(1-x)^{1-s}\right]$, which is decreasing in y. Note that $G_y(x,x) = 0$. For $0 \leq y < x$, $G(x,y)$ is increasing in y and thus $g_i(x) = G(x, i/n)$ is increasing in i. Because $f_i(x) = -\sqrt{2n[s(1-s)]^{-1} g_i(x)}$, $f_i(x)$ is increasing in i for $x > i/n$. For $x \leq y \leq 1$, $G(x,y)$ is decreasing in y and thus $f_i(x) = \sqrt{2n[s(1-s)]^{-1} G(x, i/n)}$ is increasing in i for $x \leq i/n$. By Lemma A8, $f_1^{-1}(t) \leq f_2^{-1}(t) \leq \ldots \leq f_n^{-1}(t)$.

Lastly, we will evaluate $f_i^{-1}(t)$ for $t \to \infty$. For $x \to 0$ and $x < 1/n$, we have $f_n(x) = \sqrt{2n[s(1-s)]^{-1}(1 - x^{1-s})}$ and thus

$$f_n^{-1}(t) = \left[1 + (2n)^{-1} s(s-1) t^2\right]^{-1/(s-1)} = O(t^{-2/(s-1)}).$$

More generally, note that $f_i(x) = \sqrt{2n[s(1-s)]^{-1} g_i(x)}$ and $g_i(x) = 1 - (i/n)^s x^{1-s} - (1 - i/n)^s (1-x)^{1-s} = 1 - (1-i/n)^s - (i/n)^s x^{1-s} + O(x)$. We have

$$x^{1-s}(1 + O(x^s)) = (n/i)^s \left[C + (2n)^{-1} s(s-1) f_i(x)^2\right]$$

where $C = 1 - (1 - i/n)^s > 0$. This implies that

$$\begin{aligned} x(1 + O(x^s)) =& (n/i)^{s/(1-s)} \left[C + (2n)^{-1} s(s-1) f_i(x)^2\right]^{1/(1-s)} \\ =& n^{-1} \left(\frac{(s-1)\,s}{2i^s}\right)^{\frac{1}{1-s}} f_i(x)^{-2/(s-1)} + O(f_i(x)^{-2s/(s-1)}) \end{aligned}$$

So, for $x = o(1)$ or $t \to \infty$

$$f_1^{-1}(t) = n^{-1} \left(\frac{(s-1)\,s}{2i^s}\right)^{\frac{1}{1-s}} t^{-2/(s-1)} + O(t^{-2s/(s-1)}).$$

By Theorem 4, the tail probability is

$$\begin{aligned} \mathbb{P}(T_{\phi_s}(\boldsymbol{P}) > t) =& n f_1^{-1}(t) + o(f_n^{-1}(t)) \\ =& \left(\frac{(s-1)\,s}{2}\right)^{-\frac{1}{s-1}} t^{-\frac{2}{s-1}} + o(t^{-\frac{2}{s-1}}), \quad t \to \infty. \end{aligned}$$

□

S4.9 Proof of Theorem 5

Proof. By definition, $T_f(\boldsymbol{P}) = \sup_{1 \le i \le n} f_i(P_{(i)})$. Under Assumption 1, the tail probability can be written as

$$\mathbb{P}(T_f(\boldsymbol{P}) > t) = 1 - \mathbb{P}(\sup_{1 \le i \le n} f_i(P_{(i)}) < t) \tag{S4.37}$$

$$= 1 - \mathbb{P}(P_{(i)} > f_i^{-1}(t), i = 1, ..., n). \tag{S4.38}$$

For a fixed t, let $a_i = f_i^{-1}(t)$. Because $a_1 \le a_2 \le ... \le a_n$, it follows from (S3.2) that

$$\begin{aligned} &\mathbb{P}\left(P_{(1)} \ge a_1, ..., P_{(n)} \ge a_n\right) \\ =& 1 - a_n + \sum_{i=1}^{n} \sum_{j=1}^{n-1} \mathbb{E}\left[\left\{\frac{I(P_i \le a_{j+1})}{j+1} - \frac{I(P_i \le a_j)}{j}\right\} \times \mathbb{P}\left(P_{(j)}^{(-i)} \ge a_{j+1}, .., P_{(n-1)}^{(-i)} \ge a_n | P_i\right)\right] \\ =& 1 - a_n + \sum_{i=1}^{n} \sum_{j=1}^{n-1} \left[\frac{1}{j+1} \mathbb{P}\left(P_i \le a_{j+1}, P_{(j)}^{(-i)} \ge a_{j+1}, .., P_{(n-1)}^{(-i)} \ge a_n\right)\right. \\ &\left. - \frac{1}{j} \mathbb{P}\left(P_i \le a_j, P_{(j)}^{(-i)} \ge a_{j+1}, .., P_{(n-1)}^{(-i)} \ge a_n\right)\right], \end{aligned}$$

where recall that $P_{(1)}^{(-i)}, \ldots, P_{(n-1)}^{(-i)}$ are the order statistics of the variables $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n$.

For any fixed j, let $E_{i_j,i_{j+1},\ldots,i_{n-1}} = \left\{P_{(j)}^{(-i)} = P_{i_j}, P_{(j+1)}^{(-i)} = P_{i_{j+1}}, \ldots, P_{(n-1)}^{(-i)} = P_{i_{n-1}}\right\}$, where $i_j, i_{j+1}, \ldots, i_{n-1}$ are distinct indices from the set $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$, and write $\mathbb{P}_E(\cdot)$ as the conditional probability $\mathbb{P}(\cdot \mid E_{i_j,i_{j+1},\ldots,i_{n-1}})$ for simplicity. Then

$$
\begin{aligned}
&\mathbb{P}\left(P_i \leq a_{j+1}, P_{(j)}^{(-i)} \geq a_{j+1}, .., P_{(n-1)}^{(-i)} \geq a_n\right) \\
=& \sum_{i_1,i_2,\ldots,i_{n-j} \in \{1,2,\ldots,i-1,i+1,\ldots,n\}} \mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, .., P_{i_{n-1}} \geq a_n\right) \mathbb{P}\left(E_{i_j,i_{j+1},\ldots,i_{n-1}}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, .., P_{i_{n-1}} \geq a_n\right) \\
=&\mathbb{P}_E\left(P_i \leq a_{j+1}\right) - \mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \leq a_{j+1}\right) - \mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, P_{i_{j+1}} \leq a_{j+2}\right) \\
&- \ldots - \mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, .., P_{i_{n-2}} \geq a_{n-1}, P_{i_{n-1}} \leq a_n\right),
\end{aligned}
$$

For $k = 1, \ldots, n-1-j$,

$$
\begin{aligned}
&\mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, ..P_{i_{j+k-1}} \geq a_{j+k}, P_{i_{j+k}} \leq a_{j+k+1}\right) \\
\leq&\mathbb{P}_E\left(P_i \leq a_{j+k+1}, P_{i_{j+k}} \leq a_{j+k+1}\right) \\
\leq&\frac{\mathbb{P}\left(P_i \leq a_{j+k+1}, P_{i_{j+k}} \leq a_{j+k+1}\right)}{\mathbb{P}\left(E_{i_j,i_{j+1},\ldots,i_{n-1}}\right)}
\end{aligned}
$$

by noting that a_j is non-decreasing in j. Because (4.17) can be equivalently expressed in terms of the joint distribution of the p-values as

$$
\mathbb{P}(P_i \leq p, P_j \leq p) = o(p), \quad \forall p \to 0, \tag{S4.39}
$$

we obtain $\mathbb{P}\left(P_i \leq a_{j+k+1}, P_{i_{j+k}} \leq a_{j+k+1}\right) = o(a_{j+k+1})$ for $k = 0, 1, \ldots, n-1-j$. Consequently,

$$
\begin{aligned}
&\mathbb{P}_E\left(P_i \leq a_{j+1}, P_{i_j} \geq a_{j+1}, .., P_{i_{n-1}} \geq a_n\right) \mathbb{P}\left(E_{i_j,i_{j+1},\ldots,i_{n-1}}\right) \\
=&\mathbb{P}\left(P_i \leq a_{j+1}, E_{i_j,i_{j+1},\ldots,i_{n-1}}\right) + o(a_n).
\end{aligned}
$$

This implies that

$$\begin{aligned}
&\mathbb{P}\left(P_i \leq a_{j+1}, P_{(j)}^{(-i)} \geq a_{j+1}, .., P_{(n-1)}^{(-i)} \geq a_n\right) \\
=&\sum_{i_1,i_2,\ldots,i_{n-j}\in\{1,2,\ldots,i-1,i+1,\ldots,n\}} \left(\mathbb{P}\left(P_i \leq a_{j+1}, E_{i_j,i_{j+1},\ldots,i_{n-1}}\right) + o(a_n)\right) \\
=&\mathbb{P}\left(P_i \leq a_{j+1}\right) + o(a_n) \\
=&a_{j+1} + o(a_n).
\end{aligned}$$

Similarly,

$$\mathbb{P}\left(P_i \leq a_j, P_{(j)}^{(-i)} \geq a_{j+1}, .., P_{(n-1)}^{(-i)} \geq a_n\right) = a_j + o(a_n).$$

As a result,

$$\begin{aligned}
\mathbb{P}(T_f(\boldsymbol{P}) > t) =& a_n - \sum_{i=1}^{n}\sum_{j=1}^{n-1}\left(\frac{a_{j+1}}{j+1} - \frac{a_j}{j} + o(a_n)\right) \\
=& a_n - n\sum_{j=1}^{n-1}\left(\frac{a_{j+1}}{j+1} - \frac{a_j}{j}\right) + o(a_n) \\
=& na_1 + o(a_n)
\end{aligned}$$

which is $nf_1^{-1}(t) + o(f_n^{-1}(t))$.

□

S4.10 Proof of Remark 1

Proof. 1. Product Copula:

$$C(u,u) - (2u-1) = (1-u)(1-u) = o(1-u).$$

2. Farlie-Gumbel-Morgenstern (FGM) Copula: for $\theta \in [-1, 1]$,

$$\begin{aligned} C(u,u) - (2u-1) =& u^2 \left(1 + \theta (1-u)^2\right) - (2u-1) \\ =& (1-u)^2 (1+\theta u^2) \\ =& o(1-u). \end{aligned}$$

3. Ali-Mikhail-Haq (AMH) Copula: for $\theta \in [-1, 1]$,

$$\begin{aligned} C(u,u) - (2u-1) =& \frac{u^2}{1-\theta(1-u)^2} - (2u-1) \\ =& (1-u)^2 \frac{1-\theta(1-2u)}{1-\theta(1-u)^2} \\ =& o(1-u). \end{aligned}$$

4. Survival Copula: for $\theta \in [0, 1]$,

$$C(u,u) - (2u-1) = u^2 \left(\exp(-\theta(\ln u)^2) - 1\right) + (1-u)^2$$

Because $\ln u = \ln(1-(1-u)) = -(1-u) + O((1-u)^2)$ for any i,

$$\exp(-\theta(\ln u)^2) - 1 = -\theta(1-u)^2 + o((1-u)^2)$$

Clearly,

$$C(u,u) - (2u-1) = (1-\theta)(1-u)^2 + o((1-u)^2) = o(1-u).$$

5. Gaussian copula: Introduce an auxiliary vector $(Z_1, Z_2)^\top$ which follows a bivariate normal distribution with mean vector zero and covariance matrix equal to the correlation

matrix R with $ent_{11}(R) = ent_{22}(R) = 1$ and $ent_{12}(R) = ent_{21}(R) = \rho_{ij}$. We have

$$
\begin{aligned}
& C(u,u) - (2u-1) \\
=& \mathbb{P}(Z_1 \leq \Phi^{-1}(u), Z_2 \leq \Phi^{-1}(u)) - (2u-1) \\
=& \mathbb{P}(Z_1 \leq \Phi^{-1}(u)) - \mathbb{P}(Z_2 > \Phi^{-1}(u)) + \mathbb{P}(Z_1 > \Phi^{-1}(u), Z_2 > \Phi^{-1}(u)) \\
& - (2u-1) \\
=& \mathbb{P}(Z_1 > \Phi^{-1}(u), Z_2 > \Phi^{-1}(u)).
\end{aligned}
$$

It follows from Lemma A6 that $C(u,u) - (2u-1) = o(\min(\mathbb{P}(Z_1 > \Phi^{-1}(u)), \mathbb{P}(Z_2 > \Phi^{-1}(u)))) = o(1-u)$.

□

S4.11 Proof of Remark 2

Proof. For the first four copula functions mentioned in Remark 1, it is straightforward to derive that by taking their formulas, $C_P(u,u) \sim u^2 = o(u)$, as $u \to 0$.

We just need to show that the Gaussian copula and Cuadras-Augé copula satisfy (4.19), and the Cuadras-Augé copula does not satisfy (4.17).

(1) Taking for the formula of the Gaussian copula,

$$
\begin{aligned}
C_P(u,u) =& \mathbb{P}(Z_1 \leq \Phi^{-1}(u), Z_2 \leq \Phi^{-1}(u)) \\
=& \mathbb{P}(Z_1 \geq \Phi^{-1}(1-u), Z_2 \geq \Phi^{-1}(1-u))
\end{aligned}
$$

where $(Z_1, Z_2)^\top$ follows a bivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix R with $ent_{11}(R) = ent_{22}(R) = 1$ and $ent_{12}(R) = ent_{21}(R) = \rho_{ij}$. The second equation follows from the symmetry of this bivariate normal distribution. It follows from Lemma A6 that $C_P(u,u) = o(\min(\mathbb{P}(Z_1 > \Phi^{-1}(1-u)), \mathbb{P}(Z_2 >$

$\Phi^{-1}(1-u)))) = o(u)$.

(2) For the Cuadras-Augé copula, if $C_P(u,u)$ takes its formula, we have

$$C_P(u,u) = u^{2-\theta} = o\left(u\right), \quad u \to 0.$$

So (4.19) is satisfied.

However, if $C_X(u,u)$ takes this formula, then (4.17) is not satisfied. This is because

$$C_X(1-u,1-u) - (1-2u) = (1-u)^{2-\theta} - (1-2u)\,.$$

As $(1-u)^{1-\theta} = 1-(2-\theta)u + O(u^2)$ for $u \to 0$, we obtain

$$\begin{aligned} C_X(1-u,1-u) - (1-2u) =& \left[1-(2-\theta)u + O(u^2)\right] - (1-2u) \\ =& \theta u + o(u). \end{aligned}$$

Clearly, (4.17) is not satisfied for any fixed $\theta \in (0,1)$. □

S4.12 Proof of Theorem 6

Let a_α and b_α be two nonnegative quantities. We write $a_\alpha \lesssim b_\alpha$, if there exists a constant $C > 0$, independent of α, such that $a_\alpha \le C b_\alpha$ for all sufficiently small α; write $a_\alpha \gtrsim b_\alpha$, if there exists a constant $c > 0$, independent of α, such that $a_\alpha \ge c b_\alpha$ for all sufficiently small α; and write $a_\alpha \asymp b_\alpha$, if both $a_\alpha \lesssim b_\alpha$ and $a_\alpha \gtrsim b_\alpha$.

Proof. From Theorem 4 and Assumption 2, c_α satisfies

$$f_1^{-1}(c_\alpha) = \frac{\alpha}{n}(1+o(1)), \quad \alpha \to 0. \tag{S4.40}$$

Let $a_\alpha = f_1^{-1}(c_\alpha)$. Then $a_\alpha = \alpha/n(1+o(1))$ and $a_\alpha \asymp \alpha$.

First note that

$$\{T_f(\boldsymbol{P}) > c_\alpha\} = \bigcup_{i=1}^{n} \{f_i(P_{(i)}) > c_\alpha\} = \bigcup_{i=1}^{n} \{P_{(i)} < f_i^{-1}(c_\alpha)\}.$$

Let $R_\alpha = \bigcup_{i=2}^n \{P_{(i)} < f_i^{-1}(c_\alpha)\}$. Under Assumption 2, $f_i^{-1}(c_\alpha) \asymp f_1^{-1}(c_\alpha) \asymp \alpha$, $i \geq 2$. So there exists a constant $C > 0$ such that for small α, $f_i^{-1}(c_\alpha) \leq C\alpha$, for $i \geq 2$. Therefore $R_\alpha \subseteq \{P_{(2)} \leq C\alpha\}$. It follows that

$$\mathbb{P}_{H_1}(P_{(1)} < a_\alpha) \leq \mathbb{P}_{H_1}(T_f(\boldsymbol{P}) > c_\alpha) \leq \mathbb{P}_{H_1}(P_{(1)} < a_\alpha) + \mathbb{P}_{H_1}(P_{(2)} \leq C\alpha). \tag{S4.41}$$

We next show $\mathbb{P}_{H_1}(P_{(2)} \leq C\alpha)$ is negligible. Let $u = C\alpha$ and $z_x = \Phi^{-1}(1-x)$. Note that the event $\{P_{(2)} \leq u\}$ means at least two p-values are below u. Hence, by the union bound,

$$\mathbb{P}_{H_1}(P_{(2)} \leq u) \leq \sum_{1 \leq i < j \leq n} \mathbb{P}_{H_1}(P_i \leq u, P_j \leq u). \tag{S4.42}$$

There are three types of pairs. (1) The pair is centered bivariate normal with correlation $\rho < 1$, so

$$\mathbb{P}_{H_1}(P_i \leq u, P_j \leq u) = \mathbb{P}_{H_1}(X_i \geq z_u, X_j \geq z_u) \lesssim z_u^{-2} \exp\left\{-\frac{z_u^2}{1+\rho}\right\}, \tag{S4.43}$$

for $i, j > m$ (see Lemma A5). (2) For one signal and one null coordinate,

$$\mathbb{P}_{H_1}(P_i \leq u, P_j \leq u) = \mathbb{P}_{H_1}(X_i \geq z_u, X_j \geq z_u) \leq \mathbb{P}_{H_1}(X_i - \delta \geq z_u - \delta, X_j \geq z_u - \delta),$$

for $i = 1, \ldots, m$ and $j = m+1, \ldots, n$. Thus,

$$\mathbb{P}_{H_1}(P_i \leq u, P_j \leq u) \lesssim (z_u - \delta)^{-2} \exp\left\{-\frac{(z_u - \delta)^2}{1+\rho}\right\}. \tag{S4.44}$$

(3) For two signals,

$$\mathbb{P}_{H_1}(P_i \leq u, P_j \leq u) = \mathbb{P}_{H_1}(X_i \geq z_u, X_j \geq z_u) \lesssim (z_u - \delta)^{-2} \exp\left\{-\frac{(z_u - \delta)^2}{1+\rho}\right\}. \tag{S4.45}$$

From (S4.42) - (S4.45), we have

$$\mathbb{P}_{H_1}(P_{(2)} \leq u) \lesssim (z_u - \delta)^{-2} \exp\left\{-\frac{(z_u - \delta)^2}{1+\rho}\right\}. \tag{S4.46}$$

On the other hand, since X_1 is a signal coordinate,

$$\mathbb{P}_{H_1}(P_{(1)} \leq \alpha/n) \geq \mathbb{P}_{H_1}(P_1 \leq \alpha/n) = \mathbb{P}_{H_1}(X_1 \geq z_{\alpha/n}).$$

Under H_1, $X_1 \sim N(\delta, 1)$, so by Mills' ratio,

$$\mathbb{P}_{H_1}(P_{(1)} \leq \alpha/n) \geq \bar{\Phi}(z_{\alpha/n} - \delta) \asymp \frac{1}{z_{\alpha/n} - \delta} \exp\left(-\frac{(z_{\alpha/n} - \delta)^2}{2}\right). \tag{S4.47}$$

Combining this with (S4.46) gives

$$\frac{\mathbb{P}_{H_1}(P_{(2)} \leq C\alpha)}{\mathbb{P}_{H_1}(P_{(1)} \leq \alpha/n)} \lesssim \frac{z_{\alpha/n} - \delta}{(z_{C\alpha} - \delta)^2} \exp\left\{-\frac{(z_{C\alpha} - \delta)^2}{1+\rho} + \frac{(z_{\alpha/n} - \delta)^2}{2}\right\}. \tag{S4.48}$$

Note that

$$-\frac{(z_{C\alpha} - \delta)^2}{1+\rho} + \frac{(z_{\alpha/n} - \delta)^2}{2} = -\frac{z_{C\alpha}^2}{1+\rho} + \frac{z_{\alpha/n}^2}{2} + \delta\left(\frac{2z_{C\alpha}}{1+\rho} - z_{\alpha/n}\right) + \delta^2\left(\frac{1}{2} - \frac{1}{1+\rho}\right).$$

Let $L_\alpha = \log(1/\alpha)$. From Mills' ratio, we have $z_{\alpha/n} = \sqrt{2L_\alpha}(1 + o(1))$ and $z_{C\alpha} = \sqrt{2L_\alpha}(1 + o(1))$. Therefore $\frac{z_{\alpha/n} - \delta}{(z_{C\alpha} - \delta)^2} = O(L_\alpha^{-1/2})$,

$$-\frac{z_{C\alpha}^2}{1+\rho} + \frac{z_{\alpha/n}^2}{2} = -2L_\alpha\left(\frac{1}{1+\rho} - \frac{1}{2}\right) + o(L_\alpha) = -\frac{1-\rho}{1+\rho}L_\alpha + o(L_\alpha),$$

$\delta\left(\frac{2z_{C\alpha}}{1+\rho} - z_{\alpha/n}\right) = O(\sqrt{L_\alpha})$, and $\delta^2\left(\frac{1}{2} - \frac{1}{1+\rho}\right) = O(1)$. It follows that

$$-\frac{(z_u - \delta)^2}{1+\rho} + \frac{(z_{\alpha/n} - \delta)^2}{2} = -\frac{1-\rho}{1+\rho}L_\alpha + O(\sqrt{L_\alpha}) + o(L_\alpha).$$

Because $0 < \rho < 1$ and $L_\alpha \to \infty$, the right hand side of (S4.48) tends to 0. Thus,

$$\mathbb{P}_{H_1}(P_{(2)} \leq C\alpha) = o(\mathbb{P}_{H_1}(P_{(1)} \leq \alpha/n)). \tag{S4.49}$$

We now prove that $\mathbb{P}_{H_1}(P_{(1)} \leq a_\alpha) \sim \mathbb{P}_{H_1}(P_{(1)} \leq \alpha/n)$. Since $a_\alpha/(\alpha/n) \to 1$, by the mean value theorem, $z_{a_\alpha} - z_{\alpha/n} = \frac{\alpha/n - a_\alpha}{\phi(\xi_\alpha)}$ for some ξ_α between z_{a_α} and $z_{\alpha/n}$. Since $\bar{\Phi}(z_{a_\alpha}) = a_\alpha$ and $\bar{\Phi}(z_{\alpha/n}) = \alpha/n$, Mills' ratio gives

$$z_{a_\alpha} - z_{\alpha/n} = O\left(\frac{|a_\alpha - \alpha/n|}{(\alpha/n)z_{\alpha/n}}\right) = o\left(\frac{1}{z_{\alpha/n}}\right).$$

For $1 \leq i \leq m$, $\mathbb{P}_{H_1}(P_i \leq x) = \mathbb{P}_{H_1}(X_i \geq z_x) = \bar{\Phi}(z_x - \delta)$. By Mills' ratio,

$$\frac{\bar{\Phi}(z_{a_\alpha} - \delta)}{\bar{\Phi}(z_{\alpha/n} - \delta)} \sim \frac{z_{\alpha/n} - \delta}{z_{a_\alpha} - \delta} \exp\left\{-\frac{(z_{a_\alpha} - \delta)^2 - (z_{\alpha/n} - \delta)^2}{2}\right\}.$$

Since $z_{a_\alpha} - z_{\alpha/n} = o\left(\frac{1}{z_{\alpha/n}}\right)$, we have $\frac{z_{\alpha/n}-\delta}{z_{a_\alpha}-\delta} \to 1$ and $(z_{a_\alpha} - \delta)^2 - (z_{\alpha/n} - \delta)^2 = (z_{a_\alpha} - z_{\alpha/n})(z_{a_\alpha} + z_{\alpha/n} - 2\delta) = o(1)$. Therefore

$$\bar{\Phi}(z_{a_\alpha} - \delta) \sim \bar{\Phi}(z_{\alpha/n} - \delta). \tag{S4.50}$$

For $j > m$, $\mathbb{P}_{H_1}(P_j \le x) = x = o\{\bar{\Phi}(z_x - \delta)\}$, because

$$\frac{x}{\bar{\Phi}(z_x - \delta)} = \frac{\bar{\Phi}(z_x)}{\bar{\Phi}(z_x - \delta)} \to 0, \quad \text{for fixed } \delta > 0. \tag{S4.51}$$

Let $A_i(x) = \{P_i \le x\}$. Then $\{P_{(1)} \le x\} = \bigcup_{i=1}^n A_i(x)$. Using inclusion-exclusion,

$$\mathbb{P}_{H_1}(P_{(1)} \le x) = \sum_{i=1}^n \mathbb{P}_{H_1}(A_i(x)) + O\Big(\sum_{i<j} \mathbb{P}_{H_1}(A_i(x) \cap A_j(x))\Big). \tag{S4.52}$$

For $x \asymp \alpha$, the probability that two or more p-values are below x is negligible relative to the probability that one signal p-value is below x, by the same argument used above. Hence, since m and n are fixed,

$$\mathbb{P}_{H_1}(P_{(1)} \le x) \sim m\bar{\Phi}(z_x - \delta), \qquad x \asymp \alpha. \tag{S4.53}$$

Applying this with $x = a_\alpha$ and $x = \alpha/n$, and using (S4.50), gives

$$\mathbb{P}_{H_1}(P_{(1)} \le a_\alpha) \sim \mathbb{P}_{H_1}(P_{(1)} \le \alpha/n). \tag{S4.54}$$

Combining (S4.41), (S4.49), and (S4.54), we obtain

$$\mathbb{P}_{H_1}\{T_f(\boldsymbol{P}) > c_\alpha\} = \mathbb{P}_{H_1}(P_{(1)} \le a_\alpha)(1 + o(1)) = \mathbb{P}_{H_1}(P_{(1)} \le \alpha/n)(1 + o(1)).$$

□

S4.13 Proof of Theorem 7

Proof. By Theorem 5 or from (4.18), we have $\mathbb{P}(T_f(\boldsymbol{P}) > t) = nf_1^{-1}(t) + o(f_n^{-1}(t))$, $t \to \infty$. Hence the level-α critical value satisfies $f_1^{-1}(c_\alpha) = \frac{\alpha}{n}(1 + o(1))$.

Let $a_\alpha = f_1^{-1}(c_\alpha)$. Similar to the proof of Theorem 6, we have

$$\mathbb{P}_{H_1}(P_{(1)} < a_\alpha) \le \mathbb{P}_{H_1}\{T_f(\boldsymbol{P}) > c_\alpha\} \le \mathbb{P}_{H_1}(P_{(1)} < a_\alpha) + \mathbb{P}_{H_1}(P_{(2)} \le C\alpha).$$

By (5.27) and (5.28), $\mathbb{P}_{H_1}\{T_f(\boldsymbol{P}) > c_\alpha\} = \mathbb{P}_{H_1}\left(P_{(1)} \le \frac{\alpha}{n}\right)(1 + o(1))$. □

S4.14 Proof of Theorem 8

Proof. By Theorem 7, it suffices to show (5.27) and (5.28) hold.

Let $u = C\alpha$ and $z_u = \Phi^{-1}(1-u)$. For a signal coordinate, $\mathbb{P}_{H_1}(P_i \le u) = \bar{\Phi}(z_u - \delta)$, and for a null coordinate, $\mathbb{P}_{H_1}(P_j \le u) = u$. Since $\delta > 0$, $u = \bar{\Phi}(z_u) = o\{\bar{\Phi}(z_u - \delta)\}$.

Now consider pairwise events. If i and j are both signal coordinates, let $s_u = \bar{\Phi}(z_u - \delta)$. Then

$$\mathbb{P}_{H_1}(P_i \le u, P_j \le u) = 1 - 2(1 - s_u) + C_X(1 - s_u, 1 - s_u) = o(s_u), \tag{S4.55}$$

by the copula condition: $C_X(1-s_u, 1-s_u) = 2(1-s_u) - 1 + o(s_u)$. If both are null coordinates, then

$$\mathbb{P}_{H_1}(P_i \le u, P_j \le u) = o(u) = o(s_u). \tag{S4.56}$$

If one is signal and one is null, then

$$\mathbb{P}_{H_1}(P_i \le u, P_j \le u) \le \mathbb{P}_{H_1}(P_j \le u) = u = o(s_u). \tag{S4.57}$$

Therefore, for every pair $i \ne j$, $\mathbb{P}_{H_1}(P_i \le u, P_j \le u) = o\{\bar{\Phi}(z_u - \delta)\}$. Since n is fixed,

$$\mathbb{P}_{H_1}(P_{(2)} \le u) \le \sum_{i<j} \mathbb{P}_{H_1}(P_i \le u, P_j \le u) = o\{\bar{\Phi}(z_u - \delta)\}. \tag{S4.58}$$

Also, $\mathbb{P}_{H_1}\left(P_{(1)} \le \frac{\alpha}{n}\right) \ge \bar{\Phi}(z_{\alpha/n} - \delta)$. Since $u = C\alpha$ and α/n are of the same order, $\bar{\Phi}(z_u - \delta) \asymp \bar{\Phi}(z_{\alpha/n} - \delta)$. This proves (5.27).

Finally, the same Mills' ratio argument used in Theorem 6 gives (5.28). Combining the two proves the result.

□

S4.15 Proof of Theorem 9

Proof. Assume, for simplicity, that $\omega_i > 0$ for all i. Let $W_i = \frac{P_i}{\omega_i}$, and $W_{(1)} = \min_{1\le i\le n} W_i$. Since the CCT is ACR, its upper-tail null distribution is asymptotically the same as under independence. Therefore, $\mathbb{P}_{H_0}(T_{CCT} > t) \sim \frac{1}{\pi t}$, $t \to \infty$, and hence $c_\alpha^{CCT} \sim \frac{1}{\pi\alpha}$, $\alpha \to 0$.

Fix $\eta \in (0,1)$ and define

$$A_{c,\eta}^{-} = \left\{ \max_{1\le i\le n} \frac{\omega_i}{\pi P_i} > (1+\eta)c \right\} = \left\{ W_{(1)} < \frac{1}{\pi(1+\eta)c} \right\},$$

and

$$A_{c,\eta}^{+} = \left\{ \max_{1\le i\le n} \frac{\omega_i}{\pi P_i} > (1-\eta)c \right\} = \left\{ W_{(1)} < \frac{1}{\pi(1-\eta)c} \right\}.$$

We will show that there exists a constant $K_\eta > 0$ such that, for sufficiently large c,

$$\{T_{CCT} > c\} \subseteq A_{c,\eta}^{+} \cup \left\{ P_{(2)} \le \frac{K_\eta}{c} \right\}, \tag{S4.59}$$

and

$$A_{c,\eta}^{-} \cap \left\{ P_{(n)} \le 1 - \frac{K_\eta}{c} \right\} \subseteq \{T_{CCT} > c\}. \tag{S4.60}$$

First, suppose $P_{(2)} > K_\eta/c$. Then at most one p-value is smaller than K_η/c. Let i_0 be the index attaining $P_{(1)}$. For $j \ne i_0$, we have $P_j \ge P_{(2)} > K_\eta/c$. Since $\cot(\pi p)$ is decreasing on $(0,1)$ and $\sum_i \omega_i = 1$,

$$\sum_{j\ne i_0} \omega_j \cot(\pi P_j) \le \sum_{j\ne i_0} \omega_j \cot\left(\pi \frac{K_\eta}{c}\right) \le \cot\left(\pi \frac{K_\eta}{c}\right).$$

Using $\cot x \sim x^{-1}$ as $x \downarrow 0$, choose K_η large enough so that, for sufficiently large c, $\cot\left(\pi \frac{K_\eta}{c}\right) \le (\eta/2)c$. If additionally $\max_{1\le i\le n} \frac{\omega_i}{\pi P_i} \le (1-\eta)c$, then since $\cot(\pi p) \sim 1/(\pi p)$ for small p, $\omega_{i_0}\cot(\pi P_{i_0}) \le (1-\eta/2)c$. Hence, for sufficiently large c,

$$T_{CCT} = \omega_{i_0}\cot(\pi P_{i_0}) + \sum_{i\ne i_0} \omega_i \cot(\pi P_i) \le c.$$

Therefore,

$$\{T_{CCT} > c\} \cap (A^+_{c,\eta})^c \subseteq \left\{P_{(2)} \le \frac{K_\eta}{c}\right\},$$

which proves (S4.59).

Next, suppose that $A^-_{c,\eta}$ holds and $P_{(n)} \le 1 - \frac{K_\eta}{c}$. Then there exists an index i_* such that $\frac{\omega_{i_*}}{\pi P_{i_*}} > (1+\eta)c$. Using again $\cot(\pi p) \sim 1/(\pi p)$ as $p \downarrow 0$, this implies

$$\omega_{i_*} \cot(\pi P_{i_*}) > (1+\eta/2)c$$

for sufficiently large c. For all $j \neq i_*$, the condition $P_j \le P_{(n)} \le 1 - K_\eta/c$ implies

$$\cot(\pi P_j) \ge \cot\left(\pi\left(1 - \frac{K_\eta}{c}\right)\right) = -\cot\left(\pi\frac{K_\eta}{c}\right).$$

Hence

$$\sum_{j\neq i_*} \omega_j \cot(\pi P_j) \ge -\sum_{j\neq i_*} \omega_j \cot\left(\pi\frac{K_\eta}{c}\right) \ge -\cot\left(\pi\frac{K_\eta}{c}\right) \ge -\frac{\eta}{2}c,$$

where K_η is chosen large enough. Therefore, $T_{CCT} > (1+\eta/2)c - \frac{\eta}{2}c = c$. This proves (S4.60).

In view of (S4.59) and (S4.60), we have

$$\mathbb{P}_{H_1}(A^-_{c,\eta}) - \mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right) \le \mathbb{P}_{H_1}(T_{CCT} > c) \le \mathbb{P}_{H_1}(A^+_{c,\eta}) + \mathbb{P}_{H_1}\left(P_{(2)} \le \frac{K_\eta}{c}\right).$$

Next, we will evaluate $\mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right)$ and $\mathbb{P}_{H_1}\left(P_{(2)} \le \frac{K_\eta}{c}\right)$.

Let $W_{(1)} = \min_{1\le i\le n} \frac{P_i}{\omega_i}$. Since $\omega_i > 0$ for all i, define $\omega_* = \max_{1\le i\le m} \omega_i > 0$. Then

$$\mathbb{P}_{H_1}\left(W_{(1)} \le \frac{1}{c}\right) \ge \mathbb{P}_{H_1}\left(P_i \le \frac{\omega_*}{c}\right) = \bar{\Phi}(z_{\omega_*/c} - \delta),$$

for some signal coordinate $i \le m$, where $z_x = \Phi^{-1}(1-x)$. By Mill's ratio, $\bar{\Phi}(z_{\omega_*/c} - \delta) \asymp \frac{1}{z_{\omega_*/c}-\delta} \exp\left(-\frac{(z_{\omega_*/c}-\delta)^2}{2}\right)$ and $z_{\omega_*/c} = \sqrt{2\log c}(1+o(1))$. Then $\bar{\Phi}(z_{\omega_*/c}-\delta) \asymp \frac{1}{c}\exp\{\delta\sqrt{2\log c}(1+o(1))\}$. Thus

$$\mathbb{P}_{H_1}\left(W_{(1)} \le \frac{1}{c}\right) \gtrsim \frac{1}{c}\exp\{\delta\sqrt{2\log c}(1+o(1))\}. \tag{S4.61}$$

Let $u_c = \frac{K_\eta}{c}$. From (S4.46), we have

$$\mathbb{P}_{H_1}(P_{(2)} \leq \frac{K_\eta}{c}) \lesssim (z_{K_\eta/c} - \delta)^{-2} \exp\left\{-\frac{(z_{K_\eta/c} - \delta)^2}{1+\rho}\right\}.$$

Then

$$\frac{\mathbb{P}_{H_1}(P_{(2)} \leq K_\eta/c)}{\mathbb{P}_{H_1}(W_{(1)} \leq 1/c)} \lesssim \frac{(z_{K_\eta/c} - \delta)^{-2} \exp\left\{-\frac{(z_{K_\eta/c}-\delta)^2}{1+\rho}\right\}}{(z_{\omega_*/c} - \delta)^{-1} \exp\left(-\frac{(z_{\omega_*/c}-\delta)^2}{2}\right)}.$$

Note that n is fixed, $z_{\omega_*/c} = \sqrt{2\log c}(1+o(1))$, and $z_{u_c} = \sqrt{2\log c}(1+o(1))$. Similar to the proof of (S4.48), we have

$$\mathbb{P}_{H_1}\left(P_{(2)} \leq \frac{K_\eta}{c}\right) = o\left\{\mathbb{P}_{H_1}\left(W_{(1)} \leq \frac{1}{c}\right)\right\}.$$

We need to show

$$\mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right) = o\left\{\mathbb{P}_{H_1}\left(W_{(1)} \leq \frac{1}{c}\right)\right\}. \tag{S4.62}$$

Let $a_c = \Phi^{-1}\left(\frac{K_\eta}{c}\right)$. Then $a_c \sim -\sqrt{2\log c}$, and $\left\{P_{(n)} > 1 - \frac{K_\eta}{c}\right\} = \{\min_{1\leq i\leq n} X_i < a_c\}$. By the union bound,

$$\mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right) \leq \sum_{i=1}^n \mathbb{P}_{H_1}(X_i < a_c) = \sum_{i=1}^n \Phi(a_c - \mu_i) \leq n\Phi(a_c).$$

By Mills' ratio, $\Phi(a_c) \asymp \frac{1}{c}$. Thus $\mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right) = O(c^{-1})$. Combining this with (S4.61),

$$\frac{\mathbb{P}_{H_1}\left(P_{(n)} > 1 - \frac{K_\eta}{c}\right)}{\mathbb{P}_{H_1}(W_{(1)} \leq 1/c)} \lesssim \exp\{-\delta\sqrt{2\log c}(1+o(1))\} \to 0.$$

This proves (S4.62).

Therefore, $\mathbb{P}_{H_1}(A^-_{c,\eta})(1+o(1)) \leq \mathbb{P}_{H_1}(T_{CCT} > c) \leq \mathbb{P}_{H_1}(A^+_{c,\eta})(1+o(1))$. That is,

$$\mathbb{P}_{H_1}\left(W_{(1)} < \frac{1}{\pi(1+\eta)c}\right)(1+o(1)) \leq \mathbb{P}_{H_1}(T_{CCT} > c) \leq \mathbb{P}_{H_1}\left(W_{(1)} < \frac{1}{\pi(1-\eta)c}\right)(1+o(1)).$$

Letting first $c \to \infty$ and then $\eta \downarrow 0$, we obtain

$$\mathbb{P}_{H_1}(T_{CCT} > c) = \mathbb{P}_{H_1}\left(\min_{1\leq i\leq n} \frac{P_i}{\omega_i} \leq \frac{1}{\pi c}\right)(1+o(1)).$$

Taking $c = c_\alpha^{CCT} \sim 1/(\pi\alpha)$ yields

$$\beta_{CCT}(\delta) = \mathbb{P}_{H_1}(T_{CCT} > c_\alpha^{CCT}) = \mathbb{P}_{H_1}\left(\min_{1\le i\le n}\frac{P_i}{\omega_i} \le \alpha\right)(1+o(1)) = \mathbb{P}_{H_1}\left(\bigcup_{i=1,\dots,n}\{P_i \le \omega_i\alpha\}\right)(1+o(1)).$$

□

S4.16 Proof of Remark 4

Proof. The proof is nearly identical to the proof of Theorem 6. We only indicate the necessary modifications.

Let $\delta^* = \max_{1\le i\le m}\delta_i > 0$. Note that for one signal and one null coordinate, i.e., $1 \le i \le m$ and $m < j \le n$, $\{X_i - \delta_i \ge z_u - \delta_i, X_j \ge z_u\} \subset \{X_i - \delta_i \ge z_u - \delta^*, X_j \ge z_u - \delta^*\}$; and for two signal coordinates, $\{X_i \ge z_u, X_j \ge z_u\} \subset \{X_i - \delta_i \ge z_u - \delta^*, X_j - \delta_j \ge z_u - \delta^*\}$. Thus (S4.44) and (S4.45), and thus (S4.46) are still valid with δ replaced by δ^*. For a signal coordinate $i \le m$, $\mathbb{P}_{H_1}(P_i \le u) = \mathbb{P}_{H_1}(X_i \ge z_u) = \bar\Phi(z_u - \delta_i)$. Thus, $\mathbb{P}_{H_1}\left(P_{(1)} \le \frac{\alpha}{n}\right) \ge \max_{1\le i\le m}\bar\Phi(z_{\alpha/n} - \delta_i) \ge \bar\Phi(z_{\alpha/n} - \delta_*)$. In view of (S4.48), we have (S4.49) holds true. Moreover, for each signal coordinate $i \le m$, $\mathbb{P}_{H_1}(P_i \le x) = \bar\Phi(z_x - \delta_i)$. By (S4.50), $\bar\Phi(z_{a_\alpha} - \delta_i) \sim \bar\Phi(z_{\alpha/n} - \delta_i)$. For $j > m$, $\mathbb{P}_{H_1}(P_j \le x) = x = o\{\bar\Phi(z_x - \delta_i)\}$, $i \le m$, due to (S4.51). Same as the proof of (S4.53), we can obtain, for $x \asymp \alpha$,

$$\mathbb{P}_{H_1}(P_{(1)} \le x) \sim \sum_{i=1}^m \bar\Phi(z_x - \delta_i). \tag{S4.63}$$

Applying this with $x = a_\alpha$ and $x = \alpha/n$ yields (S4.54), and thus, Combining the preceding displays gives (5.26). Thus Theorem 6 remains valid under the revised alternative.

By (S4.63), under the revised alternative, the common leading approximation is

$$\mathbb{P}_{H_1}\left(P_{(1)} \le \frac{\alpha}{n}\right) \sim \sum_{i=1}^m \bar\Phi\left(z_{\alpha/n} - \delta_i\right), \tag{S4.64}$$

which reduces to (5.25) when $\delta_1 = \cdots = \delta_m = \delta$. □

S4.17 Proof of Remark 5

Proof. The proof is nearly identical to the proof of Theorem 8. We only indicate the necessary modifications.

For each signal coordinate $i \le m$, define $s_{i,u} = \mathbb{P}_{H_1}(P_i \le u) = \bar{\Phi}(z_u - \delta_i)$. For a null coordinate $j > m$, $\mathbb{P}_{H_1}(P_j \le u) = u = \bar{\Phi}(z_u) = o\{\bar{\Phi}(z_u - \delta_i)\} = o(s_{i,u})$, $i = 1, \dots, m$. If i and j are both signal coordinates, (S4.55) becomes

$$\begin{aligned}\mathbb{P}_{H_1}(P_i \le u, P_j \le u) =& \mathbb{P}(\Phi_{\delta_i}(X_i) > 1 - s_{i,u}, \Phi_{\delta_j}(X_j) > 1 - s_{j,u}) \\ =& s_{i,u} + s_{j,u} - 1 + C_X(1 - s_{i,u}, 1 - s_{j,u}) \\ =& s_{i,u} + s_{j,u} - 1 + 1 - s_{i,u} - s_{j,u} + o\{\min(s_{i,u}, s_{j,u})\} \\ =& o\{\min(s_{i,u}, s_{j,u})\}.\end{aligned}$$

Moreover, (S4.56) becomes $\mathbb{P}_{H_1}(P_i \le u, P_j \le u) = o(u) = o(s_{k,u})$, $k \le m, i, j > m$; and (S4.57) becomes $\mathbb{P}_{H_1}(P_i \le u, P_j \le u) \le \mathbb{P}_{H_1}(P_j \le u) = u = o(s_{i,u})$, $i \le m, j > m$. Thus (S4.58) becomes

$$\mathbb{P}_{H_1}(P_{(2)} \le u) = o\left\{\sum_{i=1}^{m} s_{i,u}\right\}.$$

By virtue of (S4.52) and the probabilities of pairwise events, (S4.63) is true. Applying (S4.63) with $x = a_\alpha$ and $x = \alpha/n$ yields (5.28). Together with the pairwise negligibility result above, (5.27) also holds. Therefore, Theorem 8 remains valid under the revised alternative. In particular, (S4.64) remains valid as well. $\square$

S5 Numerical Studies

S5.1 Correlation Robustness

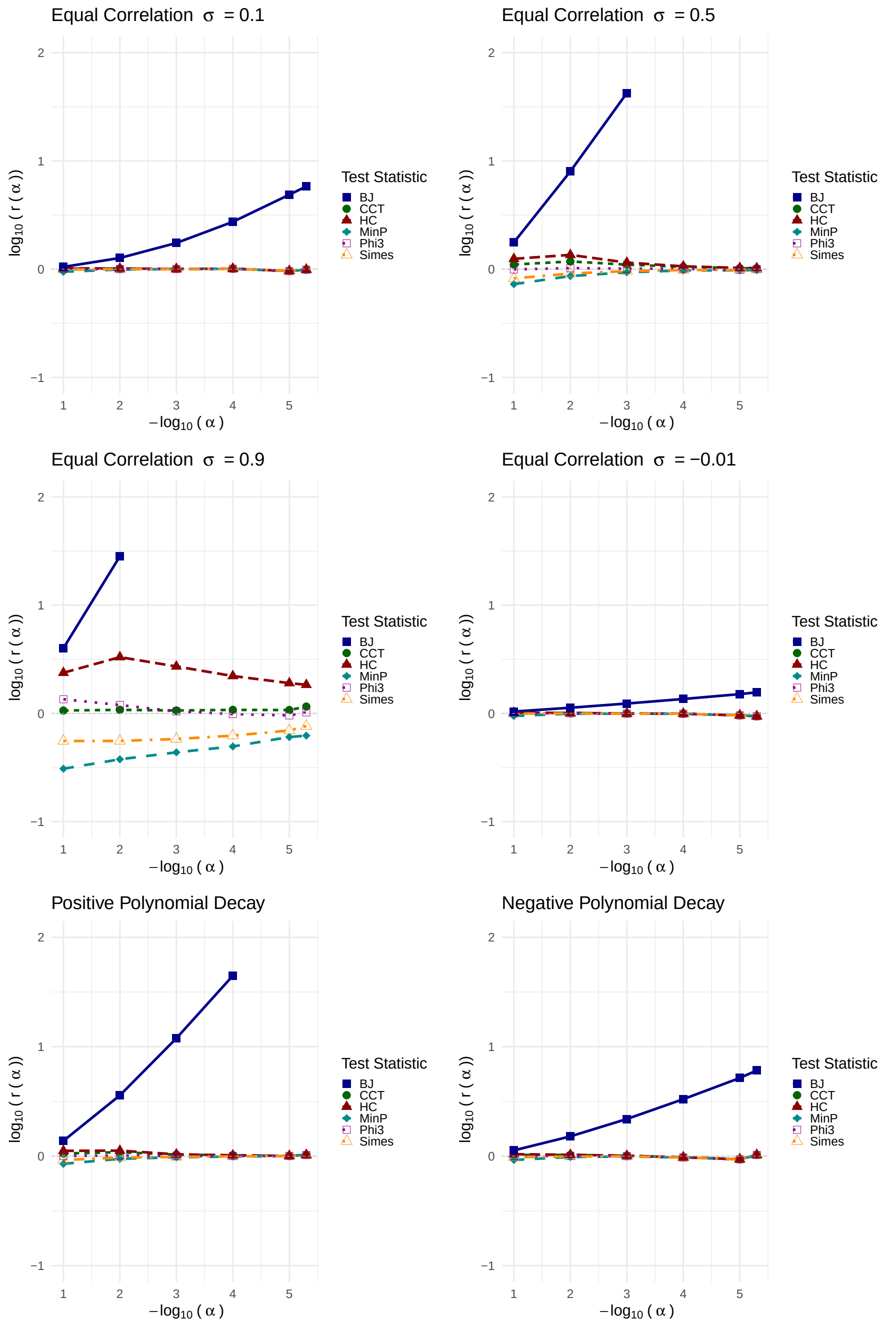

Figure S1: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ under various correlation structures, and $n = 10$. Correlation-robustness is illustrated by $\log(r(\alpha))$ approaching 0 as $-\log_{10} \alpha$ increases (i.e., as α decreases).

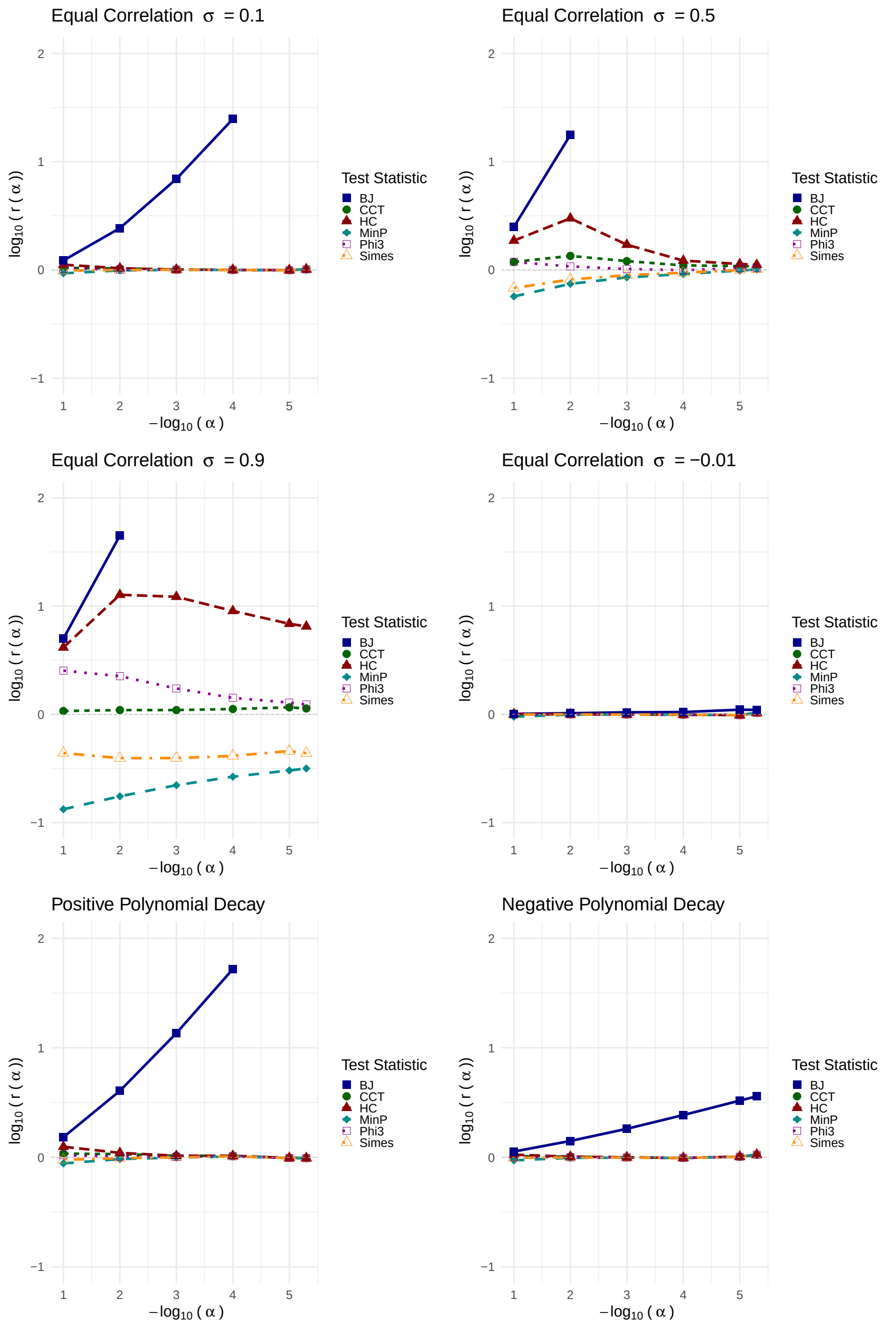

Figure S2: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ under various correlation structures, and $n = 50$. Correlation-robustness is illustrated by $\log(r(\alpha))$ approaching 0 as $-\log_{10}\alpha$ increases (i.e., as α decreases).

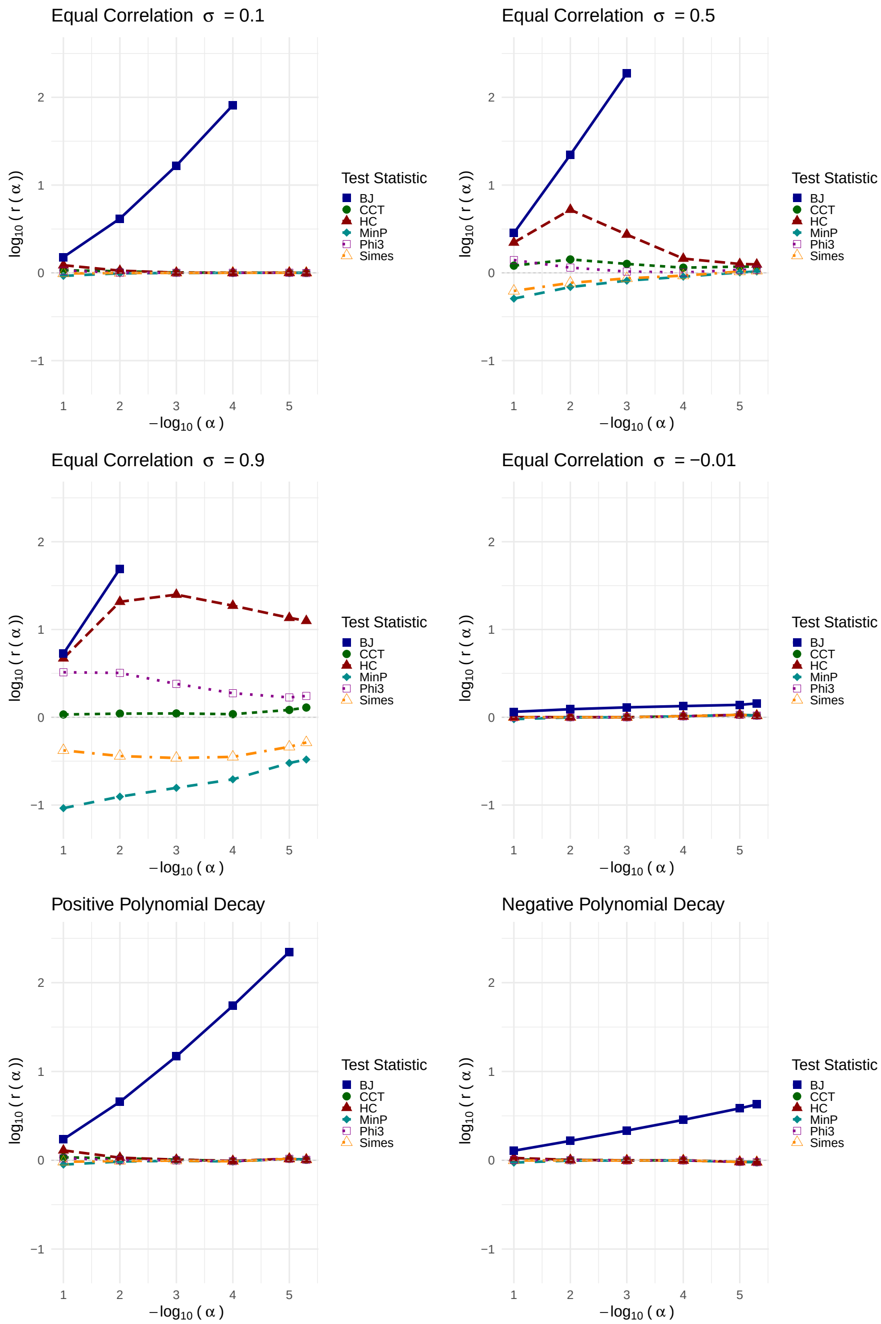

Figure S3: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ under various correlation structures, and $n = 100$. Correlation-robustness is illustrated by $\log(r(\alpha))$ approaching 0 as $-\log_{10}\alpha$ increases (i.e., as α decreases).

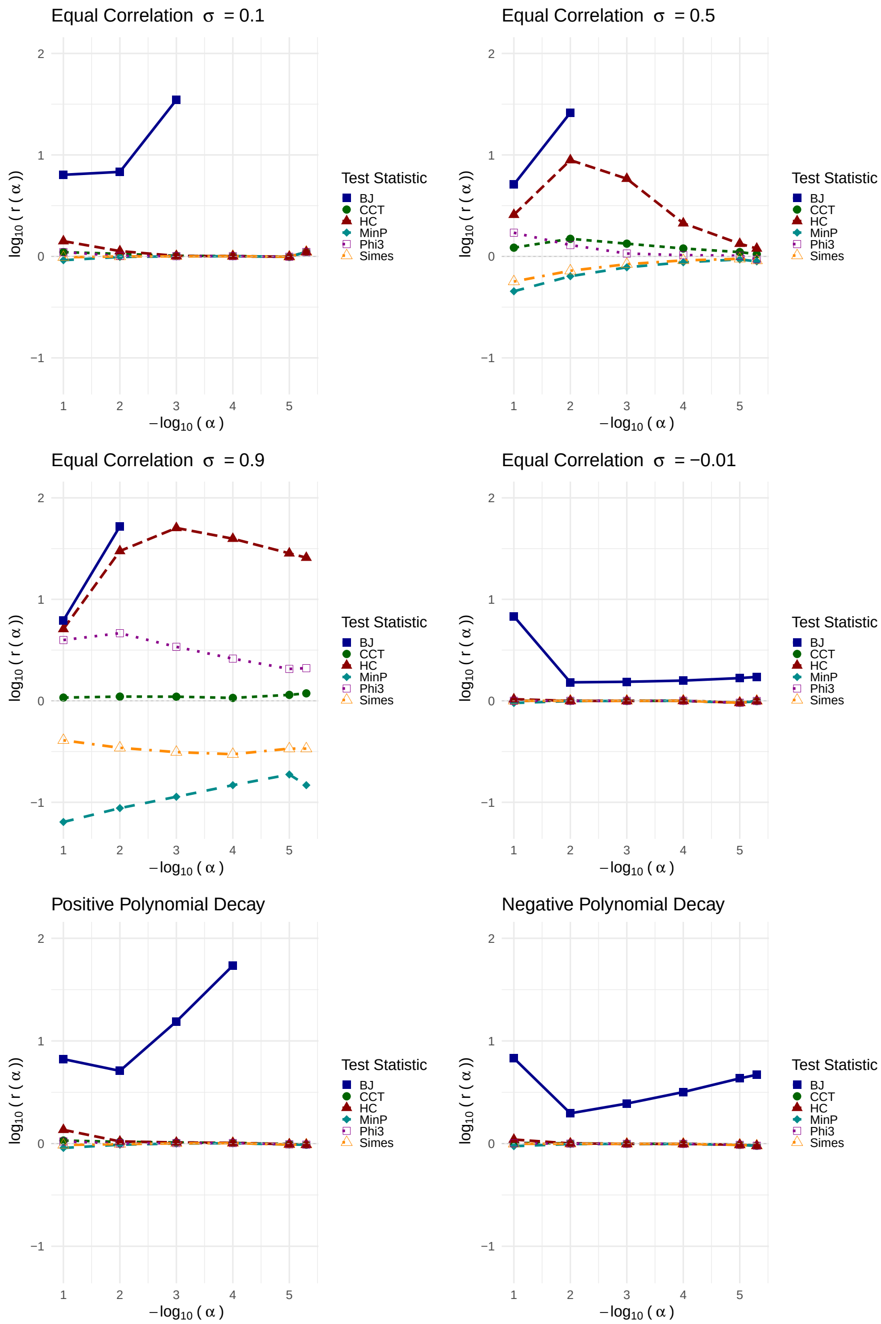

Figure S4: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ under various correlation structures, and $n = 200$. Correlation-robustness is illustrated by $\log(r(\alpha))$ approaching 0 as $-\log_{10}\alpha$ increases (i.e., as α decreases).

Going beyond the Gaussian assumption for input statistics, we further assess the correlation robustness of the tests in realistic data analysis settings—specifically, when input statistics are derived from genotype-phenotype association analyses.

We focus on quantitative trait association analysis, where input statistics are obtained from the linear regression model in (S5.2). Genotype data are simulated using the HapSim algorithm (Montana, 2005), generating genotypes (0, 1, or 2) based on specified minor allele frequencies (MAF) and LD matrices for the genes *CCDC170* and *FSHR*. The MAF and LD patterns are based on SNPs defined by the GEFOS2012FN study and real genotype data from the 1000 Genomes Project (see Section S5.3 for details).

Under the null hypothesis of no association, individual SNP p-values are computed using marginal score test statistics (McCulloch, 2000; Schaid et al., 2002; Barnett et al., 2017), and these p-values are used as input for gene-level p-value combination tests. We conduct 10^7 simulations, regenerating genotype data every 50,000 simulations to improve computational efficiency.

Figure S5 demonstrates the ACR property. The observed trends are consistent with those in Figure 1 of the main text. Among the tests, the $\phi 3$ test shows the strongest correlation robustness, with $r(\alpha)$ values remaining closest to 0 as α decreases (i.e., as $-\log_{10}(\alpha)$ increases).

While the preceding results demonstrate correlation robustness through type I error control, we now provide direct empirical evidence showing that the underlying asymptotic tail independence condition is reasonable in GWAS data. Specifically, we estimate empirical upper tail-dependence coefficients (TDCs) (Garcin, 2024; AghaKouchak et al., 2014) from genotype-phenotype association simulations. The simulation settings follow those for the continuous and binary traits in (S5.2) and (S5.3). As above, genotype data are simulated using the HapSim algorithm (Montana, 2005), generating genotypes (0, 1, or 2) based on

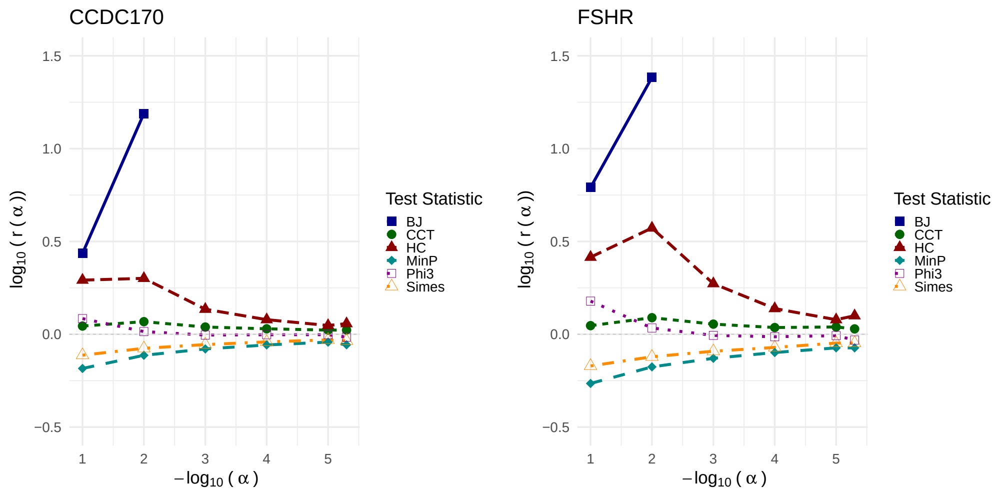

Figure S5: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ based on simulations of a genetic association study of quantitative trait, using genotype data from the genes *CCDC170* (left panel) and *FSHR* (right panel).

specified minor allele frequencies (MAF) and LD matrices for 9 genes spanning a range of sizes and LD strengths.

Specifically, for each pair of SNPs (i,j) within a gene, the empirical upper TDC at threshold level q is estimated by

$$\widehat{\lambda}_{ij}(q) = \frac{\sum_{b=1}^{B} \mathbf{1}\{X_i^{(b)} > \Phi^{-1}(q),\ X_j^{(b)} > \Phi^{-1}(q)\}}{\sum_{b=1}^{B} \mathbf{1}\{X_i^{(b)} > \Phi^{-1}(q)\}},$$

where $X_i^{(b)}$ and $X_j^{(b)}$ are the b-th Monte Carlo replicates of the marginal score test statistics for SNPs i and j, respectively. Since these statistics are asymptotically $N(0,1)$ under the null, we use $\Phi^{-1}(q)$ as the common marginal quantile function. The quantity $\widehat{\lambda}_{ij}(q)$ estimates the conditional probability that X_j exceeds the q-th quantile given that X_i exceeds the same quantile, and thus measures the strength of upper tail dependence. Under asymptotic tail independence, $\widehat{\lambda}_{ij}(q) \to 0$ as $q \to 1$. For each gene, the overall tail dependence is summarized by averaging $\widehat{\lambda}_{ij}(q)$ over all SNP pairs. We evaluate TDCs at $q \in \{0.5, 0.55, \ldots, 0.95, 0.975, 0.99, 0.999\}$.

Figure S6 (continuous trait) and Figure S7 (binary trait) present the results for 9 genes representing three levels of LD strength: weak (C12orf29, Herdin distance 0.253; SOST, 0.231), moderate (MIR8084, 0.308; CFHR4, 0.357; IL7, 0.548; C11orf40, 0.549), and strong (IPO7, 0.629; CCDC170, 0.633; FSHR, 0.759). The results show a clear decreasing trend in the estimated TDCs as q increases across all genes and LD levels, providing empirical evidence consistent with asymptotic tail independence. For most genes, the average TDC is near 0 at $q = 0.999$. For IPO7, the average TDC remains elevated at $q = 0.999$; the overall decreasing trend is nevertheless consistent with asymptotic tail independence.

Next, we present simulation results illustrating the loss of correlation robustness when the upper-tail independence condition is violated. In this setting, the input statistics $\boldsymbol{X}$ have standard normal marginal distributions, but their dependence is structured by the bivariate Cuadras-Augé copula, which induces upper-tail dependence:

$$C(u_i, u_j) = (\min(u_i, u_j))^{\theta} (u_i u_j)^{1-\theta}, \quad \theta \in (0, 1).$$

Data are simulated using the `crBivariateCA` function from the R package `Distributacalcul`, with the `dependencyParameter` set to θ. As shown in Figure S8, for $\theta = 0.1$, 0.5, or 0.9, all tests are no longer correlation-robust, as their $r(\alpha)$ values diverge for small α.

To assess the correlation robustness of the HC, MinP, Simes, $\phi 3$, and CCT tests in the GWAS setting, we examine their α^{sr} values in SNP-set tests across 22,648 human genes. Figure S9 shows the distribution of SNP counts per gene. The distribution is highly right-skewed, with the majority of genes containing fewer than 50 SNPs.

To provide an overall picture of how correlation influences α^{sr}, we group the genes based on the strength of their correlations, measured by the Herdin distance (Herdin et al., 2005), which computes the distance between a given correlation matrix and the identity matrix, thereby quantifying the deviation from independence. Specifically, for a correlation matrix

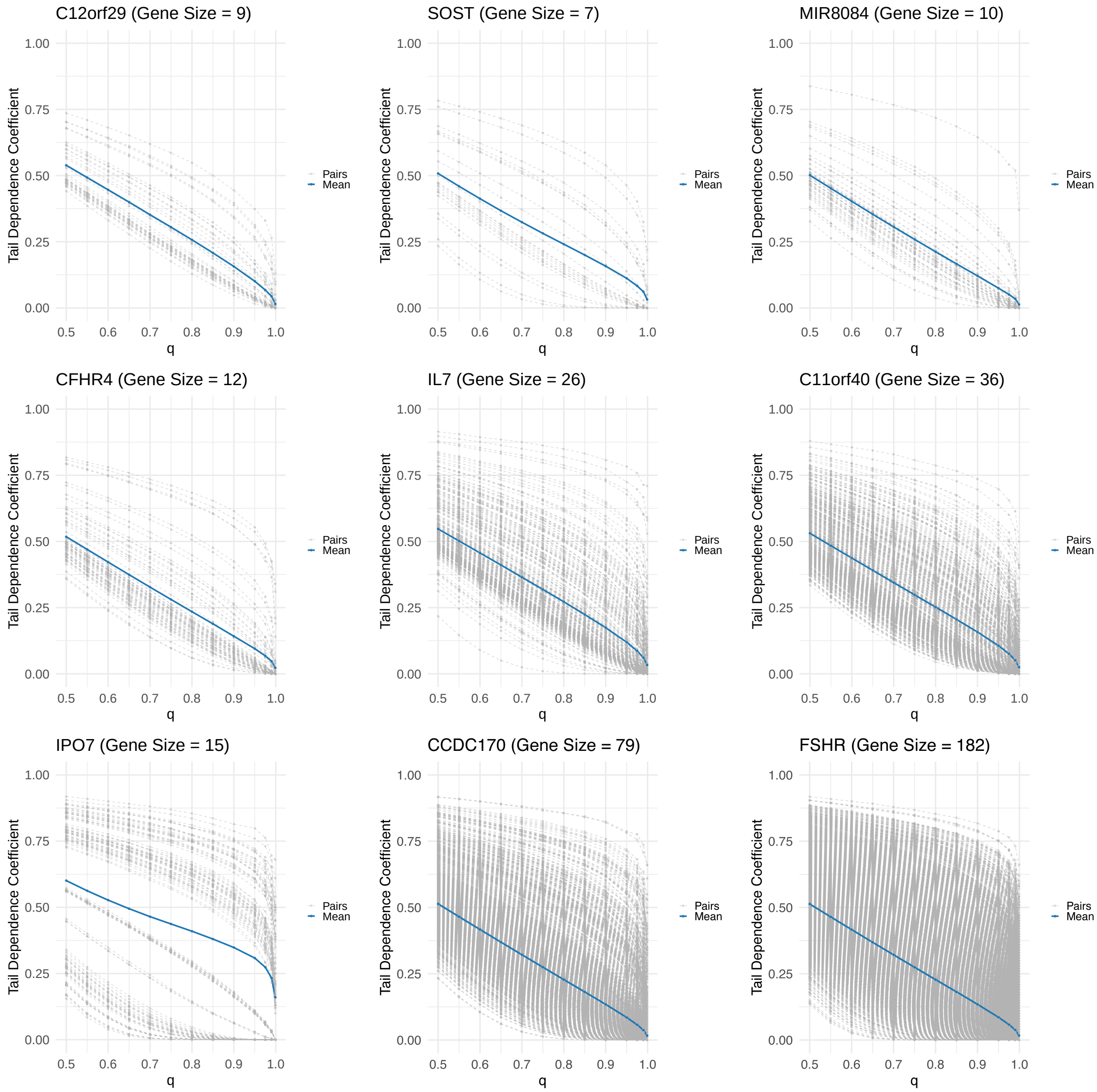

Figure S6: Empirical upper tail-dependence coefficients (TDC) under the continuous trait model for 9 genes with different levels of linkage disequilibrium (LD): weak (C12orf29, Herdin distance 0.253; SOST, 0.231), moderate (MIR8084, 0.308; CFHR4, 0.357; IL7, 0.548; C11orf40, 0.549), and strong (IPO7, 0.629; CCDC170, 0.633; FSHR, 0.759). In each panel, grey dashed lines show pairwise TDC estimates $\widehat{\lambda}_{ij}(q)$ for all SNP pairs within the gene, and the blue solid line shows the average across all pairs. The x-axis shows $q \in \{0.5, 0.55, \ldots, 0.95, 0.975, 0.99, 0.999\}$.

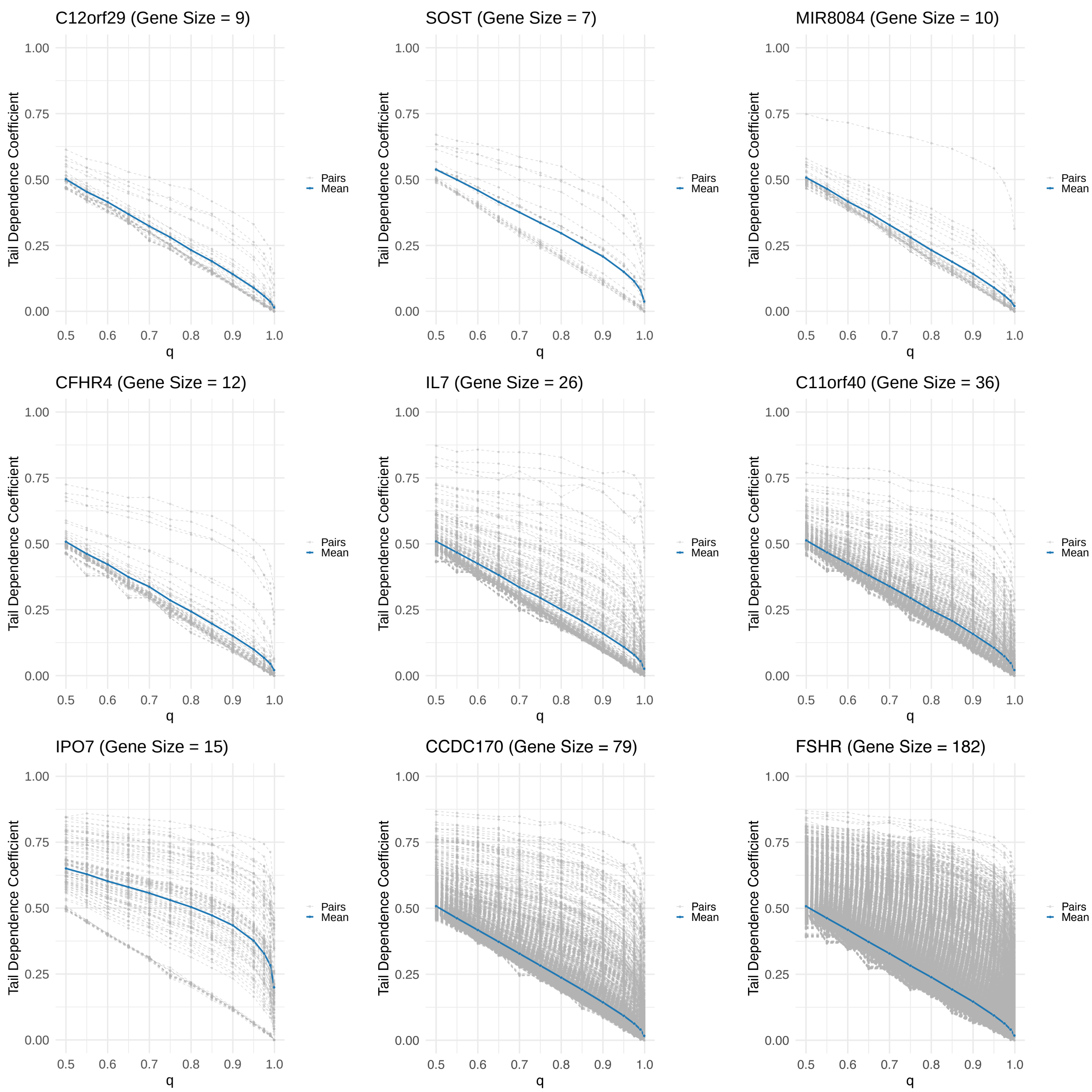

Figure S7: Empirical upper tail-dependence coefficients (TDC) under the binary trait model for 9 genes with different levels of linkage disequilibrium (LD): weak (C12orf29, Herdin distance 0.253; SOST, 0.231), moderate (MIR8084, 0.308; CFHR4, 0.357; IL7, 0.548; C11orf40, 0.549), and strong (IPO7, 0.629; CCDC170, 0.633; FSHR, 0.759). In each panel, grey dashed lines show pairwise TDC estimates $\widehat{\lambda}_{ij}(q)$ for all SNP pairs within the gene, and the blue solid line shows the average across all pairs. The x-axis shows $q \in \{0.5, 0.55, \ldots, 0.95, 0.975, 0.99, 0.999\}$.

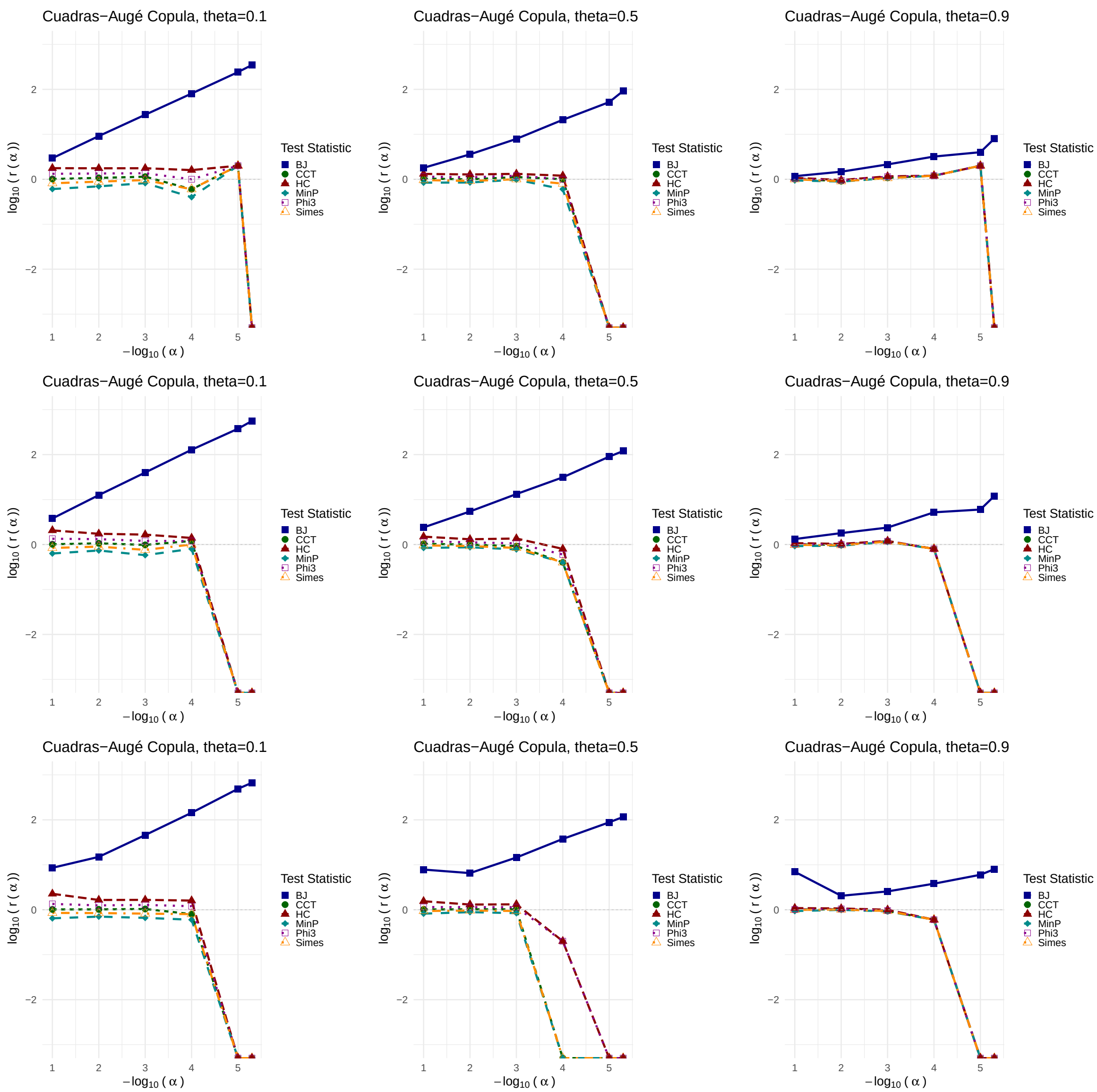

Figure S8: Plots of $\log(r(\alpha))$ against $-\log_{10}(\alpha)$ under the bivariate Cuadras-Augé copula. Rows 1, 2, and 3 correspond to $n = 10$, 100, and 200, respectively. The left, middle, and right columns show results for $\theta = 0.1$, 0.5, and 0.9, respectively.

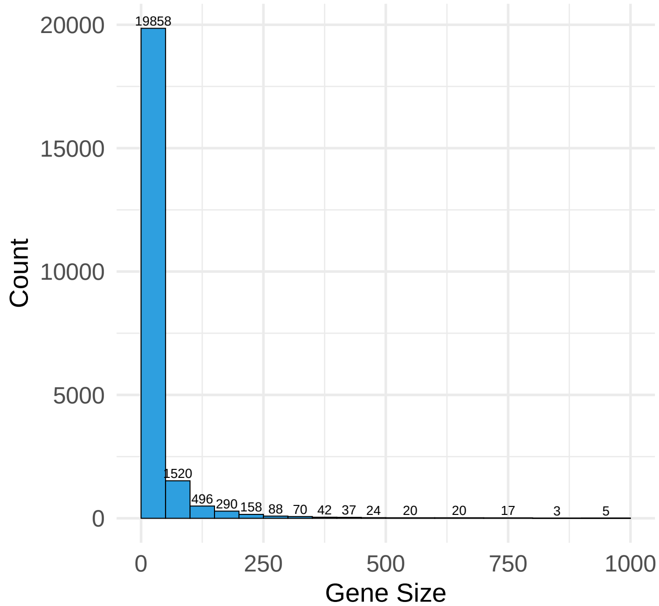

Figure S9: Distribution of the number of SNPs per gene across 22,648 human genes from the GEFOS 2012 study.

Σ, the Herdin distance is defined as

$$D = 1 - \frac{tr(\Sigma I)}{||\Sigma||_f ||I||_f} = 1 - \frac{tr(\Sigma)}{\sqrt{n}||\Sigma||_f}, \tag{S5.1}$$

where $||\cdot||_f$ denotes the Frobenius norm. We classify the 22,648 genes into three groups based on the value of D: weak correlation (i.e., $D \leq 0.3$), medium correlation (i.e., $0.3 < D \leq 0.6$), and strong correlation (i.e., $D > 0.6$). Of the total, 5,720 genes (25%) have $D \leq 0.3$, 13,980 genes (62%) have $0.3 < D \leq 0.6$, and 2,948 genes (13%) have $D > 0.6$.

Figure S10 shows the distribution of genes identified by the five tests, as reported in Table 1, across three gene classes: weak, medium, and strong correlations. For each test method and gene class, we color-code the number of genes with α^{sr} falling into six categories: $\alpha^{sr} < 10^{-5}$, $10^{-5} \leq \alpha^{sr} < 10^{-4}$, $10^{-4} \leq \alpha^{sr} < 10^{-3}$, $10^{-3} \leq \alpha^{sr} < 10^{-2}$, $10^{-2} \leq \alpha^{sr} < 10^{-1}$, and $\alpha^{sr} \geq 10^{-1}$. The bar height represents the number of genes in each category. The

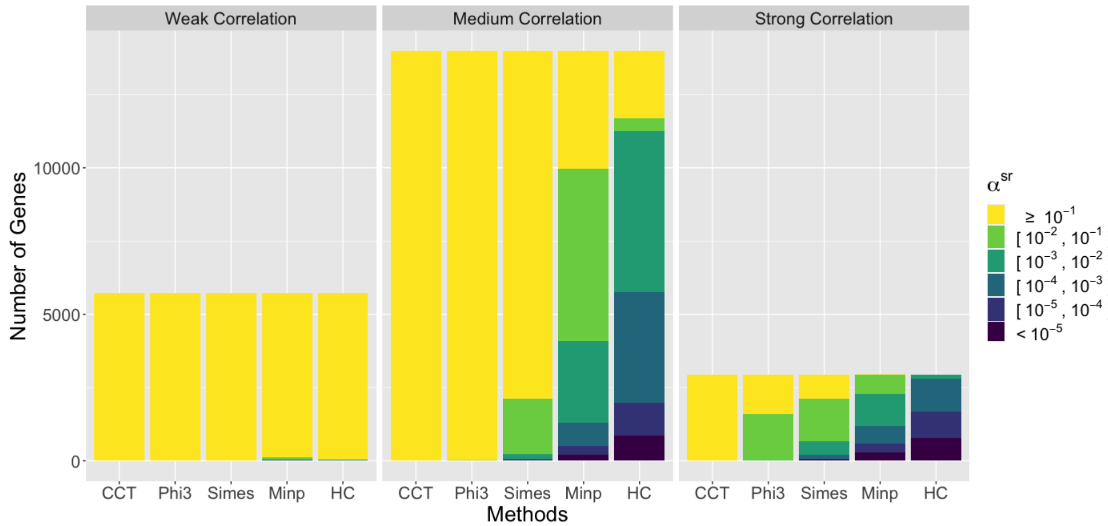

Figure S10: Distribution of α^{sr} values across three gene groups, classified by the Herdin distance-based strength of SNP correlations (LD): weak (5,720 genes), medium (13,980 genes), and strong (2,948 genes). Each bar in the plot represents a different testing method.

results indicate that stronger correlations tend to require smaller values of α^{sr}, with genes associated with lower α^{sr} values being more prevalent in groups with stronger correlations. When comparing the five tests, we observe that their robustness to correlation decreases in the following order: CCT, $\phi 3$, Simes, minP, and HC. This is evidenced by the increasing bar areas representing smaller α^{sr} within each correlation group. We also consider other measures of correlation strength, based on the Frobenius norm and the spectral norm (both scaled by gene size n), and obtain similar results, which are reported in Figures S11 and S12.

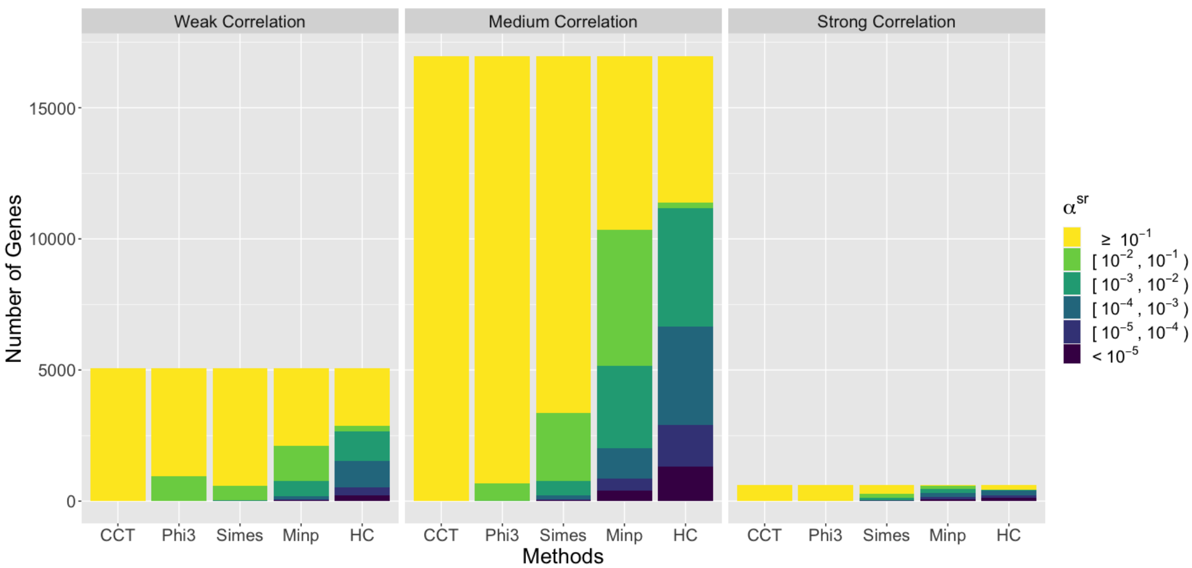

Figure S11: Distribution of α^{sr} values across three gene groups, classified by the strength of SNP correlations (LD), as measured by the gene size-scaled Frobenius norm for the distance between the correlation/LD matrix and the identity matrix. Specifically, the distance is defined as $D_f = \frac{||\Sigma - I||_f}{n} = \frac{\sqrt{\sum_{i \neq j} \sigma_{ij}^2}}{n}$. The 22,648 genes are categorized into three groups based on D_f: weak correlation ($D_f \leq 0.3$, 5061 genes), medium correlation ($0.3 < D_f \leq 0.6$, 16980 genes), and strong correlation ($D_f > 0.6$, 607 genes). Each bar in the plot represents a different testing method.

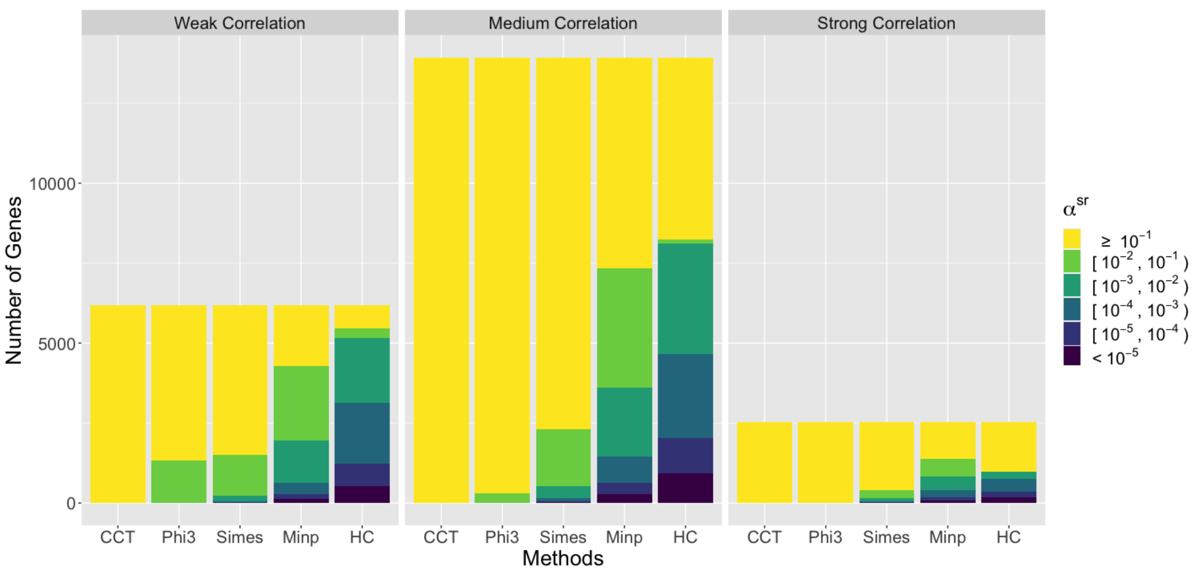

Figure S12: Distribution of α^{sr} values across three gene groups, classified by the strength of SNP correlations (LD), as measured by the gene size-scaled spectral norm of the correlation/LD matrix: $D_s = \frac{||\Sigma||_s}{n} = \frac{\text{the largest eigen value of } \Sigma}{n}$. The 22,648 genes are categorized into three groups based on D_s: weak correlation ($D_s \leq 0.3$, 6177 genes), medium correlation ($0.3 < D_s \leq 0.6$, 13930 genes), and strong correlation ($D_s > 0.6$, 2541 genes). Each bar in the plot represents a different testing method.

S5.2 On p-value Calculation of the MinP test

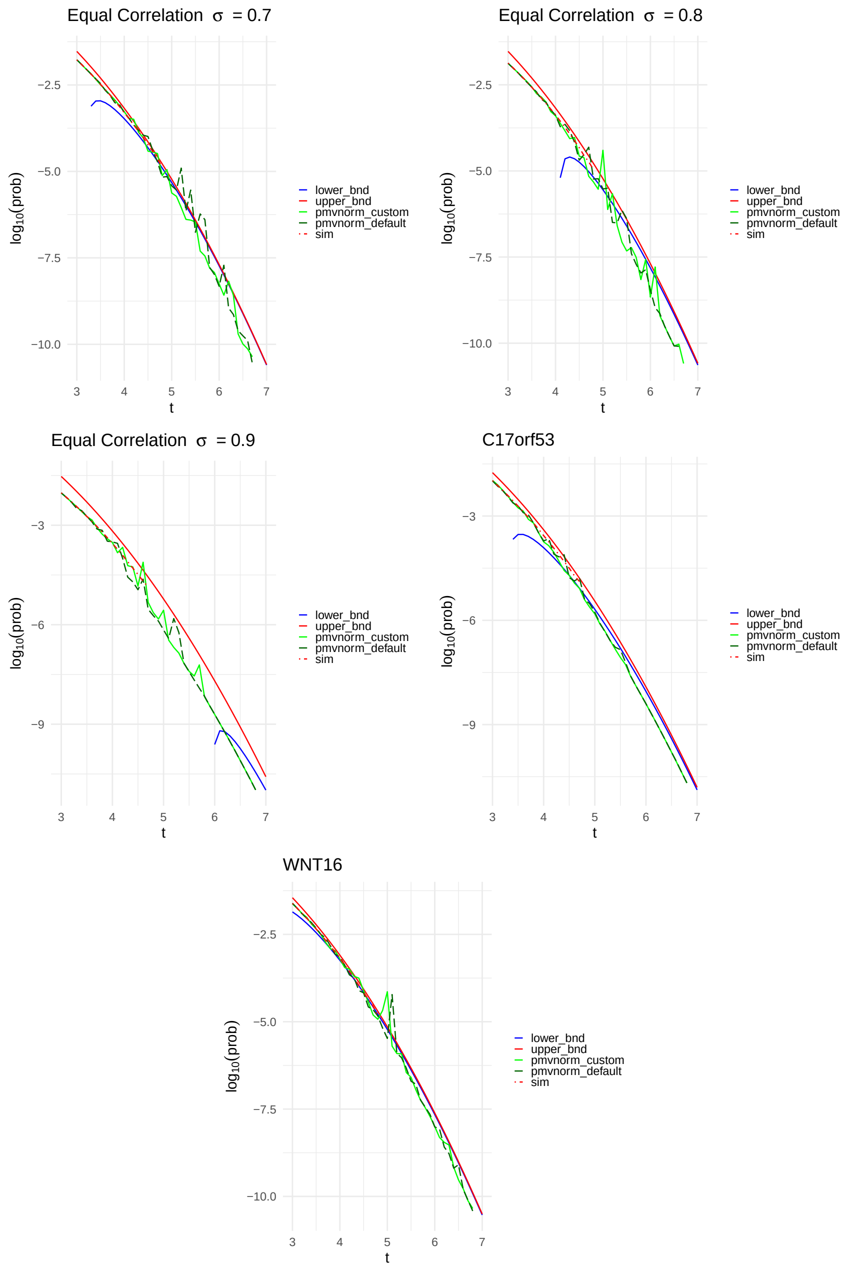

Figure S13: Plot $\log_{10} \mathbb{P}\left(T_{maxhN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right)$ against t under the equal-correlation structure with $\sigma = 0.7$, 0.8, and 0.9 ($n = 10$) and LD structures of genes *C17of53* ($n = 6$) and *WNT16* ($n = 12$). $\mathbb{P}\left(T_{maxhN}(\boldsymbol{X}) > t|\boldsymbol{\Sigma}\right)$ is evaluated by simulations ("sim"), and using `pmvnorm` ("default" and "custom" with the error tolerance parameter "abseps" set to the default 10^{-6} and 10^{-15}, respectively), along with its upper and lower bounds from Corollary 2 ("lower_bnd" and "upper_bound").

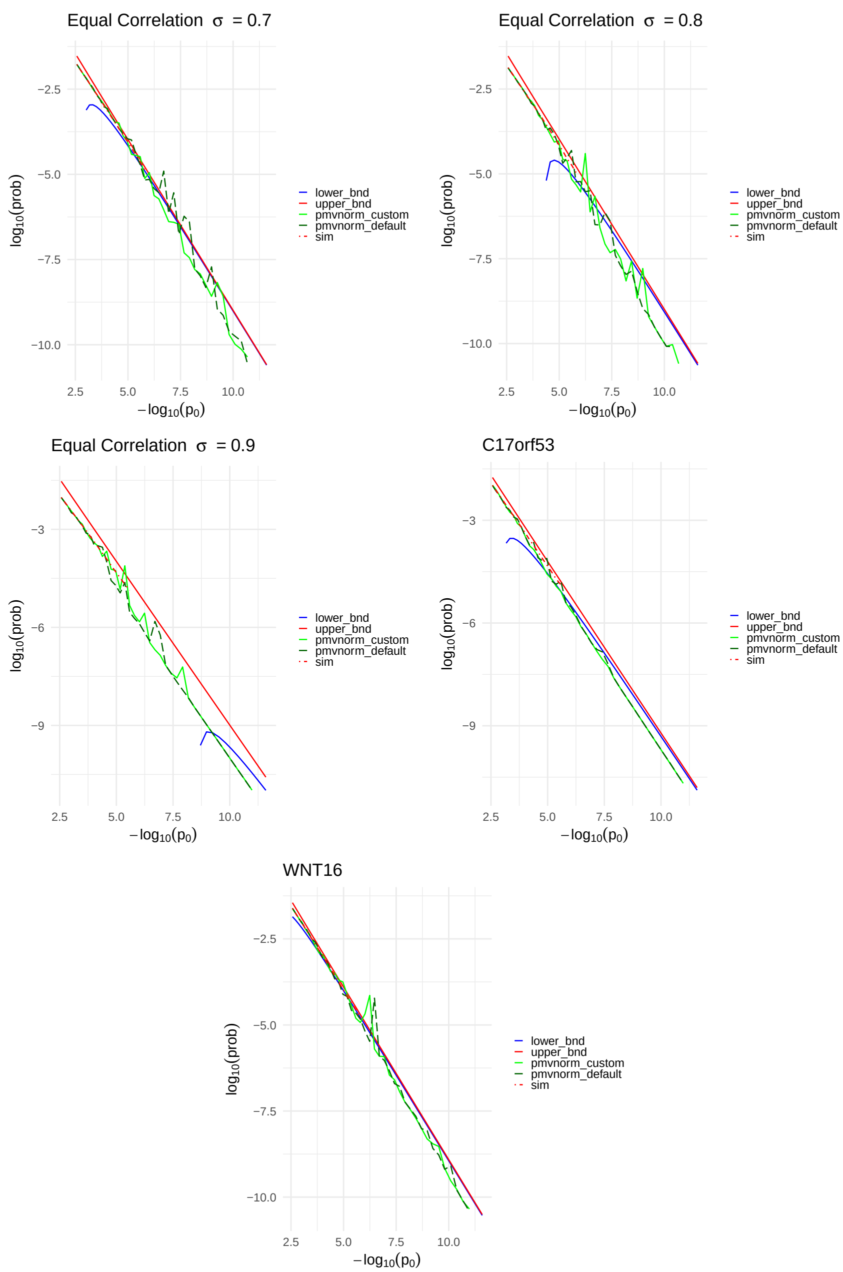

Figure S14: Plot $\log_{10} \mathbb{P}\left(T_{minP} < p_0 | \mathbf{\Sigma}\right)$ against $-\log_{10}(p_0)$ under the equal-correlation structure with $\sigma = 0.7$, 0.8, and 0.9 ($n = 10$) and LD structures of genes *C17of53* ($n = 6$) and *WNT16* ($n = 12$). $\mathbb{P}\left(T_{minP} < p_0 | \mathbf{\Sigma}\right)$ is evaluated by simulations ("sim"), and using `pmvnorm` ("default" and "custom" with the error tolerance parameter "abseps" set to the default 10^{-6} and 10^{-15}, respectively), along with its upper and lower bounds from Corollary 2 ("lower_bnd" and "upper_bound").

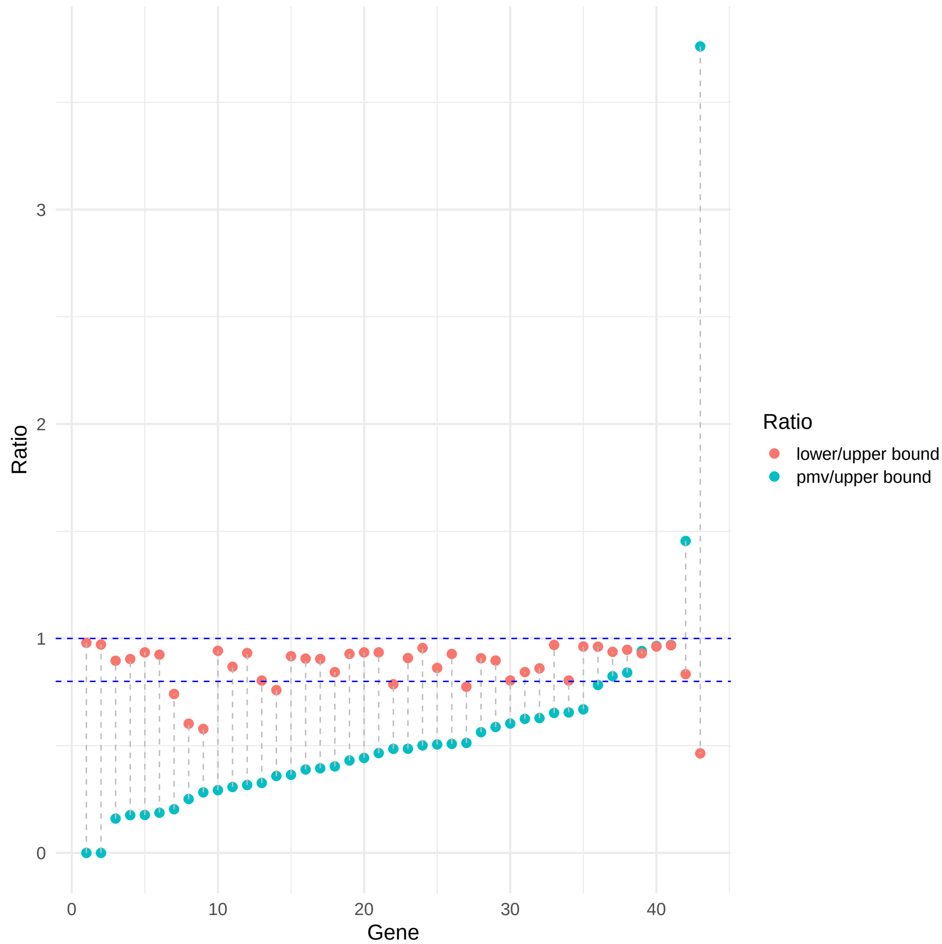

Figure S15: The lower-to-upper bound ratio (red dots) versus the `pmvnorm`-to-upper bound ratio (blue dots) for the 43 top-hit genes identified by the CCT test. The lower and upper bounds are given in (3.15).

A key benefit of using the lower and upper bounds for computing tail probabilities is their computational efficiency. The bounds can be calculated in closed form, while `pmvnorm` requires numerical integration. Computation time for `pmvnorm` increases rapidly as n grows, whereas the bounds can be computed over 100 times faster (see computation speed comparisons in Figure S16). Additionally, a limitation of the current `pmvnorm` function is that it does not support n values greater than 1000.

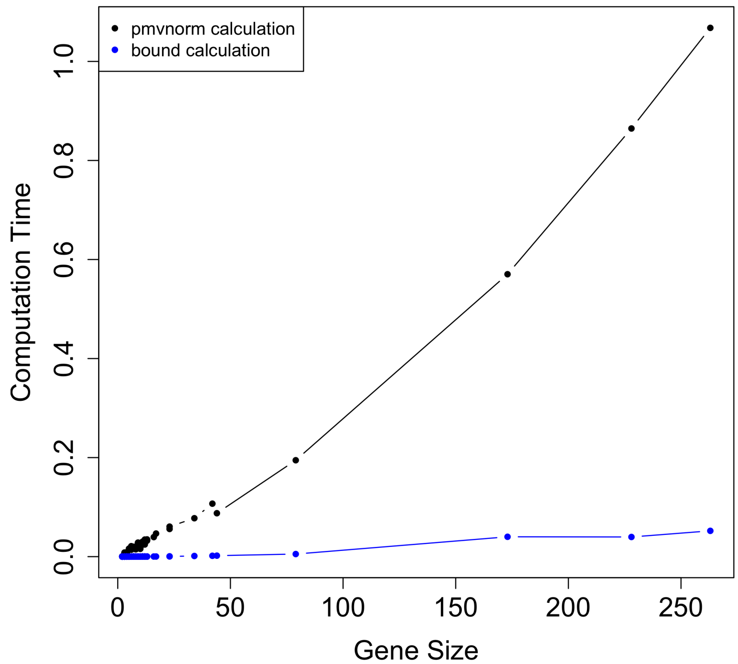

Figure S16: Computation time (in seconds) for 196 GEFOS2012-FN genes, for which the bounds from Corollary 2 provide accurate p-value calculations for the minP test (the lower-to-upper bound ratio is between 0.8 and 1). The X-axis represents gene sizes (number of SNPs, n). The black curve shows the time for tail probability computation using the `pmvnorm` function in R, while the blue curve represents the time for computing the bounds in (3.15).

S5.3 Statistical Power

S5.3.1 Gaussian Mean Model and GWAS-Based Simulations

We assess the statistical power of supremum-based p-value combination tests under the Gaussian model, linear models in a GWAS setting, and a non-Gaussian model defined by pairwise Farlie-Gumbel-Morgenstern copula. Our goals are twofold: (1) to identify which tests exhibit relatively higher power while maintaining type I error control (with the α-level threshold determined numerically from extensive null hypothesis simulations), and (2) to examine how power changes when the threshold is calculated assuming independence, which highlights the impact of this assumption on power calculations.

Under the Gaussian model, the null hypothesis is given in (3.10) in the manuscript, while the alternative is $H_1 : \boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)' \neq \mathbf{0}$. In the simulations, we generate k nonzero means $\mu_{i_1}, ..., \mu_{i_k} = \mu \neq 0$, with the indices $\{i_1, ..., i_k\}$ randomly selected from $\{1, ..., n\}$.

Figure S17 compares the statistical power of six tests (i.e., minP, Simes, HC, $\phi 3$, CCT, and BJ) at $\mu = 3$ across different signal densities (i.e., the number of signals k) and correlation structures defined by $\boldsymbol{\Sigma}$. The solid curves represent the "true" power, where type I errors are well controlled using empirical thresholds.

We have the following key observations regarding the "true" power. First, the correlation-robust tests (i.e., minP, Simes, HC, $\phi 3$, and CCT) perform similarly to each other but differ significantly from the BJ test, which is not correlation-robust. When correlations are strong (e.g., equal correlation with $\sigma = 0.5$ or 0.9), BJ exhibits lower power. In contrast, when correlations are moderate or weak, or show positive or negative polynomial decay, BJ is less powerful for detecting sparse signals (i.e., when k is small) but tends to be more powerful for detecting dense signals. Second, among the correlation-robust tests, the minP test is the most powerful when correlations are very strong (e.g., equal correlation with $\sigma = 0.9$). However, under other correlation structures, the HC test demonstrates greater power. Third, the minP and Simes tests perform relatively similarly, with the Simes test surpassing the minP test when correlations are weaker or exhibit positive or negative polynomial decay. Fourth, the CCT and $\phi 3$ tests show comparable performance, with the CCT test often being slightly more powerful. Except under strong correlations, their power surpasses that of the minP-Simes group but remains below that of the HC test.

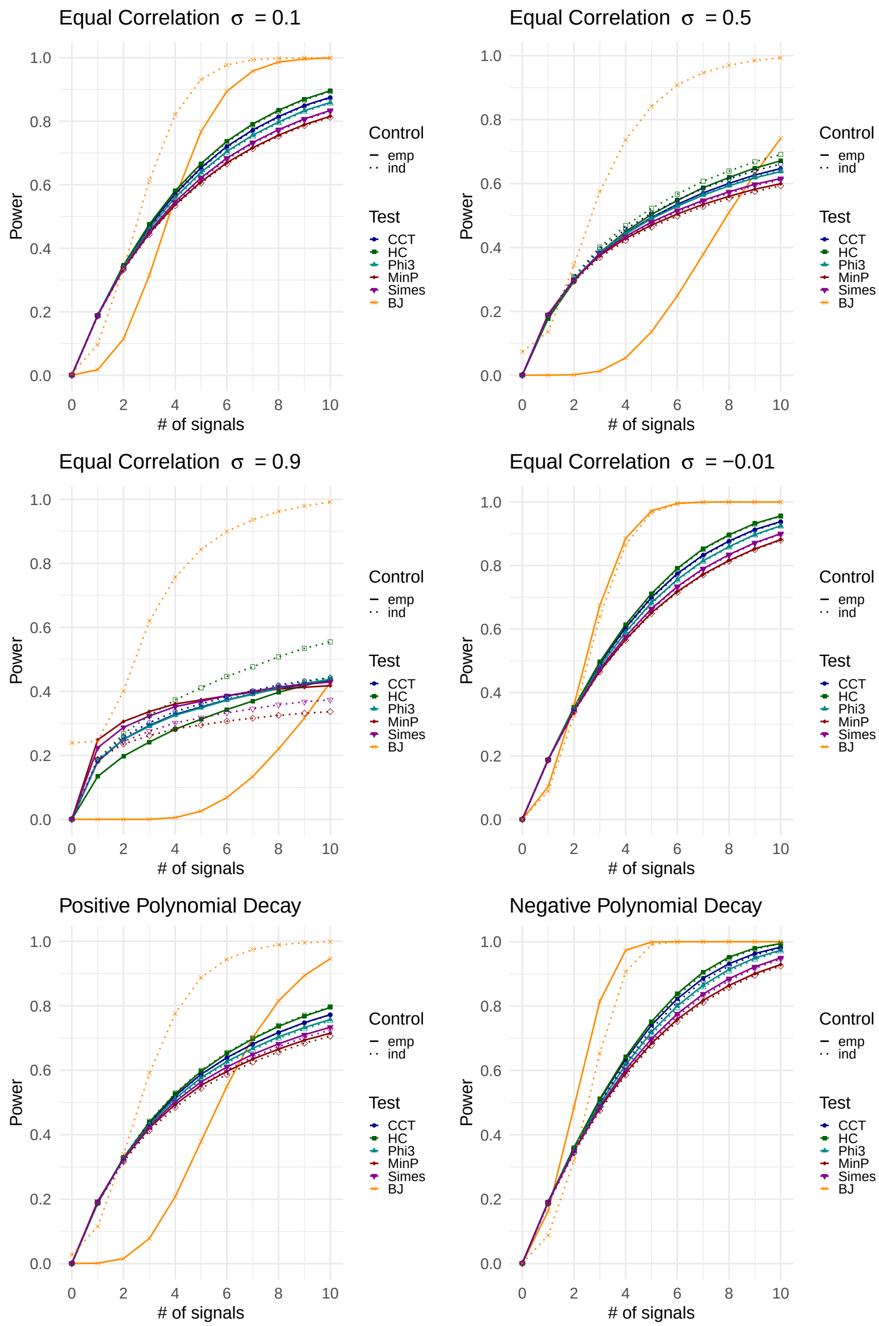

Figure S17: Power study based on a Gaussian model with different correlation structures. The signal strength is fixed at $\mu = 3$, and the number of causal signals varies along the x-axis. The significance level is set at $\alpha = 5 \times 10^{-4}$. The number of input p-values combined is $n = 10$, and the number of simulations is 10^6. Solid curves labeled "emp" represent the true power when α is controlled using an empirical method. Dotted curves labeled "ind" represent the nominal power under the assumption of independence.

Next, we turn our attention to the comparison between the true power and the nominal power (indicated by dotted curves, where the α-level threshold is obtained under the assumption of independence). For each test, the gap between its solid and dotted curves represents the change in power, providing insight into the test's robustness in terms of statistical power. As expected, the changes in power are smaller under weaker correlations. Further interesting observations include: 1) Overall, the correlation-robust tests show relatively small changes in power unless the correlations are very strong (e.g., equal correlation with $\sigma = 0.9$). In contrast, the BJ test exhibits significant changes in power, confirming its lack of robustness to correlations. If the BJ test procedure ignores correlations and assumes independence, its power will be significantly inflated when correlations are positive (since the true power is much lower) and deflated when the correlations are negative. 2) Among the correlation-robust tests, strong correlations result in slightly lower power for the minP and Simes tests, and slightly higher power for the HC test, when independence is assumed. This aligns with the findings in Section 6.1, where assuming independence tends to make the minP and Simes tests slightly conservative, while the HC test tends to behave slightly liberally. The CCT and $\phi 3$ tests show smaller differences between their solid and dotted curves, indicating that they are more robust to correlation.

Figures S19 and S20 display similar results for $\mu = 2$, with consistent patterns observed.

The second setting for the power study is consistent with GWAS. Specifically, we simulate continuous and binary traits using regression and logit models, respectively:

$$Y_j = \boldsymbol{G}'_{j\cdot}\boldsymbol{\beta} + \epsilon_j, \qquad \epsilon_j \overset{i.i.d.}{\sim} N(0,1), \tag{S5.2}$$

$$\text{logit}(\mathbb{P}(Y_j = 1)) = \boldsymbol{G}'_{j\cdot}\boldsymbol{\beta}, \quad j = 1, \ldots, N, \tag{S5.3}$$

where Y_j represents the phenotypic trait of the jth subject, and $\boldsymbol{G}_{j\cdot} = (G_{j1}, \ldots, G_{jn})'$ is the genotype vector of n SNPs associated with the jth subject. In the simulations, we set $N = 10,000$ for both continuous and binary traits. For binary traits, we assume a 1:1 case-control design with 5,000 cases and 5,000 controls. For the Type I error and TDC evaluations under the null hypothesis, we directly generate 5,000 cases and 5,000 controls. For power simulations under the alternative hypothesis, we first generate an initial population of 100,000 subjects to ensure a balanced sample, from which we randomly select 5,000 cases and 5,000 controls.

The nonzero elements of $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_n)'$ represent the causal genetic effects (i.e., the "signals") of the corresponding SNPs. The global hypothesis for the SNP-set test is as follows:

$$H_0 : \boldsymbol{\beta} = \mathbf{0} \text{ versus } H_1 : \boldsymbol{\beta} \neq \mathbf{0}.$$

The genotype data $\boldsymbol{G}_{j\cdot}$ are simulated using the HapSim algorithm (Montana, 2005), where the genotype values are 0, 1, or 2, based on given minor allele frequencies (MAF) and an LD matrix (i.e., $\boldsymbol{\Sigma}$). The MAF and $\boldsymbol{\Sigma}$ are set using gene-based SNP sets (as defined by GEFOS2012FN) and real genotype data from the 1000 Genomes Project, following the same method described previously.

We considered the genes *FSHR* and *CCDC170*, both associated with osteoporosis and bone mineral density (BMD), representing a variety of gene sizes, MAFs, and LD structures (`https://www.genecards.org`). In our data, gene *FSHR* contains 182 SNPs, including 15 rare variants with MAFs less than 0.05, and the Herdin distance between its LD matrix and the identity matrix is 0.7593. Gene *CCDC170* contains 79 SNPs, two of which are rare variants, with a Herdin distance of 0.6333.

When simulating traits under H_1, the causal SNPs are randomly located, and their effect sizes are represented by nonzero coefficients, $\beta_i = \beta$. After generating the trait values using Models (S5.2) and (S5.3), individual SNP p-values are calculated using the marginal score test statistics (McCulloch, 2000; Schaid et al., 2002; Barnett et al., 2017). These p-values serve as input for the p-value combination tests for the corresponding gene.

Figure S18 shows the true power (with type I error controlled by the empirical method, represented by solid curves) versus the nominal power (assuming independence, represented by dotted curves) for continuous traits, with SNP genotypes simulated based on the FSHR gene. We consider three levels of signal sparseness: 1, 5, and 30, representing the number of nonzero elements in $\boldsymbol{\beta}$, along with two significance levels: $\alpha = 0.005$ (left panels) and 0.0005 (right panels). The x-axis in each panel reflects varying genetic effect sizes (i.e., the values of nonzero $\boldsymbol{\beta}$ elements). Here, we focus on the ACR tests: minP, Simes, HC, $\phi3$, and CCT.

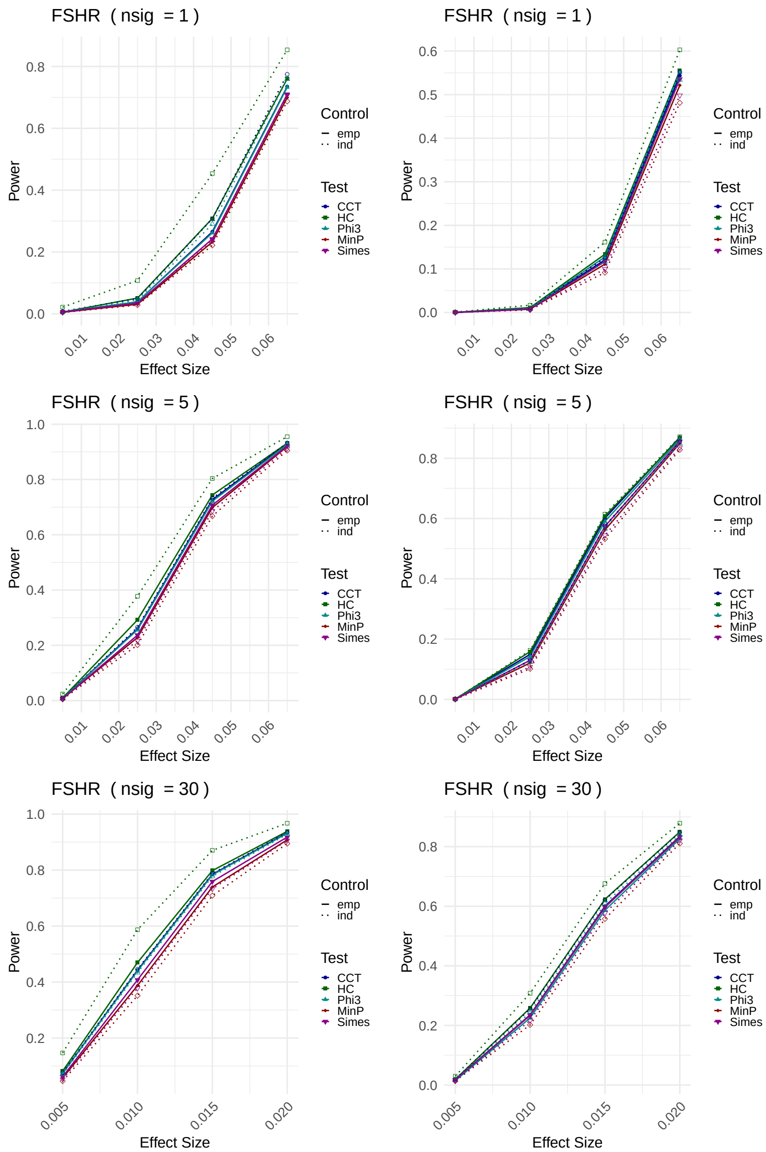

Figure S18: Power study for continuous traits with SNP genotypes simulated from the gene *FSHR*. The x-axis represents varying effect sizes. The number of causal SNPs is 1 (row 1), 5 (row 2), and 30 (row 3). The significance levels are $\alpha = 0.005$ (left column) and 0.0005 (right column). The number of repeated simulations is 10^5 for obtaining the error control threshold, and 1000 for calculating power.

The following observations are made. 1) The difference between the true and nominal powers is smaller at lower α levels, reinforcing the idea that these tests are ACR. This pattern aligns with the Gaussian model, where the nominal power of the HC test is higher than its true power, while the minP and Simes tests exhibit slightly lower nominal power. 2) For the true power, there are no substantial differences among the tests, except that the HC test demonstrates the highest power, followed by the CCT and ϕ3 tests. The minP and Simes tests have relatively lower power. 3) As the signals become denser (e.g., when the number of signals equals 30), the differences among the tests become more pronounced. This observation is consistent with previous findings that the HC test is more effective at detecting denser signals than the minP test (Li and Siegmund, 2015; Zhang and Wu, 2022). 4) When the significance threshold is more stringent (e.g., $\alpha = 0.0005$), the differences among the tests are reduced.

Additional numerical results for different angles of comparison (e.g., varying the number of signals on the x-axis), for binary traits, and for the gene *CCDC170* are provided in Figures S21 – S27. These results exhibit similar patterns to those summarized above, except that the power differences among the tests are more subtle for the binary trait.

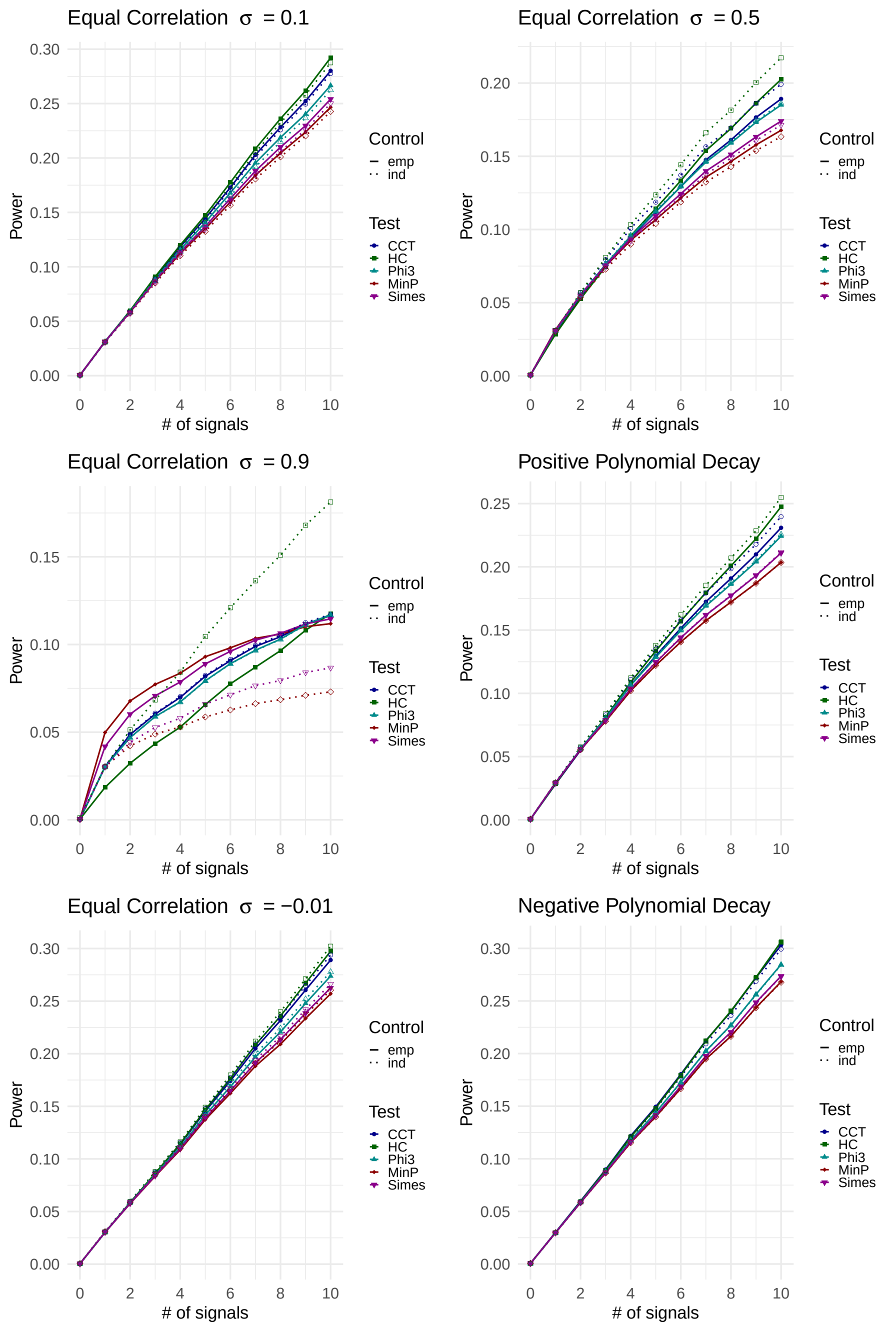

Figure S19: Power study based on Gaussian Mean Model with different correlation structures. Fix the signal strength $\mu = 2$, and vary the number of causal signals. $\alpha = 5 \times 10^{-4}$. emp: solid curves when α is controlled by empirically method. ind: doted curved when α controlled under independence assumption.The number of input p-value combined is $n = 10$. The number of simulation $N = 10^6$.

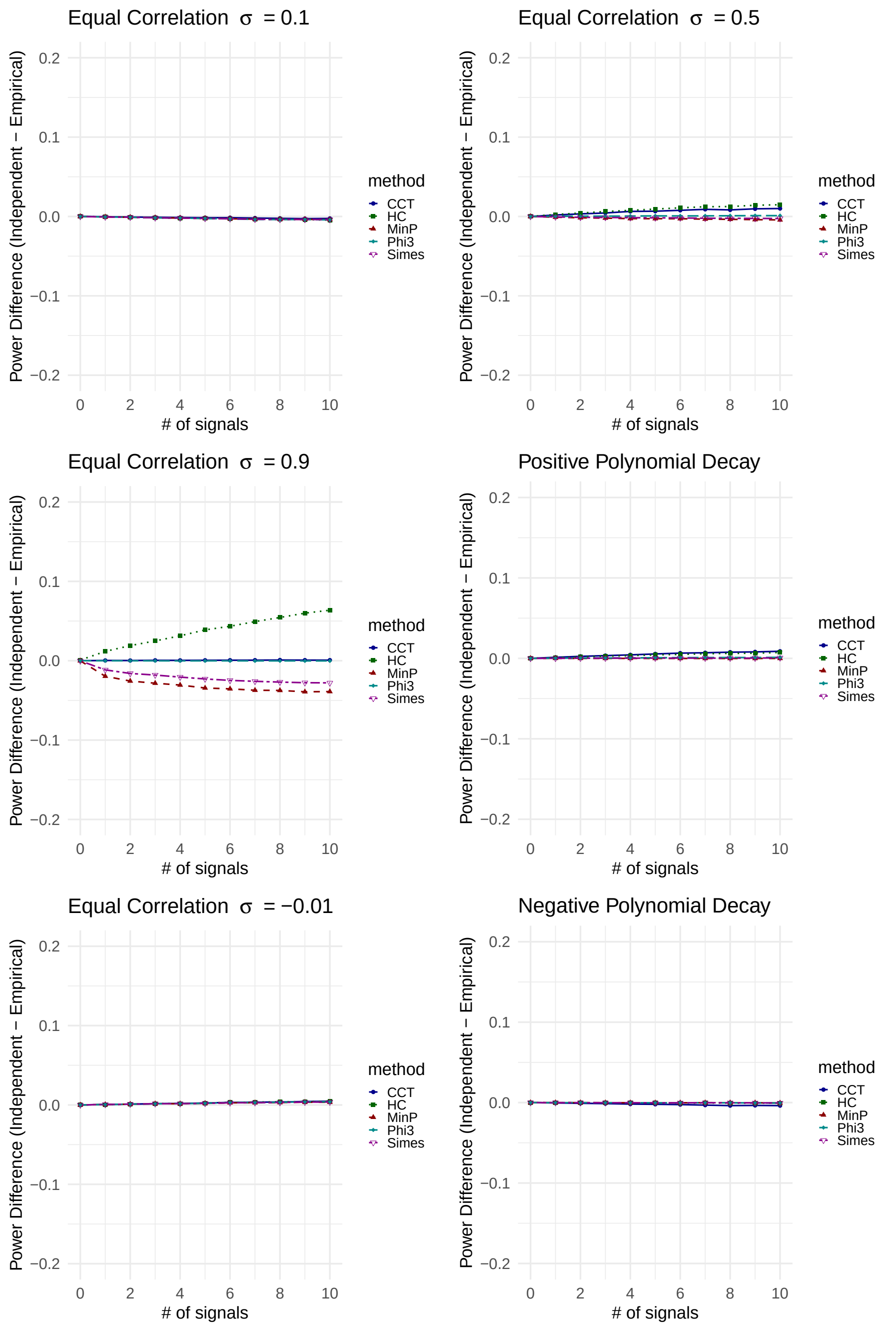

Figure S20: Visualize the power difference between power curves obtained by empirical and independence-assumption-based Type I Error Control in Figure S19.

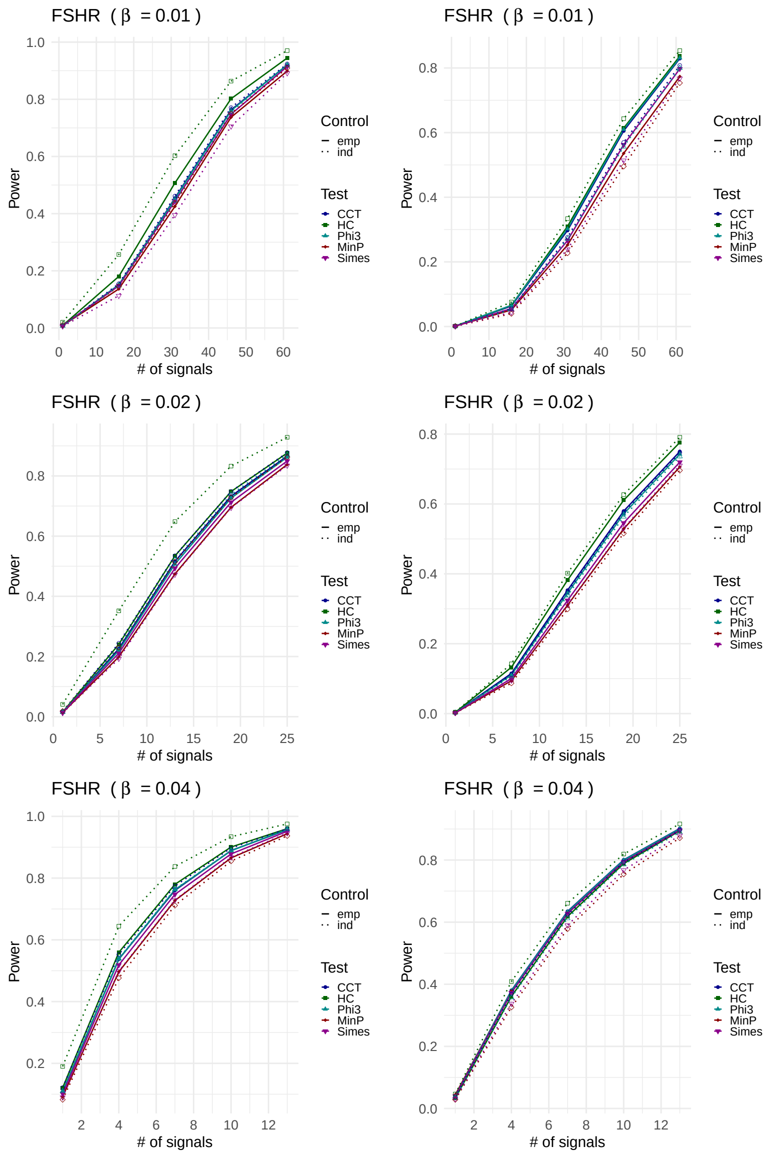

Figure S21: Power study for continuous trait with SNP genotypes simulated from gene *FSHR*. Vary the number of causal SNPs (x-axis). Effect size: $\beta = 0.01$ (row 1), 0.02 (row 2), and 0.04 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

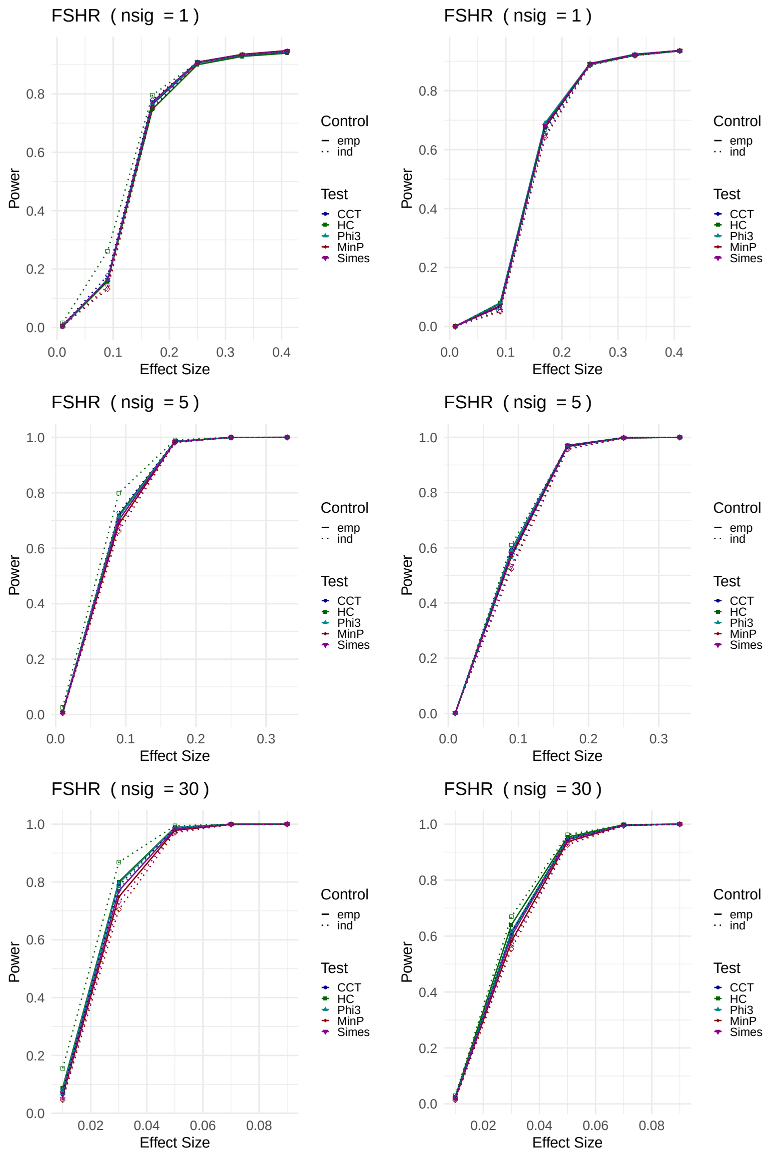

Figure S22: Power study for binary trait with SNP genotypes simulated from gene *FSHR*. Vary effect size (x-axis). Number of causal SNPs: 1 (row 1), 5 (row 2), and 30 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

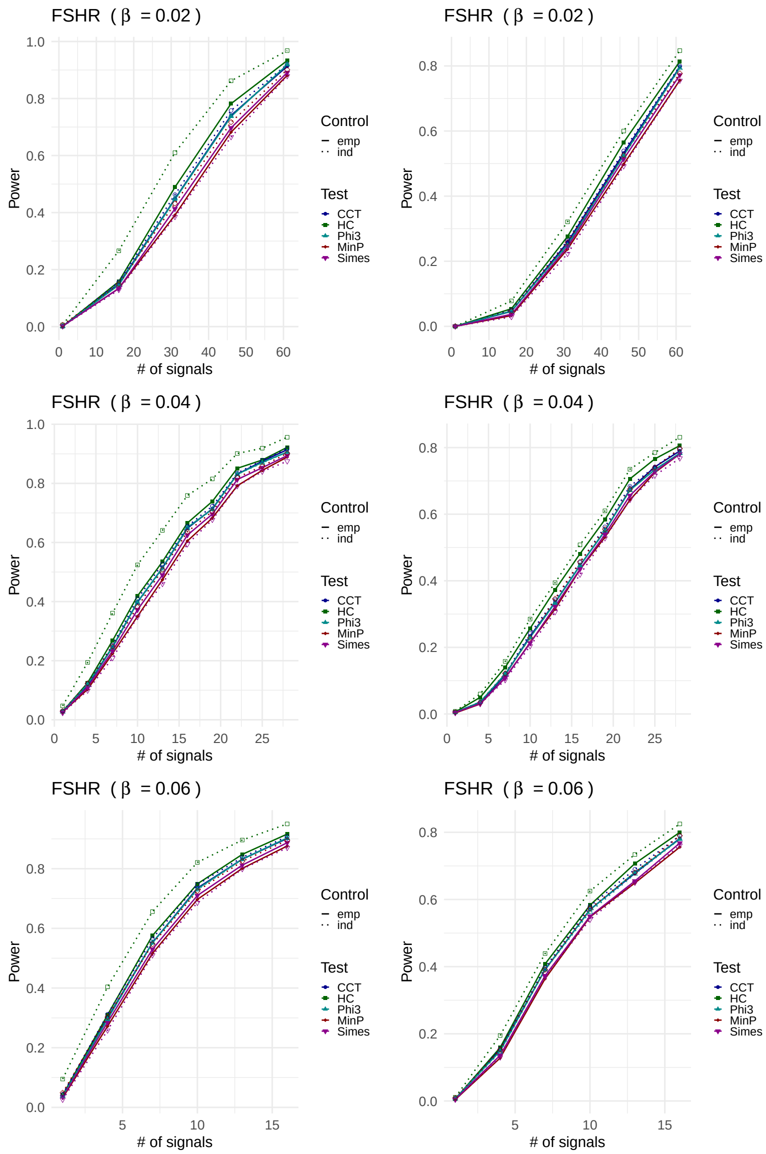

Figure S23: Power study for binary trait with SNP genotypes simulated from gene *FSHR*. Vary the number of causal SNPs (x-axis). Effect size: $\beta = 0.02$ (row 1), 0.04 (row 2), and 0.06 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

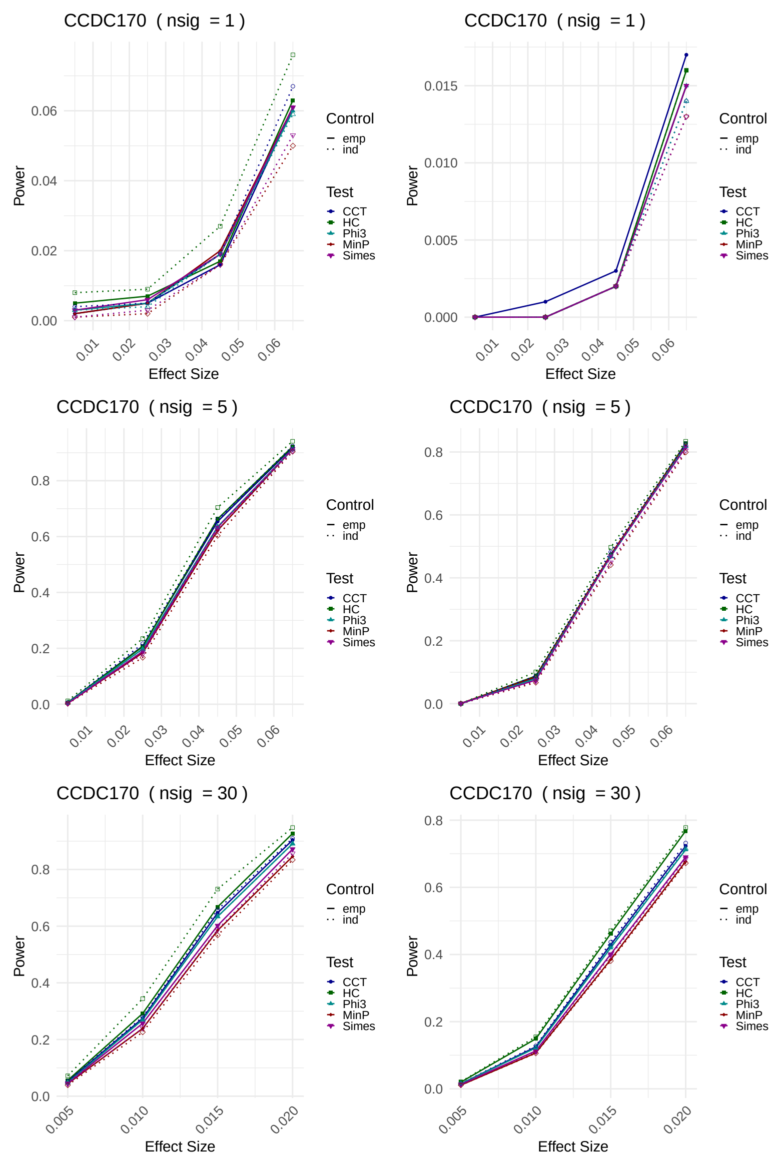

Figure S24: Power study for continuous trait with SNP genotypes simulated from gene *CCDC170*. Vary effect size (x-axis). Number of causal SNPs: 1 (row 1), 5 (row 2), and 30 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

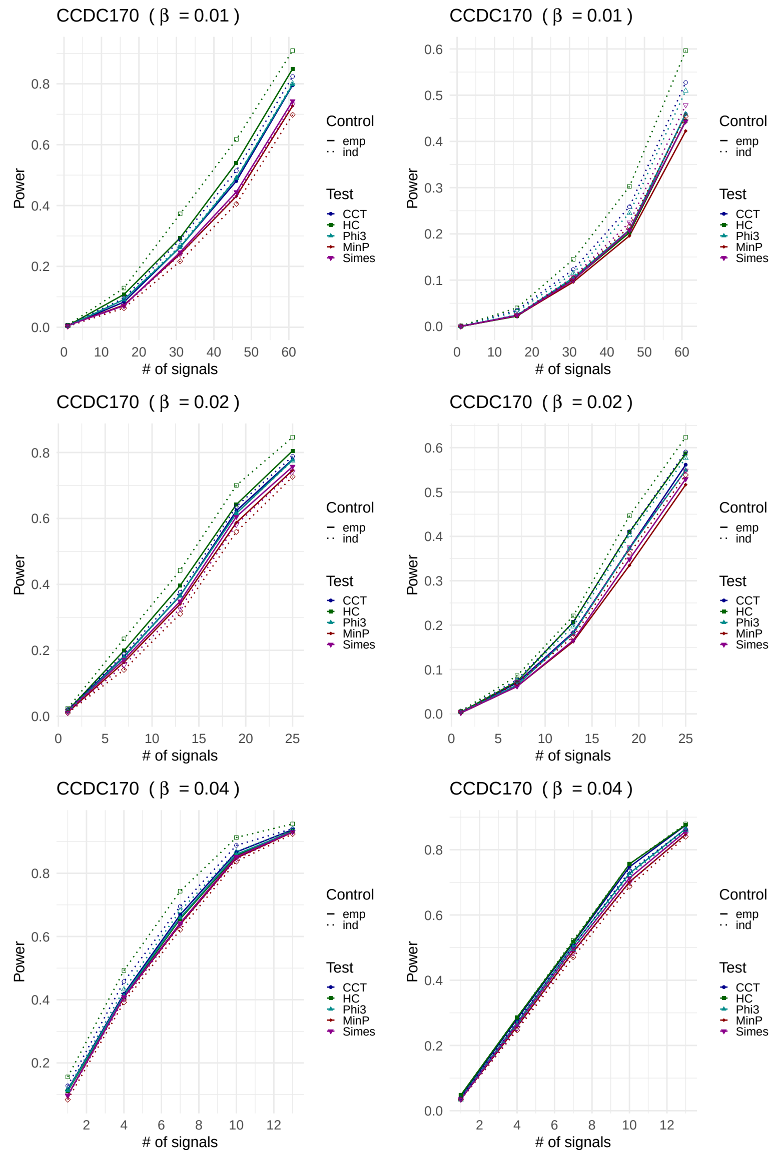

Figure S25: Power study for continuous trait with SNP genotypes simulated from gene *CCDC170*. Vary the number of causal SNPs (x-axis). Effect size: $\beta = 0.01$ (row 1), 0.02 (row 2), and 0.04 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

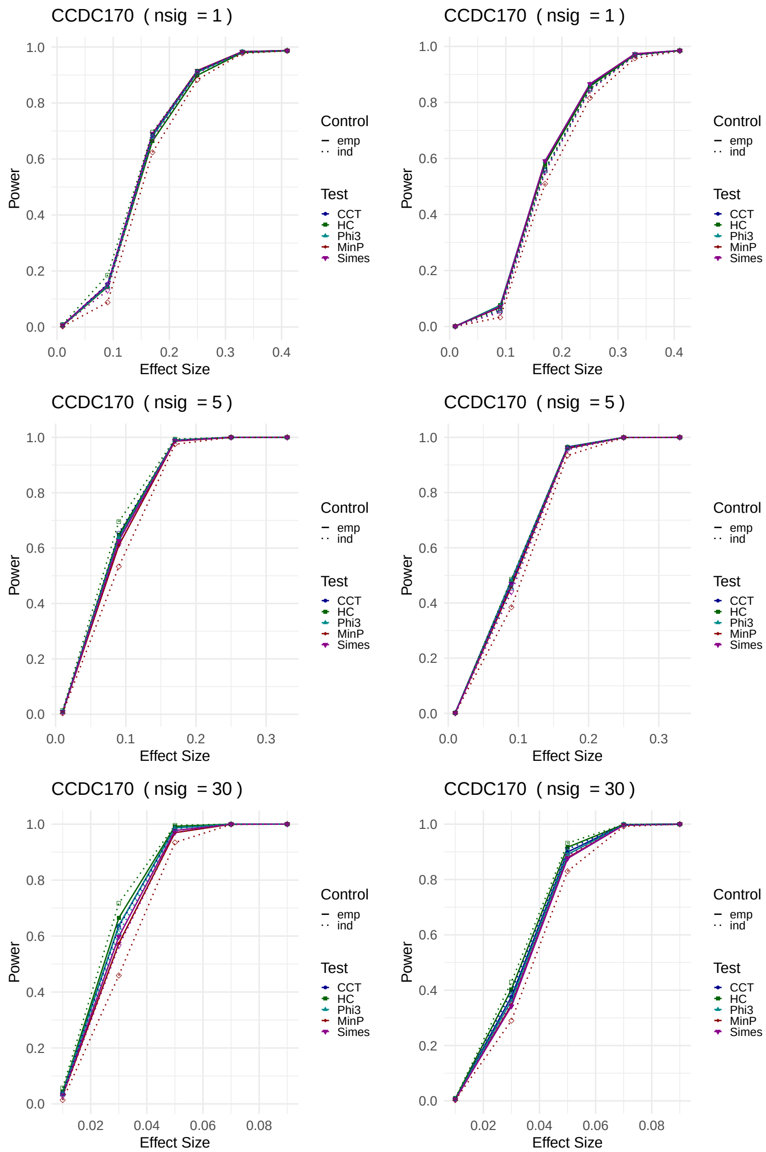

Figure S26: Power study for binary trait with SNP genotypes simulated from gene *CCDC170*. Vary effect size (x-axis). Number of causal SNPs: 1 (row 1), 5 (row 2), and 30 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

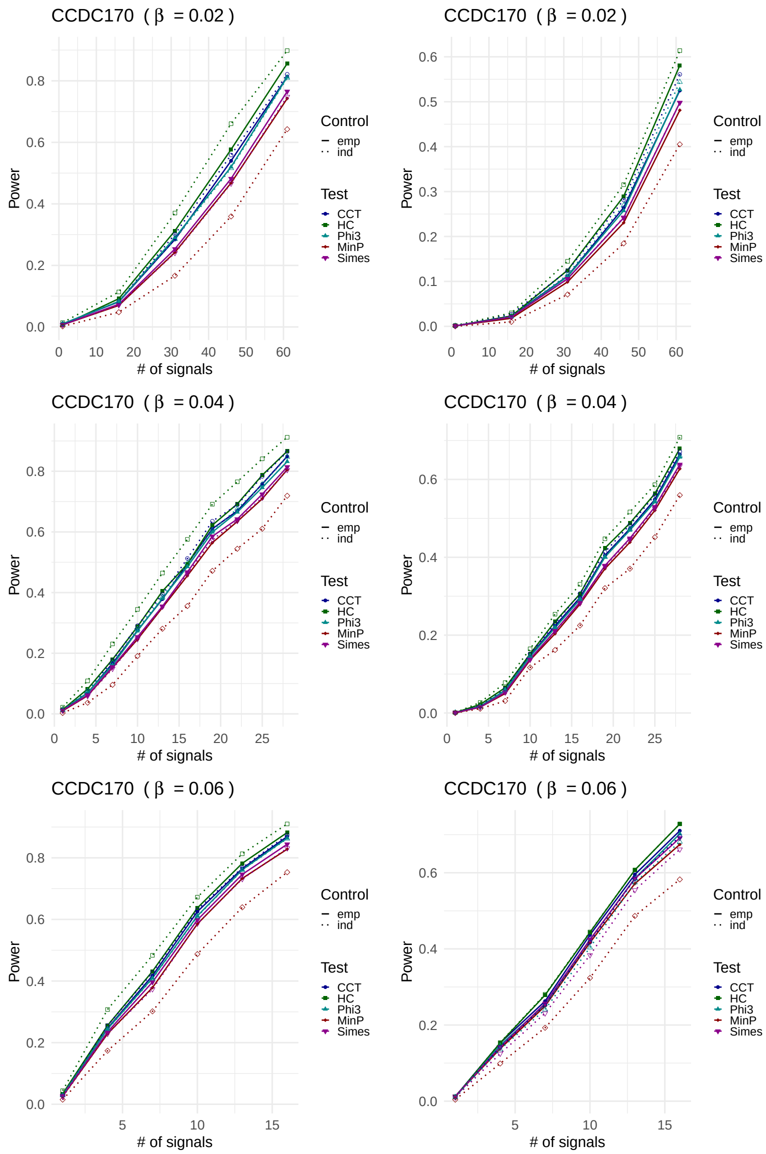

Figure S27: Power study for binary trait with SNP genotypes simulated from gene *CCDC170*. Vary the number of causal SNPs (x-axis). Effect size: $\beta = 0.02$ (row 1), 0.04 (row 2), and 0.06 (row 3). Significant level: $\alpha = 0.005$ (left column) and 0.0005 (right). The numbers of repeated simulation for getting error control threshold and power are $N = 10^5$ and $N = 1000$, respectively.

S5.3.2 Power Comparisons with Summation-Based Statistics

We extend the power study to include summation-based p-value combination methods proposed in the recent literature, including the heavy-tailed transformation statistics of Fang et al. (2023) and the averaging combination statistics of Vovk and Wang (2020). These methods are compared with the supremum-based tests (minP, Simes, HC, ϕ_3, BJ) and CCT.

In the heavy-tailed transformation framework, the test statistic takes the form:

$$T_g(\boldsymbol{P}) = \sum_{i=1}^{n} g(P_i) = \sum_{i=1}^{n} F_U^{-1}(1 - P_i),$$

where F_U^{-1} is the inverse CDF of a random variable U. Following Fang et al. (2023), we focus on the Box-Cox (BC) transformation, where $g(P_i) = 1/P_i^{\eta}$ corresponding to $U \sim \text{Pareto}(1/\eta, 1)$ with $\eta > 0$, which gives the tail probability $P(U > t) = t^{-1/\eta}$ for $t > 1$, indicating a heavy tail. A larger η corresponds to a heavier tail. We consider three specific cases with $\eta = 1.5, 1, 0.5$, denoted $BC_{1.5}$, HM, and $BC_{0.5}$, respectively, because BC_1 is equivalent to the harmonic mean (HM) statistic.

In the averaging combination framework, the test statistic takes the form:

$$M_{r,n}(\boldsymbol{P}) := \left(\frac{P_1^r + \cdots + P_n^r}{n}\right)^{1/r}, \quad r \neq 0; \qquad M_{0,n}(\boldsymbol{P}) := \left(\prod_{i=1}^{n} P_i\right)^{1/n}.$$

Our simulations include the harmonic mean (HM, $r = -1$), geometric mean (GM, $r = 0$), and the arithmetic mean (AM, $r = 1$). For fixed n, the GM is equivalent to Fisher's combination test, since the latter can be considered as applying a monotone $-2\log$ transformation to the GM statistic.

Similar to the previous power study, we evaluate both the true power and the nominal power of the tests. The true power is obtained when the type I error is well controlled by the α-level threshold determined from extensive null distribution simulations that properly account for correlations; the nominal power is obtained using the independence-based threshold. The distance between the true and nominal power curves indicates the change in power when the test procedure ignores correlations and assumes independence, reflecting the extent of the test's robustness to correlation. Since closed-form null distributions are generally not available for these statistics, the independence-based type I error control threshold is also obtained from the simulated null distribution under independence.

Gaussian mean model. Following the same simulation settings of Figure S17, Figure S28 displays the true and nominal power of all eleven tests under six correlation structures. Several observations can be made.

First, the asymptotically correlation-robust tests (i.e., minP, Simes, HC, ϕ_3, CCT, and the BC family $BC_{1.5}$, HM, $BC_{0.5}$, proven in theory) exhibit similar patterns of true power curves. Their curves tend to cluster together, all increasing in a sublinear manner as the number of signals increases — that is, the marginal gain in power diminishes as more signals are added. This pattern differs substantially from BJ, GM, and AM, which are not correlation-robust. The curves of BJ, GM, and AM show a sigmoidal shape, with power increasing steeply at moderate numbers of signals. Under weak-to-moderate correlation, these tests can achieve higher true power than the correlation-robust tests when signals are dense, but lower power when signals are sparse. However, this advantage diminishes as correlation increases: at $\sigma = 0.9$, the true power curves of BJ, GM, and AM are dominated by the correlation-robust tests across nearly all signal levels. A common trend among all tests is that their true power decreases as correlation strengthens when other factors are held constant; this can be seen by comparing the same curves across different panels of varying correlation strength.

Second, within the BC family, the true and nominal power curves (solid and dotted lines, respectively) are relatively close, confirming their robustness to correlation. However, the degree of robustness differs: HM shows the smallest gap between true and nominal power, followed by $BC_{1.5}$, while $BC_{0.5}$ shows the largest gap, especially under strong correlation. This is broadly consistent with the results in Fang et al. (2023) showing that heavier-tailed transformations yield stronger asymptotic robustness, though the relationship is not strictly monotone (HM is more robust than $BC_{1.5}$ despite having a lighter tail).

Third, the relative true power advantages within the BC family are associated with signal sparsity and correlation strength. As shown in Fang et al. (2023) (cf. their Figure 1), heavier-tailed statistics (e.g., HM and $BC_{1.5}$) transform small p-values into much larger values than lighter-tailed statistics (e.g., $BC_{0.5}$), making them more driven by extreme values. This property is advantageous when signals are sparse and strong but can be disadvantageous when signals are dense and weak. The panel for $\sigma = 0.5$ in Figure S28 shows a pattern consistent with this property: $BC_{1.5}$ and HM have higher power than $BC_{0.5}$ when signals are sparse, but lower power when signals are dense. Meanwhile, the relative advantage also depends on correlation patterns. Under strong correlation ($\sigma = 0.9$), the entire true power curve of $BC_{0.5}$ is lower than those of HM and $BC_{1.5}$, indicating that heavier-tailed statistics maintain their advantage across signal densities when dependence is strong. Under weak correlation ($\sigma = 0.1$), $BC_{0.5}$ has similar power to $BC_{1.5}$ and HM even when signals are sparse, but indeed has higher power when signals are dense. Interestingly, HM and $BC_{1.5}$ have similar true power in most settings, with HM being slightly higher.

Fourth, HM and $BC_{1.5}$ are mostly less powerful than the supremum-based tests (minP, Simes, HC, ϕ_3) and CCT under the settings of Figure S28. However, the difference is not substantial under the settings of Figure S29. This is an interesting supplementary observation consistent with Fang et al. (2023), which showed that HM and CCT are "approximately identical" for small p-values (cf. Proposition 2 and Table 4). The similarity between HM and CCT is evident in Figure S29, where their true power curves nearly overlap, but less so in Figure S28, where CCT has a visible advantage. This may be because the asymptotic equivalence in Fang et al. (2023) applies in the extreme tail regime of very small p-values, while the Gaussian model simulations produce p-values that are not sufficiently extreme for the approximation to be tight.

Fifth, AM and GM are sensitive to correlation, evidenced by the substantial departure between their true and nominal power curves. Their true power pattern is similar to BJ: they can be more powerful when correlations are weak and signals are dense, but their relative advantage drops significantly as correlation increases. The AM statistic is the least powerful in most settings.

GWAS-based linear model. Figure S29 presents the power comparisons under the linear model with genotypes simulated from the FSHR gene, using the same settings as in Figure S18. The results are broadly consistent with the findings under the Gaussian model. The correlation-robust tests are closely clustered in terms of true power. The gap between nominal and true power is larger for BJ, GM, and AM, and it becomes larger at the more stringent significance level (from $\alpha = 5 \times 10^{-3}$ to 5×10^{-4}), indicating that they are not robust to correlations. Compared with the correlation-robust tests, the BJ and GM statistics can have lower power when signals are sparse (see the panels with the number of signals = 1), but their power can increase considerably to a similar level when signals become denser (see the panels with the number of signals = 30). The AM statistic is the least powerful across the settings considered here.

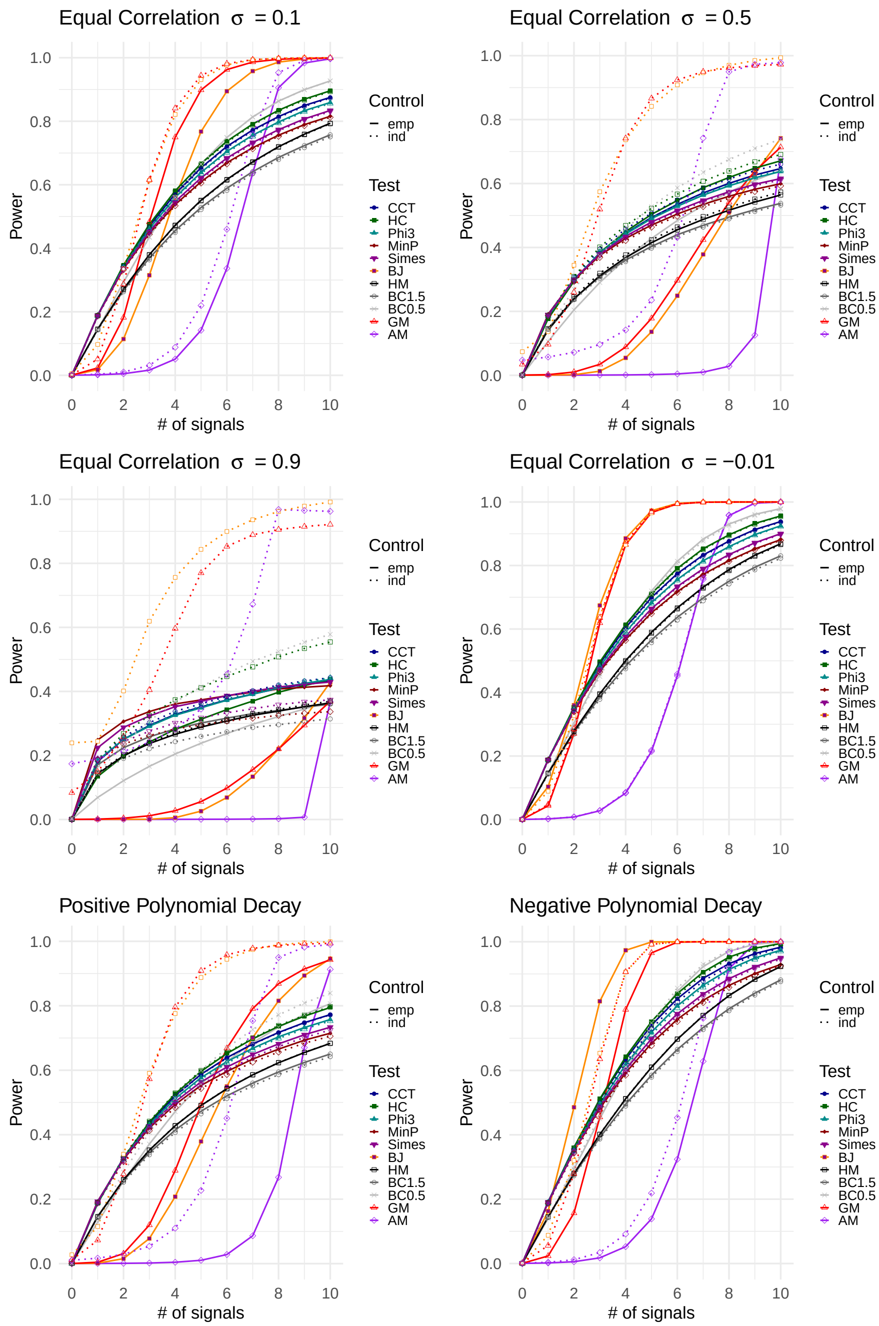

Figure S28: Power comparisons including summation-based statistics under the Gaussian mean model with six correlation structures. The signal strength is $\mu = 3$, the significance level is $\alpha = 5 \times 10^{-4}$, $n = 10$, and 10^6 simulations are performed. Solid curves ("emp") show the true power with α controlled empirically by simulation; dotted curves ("ind") show the nominal power under the independence assumption. The simulation settings are the same as in Figure S17.

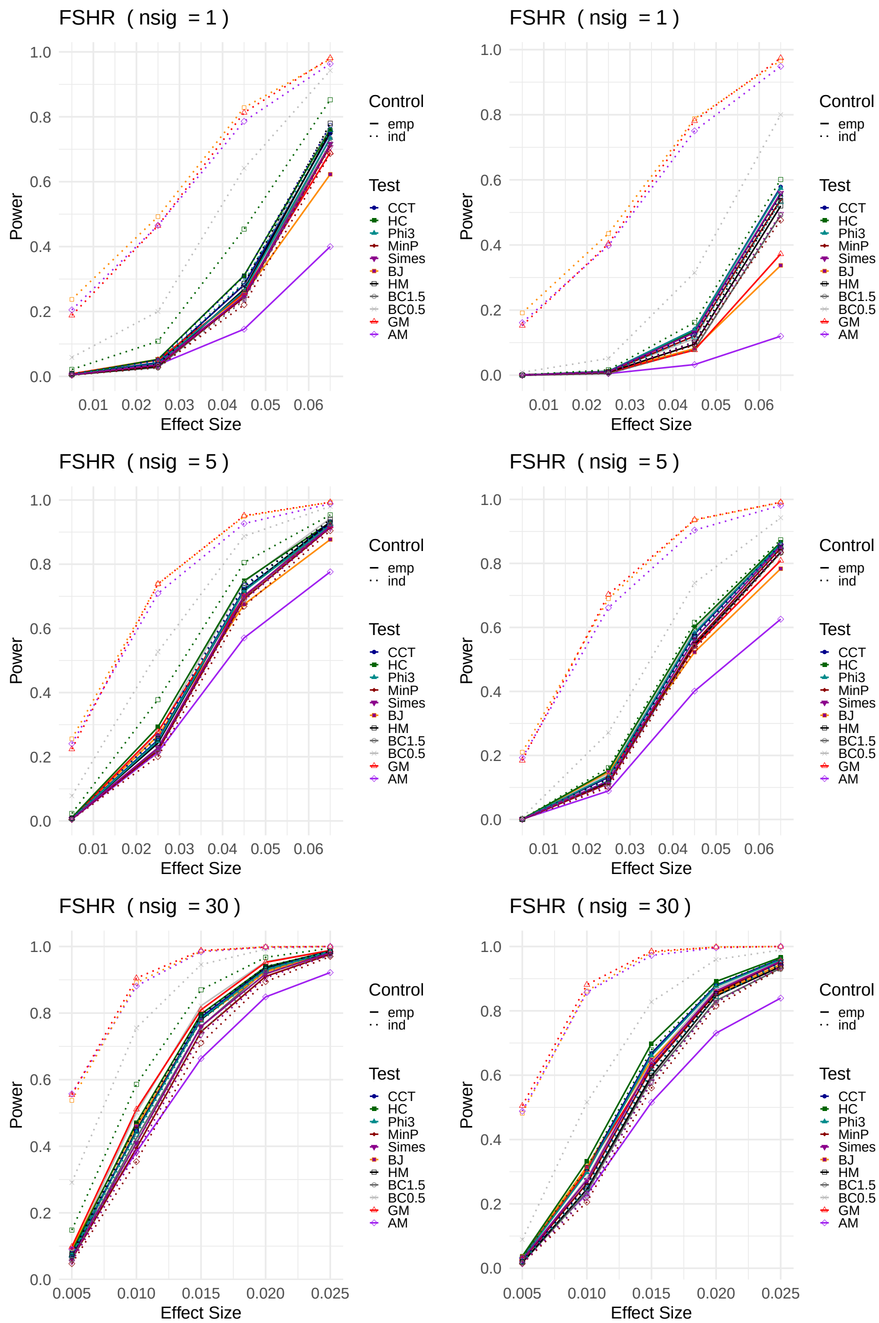

Figure S29: Power comparisons including summation-based statistics for continuous traits with SNP genotypes simulated from the FSHR gene. The number of causal SNPs (nsig) is $k = 1$ (row 1), 5 (row 2), and 30 (row 3). The significance levels are $\alpha = 0.005$ (left) and 0.0005 (right). Solid curves ("emp") show the true power with α controlled empirically by simulation; dotted curves ("ind") show the nominal power under the independence assumption. The simulation settings are the same as in Figure S18.

S5.3.3 Non-Gaussian Dependence: FGM Copula

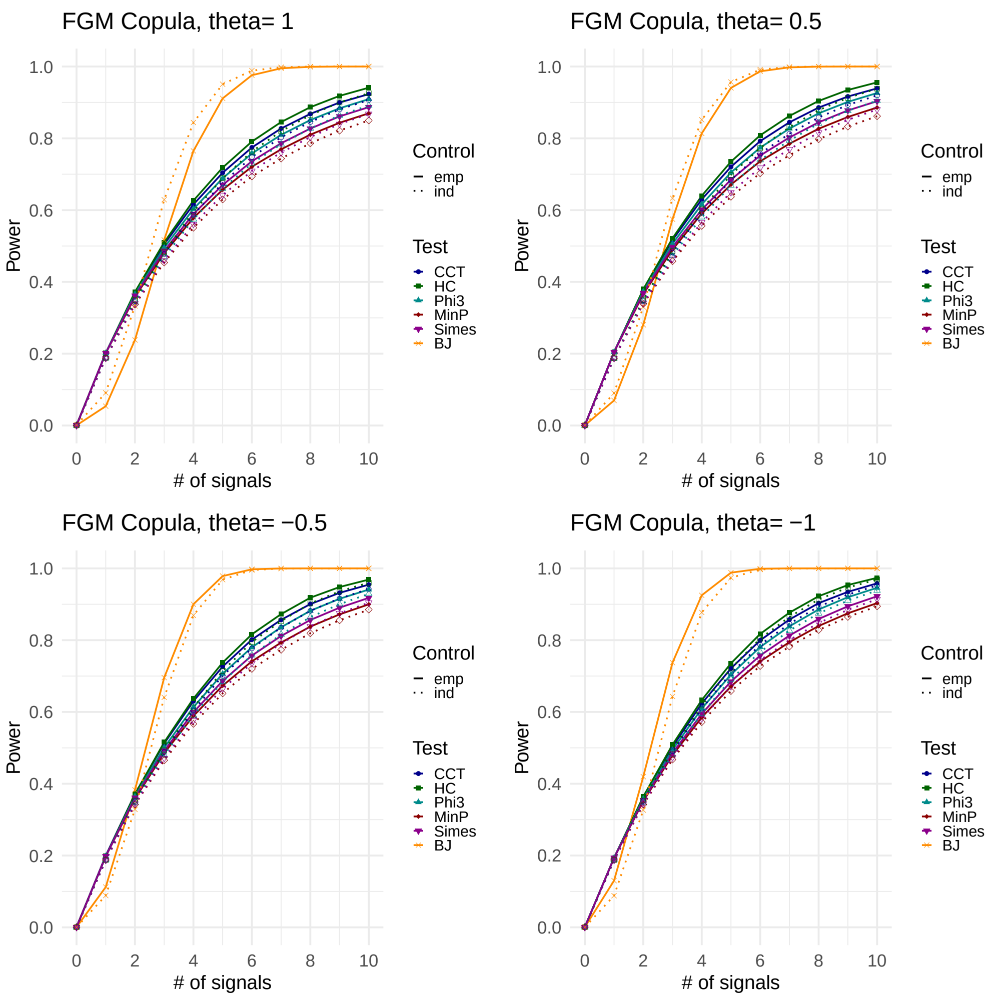

Figure S30: Power study under the bivariate FGM copula. The number of signals (k) among $n = 10$ components varies along the x-axis. The signal strength is $\mu = 3$; the significance level is $\alpha = 5 \times 10^{-4}$; 10^6 simulations are performed. Solid curves ("emp") show empirical power with type I error controlled by simulation; dotted curves ("ind") show nominal power using the testing threshold calculated under the assumption of independence.

Finally, we examine the statistical power of the tests under a non-Gaussian copula model. The goal is to assess how well the tests perform when the input p-values are generated from a non-Gaussian distribution, specifically using a copula that captures dependence between variables. Specifically, we conduct a simulation study using the Farlie-Gumbel-Morgenstern (FGM) copula as described in Remark 1. The input statistics $\boldsymbol{X}$ have normal marginal distributions, $X_i \sim N(\mu_i, 1)$, $i = 1, ..., n$, and their pairwise dependence is structured by the bivariate FGM copula:

$$C(u_i, u_j) = u_i u_j \left[1 + \theta(1 - u_i)(1 - u_j)\right], \quad \theta \in [-1, 1].$$

Under the null hypothesis, $\mu_i = 0$ for all i. Under the alternative, k randomly chosen components have a nonzero mean value, i.e., $\mu_{i_1}, ..., \mu_{i_k} = \mu \neq 0$.

Figure S30 displays the statistical power of six tests—minP, Simes, HC, $\phi 3$, CCT, and BJ—at $\mu = 3$ as the number of signals (k) varies, under FGM copula dependence with $\theta = -1, -0.5, 0.5, 1$ and significance level $\alpha = 5 \times 10^{-4}$. Solid curves represent the empirical (true) power, where the α-level threshold is determined by simulation; dotted curves represent the nominal power, where the threshold is calculated under the independence assumption. The BJ test shows the largest difference between true and nominal power, highlighting its sensitivity to dependence, while the other tests remain robust under FGM copula dependence. For true power, BJ is more powerful when signals are dense (large k). Among the correlation-robust tests, the power ranking as k increases is HC, CCT, $\phi 3$, Simes, and minP.

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