Two Sample Tests for Bivariate Heteroscedastic Extremes with a Changing Tail Copula

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Supplementary Material

This supplement collects the proofs of the theoretical results in the article.

S1 Convergence of the Bivariate Sequential Tail Empirical Process

S1.1 Convergence of the Simple Bivariate Sequential Tail Empirical Process

In this section, we prove Proposition S1 and S2, which are basic tools in the proof of Theorem 1. We introduce the following notations. For $1 \leq i \leq n$ and $i, n \in \mathbb{N}$, $V_{X,i}^{(n)}$ and $V_{Y,i}^{(n)}$ follows a marginally standard uniform distribution, with their survival copula given by $C_{n,i}$. Let $\delta_{(x,y,z)}$ denote the probability distribution that is degenerate at $(x, y, z) \in \mathbb{D}_T$, and define

$$Z_{n,i} = \frac{1}{\sqrt{k}} \delta_{\left((n/k)V_{X,i}^{(n)}/c_1(i/n),(n/k)V_{Y,i}^{(n)}/c_2(i/n),i/n\right)}.$$

Consider the class

$$\mathcal{F} = \{f_{x,y,z} := q^{-1}(x, y)\mathbf{1}\left([0, x) \times [0, y) \times [0, z]\right) \mid x, y, z \in \mathbb{D}_T\}$$

and equip $\mathcal{F}$ with the semi-metric ρ,

$$\rho\left(f_{x,y,z}, f_{u,v,w}\right) = \sqrt{\mathbf{E}\left(\frac{W(x, y, z)}{q(x, y)} - \frac{W(u, v, w)}{q(u, v)}\right)^2}.$$

We prove the result for $\mathbb{D}_1$ and denote $\mathbb{D}_1$ as $\mathbb{D}$. The proof for a general constant T is similar. Denote $\|Z_{n,i}\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |Z_{n,i}(f)|$. For arbitrary $\varepsilon > 0$, let N_ε represent the minimal number of sets in a partition $\mathcal{F} = \cup_{j=1}^{N_\varepsilon}\mathcal{F}_{\varepsilon,j}$, such that for every n and partitioning set $\mathcal{F}_{\varepsilon,j}$,

$$\mathbf{E}\sum_{i=1}^{n} \sup_{f,g\in\mathcal{F}_{\varepsilon,j}} |Z_{n,i}(f) - Z_{n,i}(g)|^2 \le \varepsilon^2.$$

We denote $d = \sup_{[0,1]} \max(c_1(t), c_2(t))$.

Proposition S1. *Under Assumptions 1 - 4, there exists a Gaussian process W, with covariance structure defined in* (3.12), *such that*

$$\sup_{(x,y,z)\in\mathbb{D}} \left|\sum_{i=1}^{n} \left(Z_{n,i} - \mathbf{E}Z_{n,i}\right)\left(f_{x,y,z}\right) - \frac{W(x, y, z)}{q(x, y)}\right| \xrightarrow{\mathbf{P}} 0. \tag{S1.1}$$

Moreover, W/q is uniformly continuous concerning the semi-metric ρ.

Proof. We prove (S1.1) by using Einmahl and Segers (2021, Theorem 3). It means that we only need to verify the following conditions

(a) $\sum_{i=1}^{n} \mathbf{E} \|Z_{n,i}\|_{\mathcal{F}} \mathbf{1} \left\{ \|Z_{n,i}\|_{\mathcal{F}} > \lambda \right\} \to 0$ as $n \to \infty$, for every $\lambda > 0$.

(b) For every finite set of points $(x_1, y_1, z_1), \ldots, (x_m, y_m, z_m) \in \mathbb{D}$, the sequence

$$\left\{ \left(\sum_{i=1}^{n} \left(Z_{n,i} - \mathbf{E} Z_{n,i}\right) f_{x_1,y_1,z_1}, \ldots, \sum_{i=1}^{n} \left(Z_{n,i} - \mathbf{E} Z_{n,i}\right) f_{x_m,y_m,z_m} \right) \right\}_{n \in \mathbb{Z}^+}$$

converges weakly in $\mathbb{R}^m$.

(c) The bracketing integral $\int_0^1 \sqrt{\log N_\varepsilon}\, d\varepsilon$ is finite.

For the sake of convenience, we denote

$$\begin{aligned}
\mathcal{F}^{(0)} =& \{q^{-1}(x,y) \mathbf{1}\left([0,x) \times [0,y) \times [0,z]\right) \mid 0 \le z \le 1, 0 \le x, y \le 1\}, \\
\mathcal{F}^{(1)} =& \{q^{-1}(x,y) \mathbf{1}\left([0,x) \times [0,y) \times [0,z]\right) \mid 0 \le z \le 1, x = \infty, 0 \le y \le 1\}, \\
\mathcal{F}^{(2)} =& \{q^{-1}(x,y) \mathbf{1}\left([0,x) \times [0,y) \times [0,z]\right) \mid 0 \le z \le 1, y = \infty, 0 \le x \le 1\}.
\end{aligned}$$

For (a), we observe that $\|Z_{n,i}\|_{\mathcal{F}} \mathbf{1} \left\{ \|Z_{n,i}\|_{\mathcal{F}} > \lambda \right\}$ is bounded by

$$\|Z_{n,i}\|_{\mathcal{F}^{(0)}} \mathbf{1} \left\{ \|Z_{n,i}\|_{\mathcal{F}^{(0)}} > \lambda \right\} + \|Z_{n,i}\|_{\mathcal{F}^{(1)}} \mathbf{1} \left\{ \|Z_{n,i}\|_{\mathcal{F}^{(1)}} > \lambda \right\} + \|Z_{n,i}\|_{\mathcal{F}^{(2)}} \mathbf{1} \left\{ \|Z_{n,i}\|_{\mathcal{F}^{(2)}} > \lambda \right\}.$$

On the class $\mathcal{F}^{(0)}$, it holds that

$$\begin{aligned}
Z_{n,i}\left(f_{x,y,z}\right) &= \frac{k^{-1/2}}{q(x,y)} \mathbf{1}_{\left\{ V_{X,i}^{(n)} < c_1(i/n)kx/n, V_{Y,i}^{(n)} < c_2(i/n)ky/n, i/n \le z \right\}} \\
&\le k^{-1/2} \left((n/k) \left(V_{Y,i}^{(n)}/c_2(i/n) \vee V_{X,i}^{(n)}/c_1(i/n) \right) \right)^{-\eta} \\
&\le k^{-1/2} \left((n/k) \left(V_{X,i}^{(n)}/c_1(i/n) \right) \right)^{-\eta}.
\end{aligned}$$

Moreover, it holds that

$$\|Z_{n,i}\|_{\mathcal{F}^{(1)}} = k^{-1/2}\left((n/k)\left(V_{X,i}^{(n)}/c_1(i/n)\right)\right)^{-\eta}, \quad \|Z_{n,i}\|_{\mathcal{F}^{(2)}} = k^{-1/2}\left((n/k)\left(V_{Y,i}^{(n)}/c_2(i/n)\right)\right)^{-\eta}.$$

Hence,

$$\|Z_{n,i}\|_{\mathcal{F}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}} > \lambda\right\} \leq 2\|Z_{n,i}\|_{\mathcal{F}^{(1)}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}^{(1)}} > \lambda\right\} + 2\|Z_{n,i}\|_{\mathcal{F}^{(2)}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}^{(2)}} > \lambda\right\}.$$

Without loss of generality, it suffices to show the result for $\|Z_{n,i}\|_{\mathcal{F}^{(1)}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}^{(1)}} > \lambda\right\}$ such that

$$\begin{aligned}
\sum_{i=1}^{n}\mathbf{E}\|Z_{n,i}\|_{\mathcal{F}^{(1)}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}^{(1)}} > \lambda\right\} &\leq \frac{1}{\sqrt{k}}\sum_{i=1}^{n}\mathbf{E}\left(\frac{c_1(i/n)k}{nV_{X,i}^{(n)}}\right)^{\eta}\mathbf{1}\Big\{V_{X,i}^{(n)} \leq k(\sqrt{k}\lambda)^{-1/\eta}c_1(i/n)/n\Big\} \\
&= \sum_{i=1}^{n}\frac{kc_1(i/n)}{n\sqrt{k}}\int_0^{(\sqrt{k}\lambda)^{-1/\eta}} x^{-\eta}dx \\
&= \frac{1}{1-\eta}\sum_{i=1}^{n}\frac{c_1(i/n)}{n}k^{1-1/2\eta}\lambda^{1-1/\eta} \\
&\to 0.
\end{aligned}$$

To prove (b), we apply the Lindeberg-Feller Theorem. Given that

$$\sup_{j\in 1,\ldots,m}\sum_{i=1}^{n}\mathbf{E}\|Z_{n,i}f_{x_j,y_j,z_j}\|^2\mathbf{1}\left\{\left\|Z_{n,i}f_{x_j,y_j,z_j}\right\| > \lambda\right\} \leq \frac{1}{\sqrt{k}}\sum_{i=1}^{n}\mathbf{E}\left\|Z_{n,i}\right\|_{\mathcal{F}}\mathbf{1}\left\{\|Z_{n,i}\|_{\mathcal{F}} > \lambda\right\},$$

it follows that $\sup_{j\in 1,\ldots,m}\sum_{i=1}^{n}\mathbf{E}\|Z_{n,i}f_{x_j,y_j,z_j}\|^2\mathbf{1}\left\{\left\|Z_{n,i}f_{x_j,y_j,z_j}\right\| > \lambda\right\} \to 0$ for every $\lambda > 0$. Then, it suffices to verify the convergence of the asymptotic covariance. Suppose $u = d\max(a,b)$. Notice that

$$t\,C_{n,i}\left(\frac{c_1(i/n)a}{t}, \frac{c_2(i/n)b}{t}\right) = ut\,C_{n,i}\left(\frac{c_1(i/n)a/u}{t}, \frac{c_2(i/n)b/u}{t}\right)$$

$$= R\left(c_1(i/n)a, c_2(i/n)b, \frac{i}{n}\right) + d(a \vee b)O(t^{-\alpha}).$$

We sketch the convergence of the asymptotic covariance by verifying when $0 < x, y < 1$, $0 \le z \le 1$ that as $n \to \infty$,

$$\operatorname{cov}\left(\sum_{i=1}^{\lfloor nz_1 \rfloor} \frac{\mathbf{1}\left\{V_{X,i}^{(n)} < c_1(i/n)kx_1/n, V_{Y,i}^{(n)} < c_2(i/n)ky_1/n\right\} - C_{n,i}\left(\frac{kx_1c_1(i/n)}{n}, \frac{ky_1c_2(i/n)}{n}\right)}{\sqrt{k}(x_1 \vee y_1)^\eta},\right.$$
$$\left.\sum_{i=1}^{\lfloor nz_2 \rfloor} \frac{\mathbf{1}\left\{V_{X,i}^{(n)} < c_1(i/n)kx_2/n, V_{Y,i}^{(n)} < c_2(i/n)ky_2/n\right\} - C_{n,i}\left(\frac{kx_2c_1(i/n)}{n}, \frac{ky_2c_2(i/n)}{n}\right)}{\sqrt{k}(x_2 \vee y_2)^\eta}\right)$$
$$= \frac{1}{n(x_1 \vee y_1)^\eta (x_2 \vee y_2)^\eta} \sum_{i=1}^{\lfloor n(z_1 \wedge z_2) \rfloor} \frac{n}{k} C_{n,i}\left(\frac{k(x_1 \wedge x_2)c_1(i/n)}{n}, \frac{k(y_1 \wedge y_2)c_2(i/n)}{n}\right) + O\left(\frac{k}{n}\right)$$
$$= \frac{1}{n(x_1 \vee y_1)^\eta (x_2 \vee y_2)^\eta} \sum_{i=1}^{\lfloor n(z_1 \wedge z_2) \rfloor} R\left(c_1(i/n)(x_1 \wedge x_2), c_2(i/n)(y_1 \wedge y_2), \frac{i}{n}\right)$$
$$+ O\left(\frac{k^\alpha}{n^\alpha}\right) + O\left(\frac{k}{n}\right)$$
$$= \frac{1}{(x_1 \vee y_1)^\eta (x_2 \vee y_2)^\eta} R'(x_1 \wedge x_2, y_1 \wedge y_2, z_1 \wedge z_2) + o(1).$$

For (c), we provide a detailed proof in Lemma S1. □

We next prove for sufficiently small $\varepsilon > 0$, there exists a covering where the number of sets is bounded by ε^{-7}. It follows that the bracketing integral $\int_0^1 \sqrt{\log N_\varepsilon}\, d\varepsilon$ is finite.

Lemma S1. *Let $\varepsilon > 0$, $a = \varepsilon^{2/(1-2\eta)}/(16d)$, and $\theta = 1 - (\varepsilon^2/10d)$. Denote the covering*

$$\mathcal{F} = \mathcal{F}^1(a) \cup \mathcal{F}^2(a) \cup \left(\bigcup_{m=0}^{\lceil \log a / \log \theta \rceil} \bigcup_{l=0}^{\lceil \log a / \log \theta \rceil} \bigcup_{j=0}^{\lceil \log a / \log \theta \rceil} \mathcal{F}(l, m, j) \cup \mathcal{F}^1(l, j) \cup \mathcal{F}^2(m, j) \right),$$

where

$$\mathcal{F}^1(a) = \{f_{x,y,z} \in \mathcal{F} \mid x \wedge y \leq a, z \in [0,1]\},$$

$$\mathcal{F}^2(a) = \{f_{x,y,z} \in \mathcal{F} \mid a < x, y \leq 1, \ \text{or}\ x = \infty, a < y \leq 1, \ \text{or}\ y = \infty, a < x \leq 1, z \in [0,a]\},$$

$$\mathcal{F}^1(l,j) = \left\{f_{x,y,z} \in \mathcal{F} \mid \theta^{l+1} \leq x \leq \theta^l, y = \infty, \theta^{j+1} \leq z \leq \theta^j\right\},$$

$$\mathcal{F}^2(m,j) = \left\{f_{x,y,z} \in \mathcal{F} \mid \theta^{m+1} \leq y \leq \theta^m, x = \infty, \theta^{j+1} \leq z \leq \theta^j\right\},$$

$$\mathcal{F}(l,m,j) = \left\{f_{x,y,z} \in \mathcal{F} \mid \theta^{l+1} \leq x \leq \theta^l, \theta^{m+1} \leq y \leq \theta^m, \theta^{j+1} \leq z \leq \theta^j\right\}.$$

Under Assumptions 1-4, it holds that for every $n > 0$,

$$\sum_{i=1}^n \mathbf{E} \sup_{f,g \in p} |Z_{n,i}(f) - Z_{n,i}(g)|^2 \leq \varepsilon^2,$$

for partitioning sets $p \in \mathcal{P}_\varepsilon := \{\mathcal{F}(l,m,j), \mathcal{F}^1(l,j), \mathcal{F}^2(m,j), \mathcal{F}^1(a), \mathcal{F}^2(a), l, m, j \in \mathbb{N}\}$.

Proof. For $\mathcal{F}^1(a)$, it holds that

$$\begin{aligned}
&\sum_{i=1}^n \mathbf{E} \sup_{f,g \in \mathcal{F}^1(a)} (Z_{n,i}(f) - Z_{n,i}(g))^2 \\
&\leq 4 \sum_{i=1}^n \mathbf{E} \sup_{f \in \mathcal{F}^1(a)} Z_{n,i}^2(f) \\
&\leq \frac{4}{k} \sum_{i=1}^n \mathbf{E} \left(\frac{nV_{X,i}^{(n)}}{kc_1(i/n)} \right)^{-2\eta} \mathbf{1}\left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < a \right\} + \mathbf{E} \left(\frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} \right)^{-2\eta} \mathbf{1}\left\{ \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < a \right\} \\
&= \frac{4}{k} \sum_{i=1}^n \int_0^{akc_1(i/n)/n} \left(\frac{nx}{kc_1(i/n)} \right)^{-2\eta} dx + \int_0^{akc_2(i/n)/n} \left(\frac{ny}{kc_2(i/n)} \right)^{-2\eta} dy \\
&= \frac{4a^{1-2\eta}}{1-2\eta} \sum_{i=1}^n \frac{c_1(i/n) + c_2(i/n)}{n} \\
&\leq \varepsilon^2.
\end{aligned}$$

For $\mathcal{F}^2(a)$, without loss of generality, we assume $x \neq \infty$ and derive that

$$\sum_{i=1}^{n} \mathbf{E} \sup_{f,g\in\mathcal{F}^2(a)} (Z_{n,i}(f) - Z_{n,i}(g))^2 \leq \frac{4}{k} \sum_{i=1}^{n} \mathbf{E} \sup_{\substack{a<x\leq 1 \\ z\in[0,a]}} a^{-2\eta} \mathbf{1}\Big\{ V_{X,i}^{(n)} < c_1(i/n)kx/n, i/n \leq z \Big\} \leq \varepsilon^2.$$

For $\mathcal{F}(l,m,j)$, it holds that

$$\begin{aligned}
&\sum_{i=1}^{n} \mathbf{E} \sup_{f,g\in\mathcal{F}(l,m,j)} (Z_{n,i}(f) - Z_{n,i}(g))^2 \\
\leq &\sum_{i=1}^{n} \mathbf{E} \left(\sup_{f\in\mathcal{F}(l,m,j)} Z_{n,i}(f) - \inf_{f\in\mathcal{F}(l,m,j)} Z_{n,i}(f) \right)^2 \\
\leq &\frac{1}{k} \sum_{i=1}^{n} \mathbf{E} \Bigg(\left(\theta^{l+1} \vee \theta^{m+1}\right)^{-\eta} \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^l, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^m, \frac{i}{n} \leq \theta^j \right\} \\
&- \left(\theta^l \vee \theta^m\right)^{-\eta} \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^{l+1}, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^{m+1}, \frac{i}{n} \leq \theta^{j+1} \right\} \Bigg)^2 \\
\leq &\frac{2}{k} \sum_{i=1}^{\lfloor n\theta^j \rfloor} \mathbf{E} \Bigg(\left(\theta^{l+1} \vee \theta^{m+1}\right)^{-\eta} \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^l, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^m \right\} \\
&- \left(\theta^l \vee \theta^m\right)^{-\eta} \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^{l+1}, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^{m+1} \right\} \Bigg)^2 \\
&+ \frac{2}{k} \sum_{i=\lfloor n\theta^{j+1} \rfloor+1}^{\lfloor n\theta^j \rfloor} \mathbf{E} \left(\left(\theta^l \vee \theta^m\right)^{-\eta} \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^{l+1}, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^{m+1} \right\} \right)^2 \\
:= &I_1 + I_2.
\end{aligned}$$

Without loss of generality, we prove the case when $l \leq m$ and $\varepsilon < 1$ for the convergence of I_1,

$$\begin{aligned}
I_1 \leq &\frac{2}{k} \sum_{i=1}^{\lfloor n\theta^j \rfloor} \mathbf{E} \Bigg(\mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^l, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^m \right\} \left(\frac{1}{\theta^{\eta(l+1)}} - \frac{1}{\theta^{\eta l}} \right) \\
&+ \left(\mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^l, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^m \right\} - \mathbf{1} \left\{ \frac{nV_{X,i}^{(n)}}{kc_1(i/n)} < \theta^{l+1}, \frac{nV_{Y,i}^{(n)}}{kc_2(i/n)} < \theta^{m+1} \right\} \right) \frac{1}{\theta^{\eta l}} \Bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4}{k}\sum_{i=1}^{\lfloor n\theta^j\rfloor}\left(C_{n,i}\left(\frac{kc_1(i/n)}{n}\theta^l, \frac{kc_2(i/n)}{n}\theta^m\right)\frac{1}{\theta^{2\eta l}}\left(\frac{1}{\theta^\eta}-1\right)^2\right.\\
&\quad\left.+\left[C_{n,i}\left(\frac{kc_1(i/n)}{n}\theta^l, \frac{kc_2(i/n)}{n}\theta^m\right)-C_{n,i}\left(\frac{kc_1(i/n)}{n}\theta^{l+1}, \frac{kc_2(i/n)}{n}\theta^{m+1}\right)\right]\frac{1}{\theta^{2\eta l}}\right)\\
&\leq \frac{4}{k}\sum_{i=1}^{\lfloor n\theta^j\rfloor}c_1(i,n)\left(\frac{k}{n}\frac{\theta^l}{\theta^{2\eta l}}\left(\frac{1}{\theta^\eta}-1\right)^2+\frac{2k}{n}\frac{\theta^l}{\theta^{2\eta l}}(1-\theta)\right)\\
&\leq d\left(4\left(\frac{1}{\theta^{1/2}}-1\right)^2+8(1-\theta)\right)\theta^j\\
&\leq \varepsilon^2.
\end{aligned}
$$

It holds for I_2 when $l \leq m$ that

$$
I_2 \leq \frac{2}{k}\sum_{i=\lfloor\theta^{j+1}n\rfloor+1}^{\lfloor\theta^j n\rfloor}\left(C_{n,i}\left(\frac{kc_1(i/n)}{n}\theta^{l+1}, \frac{kc_2(i/n)}{n}\theta^{m+1}\right)\frac{1}{\theta^{2\eta l}}\right) \leq d\,(1-\theta)\,\theta^{(1-2\eta)l+j+1} \leq \varepsilon^2.
$$

For the classes $\mathcal{F}^1(l,j)$ and $\mathcal{F}^2(m,j)$, the proofs are similar to that of $\mathcal{F}(l,m,j)$, and we obtain that

$$
\mathbf{E}\sup_{f,g\in\mathcal{F}^1(l,j)}\left(Z_{n,i}(f)-Z_{n,i}(g)\right)^2 \leq \varepsilon^2,
$$

$$
\mathbf{E}\sup_{f,g\in\mathcal{F}^2(m,j)}\left(Z_{n,i}(f)-Z_{n,i}(g)\right)^2 \leq \varepsilon^2.
$$

Hence, the proof is completed. □

We then establish the conditional weak convergence for the bootstrap process.

Proposition S2. *Under Assumptions 1, 2, 4 and 5, there exists a Gaussian* *process* $\tilde{W}(x,y,z)$ *with covariance defined in* (3.12) *that*

$$\sup_{h\in BL_1(l^\infty(\mathbb{D}))}\left|\mathbf{E}_\xi\left(h\left(\sum_{i=1}^n(\xi_i-1)\,Z_{n,i}f\right)\right)-\mathbf{E}\left(h\left(\frac{\tilde{W}}{q}\right)\right)\right|\xrightarrow{\mathbf{P}}0.$$

Proof. Firstly, we prove that under Assumptions 1, 2, 4 and 5, there exists a Gaussian process $\tilde{W}(x,y,z)$ with covariance defined in (3.12) such that

$$\sup_{(x,y,z)\in\mathbb{D}}\left|\sum_{i=1}^n(\xi_i-1)\,Z_{n,i}f_{x,y,z}-\frac{\tilde{W}(x,y,z)}{q(x,y)}\right|\xrightarrow{\mathbf{P}}0.$$

Denote for $i=1,\ldots,n$,

$$\tilde{Z}_{n,i}=k^{-1/2}(\xi_i-1)\delta_{\left((n/k)V_{X,i}^{(n)}/c_1(i/n),(n/k)V_{Y,i}^{(n)}/c_2(i/n),i/n\right)}.$$

We verify the three conditions of Einmahl and Segers (2021, Theorem 3) for $\tilde{Z}_{n,i}$. Firstly, we have

$$\begin{aligned}\sum_{i=1}^n\mathbf{E}\left\|\tilde{Z}_{n,i}\right\|_{\mathcal{F}}\mathbf{1}\left\{\left\|\tilde{Z}_{n,i}\right\|_{\mathcal{F}}>\lambda\right\}&=\sum_{i=1}^n\mathbf{E}|\xi_i-1|\left\|Z_{n,i}\right\|_{\mathcal{F}}\mathbf{1}\left\{|\xi_i-1|\left\|Z_{n,i}\right\|_{\mathcal{F}}>\lambda\right\}\\&\le\frac{1}{1-\eta}\sum_{i=1}^n\frac{c_1(i/n)}{n}k^{1-1/2\eta}\lambda^{1-1/\eta}\mathbf{E}\left[|\xi_i-1|^{1/\eta}\mathbf{1}(\xi_i\neq1)\right]\\&\to0.\end{aligned}$$

Proofs for the finite-dimensional convergence and finite bracketing integral follow a similar approach in Proposition S1 and Lemma S1, respectively, so we omitted them here. Then, we complete the first step of our proof.

For the second step, we prove the conditional weak convergence. For arbitrary $\delta>0$, recall the partition $\mathcal{P}_\delta$ for $\mathcal{F}$ proposed in Lemma S1 and denote the number of partitioning sets as $\mathcal{N}_\delta$. We pick a unique element from each partitioning set of $\mathcal{P}_\delta$, and define

$$\delta\text{-net} := \{(x_1, y_1, z_1), \ldots, (x_{\mathcal{N}_\delta}, y_{\mathcal{N}_\delta}, z_{\mathcal{N}_\delta})\}.$$

Denote M_δ as a function on $\mathbb{D}$ that it maps each (x, y, z) to a (u, v, w) such that (x, y, z) and (u, v, w) belong to the same partitioning set of $\mathcal{P}_\delta$. Then, it holds that

$$\begin{aligned}
&\sup_{h\in BL_1(l^\infty(\mathbb{D}))}\left|\mathbf{E}_\xi h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}f\right)-\mathbf{E}\left(h\left(\frac{\tilde{W}}{q}\right)\right)\right|\\
\leq&\sup_{h\in BL_1(l^\infty(\mathbb{D}))}\left|\mathbf{E}_\xi h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}(f\circ M_\delta)\right)-\mathbf{E}\left(h\left(\frac{\tilde{W}}{q}\circ M_\delta\right)\right)\right|\\
&+\sup_{h\in BL_1(\ell^\infty(\mathbb{D}))}\left|\mathbf{E}\left(h\left(\frac{\tilde{W}}{q}\circ M_\delta\right)\right)-\mathbf{E}\left(h\left(\frac{W}{q}\right)\right)\right|\\
&+\sup_{h\in BL_1(\ell^\infty(\mathbb{D}))}\left|\mathbf{E}_\xi h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}(f\circ M_\delta)\right)-\mathbf{E}_\xi h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}f\right)\right|\\
=:&I_1(\delta)+I_2(\delta)+I_3(\delta).
\end{aligned}$$

By Kosorok (2003, Lemma 3), we derive $I_1(\delta) = o_{\mathbf{P}}(1)$ as $n \to \infty$ for every $\delta > 0$. By the continuity of W/q, we obtain $\lim_{\delta\downarrow 0} I_2(\delta) = 0$. Moreover, we have that

$$\begin{aligned}
I_3(\delta)\leq&\sup_{h\in BL_1(\ell^\infty(\mathbb{D}))}\mathbf{E}_\xi\left|h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}f\circ M_\delta\right)-h\left(\sum_{i=1}^n(\xi_i-1)Z_{n,i}f\right)\right|\\
\leq&\mathbf{E}_\xi\left(\sup_{\substack{(x_1,y_1,z_1)\in p\\(x_2,y_2,z_2)\in p\\p\in\mathcal{P}_\delta}}\left|\sum_{i=1}^n(\xi_i-1)Z_{n,i}f_{x_1,y_1,z_1}-\sum_{i=1}^n(\xi_i-1)Z_{n,i}f_{x_2,y_2,z_2}\right|\right).
\end{aligned}$$

By van der Vaart and Wellner (1996, Theorem 1.5.6), for $\varepsilon, \epsilon > 0$, there exists a $\delta > 0$, such that

$$\limsup_{n\to\infty} \mathbf{P}\left(\sup_{\substack{(x_1,y_1,z_1)\in p\\(x_2,y_2,z_2)\in p\\p\in\mathcal{P}_{\delta'},0\le\delta'\le\delta}} \left| \sum_{i=1}^n (\xi_i - 1) Z_{n,i} f_{x_1,y_1,z_1} - \sum_{i=1}^n (\xi_i - 1) Z_{n,i} f_{x_2,y_2,z_2} \right| \ge \varepsilon \right) \le \epsilon.$$

Thus, we derive that for arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{0\le\delta'\le\delta} I_3(\delta') > \varepsilon \right) = 0.$$

Finally, the proof is completed by

$$I_1(\delta \wedge n^{-1}) + I_2(\delta \wedge n^{-1}) + I_3(\delta \wedge n^{-1}) = o_{\mathbf{P}}(1), \quad \text{as } n \to \infty. \qquad \square$$

S1.2 Proof of Theorem 1

Proof of the unconditional weak convergence in Theorem 1. We start the proof by noting that

$$\begin{aligned}
&\sup_{(x,y,z)\in\mathbb{D}} \left| \sqrt{k}\mathbb{F}_n(x,y,z) - \frac{W(x,y,z)}{q(x,y)} \right| \\
\le &\sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \sqrt{k}(\tilde{R}'(x,y,z) - \mathbf{E}\tilde{R}'(x,y,z)) - W(x,y,z) \right| \\
&+ \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \sqrt{k}\mathbf{E}\tilde{R}'(x,y,z) - \sum_{i=1}^n \mathbf{E} Z_{n,i} f_{x,y,z} \right| \\
&+ \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \sum_{i=1}^n \mathbf{E} Z_{n,i} f_{x,y,z} - \sqrt{k}\int_0^z R(x,y;t)\,dt \right| \\
=: &J_1 + J_2 + J_3.
\end{aligned}$$

First, we prove that as $n \to \infty$, $J_1 \xrightarrow{\mathbf{P}} 0$, that is,

$$\sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \sqrt{k}(\tilde{R}'(x,y,z) - \mathbf{E}\tilde{R}'(x,y,z)) - W(x,y,z) \right| \xrightarrow{\mathbf{P}} 0. \quad \text{(S1.2)}$$

Assumption 1 implies that there exist real number $\tau > 0$ such that for all $0 \le x \le 1$, $n \in \mathbb{N}$, $1 \le i \le n$, and $j = 1, 2$,

$$\frac{kx}{n} c_j \left(\frac{i}{n}\right) \left[1 - \frac{\tau}{b} A_j \left\{\frac{n}{kx}\right\}\right] < 1 - F_{n,i}^{(j)} \left(U_j\left(\frac{n}{kx}\right), \infty\right) < \frac{kx}{n} c_j \left(\frac{i}{n}\right) \left[1 + \frac{\tau}{b} A_j \left\{\frac{n}{kx}\right\}\right].$$

For $j = 1, 2$, let

$$\delta_{j,n} = \sup_{0<t\le 1} \frac{\tau}{b} A_j \left(\frac{n}{kt}\right) = \frac{\tau}{b} A_j \left(\frac{n}{k}\right).$$

Without loss of generality, we show that the supermum of (S1.2) is bounded above by $o_{\mathbf{P}}(1)$. The proof that the infimum is bounded below by $o_{\mathbf{P}}(1)$ is similar. It holds that as $n \to \infty$,

$$\begin{aligned}
&\sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left(\sqrt{k}\left(\tilde{R}'(x,y,z) - \mathbf{E}\tilde{R}'(x,y,z)\right) - W(x,y,z)\right) \\
\le &\sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \sum_{i=1}^{n} (Z_{n,i} - \mathbf{E}Z_{n,i}) \left(f_{x(1+\delta_{1,n}),y(1+\delta_{2,n}),z}\right) - W(x(1+\delta_{1,n}), y(1+\delta_{2,n}), z) \\
&+ \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \sum_{i=1}^{n} \mathbf{E}Z_{n,i} f_{x(1+\delta_{1,n}),y(1+\delta_{2,n}),z} - \mathbf{E}Z_{n,i} f_{x(1-\delta_{1,n}),y(1-\delta_{2,n}),z} \\
&+ \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left(W(x(1+\delta_{1,n}), y(1+\delta_{2,n}), z) - W(x,y,z)\right) \\
=: &I_1 + I_2 + I_3.
\end{aligned}$$

By Proposition S1, we have $I_1 \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. By the uniform continuity of W/q, we have $I_3 \xrightarrow{\mathbf{P}} 0$. For I_2, since c_j is bounded on $[0, 1]$, it holds that

$$
\begin{aligned}
I_2 \leq & \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nz \rfloor} C_{n,i} \left(\frac{kxc_1(i/n)(1+\delta_{1,n})}{n}, \frac{kyc_2(i/n)(1+\delta_{2,n})}{n} \right) \right. \\
& \left. -C_{n,i} \left(\frac{kxc_1(i/n)(1-\delta_{1,n})}{n}, \frac{kyc_2(i/n)(1-\delta_{2,n})}{n} \right) \right| \\
\leq & 2\sqrt{k} d(\delta_{1,n}+\delta_{2,n}) \\
= & o(1).
\end{aligned}
$$

Thus, the proof of (S1.2) is completed. The proof for $J_2 \xrightarrow{\mathbf{P}} 0$ is similar to that of I_2, by noting that

$$
\begin{aligned}
J_2 \leq & \sup_{(x,y,z)\in\mathbb{D}} \frac{1}{q(x,y)} \left| \frac{1}{\sqrt{k}} \sum_{i=1}^{\lfloor nz \rfloor} C_{n,i} \left(\frac{kxc_1(i/n)(1+\delta_{1,n})}{n}, \frac{kyc_2(i/n)(1+\delta_{2,n})}{n} \right) \right. \\
& \left. -C_{n,i} \left(\frac{kxc_1(i/n)(1-\delta_{1,n})}{n}, \frac{kyc_2(i/n)(1-\delta_{2,n})}{n} \right) \right|.
\end{aligned}
$$

Finally, we show that as $n \to \infty$, $J_3 \to 0$. Hence, we have verified the case when $0 \leq x, y \leq 1$ and $z \in [0,1]$. The proofs for $0 \leq x \leq 1$, $y = \infty$, $z \in [0,1]$ and $0 \leq y \leq 1$, $x = \infty$, $z \in [0,1]$ are similar. It holds for θ_i determined by the mean value theorem that

$$
\begin{aligned}
& \sup_{\substack{0\leq x,y\leq 1 \\ z\in[0,1]}} \frac{1}{q(x,y)} \left| \sum_{i=1}^{n} \mathbf{E} Z_{n,i} f_{x,y,z} - \sqrt{k} R'(x,y,z) \right| \\
= & \sup_{\substack{0\leq x,y\leq 1 \\ z\in[0,1]}} \frac{\sqrt{k}}{q(x,y)} \left| \sum_{i=1}^{\lfloor nz \rfloor} \frac{1}{n} \frac{n}{k} C_{n,i} \left(\frac{kc_1(i/n)x}{n}, \frac{kc_2(i/n)y}{n} \right) - R'(x,y,z) \right| \\
= & \sup_{\substack{0\leq x,y\leq 1 \\ z\in[0,1]}} \frac{\sqrt{k}}{q(x,y)} \left| \sum_{i=1}^{\lfloor nz \rfloor} \frac{1}{n} R \left(c_1\left(\frac{i}{n}\right) x, c_2\left(\frac{i}{n}\right) y, \frac{i}{n} \right) - R'\left(x, y, \frac{\lfloor nz \rfloor}{n}\right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&+O\left(\frac{\sqrt{k}}{n}\right)+O\left(\frac{k^{\alpha+1/2}}{n^{\alpha}}\right)\\
=&\sup_{\substack{0\le x,y\le 1\\ z\in[0,1]}}\frac{\sqrt{k}/n}{q(x,y)}\sum_{i=1}^{\lfloor nz\rfloor}\left|R\left(c_1\left(\frac{i}{n}\right)x,c_2\left(\frac{i}{n}\right)y,\frac{i}{n}\right)-R\left(c_1\left(\frac{\theta_i}{n}\right)x,c_2\left(\frac{\theta_i}{n}\right)y,\frac{\theta_i}{n}\right)\right|\\
&+O\left(\frac{\sqrt{k}}{n}\right)+O\left(\frac{k^{\alpha+1/2}}{n^{\alpha}}\right)\\
\le&\sup_{0\le x,y\le 1}\sum_{i=1}^{n}\frac{\sqrt{k}/n}{q(x,y)}\left(x\left|c_1\left(\frac{i}{n}\right)-c_1\left(\frac{\theta_i}{n}\right)\right|\right)\vee\left(y\left|c_2\left(\frac{i}{n}\right)-c_2\left(\frac{\theta_i}{n}\right)\right|\right)\\
&+\sup_{\substack{0\le x,y\le 1\\ z\in[0,1]}}\frac{\sqrt{k}/n}{q(x,y)}\sum_{i=1}^{\lfloor nz\rfloor}\left|R\left(c_1\left(\frac{i}{n}\right)x,c_2\left(\frac{i}{n}\right)y,\frac{i}{n}\right)-R\left(c_1\left(\frac{i}{n}\right)x,c_2\left(\frac{i}{n}\right)y,\frac{\theta_i}{n}\right)\right|\\
&+O\left(\frac{\sqrt{k}}{n}\right)+O\left(\frac{k^{\alpha+1/2}}{n^{\alpha}}\right).
\end{aligned}
$$

For $0\le y<x\le 1$, it holds that

$$
\begin{aligned}
\frac{1}{q(x,y)}\left|R\left(x,y,i/n\right)-R\left(x,y,\theta_i/n\right)\right|&=x^{1-\eta}\left|R\left(1,x^{-1}y,i/n\right)-R\left(1,x^{-1}y,\theta_i/n\right)\right|\\
&\le\sup_{y\in[0,1]}\left|R\left(1,y,i/n\right)-R\left(1,y,\theta_i/n\right)\right|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sup_{\substack{0\le x,y\le 1\\ z\in[0,1]}}\frac{\sqrt{k}/n}{q(x,y)}\sum_{i=1}^{\lfloor nz\rfloor}\left|R\left(c_1\left(\frac{i}{n}\right)x,c_2\left(\frac{i}{n}\right)y,\frac{i}{n}\right)-R\left(c_1\left(\frac{i}{n}\right)x,c_2\left(\frac{i}{n}\right)y,\frac{\theta_i}{n}\right)\right|\\
\le&\sup_{\substack{0\le x,y\le 1\\ z\in[0,1]}}\frac{\sqrt{k}}{n}\sum_{i=1}^{\lfloor nz\rfloor}\frac{q(c_1(i/n)x/d,c_2(i/n)y/d)d}{q(x,y)q(c_1(i/n)x/d,c_2(i/n)y/d)}\\
&\times\left|R\left(c_1\left(\frac{i}{n}\right)\frac{x}{d},c_2\left(\frac{i}{n}\right)\frac{x}{d},\frac{i}{n}\right)-R\left(c_1\left(\frac{i}{n}\right)\frac{x}{d},c_2\left(\frac{i}{n}\right)\frac{y}{d},\frac{\theta_i}{n}\right)\right|\\
\le&d\sup_{0\le x\le 1}\frac{\sqrt{k}}{n}\sum_{i=1}^{n}\left|R\left(1,x,\frac{i}{n}\right)-R\left(1,x,\frac{\theta_i}{n}\right)\right|+\left|R\left(x,1,\frac{i}{n}\right)-R\left(x,1,\frac{\theta_i}{n}\right)\right|\to 0.
\end{aligned}
$$

Combine the results above and we complete the proof. □

Proof of the conditional weak convergence in Theorem 1. The proof is completed by using Proposition S2 and Lemma C.4 in Bücher and Dette (2013), if we can show that

$$\sup_{(x,y,z)\in\mathbb{D}}\left|\mathbb{F}_n^b(x,y,z)-\sum_{i=1}^n(\xi_i-1)\,Z_{n,i}f_{x,y,z}\right|\xrightarrow{\mathbf{P}}0.$$

We now verify the case when $(x,y,z)\in[0,1]^3$ for the above result. Note that

$$\begin{aligned}
&\left|\mathbb{F}_n^\xi(x,y,z)-\sum_{i=1}^n(\xi_i-1)\,Z_{n,i}f_{x,y,z}\right|\\
=&\frac{1}{q(x,y)}\left|\frac{1}{\sqrt{k}}\sum_{i=1}^{\lfloor nz\rfloor}(\xi_i-1)\Big(\mathbf{1}\left\{X_i^{(n)}>U_1(n/kx),Y_i>U_2(n/ky)\right\}\right.\\
&\quad\left.-\mathbf{1}\left\{V_{X,i}^{(n)}<c_1(i/n)kx/n,V_{Y,i}^{(n)}<c_2(i/n)ky/n\right\}\Big)\right|\\
\le&\frac{1}{q(x,y)}\sup_{\substack{(1-\delta_{1,n})x\le u\le(1+\delta_{1,n})x,\\(1-\delta_{2,n})y\le v\le(1+\delta_{2,n})y.}}\left|q(u,v)\tilde{Z}_{n,i}f_{u,v,z}-q(x,y)\tilde{Z}_{n,i}f_{x,y,z}\right|\\
\le&\sup_{\substack{(1-\delta_{1,n})x\le u\le(1+\delta_{1,n})x,\\(1-\delta_{2,n})y\le v\le(1+\delta_{2,n})y.}}\left|\tilde{Z}_{n,i}f_{u,v,z}-\tilde{Z}_{n,i}f_{x,y,z}\right|\\
&\quad+\sup_{\substack{(1-\delta_{1,n})x\le u\le(1+\delta_{1,n})x,\\(1-\delta_{2,n})y\le v\le(1+\delta_{2,n})y.}}\left|\frac{q(u,v)}{q(x,y)}-1\right|\left|\tilde{Z}_{n,i}f_{u,v,z}\right|\\
=&I_1+I_2.
\end{aligned}$$

It is straightforward to see that $I_1\xrightarrow{\mathbb{P}}0$ by uniform equicontinuity of $\tilde{Z}_{n,i}$, while $I_2\xrightarrow{\mathbb{P}}0$ by the fact that $\left|\tilde{Z}_{n,i}f_{u,v,z}\right|$ is $O_{\mathbf{P}}(1)$ for $(u,v,z)\in\mathbb{D}$, and the fact that

$$\sup_{\substack{(1-\delta_{1,n})x\le u\le(1+\delta_{1,n})x,\\ (1-\delta_{2,n})y\le v\le(1+\delta_{2,n})y.}} \left|\frac{q(u,v)}{q(x,y)}-1\right| \lesssim \delta_{1,n}\vee\delta_{2,n} = o\left(k^{-1/2}\right). \qquad \square$$

Remark S1. For the conditional weak convergence, it is worth noting that Assumption 3 is not necessary for the proof. Recall the relation

$$\sqrt{k}\mathbb{F}_n(x,y,z) = \frac{\sqrt{k}}{q(x,y)}\left|\tilde{R}'(x,y,z)-\mathbf{E}(\tilde{R}'(x,y,z))+\mathbf{E}(\tilde{R}'(x,y,z))-R'(x,y,z)\right|.$$

In the proof, we bound the asymptotic bias $\sqrt{k}\mathbf{E}(\tilde{R}'(x,y,z))-R'(x,y,z)$ by

$$\sup_{(x,y,z)\in\mathbb{D}_T}\frac{\sqrt{k}}{q(x,y)}\left|\sum_{i=1}^{\lfloor nz\rfloor}\frac{1}{n}\frac{n}{k}C_{n,i}\left(\frac{kc_1(i/n)x}{n},\frac{kc_2(i/n)y}{n}\right)-R'(x,y,z)\right|.$$

Assumption 3 is utilized to demonstrate that the above term converges to 0 as $n\to\infty$. Since the bootstrap estimator is based on the process

$$\sqrt{k}\mathbb{F}_n^b(x,y,z) = \frac{1}{\sqrt{k}q(x,y)}\sum_{i=1}^{\lfloor nz\rfloor}(\xi_{bi}-1)\,\mathbf{1}\left\{X_i^{(n)}>U_1(n/kx),Y_i^{(n)}>U_2(n/ky)\right\}$$

the bias item vanishes in the proof.

S2 Derivatives of the Functional Mappings

We provide the derivatives of Φ and Ψ. These derivatives are crucial in the proof of the weak convergence of the three estimators. Generally, if a statistic is a functional of the B-STEP and the functional is Hadamard differentiable, the asymptotic properties of the statistic can be derived from the weak convergence of the B-STEP using the functional delta method. The concept of Hadamard differentiability is introduced in Chapter 3.9 of van der Vaart and Wellner (1996). To proceed, we denote the following two classes:

$$\mathcal{C}_R := \{\theta \in \ell^\infty(\mathbb{D}_T) \mid \theta \text{ continuous with } \theta(\cdot,0,\cdot) = \theta(0,\cdot,\cdot) = 0\},$$
$$\mathcal{C}_{H,T} := \{\theta \in \ell^\infty([0,T]) \mid \theta \text{ continuous with } \theta(0) = 0\}.$$

Proposition S3. *Under Assumptions 1 and 2, Φ is Hadamard differentiable at R'/q tangentially to $\mathcal{C}_R$, whose derivative is the Lipschitz continuous functional,*

$$\begin{aligned}\Phi'_{R'/q}(\theta)(x,y,z) &= q(x,y)\theta(x,y,z) \\ &\quad - R'_1(x,y,z)q_1(x)\theta(x,\infty,1) - R'_2(x,y,z)q_2(y)\theta(\infty,y,1).\end{aligned}$$

Proposition S4. *Ψ is Hadamard-differentiable at $\Pi(x) = x/q_1(x)$ tangentially to $\mathcal{C}_{H,3/2}$ whose derivative is the Lipschitz continuous functional,*

$$\Psi'_\Pi(\theta)(x) = \int_0^1 \theta(t)q_1(t)\frac{dt}{t} - \theta(1).$$

Denote the identity map on $\mathbb{R}^+$ as $\mathrm{id}_{\mathbb{R}^+}$. In this section, the derivatives g'_f of g are derived at one fixed element f. For a clear representation, we denote g' as g'_f and omit the specific function f at subscript. We consider the following function classes in this section:

$$\mathcal{B}_1([0,T]) := \{f : [0,T] \mapsto [0,\infty) \mid f \text{ is a bounded non-decreasing function with } f(0+) = 0\},$$

$$\mathcal{B}_1^{\leftarrow}([0,T]) := \{f : [0,\infty) \mapsto [0,T] \mid f \text{ is the generalized inverse function of a } h \in \mathcal{B}_1([0,T])\},$$

$$\mathcal{B}_2(\mathbb{D}_T) := \{f : \mathbb{D}_T \mapsto \mathbb{R}_+ \mid f \text{ is bounded and } f(\cdot,\infty,1) \in \mathcal{B}_1([0,T]), f(\infty,\cdot,1) \in \mathcal{B}_1([0,T])\}.$$

In this section, we still prove the result for $\mathbb{D}$, and the proof for a general T follows similarly. Throughout this section, we will repeatedly apply the chain rule (van der Vaart and Wellner, 1996, Lemma 3.9.3).

Proof of Proposition S3. We prove the derivative by decomposing the map into four parts.

Step 1: For $\theta \in \mathcal{B}_2(\mathbb{D})$, define ϕ_1 as

$$\phi_1 : \theta(x,y,z) \mapsto q(x,y)\theta(x,y,z),$$

Denote the function $g \in \mathcal{C}_R$ and a series of functions g_t satisfying $g_t \to g$ uniformly as $t \to 0$. Moreover, assume $R'(x,y,z)/q(x,y) + tg_t(x,y,z) \in \mathcal{B}_2(\mathbb{D})$ for every $t > 0$. Then it holds that

$$\lim_{t\to 0} \sup_{(x,y,z)\in\mathbb{D}} \left| t^{-1}\left(R(x,y,z) + tq(x,y)g_t(x,y,z) - R(x,y,z)\right) - q(x,y)g(x,y,z)\right| = 0.$$

Thus, ϕ_1 is Hadamard differentiable at R'/q tangentially to $\mathcal{C}_R$, with derivative

$$\phi_1' : g(x,y,z) \mapsto q(x,y)g(x,y,z),$$

where $q \cdot g$ is continuous and satisfies $q(0,\cdot)g(0,\cdot,\cdot) = 0$ and $q(\cdot,0)g(\cdot,0,\cdot) = 0$.

Step 2: For $\theta \in \mathcal{B}_2(\mathbb{D})$, define ϕ_2 as

$$\phi_2 : \theta(x,y,z) \mapsto (\theta(x,y,z), \theta(x,\infty,1), \theta(\infty,y,1)).$$

This map is Hadamard differentiable at R' tangentially to $\mathcal{C}_R$, with derivative

$$\phi_2' : g(x,y,z) \mapsto (g(x,y,z), g(x,\infty,1), g(\infty,y,1)),$$

where $g(x,\infty,1), g(\infty,y,1) \in \mathcal{C}_{H,1}$.

Step 3: For $(\theta, \vartheta_1, \vartheta_2) \in (\mathcal{B}_2(\mathbb{D}) \times \mathcal{B}_1([0,1]) \times \mathcal{B}_1([0,1]))$, define ϕ_3 as

$$\phi_3 : (\theta(x,y,z), \vartheta_1(x), \vartheta_2(y)) \mapsto (\theta(x,y,z), \vartheta_1^{\leftarrow}(x), \vartheta_2^{\leftarrow}(y)).$$

By Schmidt and Stadtmuller (2006, Theorem 5), ϕ_3 is Hadamard differentiable at $(R', \mathrm{id}_{\mathbb{R}+}, \mathrm{id}_{\mathbb{R}+})$ tangentially to $\mathcal{C}_R \times \mathcal{C}_{H,1} \times \mathcal{C}_{H,1}$, with derivative

$$\phi_3' : (g(x,y,z), f_1(x), f_2(y)) \mapsto (g(x,y,z), -f_1(x), -f_2(y)).$$

Step 4: For $(\theta, \vartheta_1, \vartheta_2) \in (\mathcal{B}_2(\mathbb{D}) \times \mathcal{B}_1^{\leftarrow}([0,1]) \times \mathcal{B}_1^{\leftarrow}([0,1]))$, define ϕ_4 as

$$\phi_4 : (\theta(x,y,z), \vartheta_1(x), \vartheta_2(y)) \mapsto \begin{cases} \theta(\vartheta_1(x), \vartheta_2(y), z), & 0 \le x, y \le \infty, 0 \le z \le 1, \\ \theta(\vartheta_1(x), \infty, z), & 0 \le x \le \infty, y = \infty, 0 \le z \le 1, \\ \theta(\infty, \vartheta_2(y), z), & 0 \le y \le \infty, x = \infty, 0 \le z \le 1. \end{cases}$$

This map is Hadamard differentiable at $(R', \mathrm{id}_{\mathbb{R}^+}, \mathrm{id}_{\mathbb{R}^+})$ tangentially to $\mathcal{C}_R \times \mathcal{C}_{H,1} \times \mathcal{C}_{H,1}$, with derivative

$$\phi_4' : (g(x,y,z), f_1(x), f_2(y)) \mapsto g(x,y,z) + R_1'(x,y,z)f_1(x) + R_2'(x,y,z)f_2(y).$$

To calculate the derivative, suppose a function $(g, f_1, f_2) \in \mathcal{C}_R \times \mathcal{C}_{H,1} \times \mathcal{C}_{H,1}$, and $(g_t, f_{1t}, f_{2t}) \to (g, f_1, f_2)$ as $t \to 0$. Also, suppose for every $t > 0$, $R' + tg_t \in \mathcal{B}_2(\mathbb{D})$, $\mathrm{id}_{\mathbb{R}^+} + tf_{1t} \in \mathcal{B}_1^{\leftarrow}([0,1])$, and $\mathrm{id}_{\mathbb{R}^+} + tf_{2t} \in \mathcal{B}_1^{\leftarrow}([0,1])$. It holds that

$$\begin{aligned}
&t^{-1}\left(\phi_4(R' + tg_t, \mathrm{id}_{\mathbb{R}^+} + tf_{1t}, \mathrm{id}_{\mathbb{R}^+} + tf_{2t}) - \phi_4(R', \mathrm{id}_{\mathbb{R}^+}, \mathrm{id}_{\mathbb{R}^+})\right) \\
=&t^{-1}\left(\phi_4(R', \mathrm{id}_{\mathbb{R}^+} + tf_{1t}, \mathrm{id}_{\mathbb{R}^+} + tf_{2t}) - \phi_4(R', \mathrm{id}_{\mathbb{R}^+}, \mathrm{id}_{\mathbb{R}^+})\right) \\
&+ \phi_4(g_t, \mathrm{id}_{\mathbb{R}^+} + tf_{1t}, \mathrm{id}_{\mathbb{R}^+} + tf_{2t}) \\
=:&I_1 + I_2.
\end{aligned}$$

Note that $\mathrm{id}_{\mathbb{R}^+} + tf_{jt} \to \mathrm{id}_{\mathbb{R}^+}$ for $j = 1, 2$, and $g_t \to g$ as $t \to 0$. Moreover, by the uniform continuity of g, it follows that $\lim_{t\to 0} I_2 = [(x,y) \mapsto g(x,y,z)]$.

For I_1, we split the set $\mathbb{D}$ by

$$\begin{aligned}
\mathbb{D} :=&\{(x,y,z) \mid 0 < x \leq 1, 0 < y \leq 1, 0 \leq z \leq 1\} \\
&\cup \{(x,y,z) \mid x = 0, 0 < y \leq 1, 0 \leq z \leq 1\} \\
&\cup \{(x,y,z) \mid y = 0, 0 < x \leq 1, 0 \leq z \leq 1\} \\
&\cup \{(x,y,z) \mid x = 0, y = 0, 0 \leq z \leq 1\}
\end{aligned}$$

$$\cup \{(x,y,z) \mid x=\infty, 0\le y\le 1, 0\le z\le 1\}$$

$$\cup \{(x,y,z) \mid y=\infty, 0\le x\le 1, 0\le z\le 1\}.$$

We first verify the result on $\{(x,y,z) \mid 0<x\le 1, 0<y\le 1, 0\le z\le 1\}$. It holds for x_t and y_t determined by the mean value theorem that

$$\begin{aligned}
&I_1(x,y,z) - R_1'(x,y,z)f_{1t}(x) - R_2'(x,y,z)f_{2t}(y)\\
&= (R_1'(x_t, y+tf_{2t}(y), z) - R_1'(x,y,z))f_{1t}(x) + (R_2'(x,y_t,z) - R_2'(x,y,z))f_{2t}(y)\\
&=: J_1 + J_2.
\end{aligned}$$

It suffices to prove $J_1 \to 0$ and $J_2 \to 0$ as $t \to 0$, uniformly on $\{(x,y,z) \mid 0<x\le 1, 0<y\le 1, 0\le z\le 1\}$. Without loss of generality, we verify the result for J_1. For a sufficiency small t, we have that $|f_{1t}(x)| \le 2\varepsilon$ for all $x\le\delta$. Moreover, it holds for any $(x,y,z)\in(0,1]^3$ that

$$R_1'(x,y,z) = \int_0^z \frac{\partial R}{\partial x}(c_1(s)x, c_2(s)y, s)c_1(s)\,ds \le \sup_{z\in[0,1]} c_1(z).$$

Then, $|J_1|$ is bounded by $2d\varepsilon$ on $(0,\delta]\times(0,1]\times[0,1]$. Besides, it holds that

$$\begin{aligned}
&\sup_{(x,y,z)\in[\delta,1]\times(0,1]\times[0,1]} |J_1|\\
&\le d\int_0^z \sup_{(x,y,s)\in[\delta,1]\times(0,1]\times[0,1]} \left|\frac{\partial R}{\partial x}(x_t, y+tf_{2t}(y), s) - \frac{\partial R}{\partial x}(x,y,s)\right| ds\\
&\le d \sup_{(x,y,z)\in[\delta,1]\times(0,1]\times[0,1]} \left|\frac{\partial R}{\partial x}(x_t, y+tf_{2t}(y), z) - \frac{\partial R}{\partial x}(x,y,z)\right|.
\end{aligned}$$

Since the derivative $\partial R/\partial x$ is uniformly continuous on $[\delta,1]\times[0,1]\times[0,1]$, we derive that $\lim_{t\to 0}\sup_{(x,y,z)\in[\delta,1]\times(0,1]\times[0,1]}|J_1| = 0$. Thus, we have verified the desired result.

We next verify the result on $\{(x,y,z) \mid x = 0, 0 < y \leq 1, 0 \leq z \leq 1\}$. The arguments are similar for $\{(x,y,z) \mid y = 0, 0 < x \leq 1, 0 \leq z \leq 1\}$ and $\{(x,y,z) \mid x = 0, y = 0, 0 \leq z \leq 1\}$. Notice that $R_1'(0,y,z) = 0$ and $R_2'(0,y,z) = 0$. Additionally, we derive that as $t \to \infty$,

$$
\begin{aligned}
|I_1(0,y,z)| &= \frac{1}{t}\left|R'(tf_{1t}(0), y + tf_{2t}(y), z) - R'(0,y,z)\right| \\
&= \frac{1}{t}\left|R'(tf_{1t}(0), y + tf_{2t}(y), z) - R'(0, y + tf_{2t}(y), z)\right| \\
&= \frac{1}{t}\left|\int_0^z R(c_1(s)tf_{1t}(0), c_2(s)(y + tf_{2t}(y))) - R(0, c_2(s)(y + tf_{2t}(y)))\, ds\right| \\
&\leq \int_0^z c_1(s) f_{1t}(0)\, ds \\
&\to 0.
\end{aligned}
$$

Thus, the derivative aligns with the result $R_1'(0,y,z)f_1(0) + R_2'(0,y,z)f_2(y) = 0$.

Finally, without loss of generality, we verify the case $\{(x,y,z) \mid x = \infty, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. It holds by the uniform convergence of f_{2t} to f_2 as $t \to 0$ that

$$
I_1(\infty,y,z) = (R'(\infty, y + tf_{2t}(y), z) - R'(\infty,y,z)) = \int_0^z c_2(s) f_{2t}(y)\, ds \to C_2(z) f_2(y),
$$

which aligns with the definition of the derivative by observing that $R_1'(\infty,y,z) = 0$ and $R_2'(\infty,y,z) = C_2(z)$.

The proof is finished by the application of the chain rule to $\Phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$. □

Proof of Proposition S4. We prove the derivative of the map by decomposing the map into three parts and derive the derivatives for each part.

Step 1: For $\theta \in \mathcal{B}_1([0, 3/2])$, the first map ψ_1 is defined as

$$\psi_1 : \theta(x) \mapsto (q_1(x)\theta(x), \theta(x)),$$

which is Hadamard differentiable at Π tangentially to $\mathcal{C}_{H,3/2}$. The derivative is

$$\psi_1' : g(x) \mapsto (q_1(x)g(x), g(x)).$$

Step 2: For $(\vartheta, \theta) \in (\mathcal{B}_1([0, 3/2]) \times \mathcal{B}_1([0, 3/2]))$, the second map ψ_2 is defined as

$$\psi_2 : (\vartheta(x), \theta(x)) \mapsto (\vartheta^{\leftarrow}(1), \theta(x)),$$

which is Hadamard differentiable at $(\mathrm{id}_{\mathbb{R}^+}, \Pi)$ tangentially to $\mathcal{C}_{H,3/2} \times \mathcal{C}_{H,3/2}$. The derivative is

$$\psi_2' : (f(x), g(x)) \mapsto (-f(1), g(x)).$$

Step 3: For $(\mathcal{X}, \theta) \in ([0, 3/2] \times \mathcal{B}_1([0, 3/2]))$, the third map ψ_3 is defined as

$$\psi_3 : (\mathcal{X}, \theta(x)) \mapsto \int_0^{\mathcal{X}} \theta(x) q_1(x) \frac{dx}{x}.$$

We derive that ψ_3 is Hadamard differentiable at $(1, \Pi)$ tangentially to the set $\mathbb{R} \times \mathcal{C}_{H,3/2}$, with its derivative as

$$\psi_3' : \quad (a, g(x)) \mapsto a + \int_0^1 g(x) x^{\eta-1} \, dx.$$

Denote $(a, g) \in \mathbb{R} \times \mathcal{C}_{H,3/2}$, and the sequence $(a_t, g_t) \to (a, g)$ as $t \to 0$, satisfying $0 \leq 1 + ta_t \leq 3/2$ for every $t > 0$. Then, it holds for an x_t determined by the mean value theorem that as $t \to 0$,

$$\begin{aligned}
&\frac{1}{t}\left(\int_0^{1+ta_t} x + x^{\eta} t g_t(x) \frac{dx}{x} - \int_0^1 x \frac{dx}{x}\right) \\
=&\frac{1}{t}\int_1^{1+ta_t} 1 + x^{\eta-1} t g_t(x) \, dx + \int_0^1 g_t(x) x^{\eta-1} \, dx \\
=&a_t(1 + x_t^{\eta-1} t g_t(x_t)) + \int_0^1 g_t(x) x^{\eta-1} \, dx \\
\to&a + \int_0^1 g(x) x^{\eta-1} \, dx.
\end{aligned}$$

The proof is finished by the application of the chain rule to $\Psi = \psi_3 \circ \psi_2 \circ \psi_1$. $\square$

S3 Convergence of the Estimators

In this appendix, we present the proofs for the asymptotic properties of the estimators introduced in Section 3. Additional proofs for other theoretical results in the article are provided in the supplementary material.

The proofs are based on the functional delta method, which is a powerful tool for establishing asymptotic distributions of estimators in functional spaces. Specifically, we utilize a generalized version of the functional delta method, as presented in Bücher and Dette (2013, Theorem 3.4). This version is applicable to metrizable topological vector spaces, as discussed in van der Vaart and Wellner (1996, Lemma 3.9.3). The key requirement for the functional delta method is the Lipschitz continuity of the derivative map, which is satisfied by the maps $\Phi'_{R'/q}$ and Ψ'_{Π} in our context.

Proof of Theorem 2. For $T > 0$, according to the proofs in the supplementary material, it can be verified that (3.13) holds on $\mathbb{D}_{3T/2}$. By applying functional delta method to (3.13), we derive the weak convergence of

$$\sup_{(x,y,z)\in\tilde{\mathbb{D}}_{3T/2}} \left| \sqrt{k}\left(\Phi(\tilde{R}')(x,y,z) - \Phi(R')(x,y,z)\right) - \frac{W(x,y,z)}{q(x,y)} \right| \xrightarrow{\mathbf{P}} 0, \tag{S3.3}$$

whereby noticing the definition of the inverse function, the space $\tilde{\mathbb{D}}_{3T/2}$ is defined as

$$\begin{aligned} \tilde{\mathbb{D}}_{3T/2} = &\{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le x_n, 0 \le y \le y_n\} \\ &\cup \{(x,y,z) \mid 0 \le z \le 1, x = \infty, 0 \le y \le y_n\} \\ &\cup \{(x,y,z) \mid 0 \le z \le 1, y = \infty, 0 \le x \le x_n\}, \end{aligned}$$

where

$$x_n := \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}\left\{X_i^{(n)} > U_1\left(\frac{2n}{3Tk}\right)\right\}, \quad \text{and} \quad y_n := \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}\left\{Y_i^{(n)} > U_2\left(\frac{2n}{3Tk}\right)\right\}.$$

Next, by applying $\phi_2 \circ \phi_1$ and functional delta method to the empirical process (3.13), we derive

$$(x_n, y_n) \xrightarrow{\mathbb{P}} (3T/2, 3T/2). \tag{S3.4}$$

Thus, $T < x_n$ and $T < y_n$ holds with probability tending to 1. Then, the space $\tilde{\mathbb{D}}_{3T/2}$ can be replaced by $\mathbb{D}_T$ in the result.

We then verify the inverse of $\sum_{i=1}^n \mathbf{1}\{X_i^{(n)} > U_1\left(n/(kx)\right)\}/k$ in (S3.3), and the argument for the other dimension is the same. By the inequality

$$\frac{1}{k}\sum_{i=1}^n \mathbf{1}\{X_i^{(n)} > X_{n-\lfloor kx \rfloor, n}\} \le x \le T \quad \text{for } x \in [0, T],$$

it follows that $0 \le \frac{n}{k}\left(1 - G_1(X_{n-\lfloor kx \rfloor, n})\right) \le T$, which holds with probability tending to 1 for $x \in [0, T]$. Furthermore, since

$$\frac{1}{k}\sum_{i=1}^n \mathbf{1}\{X_i^{(n)} > X_{n-\lfloor kx \rfloor - 1, n}\} > x,$$

Thus, the generalized inverse of $\sum_{i=1}^n \mathbf{1}\{X_i^{(n)} > U_1(n/(kx))\}/k$ is given by $n\left(1 - G_1(X_{n-\lfloor kx \rfloor, n})\right)/k$. The unconditional convergence result follows by observing that on the domain $\mathbb{D}_T$,

$$\Phi(\tilde{R}')(x, y, z) = \hat{R}'(x, y, z) \quad \text{and} \quad \Phi(R')(x, y, z) = R'(x, y, z).$$

Similarly, by applying the functional delta method, we can derive the conditional weak convergence result. $\square$

Proof of Theorem 3. The unconditional weak convergence of the integrated scedasis functions $\hat{C}_1^b$ is a direct result of plugging $(k_1/k, \infty, z)$ into the process $\sqrt{k}(\hat{R}^{b\prime} - \hat{R}^{\prime})$. For the conditional weak convergence, we denote

$$\tilde{C}_1^b(z) := \frac{1}{k_1}\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}\left(X_i^{(n)} > U_b^{(1)}(n/k_1)\right), \quad \tilde{C}_2^b(z) := \frac{1}{k_2}\sum_{i=1}^{\lfloor nz \rfloor} \xi_{bi}\mathbf{1}\left(Y_i^{(n)} > U_b^{(2)}(n/k_2)\right).$$

Firstly, by plugging $(k_1/k, \infty, z)$ into the process $\sqrt{k}(\hat{R}^{b\prime} - \hat{R}^{\prime})$, we obtain that as $n \to \infty$,

$$\sup_{h \in BL_1(l^\infty([0,1]))}\left|\mathbf{E}_\xi\left(h\left(s_1 k_1(\tilde{C}_1^b - \hat{C}_1)/\sqrt{k}\right)\right) - \mathbf{E}\left(h\left(s_1 W_R(k_1/k, \infty, \cdot)\right)\right)\right| \xrightarrow{\mathbf{P}} 0.$$

Since $k/k_1 \to s_1$ as $n \to \infty$, it follows by the uniform continuity of W_R that as $n \to \infty$,

$$\sup_{h \in BL_1(l^\infty([0,1]))}\left|\mathbf{E}\left(h\left(s_1 W_R(k_1/k, \infty, \cdot)\right)\right) - \mathbf{E}\left(h\left(s_1 W_R(1/s_1, \infty, \cdot)\right)\right)\right| = o(1).$$

Moreover, it holds that as $n \to \infty$,

$$\begin{aligned} &\sup_{h \in BL_1(l^\infty([0,1]))}\left|\mathbf{E}_\xi\left(\left(h\left(s_1 k_1(\tilde{C}_1^b - \hat{C}_1)/\sqrt{k}\right)\right) - h\left(\sqrt{k}(\tilde{C}_1^b - \hat{C}_1)\right)\right)\right| \\ &\leq |s_1 - k/k_1|\left|\mathbf{E}_\xi\left(\sup_{z\in[0,1]}\left|\left(k_1(\tilde{C}_1^b(z) - \hat{C}_1(z))/\sqrt{k}\right)\right|\right)\right| = o_{\mathbf{P}}(1). \end{aligned}$$

Next, we take a map $\omega : \theta(z) \mapsto \theta(z)/\theta(1)$ for a non-decreasing function $\theta \in \ell^\infty([0,1])$ with $\theta(1) > \varepsilon > 0$ and $\theta(0+) = 0$. It holds for the function C_1 that $\omega(C_1) = C_1$, and for the function $\hat{C}_1$ that $\omega(\hat{C}_1) = \hat{C}_1$. The Hadamard derivative of ω at C_1 tangentially to $\mathcal{C}_{H,1}$ is $g(z) \mapsto g(z) - C_1(z)g(1)$. Notice that

$$
\begin{aligned}
& s_1 W_R(1/s_1,\infty,z) - C_1(z) s_1 W_R(1/s_1,\infty,z) \\
=& s_1 W(s_1^{-1},\infty,z) - s_1 C_1(z) W(s_1^{-1},\infty,1) - C_1(z)(s_1 W(s_1^{-1},\infty,1) - s_1 W(s_1^{-1},\infty,1)) \\
=& W_C^{(1)}(z).
\end{aligned}
$$

The proof is thus finished by the functional delta method. □

Proof of Theorem 4. The unconditional result can be proved By Corollary 3 of Einmahl et al. (2014). We then verify the conditional weak convergence of the bootstrap estimator. Denote

$$\mathbb{F}_{1n}^b(x) := \frac{1}{k_1 x^\eta} \sum_{i=1}^n (\xi_{bi} - 1)\, \mathbf{1}\left\{X_i^{(n)} > U_1(n/k_1 x)\right\}, \quad W_{H1}(x) := x^{-\eta} s_1 W(x/s_1,\infty,1).$$

First, plug $(k_1/(kx),\infty,1)$ into the process $\mathbb{F}_n^b$ for $x \in [0,2]$. We derive that as $n \to \infty$,

$$\sup_{h \in BL_1(l^\infty([0,2]))} \left|\mathbf{E}_\xi\left(h\left(\mathbb{F}_{1n}^b\right)\right) - \mathbf{E}\left(h\left(W_{H1}\right)\right)\right| \xrightarrow{\mathbf{P}} 0.$$

Define $x_n(x) := n(1 - G_1(U_1(n/k_1)x^{-\gamma_1}))/k_1$ for $x \in [0,3/2]$. By Corollary 3 in Einmahl et al. (2014), it holds that as $n \to \infty$,

$$\sup_{x \in [0,3/2]} |x_n(x)/x - 1| = O(|B_1(n/k)|).$$

Thus, by the uniform continuity of W_{H_1}, it holds that as $n \to \infty$,

$$\sup_{x \in [0,3/2]} |W_{H1}(x_n(x)) - W_{H1}(x)| \xrightarrow{\mathbf{P}} 0.$$

Also, it holds that as $n \to \infty$,

$$\sup_{h \in BL_1(l^\infty([0,3/2]))} \left| \mathbf{E}_\xi \left(h \left((x_n^\eta / q_1)(\mathbb{F}_{1n}^b \circ x_n) \right) \right) - \mathbf{E}_\xi \left(h \left(\mathbb{F}_{1n}^b \circ x_n \right) \right) \right|$$

$$\leq \sup_{x \in [0,3/2]} \left| \left(\frac{x_n(x)}{x} \right)^\eta - 1 \right| E_\xi \left(\sup_{x \in [0,3/2]} \left| \mathbb{F}_{1n}^b (x_n(x)) \right| \right) = o_{\mathbf{P}}(1).$$

Thus, we derive the conditional weak convergence for the process

$$\mathbf{F}_{1n}^b(x) - \mathbf{F}_{1n}(x) = \frac{x_n^\eta(x)}{q_1(x)} \mathbb{F}_{1n}^b \circ x_n(x) = \frac{1}{k_1 q_1(x)} \sum_{i=1}^n (\xi_{bi} - 1) \mathbf{1} \left\{ X_i^{(n)} > x^{-\gamma_1} U_1(n/k_1) \right\}.$$

Since $q_1(3/2)\mathbf{F}_{1n}(3/2) \xrightarrow{\mathbf{P}} 3/2$ and

$$\frac{1}{k_1} \sum_{i=1}^n \mathbf{1} \left\{ X_i^{(n)} > ((X_{n-k_1,n}/U_1(n/k_1))^{-1/\gamma_1})^{-\gamma_1} U_1(n/k_1) \right\} = 1,$$

$(X_{n-k_1,n}/U_1(n/k_1))^{-1/\gamma_1} < 3/2$ holds with probability tending to 1. Then, it follows that

$$(q_1 \mathbf{F}_{1n})^{\leftarrow}(1) = (X_{n-k_1,n}/U_1(n/k_1))^{-1/\gamma_1}.$$

Finally, the proof is completed by Proposition S4, functional delta method, and continuous mapping theorem. □

S4 Proof of Proposition 1

Proposition S5. *Under Assumptions 1-5, as $n \to \infty$ and $B \to \infty$, we have*

(a) $\mathbf{P}(k(\hat{\gamma}_1 - \hat{\gamma}_2 - \gamma_1 + \gamma_2)^2 \geq \hat{u}_{10}(1-\alpha)) \to \alpha;$

(b) $\mathbf{P}(\sup_{z\in[0,1]}\sqrt{k}\,|T_{20}(z)-C_1(z)+C_2(z)|\geq \hat{u}_{20}^{(\mathrm{KS})}(1-\alpha))\to\alpha$;

(c) $\mathbf{P}(k\int_0^1 (T_{20}(z)-C_1(z)+C_2(z))^2\,dz\geq \hat{u}_{20}^{(\mathrm{CVM})}(1-\alpha))\to\alpha$.

Proof. We prove the result for Proposition S5(a); the argument for the other two statistics is similar. First, by the continuous mapping theorem, we have

$$\sup_{h\in BL_1(\mathbb{R})}\left|\mathbf{E}_\xi[h(T^b_{H10})]-\mathbf{E}(h((\Gamma_1-\Gamma_2)^2))\right|\xrightarrow{\mathbf{P}}0\quad\text{as } n\to\infty.$$

Since $k(\hat{\gamma}_1-\hat{\gamma}_2-\gamma_1+\gamma_2)^2$ is a random variable on $\mathbb{R}$,

$$P(T^b_{H10}\in\cdot\mid\{X_i^{(n)},Y_i^{(n)}\}_{i=1,\dots,N})$$

satisfies Condition 2.1 of Bücher and Kojadinovic (2019), and the distribution of $(\Gamma_1-\Gamma_2)^2$ is continuous, the result follows from Bücher and Kojadinovic (2019, Lemma 4.2). $\square$

Proof of Proposition 1. The argument under H_{10} and H_{20} follows immeditately from Proposition S5. We next prove the statement for Proposition 1(b). By consistency of the Hill estimators, $\hat{\gamma}_j\xrightarrow{P}\gamma_j$ $j=1,2$, we have

$$\frac{T_{H10}}{k}=(\hat{\gamma}_1-\hat{\gamma}_2)^2\xrightarrow{\mathbf{P}}\delta^2. \tag{S4.5}$$

Thus, for an $\varepsilon>0$, we have for a sufficiently large n,

$$\mathbf{P}\big(T_{H10}\geq k\delta^2/2\big)\geq 1-\varepsilon/2.$$

On the other hand, Proposition S5(a) also yields that $k(\hat{\gamma}_1-\hat{\gamma}_2-(\gamma_1-\gamma_2))^2$ converges in distribution and that the bootstrap consistently estimates its $(1-\alpha)$-quantile. By the asymptotic tightness of $\hat{u}_{10}(1-\alpha)$, for any $\varepsilon > 0$, there exists a constant M such that for sufficiently large n and B,

$$\mathbf{P}\big(\hat{u}_{10}(1-\alpha) \leq M\big) \geq 1 - \varepsilon/2. \tag{S4.6}$$

Finally, take n so large that $k\delta^2/2 > M$. Then, we have

$$\mathbf{P}\big(T_{H10} \geq \hat{u}_{10}(1-\alpha)\big) \;\geq\; \mathbf{P}\big(T_{H10} \geq k\delta^2/2,\; \hat{u}_{10}(1-\alpha) \leq M\big) \;\geq\; 1 - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\mathbf{P}\big(T_{H10} \geq \hat{u}_{10}(1-\alpha)\big) \to 1.$$

The argument for testing H_{20} proceeds analogously and is omitted. $\square$

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