

Supplementary Material of “Determining The Number of Factors in Two-Way Factor Model of High-Dimensional Matrix-Variate Time Series: A White-Noise based Method for Serial Correlation Models”

Qiang Xia^a Wai Keung Li^b Rubing Liang^a

^aCollege of Mathematics and Informatics, South China Agricultural University

^bDepartment of Mathematics and Information Technology,
Education University of Hong Kong

June 3, 2026

1 Max-type and Sum-type Estimators

Assume that $\mathbf{Q} = (Q_1, \dots, Q_r) = \mathbf{R}/p^{\frac{1-\delta}{2}}$ satisfies $\mathbf{Q}^\top \mathbf{Q} = \mathbf{I}_r$, where $Q_i \in \mathbb{R}^{p \times 1}$ for each $1 \leq i \leq r$. Let $\mathbf{Q}^\dagger = (Q_{r+1}, \dots, Q_p)$ denote the orthogonal complement of \mathbf{Q} , such that $(\mathbf{Q}, \mathbf{Q}^\dagger)^\top (\mathbf{Q}, \mathbf{Q}^\dagger) = \mathbf{I}_p$. To clarify our motivation, we initially assume that \mathbf{E}_t in model (2.1) is white noise. This leads to the following expressions:

$$Q_i^\top \mathbf{Y}_t = Q_i^\top \mathbf{R} \mathbf{F}_t \mathbf{C} + Q_i^\top \mathbf{E}_t, \quad i = 1, \dots, r; \quad (\text{A.1})$$

$$Q_i^\top \mathbf{Y}_t = Q_i^\top \mathbf{E}_t, \quad i = r + 1, \dots, p. \quad (\text{A.2})$$

Equations (A.1) and (A.2) reveal that $Q_i^\top \mathbf{Y}_t$ exhibits non-white noise characteristics for $i = 1, \dots, r$, whereas $Q_i^\top \mathbf{Y}_t$ remains white noise for $i = r + 1, \dots, p$. This observation motivates the development of a new factor number estimation approach by examining the uncorrelatedness of the sequences $Q_i^\top \mathbf{Y}_t$ across $i = 1, \dots, p$.

Define $\mathcal{R}_i = (Q_i, \dots, Q_p)$ for each $i = 1, \dots, p$. Our method leverages statistical tests to formally assess the white noise hypothesis for the sequence $\{\mathcal{R}_i^\top \mathbf{Y}_t : t = 1, 2, \dots\}$. Specifically, we focus on two categories of test statistics: max-type statistics and sum-type statistics.

For any integer $h \geq 0$, we define the autocovariance matrix for \mathbf{Y}_t as:

$$\begin{aligned} \Sigma_Y(h) &:= \frac{1}{n-h} \sum_{t=1}^{n-h} \sum_{k=1}^q \text{Cov}(Y_{\cdot,k,t}, Y_{\cdot,k,t+h}) \\ &= \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbb{E}(\mathbf{Y}_t \mathbf{Y}_{t+h}^\top) \in \mathbb{R}^{p \times p}. \end{aligned}$$

The autocovariance matrix of the sequence $\{\mathcal{R}_i^\top \mathbf{Y}_t\}$ at lag h is given by

$$\Gamma_i(h) := \left(\gamma_{i,kl}(h) \right) = \mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i. \quad (\text{A.3})$$

Our experience suggests that the estimator derived from (A.3) exhibits superior performance in estimating the number of factors compared to correlation matrix-based approaches. The max-type test is formulated using the maximum norm of the autocovariance matrix:

$$T_{i,n} := \max_{1 \leq h \leq K} n^{1/2} |\Gamma_i(h)|_\infty = \max_{1 \leq h \leq K} n^{1/2} |\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i|_\infty.$$

Alternatively, the sum-type test is formulated using the Frobenius norm of the autocovariance matrix:

$$G_{i,n} := \sum_{1 \leq h \leq K} \text{Tr}(\Gamma_i^\top(h) \Gamma_i(h)) = \sum_{1 \leq h \leq K} \|\Gamma_i(h)\|_F^2 = \sum_{1 \leq h \leq K} \sum_j \sigma_j^2(\Gamma_i(h)),$$

where σ_j^2 denotes the square of the j -th largest singular value of $\Gamma_{\mathcal{R}_i, Y}(h)$. Substantial values of $T_{i,n}$ and $G_{i,n}$ suggest a potential deviation from the white noise assumption for the sequence $\{\mathcal{R}_i^\top \mathbf{Y}_t\}$. To estimate the number of factors r , we sequentially examine these statistics as i increases. Intuitively, when $i = r$, the discrepancy between $T_{i,n}$ and $T_{i+1,n}$ is anticipated to be maximized: since $\{\mathcal{R}_{r+1}^\top \mathbf{Y}_t\}$ is white noise, while $\{\mathcal{R}_r^\top \mathbf{Y}_t\}$ is not. This insight underpins our method, which is applicable to weakly correlated \mathbf{E}_t and detailed as follows.

1.1 Estimation of The Number of Row Factors

Throughout this section, we assume that \mathbf{Y}_t is a zero-mean weakly stationary time series.

Under Assumption 2.1-(C5), the following expectations hold:

$$\begin{aligned}\mathbb{E}(\mathbf{R}\mathbf{F}_t\mathbf{C}^\top\mathbf{E}_{t+h}) &= \text{trs}_p(\mathbb{E}(\text{vec}(\mathbf{R}\mathbf{F}_t)\text{vec}^\top(\mathbf{E}_{t+h}\mathbf{C}))) = 0, \\ \mathbb{E}(\mathbf{E}_t\mathbf{C}\mathbf{F}_{t+h}^\top\mathbf{R}^\top) &= \text{trs}_p(\mathbb{E}(\text{vec}(\mathbf{E}_t\mathbf{C})\text{vec}^\top(\mathbf{R}\mathbf{F}_{t+h}))) \neq 0.\end{aligned}$$

Hence, for $h > 0$, under Assumption 2.2-(C4), the following holds:

$$\begin{aligned}\Sigma_Y(h) &= \mathbb{E}(\mathbf{Y}_t\mathbf{Y}_{t+h}^\top) = q^{1-\omega}\mathbb{E}(\mathbf{R}\mathbf{F}_t\mathbf{F}_{t+h}^\top\mathbf{R}^\top) + \mathbb{E}(\mathbf{E}_t\mathbf{C}\mathbf{F}_{t+h}^\top\mathbf{R}^\top) + \mathbb{E}(\mathbf{E}_t\mathbf{E}_{t+h}^\top) \\ &\approx q^{1-\omega}\mathbf{R}\mathbb{E}(\mathbf{F}_t\mathbf{F}_{t+h}^\top)\mathbf{R}^\top + \mathbb{E}(\mathbf{E}_t\mathbf{C}\mathbf{F}_{t+h}^\top)\mathbf{R}^\top.\end{aligned}\tag{A.4}$$

For a pre-specified integer $h_0 \geq 1$, we define:

$$\mathbf{M}_1 = \sum_{h=1}^{h_0} \Sigma_Y^\top(h)\Sigma_Y(h).\tag{A.5}$$

Equation (A.4) implies that each column of \mathbf{R} can be approximated by the first r eigenvectors of \mathbf{M}_1 . Consequently, the eigenspace of \mathbf{M}_1 aligns with that of \mathbf{R} , denoted as $\mathcal{M}(\mathbf{R})$. This motivates us to estimate $\mathcal{M}(\mathbf{R})$ using the eigenvectors of the sample counterpart of \mathbf{M}_1 , which is defined as:

$$\widehat{\mathbf{M}}_1 = \sum_{h=1}^{h_0} \widehat{\Sigma}_Y^\top(h)\widehat{\Sigma}_Y(h),$$

where

$$\widehat{\Sigma}_Y(h) = \frac{1}{n} \sum_{t=1}^n \mathbf{Y}_t\mathbf{Y}_{t+h}^\top.$$

Let \widehat{Q}_i denote the eigenvector of $\widehat{\mathbf{M}}_1$ corresponding to its i -th largest eigenvalue for each $i = 1, 2, \dots, p$. Subsequently, the eigenspace $\mathcal{M}(\mathbf{R})$ can be estimated using $\mathcal{M}(\widehat{\mathbf{Q}})$, where $\widehat{\mathbf{Q}} = (\widehat{Q}_1, \dots, \widehat{Q}_r)$. Consequently, we estimate \mathbf{R} and \mathbf{Q}^\dagger via $\widehat{\mathbf{R}}_o = p^{\frac{1-\delta}{2}}\widehat{\mathbf{Q}}$ and $\widehat{\mathbf{Q}}^\dagger = (\widehat{Q}_{r+1}, \dots, \widehat{Q}_p)$, respectively.

Using these estimates, we define the feasible max-type and sum-type statistics as follows:

$$\begin{aligned}\widehat{T}_{i,n} &= \max_{1 \leq h \leq K} n^{1/2} \left| \widehat{\Gamma}_i(h) \right|_{\infty}, \\ \widehat{G}_{i,n} &= \sum_{1 \leq h \leq K} \text{Tr} \left(\widehat{\Gamma}_i^{\top}(h) \widehat{\Gamma}_i(h) \right),\end{aligned}$$

where $\widehat{\Gamma}_i(h) = \widehat{\mathcal{R}}_i^{\top} \widehat{\Sigma}_Y(h) \widehat{\mathcal{R}}_i$. We propose the following ratio-based estimators for r leveraging an enhanced elbow criterion:

$$\widehat{r}_{\text{MR}_o} = \arg \max_{1 \leq i \leq r_{\max}} \frac{\widehat{T}_{i,n}}{\widehat{T}_{i+1,n}} = \arg \max_{1 \leq i \leq r_{\max}} \text{MR}_o(i), \quad (\text{A.6})$$

$$\widehat{r}_{\text{SR}_o} = \arg \max_{1 \leq i \leq r_{\max}} \frac{\widehat{G}_{i,n} - \widehat{G}_{i+1,n}}{\widehat{G}_{i+1,n} - \widehat{G}_{i+2,n}} = \arg \max_{1 \leq i \leq r_{\max}} \text{SR}_o(i), \quad (\text{A.7})$$

where $\text{MR}_o(i) = \frac{\widehat{T}_{i,n}}{\widehat{T}_{i+1,n}}$ and $\text{SR}_o(i) = \frac{\widehat{G}_{i,n} - \widehat{G}_{i+1,n}}{\widehat{G}_{i+1,n} - \widehat{G}_{i+2,n}}$, and r_{\max} denotes a user-specified upper bound. The ratios derived from max-type or sum-type statistics may exhibit considerable variability, which could compromise the performance of these ratio estimators of r .

Remark 1. Furthermore, the ER method is defined by maximizing $\text{ER}_o(i)$, which yields the estimator $\widehat{r}_{\text{ER}_o}$ as follows:

$$\widehat{r}_{\text{ER}_o} = \arg \max_{1 \leq i \leq r_{\max}} \frac{\widehat{\lambda}_i(\widehat{\mathbf{M}}_1)}{\widehat{\lambda}_{i+1}(\widehat{\mathbf{M}}_1)} =: \arg \max_{1 \leq i \leq r_{\max}} \text{ER}_o(i),$$

where $\widehat{\lambda}_i(\widehat{\mathbf{M}}_1)$ denotes the i -th largest eigenvalue of the matrix $\widehat{\mathbf{M}}_1$.

1.2 Estimation of The Number of Column Factors

Define

$$\mathbf{M}_2 = \sum_{h=1}^{h_0} \widetilde{\Sigma}_Y^{\top}(h) \widetilde{\Sigma}_Y(h) \quad \text{and} \quad \widehat{\mathbf{M}}_2 = \sum_{h=1}^{h_0} \widehat{\Sigma}_Y^{\top}(h) \widehat{\Sigma}_Y(h), \quad (\text{A.8})$$

where $\widetilde{\Sigma}_Y(h) = \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbb{E}(\mathbf{Y}_t^{\top} \mathbf{Y}_{t+h})$ and its sample counterpart is $\widehat{\Sigma}_Y(h) = \frac{1}{n} \sum_{t=1}^{n-h} \mathbf{Y}_t^{\top} \mathbf{Y}_{t+h}$. Since $\widetilde{\Sigma}_Y(h) \approx p^{1-\delta} \mathbf{C} \mathbb{E}(\mathbf{F}_t^{\top} \mathbf{F}_{t+h}) \mathbf{C}^{\top} + \mathbf{C} \mathbb{E}(\mathbf{F}_t^{\top} \mathbf{E}_{t+h}) \mathbf{R}^{\top}$, Equation (A.8) further implies that each column of \mathbf{C} can be approximated by the first c eigenvectors of \mathbf{M}_2 . Hence, we obtain $\mathcal{M}(\mathbf{M}_2) \approx \mathcal{M}(\mathbf{C})$.

Let \tilde{Q}_j and \hat{Q}_j denote the unit eigenvectors associated with the j -th largest eigenvalues of \mathbf{M}_2 and $\widehat{\mathbf{M}}_2$, respectively. Define $\Gamma_j(h) = \mathcal{C}_j^\top \widetilde{\Sigma}_Y(h) \mathcal{C}_j$, where $\mathcal{C}_j = (\tilde{Q}_j, \dots, \tilde{Q}_q)$, for $j = 1, \dots, q$.

We define the max-type and sum-type statistics as follows:

$$\widehat{T}_{j,n} = \max_{1 \leq h \leq K} n^{1/2} \left| \widehat{\Gamma}_j(h) \right|_\infty,$$

and

$$\widehat{G}_{j,n} = \sum_{1 \leq h \leq K} \text{Tr} \left(\widehat{\Gamma}_j^\top(h) \widehat{\Gamma}_j(h) \right),$$

where $\widehat{\Gamma}_j(h) = \widehat{\mathcal{C}}_j^\top \widetilde{\Sigma}_Y(h) \widehat{\mathcal{C}}_j$.

The ratio estimators are defined as:

$$\widehat{c}_{\text{MR}_o} = \arg \max_{1 \leq j \leq c_{\max}} \frac{\widehat{T}_{j,n}}{\widehat{T}_{j+1,n}} =: \arg \max_{1 \leq j \leq c_{\max}} \widetilde{\text{MR}}_o(j), \quad (\text{A.9})$$

$$\widehat{c}_{\text{SR}_o} = \arg \max_{1 \leq j \leq c_{\max}} \frac{\widehat{G}_{j,n} - \widehat{G}_{j+1,n}}{\widehat{G}_{j+1,n} - \widehat{G}_{j+2,n}} =: \arg \max_{1 \leq j \leq c_{\max}} \widetilde{\text{SR}}_o(j), \quad (\text{A.10})$$

where c_{\max} represents a user-specified upper bound.

Remark 2. Similarly, the estimators for c that we propose are the maximizers of $\widetilde{\text{ER}}_o(j)$, referred to as the following $\widehat{c}_{\text{ER}_o}$ estimator,

$$\widehat{c}_{\text{ER}_o} = \arg \max_{1 \leq i \leq c_{\max}} \frac{\widehat{\lambda}_i(\widehat{\mathbf{M}}_2)}{\widehat{\lambda}_{i+1}(\widehat{\mathbf{M}}_2)} =: \arg \max_{1 \leq i \leq c_{\max}} \widetilde{\text{ER}}_o(i),$$

where $\widehat{\lambda}_i(\widehat{\mathbf{M}}_2)$ denotes the i -th largest eigenvalue of the matrix $\widehat{\mathbf{M}}_2$.

1.3 Theoretical Properties

In this section, we investigate the large-sample properties of the proposed estimators within the asymptotic regime where p , q , and n tend to infinity, while r and c remain fixed. The subsequent theorems derive the rates of convergence and establish the consistency of the ratios of the max-type and sum-type statistics.

Theorem 1. Under Assumption 2.1 and the condition $p^\delta q^\omega n^{-1/2} = \mathbf{o}(1)$, we establish that

$$\frac{\widehat{T}_{r+1,n}}{\widehat{T}_{r,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}); \quad \frac{\widehat{T}_{c+1,n}}{\widehat{T}_{c,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}).$$

Theorem 2. Under Assumption 2.1 and the condition $p^\delta q^\omega n^{-1/2} = \mathbf{o}(1)$, we establish that

$$\frac{\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n}}{\widehat{G}_{r,n} - \widehat{G}_{r+1,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}); \quad \frac{\widehat{G}_{c+1,n} - \widehat{G}_{c+2,n}}{\widehat{G}_{c,n} - \widehat{G}_{c+1,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}).$$

Theorem 3. Under Assumption 2.1 and $p^\delta q^\omega n^{-1/2} = \mathbf{o}(1)$, let $|\Sigma_e(h)|_\infty > 0$. Then the MR_o , SR_o , $\widetilde{\text{MR}}_o$ and $\widetilde{\text{SR}}_o$ estimators – specifically, $\widehat{r}_{\text{MR}_o}$ in (A.6), $\widehat{r}_{\text{SR}_o}$ in (A.7), $\widehat{c}_{\text{MR}_o}$ in (A.9) and $\widehat{c}_{\text{SR}_o}$ in (A.10) – satisfy

$$P(\widehat{r}_{\text{SR}_o} = r) \rightarrow 1; \quad P(\widehat{c}_{\text{SR}_o} = c) \rightarrow 1;$$

$$P(\widehat{r}_{\text{MR}_o} = r) \rightarrow 1; \quad P(\widehat{c}_{\text{MR}_o} = c) \rightarrow 1.$$

2 Lemmas and Proofs

Lemma 1. Let $F_{t,ij}$ be ij -th entry of \mathbf{F}_t , and $E_{t,ij}$ be ij -th entry of \mathbf{E}_t . Suppose that Assumption 1-(C1), (C3), (C4) and (C5) hold, for any $i_1, i_2 = 1, \dots, r$, $j_1, j_2 = 1, \dots, c$, $i_3 = 1, \dots, p$, $j_3 = 1, \dots, q$, and $h \geq 0$, we then have

$$\left| \frac{1}{n-h} \sum_{t=1}^{n-h} (F_{t,i_1 j_1} F_{t+h,i_2 j_2} - \text{Cov}(F_{t,i_1 j_1}, F_{t+h,i_2 j_2})) \right| = \mathbf{O}_p(n^{-1/2}), \quad (\text{B.1})$$

$$\left| \frac{1}{n-h} \sum_{t=1}^{n-h} (E_{t,i_3 j_3} F_{t+h,i_2 j_2} - \text{Cov}(E_{t,i_3 j_3}, F_{t+h,i_2 j_2})) \right| = \mathbf{O}_p(n^{-1/2}), \quad (\text{B.2})$$

$$\left| \frac{1}{n-h} \sum_{t=1}^{n-h} F_{t,i_2 j_2} E_{t+h,i_3 j_3} \right| = \mathbf{O}_p(n^{-1/2}), \quad (\text{B.3})$$

$$\left| \frac{1}{n-h} \sum_{t=1}^{n-h} \sum_{t=1}^{n-h} E_{t,i_2 j_2} E_{t+h,i_3 j_3} - \text{Cov}(E_{t,i_2 j_2}, E_{t+h,i_3 j_3}) \right| = \mathbf{O}_p(n^{-1/2}). \quad (\text{B.4})$$

Proof of Lemma 1.

By Davydov's inequality, the proof of (B.1), (B.2), (B.3) and (B.4) is similar to the proof of Lemma 3 of Wang, Liu and Chen (2019) under Assumptions 1-(C3) and (C4), under Assumptions 1-(C1), (C3) and (C4), under Assumptions 1-(C1), (C3), (C4) and (C5), and under Assumptions 1-(C1) and (C5), respectively. \square

Lemma 2. *Suppose Assumption 1 hold. It holds that for $h \geq 0$*

$$\begin{aligned}\|\widehat{\Sigma}_Y(h) - \Sigma_Y(h)\|_2 &= \mathcal{O}_p(pqn^{-1/2}), \\ \|\Sigma_Y(h)\|_2 &= \mathcal{O}(p^{1-\delta}q^{1-\omega}).\end{aligned}$$

Proof of Lemma 2. Suppose that Assumption 1-(C1)-(C5) hold. For $h \geq 0$, we have

$$\begin{aligned}\|\widehat{\Sigma}_Y(h) - \Sigma_Y(h)\|_2 &= \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{Y}_t \mathbf{Y}_{t+h}^\top - \mathbb{E} \mathbf{Y}_t \mathbf{Y}_{t+h}^\top) \right\|_2 \\ &\leq q^{1-\omega} \left\| \mathbf{R} \left(\frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{F}_t \mathbf{F}_{t+h}^\top - \mathbb{E} \mathbf{F}_t \mathbf{F}_{t+h}^\top) \right) \mathbf{R}^\top \right\|_2 + \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{R} \mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top \right\|_2 \\ &\quad + \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top \right\|_2 + \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{E}_t \mathbf{E}_{t+h}^\top - \mathbb{E} \mathbf{E}_t \mathbf{E}_{t+h}^\top) \right\|_2 \\ &= RE_1 + RE_2 + RE_3 + RE_4,\end{aligned}$$

where

$$\begin{aligned}RE_1 &= q^{1-\omega} \left\| \mathbf{R} \left(\frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{F}_t \mathbf{F}_{t+h}^\top - \mathbb{E} \mathbf{F}_t \mathbf{F}_{t+h}^\top) \right) \mathbf{R}^\top \right\|_2 \\ &\leq q^{1-\omega} \|\mathbf{R}\|_2^2 \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{F}_t \mathbf{F}_{t+h}^\top - \mathbb{E} \mathbf{F}_t \mathbf{F}_{t+h}^\top) \right\|_2 \\ &= p^{1-\delta} q^{1-\omega} \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} (\mathbf{F}_t \mathbf{F}_{t+h}^\top - \mathbb{E} \mathbf{F}_t \mathbf{F}_{t+h}^\top) \right\|_2; \\ RE_2 &= \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{R} \mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top \right\|_2 \leq \|\mathbf{R}\|_2 \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top \right\|_F \\ &= \|\mathbf{R}\|_2 \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \text{vec}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top) - \text{vec}(\mathbb{E}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top)) \right\|_2 \\ &\leq \|\mathbf{R}\|_2 \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_{t+h} \otimes \mathbf{F}_t - \mathbb{E}(\mathbf{E}_{t+h} \otimes \mathbf{F}_t) \right\|_2 \|\text{vec}(\mathbf{C}^\top)\|_F;\end{aligned}$$

$$\begin{aligned}
&\leq p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_{t+h} \otimes \mathbf{F}_t - \mathbb{E}(\mathbf{E}_{t+h} \otimes \mathbf{F}_t) \right\|_2; \\
RE_3 &= \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_t^\top \mathbf{R} \mathbf{F}_{t+h} \mathbf{C}^\top - \mathbb{E}(\mathbf{E}_t^\top \mathbf{R} \mathbf{F}_{t+h}) \mathbf{C}^\top \right\|_2 \\
&\leq \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \text{vec}(\mathbf{E}_t^\top \mathbf{R} \mathbf{F}_{t+h}) - \text{vec}(\mathbb{E}(\mathbf{E}_t^\top \mathbf{R} \mathbf{F}_{t+h})) \right\|_2 \|\mathbf{C}^\top\|_2 \\
&\leq p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{F}_{t+h}^\top \otimes \mathbf{E}_t^\top - \mathbb{E}(\mathbf{F}_{t+h}^\top \otimes \mathbf{E}_t^\top) \right\|_2; \\
RE_4 &= \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_t \mathbf{E}_{t+h}^\top - \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \right\|_2,
\end{aligned}$$

where $\text{vec}(\mathbb{E}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top)) = \mathbb{E}(\mathbf{E}_{t+h} \otimes \mathbf{F}_t) \text{vec}(\mathbf{C}^\top) = 0$ and $\text{vec}(\mathbb{E}(\mathbf{E}_t^\top \mathbf{R} \mathbf{F}_{t+h})) = \mathbb{E}(\mathbf{F}_{t+h}^\top \otimes \mathbf{E}_t^\top) \text{vec}(\mathbf{R}^\top) = 0$. By Lemma 1, we obtain that $RE_1 = \mathbf{O}_p(p^{1-\delta} q^{1-\omega} n^{-1/2})$; under Assumption 1-(C5), $RE_2 = \mathbf{O}_p(p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} n^{-1/2})$; $RE_3 = \mathbf{O}_p(p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} n^{-1/2})$; under Assumption 1-(C2), (C4), $RE_4 = \left\| \frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{E}_t \mathbf{E}_{t+h}^\top - \mathbb{E} \mathbf{E}_t \mathbf{E}_{t+h}^\top \right\|_2 = \mathbf{O}_p(pqn^{-1/2})$. Thus

$$\|\widehat{\Sigma}_Y(h) - \Sigma_Y(h)\|_2 = \mathbf{O}_p(pqn^{-1/2}).$$

Next, by Lemma 1,

$$\begin{aligned}
\|\Sigma_Y(h)\|_2 &= \|\mathbb{E}(\mathbf{Y}_t \mathbf{Y}_{t+h}^\top)\|_2 \\
&\leq q^{1-\omega} \|\mathbf{R} \mathbb{E}(\mathbf{F}_t \mathbf{F}_{t+h}^\top) \mathbf{R}^\top\|_2 + \|\mathbf{R} \mathbb{E}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top)\|_2 \\
&\quad + \|\mathbb{E}(\mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top) \mathbf{R}^\top\|_2 + \|\mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top)\|_2 \\
&\leq p^{1-\delta} q^{1-\omega} \|\mathbb{E}(\mathbf{F}_t \mathbf{F}_{t+h}^\top)\|_2 + p^{\frac{1-\delta}{2}} \|\text{vec}(\mathbb{E}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top))\|_2 \\
&\quad + p^{\frac{1-\delta}{2}} \|\text{vec}(\mathbb{E}(\mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top))\|_2 + \|\mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top)\|_2 \\
&\leq p^{1-\delta} q^{1-\omega} \|\mathbb{E}(\mathbf{F}_t \mathbf{F}_{t+h}^\top)\|_2 + p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \|\mathbb{E}(\mathbf{E}_{t+h} \otimes \mathbf{F}_t)\|_2 \\
&\quad + p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \|\mathbb{E}(\mathbf{F}_{t+h} \otimes \mathbf{E}_t)\|_2 + \|\mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top)\|_2 \\
&= \mathbf{O}_p(p^{1-\delta} q^{1-\omega}).
\end{aligned}$$

□

Lemma 3. Under Assumption 1, for $h \geq 1$ it holds that

$$\sigma_i(\Sigma_Y(h)) \asymp p^{1-\delta}q^{1-\omega}, \quad i = 1, \dots, r, \quad \sigma_i(\Sigma_Y(h)) = \mathbf{o}(p^{1-\delta}q^{1-\omega}), \quad i = r+1, \dots, p,$$

where $\sigma_i(\Sigma_Y(h))$ denotes the i -th largest singular value of $\Sigma_Y(h)$.

Proof of Lemma 3. By Assumption 1-(C4), we can obtain $\|\mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top)\|_2 = \mathbf{O}(pqn^{-1})$.

Then we have

$$\begin{aligned} \Sigma_Y(h) &= q^{1-\omega} \mathbf{R} \Sigma_f(h) \mathbf{R}^\top + \text{trs}_p((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top)) + \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \\ &\approx q^{1-\omega} \mathbf{R} \Sigma_f(h) \mathbf{R}^\top + \text{trs}_p((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top)). \end{aligned}$$

Thus, it holds that

$$\begin{aligned} \sigma_i(\Sigma_Y(h)) &\leq \sigma_i(q^{1-\omega} \mathbf{R} \Sigma_f(h) \mathbf{R}^\top) + \sigma_1(\text{trs}_p((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top))) \\ &\leq q^{1-\delta} q^{1-\omega} \sigma_i(\Sigma_f(h)) + \text{trs}_p(\sigma_1((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top))) \\ &\leq q^{1-\delta} q^{1-\omega} \sigma_i(\Sigma_f(h)) + q^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \sigma_1(\Sigma_{ef}(h)). \end{aligned}$$

and for $i = 1, \dots, r$

$$\begin{aligned} \sigma_i(\Sigma_Y(h)) &\geq \sigma_r(q^{1-\omega} \mathbf{R} \Sigma_f(h) \mathbf{R}^\top) - \sigma_1(\text{trs}_p((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top))) \\ &\geq q^{1-\delta} q^{1-\omega} \sigma_r(\Sigma_f(h)) - \text{trs}_p(\sigma_1((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top))) \\ &\geq q^{1-\delta} q^{1-\omega} \sigma_r(\Sigma_f(h)) - q^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \sigma_1(\Sigma_{ef}(h)) \\ &= \mathbf{O}(p^{1-\delta} q^{1-\omega}). \end{aligned}$$

Thus, for $i = 1, \dots, r$, $\sigma_i(\Sigma_Y(h)) \asymp p^{1-\delta} q^{1-\omega}$; for $i = r+1, \dots, p$, $\sigma_i(\Sigma_Y(h)) = \mathbf{o}(p^{1-\delta} q^{1-\omega})$. Hence, Lemma 3 is proved. □

Lemma 4. Let Assumption 1 and $p^\delta q^\omega n^{-1/2} = \mathbf{o}(1)$ hold, then

$$\begin{aligned} \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 &= \mathbf{O}_p(p^\delta q^\omega n^{-1/2}), \quad \|\widehat{\mathbf{Q}}^\dagger - \mathbf{Q}^\dagger\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2}), \\ \|\widetilde{\widehat{\mathbf{Q}}} - \widetilde{\mathbf{Q}}\|_2 &= \mathbf{O}_p(p^\delta q^\omega n^{-1/2}), \quad \|\widetilde{\widehat{\mathbf{Q}}}^\dagger - \widetilde{\mathbf{Q}}^\dagger\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2}). \end{aligned}$$

Proof of Lemma 4.

By Lemmas 2, it is easily to obtain that

$$\|\widehat{\mathbf{M}}_1 - \mathbf{M}_1\|_2 = \mathbf{O}_p\left(\|\Sigma_Y(h)\|_2 \|\widehat{\Sigma}_Y(h) - \Sigma_Y(h)\|_2\right) = \mathbf{O}_p(p^{2-\delta}q^{2-\omega}n^{-1/2}).$$

By Lemma 3 and Lemma 3 in Lam, Yao and Bathia (2011), the proof of $\|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2})$, $\|\widehat{\mathbf{Q}}^\dagger - \mathbf{Q}^\dagger\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2})$ is in the same spirit to Theorem 1 in Lam et al. (2011) and Theorem 1 in Wang et al. (2019). Then the proof for $\|\widehat{\widetilde{\mathbf{Q}}} - \widetilde{\mathbf{Q}}\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2})$, $\|\widehat{\widetilde{\mathbf{Q}}}^\dagger - \widetilde{\mathbf{Q}}^\dagger\|_2 = \mathbf{O}_p(p^\delta q^\omega n^{-1/2})$ is similar to the previous one. The details are omitted.

□

Lemma 5. *Under Assumption 1, it holds that*

$$\widehat{T}_{i,n} \asymp p^{1-\delta}q^{1-\omega}n^{1/2}, \quad i = 1, \dots, r; \quad \widehat{T}_{i,n} = \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1/2}), \quad i = r+1, \dots, p.$$

Proof of Lemma 5. In the first part, it can be showed that

$$T_{i,n} \asymp p^{1-\delta}q^{1-\omega}n^{1/2}, \quad i = 1, \dots, r; \quad T_{i,n} = \mathbf{O}(pqn^{-1/2}), \quad i = r+1, \dots, p.$$

As $i = 1, \dots, r$, from (B.5) in Lemma 6, we have

$$\Gamma_i(h) = \mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i = \begin{pmatrix} \mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir} & \mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{rp} \\ \mathcal{R}_{rp}^\top \Sigma_Y(h) \mathcal{R}_{ir} & \mathcal{R}_{rp}^\top \Sigma_Y(h) \mathcal{R}_{rp} \end{pmatrix},$$

where $\mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir} = q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \Sigma_F(h) \mathbf{R}^\top \mathcal{R}_{ir} + \mathcal{R}_{ir}^\top \text{trs}_p\left(\left(\mathbf{C}^\top \otimes \mathbf{I}_p\right) \Sigma_{ef}(h) \left(\mathbf{I}_q \otimes \mathbf{R}^\top\right)\right) \mathcal{R}_{ir}$ with $\mathcal{R}_{ir} = (Q_i, \dots, Q_r)$ and $\mathcal{R}_{rp} = (Q_{r+1}, \dots, Q_p)$. By Lemma 3, $\mathcal{R}_i^\top \mathcal{R}_i = \mathbf{I}_{p-i+1}$ and $\mathcal{R}_{ir}^\top \mathcal{R}_{ir} = \mathbf{I}_{r-i+1}$, it holds that

$$T_{i,n} = \max_{1 \leq h \leq K} n^{1/2} |\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i|_\infty \leq \sum_{h=1}^K n^{1/2} \sigma_1\left(\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i\right) = \mathbf{O}(p^{1-\delta}q^{1-\omega}n^{1/2})$$

and

$$T_{i,n} = \max_{1 \leq h \leq K} n^{1/2} |\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i|_\infty \geq \max_{1 \leq h \leq K} n^{1/2} |\mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir}|_\infty$$

$$\geq \sigma_1(\mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir}) / (r - i + 1) = \mathbf{O}(p^{1-\delta} q^{1-\omega} n^{1/2}).$$

Then, we obtain that $T_{i,n} \asymp p^{1-\delta} q^{1-\omega} n^{1/2}$ for $i = 1, \dots, r$. Furthermore, for $i = r+1, \dots, p$, $T_{i,n} = n^{1/2} \|\mathcal{R}_i^\top \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \mathcal{R}_i\|_2 = \mathbf{O}(pqn^{-1/2})$.

In the second part, we can obtain that

$$\widehat{T}_{i,n} \asymp p^{1-\delta} q^{1-\omega} n^{1/2}, \quad i = 1, \dots, r; \quad \widehat{T}_{i,n} = \mathbf{O}_p(p^{1+\delta} q^{1+\omega} n^{-1/2}), \quad i = r+1, \dots, p.$$

For $i = 1, \dots, r$, by Lemma 4 and Assumption 1, we have

$$\begin{aligned} |\widehat{T}_{i,n} - T_{i,n}| &= \left| \max_{1 \leq h \leq K} n^{1/2} |\widehat{\mathcal{R}}_i^\top \widehat{\Sigma}_Y(h) \widehat{\mathcal{R}}_i|_\infty - \max_{1 \leq h \leq K} n^{1/2} |\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i|_\infty \right| \\ &\leq \left| \max_{1 \leq h \leq K} n^{1/2} \left(|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \right. \right. \\ &\quad + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty + |(\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \\ &\quad + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) \mathcal{R}_i|_\infty + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \\ &\quad \left. + |(\mathcal{R}_i^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) \mathcal{R}_i|_\infty \right) \Big| \\ &\leq \sum_{h=1}^K n^{1/2} \left(\|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 + 2 \|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) \mathcal{R}_i\|_2 \right. \\ &\quad \left. + \|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 + 2 \|\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 \right) \\ &= \mathbf{O}_p(pq). \end{aligned}$$

Then, by the result of the first part, it holds that $\widehat{T}_{i,n} \asymp T_{i,n} \asymp p^{1-\delta} q^{1-\omega} n^{1/2}$.

As $i = r+1, \dots, p$, by Lemma 2, $\mathcal{R}_i^\top \mathbf{R} = 0$ and $\mathbf{R}^\top \mathcal{R}_i = 0$, we can obtain that

$$\begin{aligned} |\widehat{T}_{i,n}| &= \max_{1 \leq h \leq K} n^{1/2} |\widehat{\mathcal{R}}_i^\top \widehat{\Sigma}_Y(h) \widehat{\mathcal{R}}_i|_\infty \\ &\leq \max_{1 \leq h \leq K} n^{1/2} \left(|\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i|_\infty + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \right. \\ &\quad + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty + |(\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \\ &\quad \left. + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) \mathcal{R}_i|_\infty + |(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)|_\infty \right) \\ &\leq \sum_{h=1}^K n^{1/2} \left(\|\mathcal{R}_i^\top \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \mathcal{R}_i\|_2 + \|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 \right. \\ &\quad \left. + \|(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 + 2 \|\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)\|_2 \right) \end{aligned}$$

$$= \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1/2}).$$

Thus, Lemma 5 is proved. □

Proof of Theorem 1. By Lemma 5, we have $\widehat{T}_{i,n} \asymp p^{1-\delta}q^{1-\omega}n^{1/2}$ for $i = 1, \dots, r$, and $\widehat{T}_{r+1,n} = \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1/2})$. This also implies

$$\begin{aligned} \frac{\widehat{T}_{i+1,n}}{\widehat{T}_{i,n}} &\asymp 1 \quad \text{for } i = 1, \dots, r-1, \\ \frac{\widehat{T}_{r+1,n}}{\widehat{T}_{r,n}} &= \mathbf{O}_p(p^{2\delta}q^{2\omega}n^{-1}). \end{aligned}$$

□

Lemma 6. *Under Assumption 1, it holds that*

$$\widehat{G}_{i,n} \asymp p^{2-2\delta}q^{2-2\omega}, \quad i = 1, \dots, r; \quad \widehat{G}_{i,n} = \mathbf{O}_p(p^2q^2n^{-1}), \quad i = r+1, \dots, p.$$

Proof of Lemma 6. In the first part, we show that

$$G_{i,n} \asymp p^{2-2\delta}q^{2-2\omega}, \quad i = 1, \dots, r; \quad G_{i,n} = \mathbf{O}(p^2q^2n^{-2}), \quad i = r+1, \dots, p.$$

As $i = 1, \dots, r$, denote $\mathcal{R}_{ir} = (Q_i, \dots, Q_r)$ and $\mathcal{R}_{rp} = (Q_{r+1}, \dots, Q_p)$, where Q_i is defined the same as in Subsection 2.2. Then $\mathcal{R}_i = (\mathcal{R}_{ir}, \mathcal{R}_{rp})$. Thus, $\mathcal{R}_{ir}^\top \mathbf{R} / p^{\frac{1-\delta}{2}} = (0, \mathbf{I}_{r-i+1})$ and $\mathcal{R}_{rp}^\top \mathbf{R} = 0$. Note that

$$\Gamma_i(h) = \mathcal{R}_i^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_i = \begin{pmatrix} \mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} & \mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} \\ \mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} & \mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} \end{pmatrix} \quad (\text{B.5})$$

and

$$\boldsymbol{\Sigma}_Y(h) \approx q^{1-\omega} \mathbb{E}(\mathbf{R} \mathbf{F}_t \mathbf{F}_{t+h}^\top \mathbf{R}^\top) + \mathbb{E}(\mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top).$$

We have

$$\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} = q^{1-\omega} \mathbb{E}(\mathcal{R}_{ir}^\top \mathbf{R} \mathbf{F}_t \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{ir}) + \mathbb{E}(\mathcal{R}_{ir}^\top \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{ir})$$

$$\begin{aligned}
&= q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \boldsymbol{\Sigma}_F(h) \mathbf{R}^\top \mathcal{R}_{ir} + \mathcal{R}_{ir}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir}; \\
\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} &= q^{1-\omega} \mathbb{E} \left(\mathcal{R}_{ir}^\top \mathbf{R} \mathbf{F}_t \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{rp} \right) + \mathbb{E} \left(\mathcal{R}_{ir}^\top \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{rp} \right) = 0; \\
\mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} &= q^{1-\omega} \mathbb{E} \left(\mathcal{R}_{rp}^\top \mathbf{R} \mathbf{F}_t \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{ir} \right) + \mathbb{E} \left(\mathcal{R}_{rp}^\top \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{ir} \right) \\
&= \mathcal{R}_{rp}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir}; \\
\mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} &= q^{1-\omega} \mathbb{E} \left(\mathcal{R}_{rp}^\top \mathbf{R} \mathbf{F}_t \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{rp} \right) + \mathbb{E} \left(\mathcal{R}_{rp}^\top \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top \mathbf{R}^\top \mathcal{R}_{rp} \right) = 0.
\end{aligned}$$

Meanwhile, it can be easily found that $\mathfrak{R}(\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir}) \leq \min\{r, c\}$ and $\mathfrak{R}(\mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp}) \leq \min\{r, c\}$, where \mathfrak{R} represents the rank of a matrix. Denote $J = \mathfrak{R}(\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir})$. Hence, for $i = 1, \dots, r$, we can obtain

$$\begin{aligned}
G_{i,n} &= \sum_{h=1}^K \text{Tr} \left(\Gamma_i^\top(h) \Gamma_i(h) \right) \\
&= \sum_{h=1}^K \text{Tr} \left(\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y^\top(h) \mathcal{R}_{ir} \mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} \right) + \sum_{h=1}^K \text{Tr} \left(\mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y^\top(h) \mathcal{R}_{rp} \mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} \right) \\
&= \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\mathcal{R}_{ir}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{ir} \right) + \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\mathcal{R}_{rp}^\top \boldsymbol{\Sigma}_Y(h) \mathcal{R}_{rp} \right) \\
&= \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left\{ q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \boldsymbol{\Sigma}_f(h) \mathbf{R}^\top \mathcal{R}_{ir} + \mathcal{R}_{ir}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir} \right\} \\
&\quad + \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left\{ \mathcal{R}_{rp}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{rp} \right\} \\
&\leq \sum_{h=1}^K \sum_{j=1}^J \left\{ \sigma_j \left(q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \boldsymbol{\Sigma}_f(h) \mathbf{R}^\top \mathcal{R}_{ir} \right) + \sigma_1 \left[\mathcal{R}_{ir}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir} \right] \right\}^2 \\
&\quad + \sum_{h=1}^K \sum_{j=1}^J \left\{ \sigma_1 \left[\mathcal{R}_{rp}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \boldsymbol{\Sigma}_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{rp} \right] \right\}^2 \\
&\leq \sum_{h=1}^K \sum_{j=1}^J \left\{ p^{1-\delta} q^{1-\omega} \sigma_j \left(\boldsymbol{\Sigma}_F(h) \right) + p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \text{trsp} \left[\sigma_1 \left(\boldsymbol{\Sigma}_{ef}(h) \right) \right] \right\}^2 \\
&\quad + \sum_{h=1}^K \sum_{j=1}^J \left\{ p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \text{trsp} \left[\sigma_1 \left(\boldsymbol{\Sigma}_{ef}(h) \right) \right] \right\}^2 \\
&= \mathbf{O} \left(p^{2-2\delta} q^{2-2\omega} \right)
\end{aligned}$$

and

$$\begin{aligned}
G_{i,n} &= \sum_{h=1}^K \text{Tr} \left(\Gamma_i^\top(h) \Gamma_i(h) \right) \\
&= \sum_{h=1}^K \text{Tr} \left(\mathcal{R}_{ir}^\top \Sigma_Y^\top(h) \mathcal{R}_{ir} \mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir} \right) + \sum_{h=1}^K \text{Tr} \left(\mathcal{R}_{ir}^\top \Sigma_Y^\top(h) \mathcal{R}_{rp} \mathcal{R}_{rp}^\top \Sigma_Y(h) \mathcal{R}_{ir} \right) \\
&= \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\mathcal{R}_{ir}^\top \Sigma_Y(h) \mathcal{R}_{ir} \right) + \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\mathcal{R}_{rp}^\top \Sigma_Y(h) \mathcal{R}_{ir} \right) \\
&= \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left\{ q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \Sigma_f(h) \mathbf{R}^\top \mathcal{R}_{ir} + \mathcal{R}_{ir}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir} \right\} \\
&\quad + \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left\{ \mathcal{R}_{rp}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir} \right\} \\
&\geq \sum_{h=1}^K \sum_{j=1}^J \left\{ \sigma_j \left(q^{1-\omega} \mathcal{R}_{ir}^\top \mathbf{R} \Sigma_f(h) \mathbf{R}^\top \mathcal{R}_{ir} \right) - \sigma_1 \left[\mathcal{R}_{ir}^\top \text{trsp} \left((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_q \otimes \mathbf{R}^\top) \right) \mathcal{R}_{ir} \right] \right\}^2 \\
&\geq \sum_{h=1}^K \sum_{j=1}^J \left\{ p^{1-\delta} q^{1-\omega} \sigma_j \left(\Sigma_f(h) \right) - p^{\frac{1-\delta}{2}} q^{\frac{1-\omega}{2}} \text{trsp} \left[\sigma_1 \left(\Sigma_{ef}(h) \right) \right] \right\}^2 \\
&= \mathbf{O}(p^{2-2\delta} q^{2-2\omega}).
\end{aligned}$$

Therefore, it holds that $G_{i,n} \asymp p^{2-2\delta} q^{2-2\omega}$ for $i = 1, \dots, r$. Furthermore, for $i = r+1, \dots, p$, it is always $G_{i,n} = \sigma_1^2 \left(\mathcal{R}_i^\top \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \mathcal{R}_i \right) = \mathbf{O}(p^2 q^2 n^{-2})$.

In the second part, we can prove that

$$\widehat{G}_{i,n} \asymp p^{2-2\delta} q^{2-2\omega}, \quad i = 1, \dots, r; \quad \widehat{G}_{i,n} = \mathbf{O}_p(p^2 q^2 n^{-1}), \quad i = r+1, \dots, p.$$

Recall $\widehat{G}_{i,n} = \sum_{h=1}^K \text{Tr} \left(\widehat{\Gamma}_i^\top(h) \widehat{\Gamma}_i(h) \right) = \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\widehat{\Gamma}_i(h) \right) = \sum_{h=1}^K \sum_{j=1}^J \lambda_j \left(\widehat{\Gamma}_i^\top(h) \widehat{\Gamma}_i(h) \right)$, where λ_j denotes the j -th largest eigenvalue, and $J = \min\{r, c\}$. As $i = 1, \dots, r$, denote $\widehat{\mathcal{R}}_{ir} = (\widehat{Q}_i, \dots, \widehat{Q}_r)$ and $\widehat{\mathcal{R}}_{rp} = (\widehat{Q}_{r+1}, \dots, \widehat{Q}_p)$, where \widehat{Q}_i is defined the same as in Subsection 2.3. Then $\widehat{\mathcal{R}}_i = (\widehat{\mathcal{R}}_{ir}, \widehat{\mathcal{R}}_{rp})$. Thus, $\widehat{\mathcal{R}}_{ir}^\top \widehat{\mathbf{R}} / p^{\frac{1-\delta}{2}} = (0, \mathbf{I}_{r-i+1})$ and $\widehat{\mathcal{R}}_{rp}^\top \widehat{\mathbf{R}} = 0$. Hence,

$$\begin{aligned}
\widehat{\Gamma}_i(h) &= \widehat{\mathcal{R}}_i^\top \widehat{\Sigma}_Y(h) \widehat{\mathcal{R}}_i \\
&= \left((\widehat{\mathcal{R}}_i - \mathcal{R}_i) + \mathcal{R}_i \right)^\top \left((\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) + \Sigma_Y(h) \right) \left((\widehat{\mathcal{R}}_i - \mathcal{R}_i) + \mathcal{R}_i \right) \\
&= (\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h)) (\widehat{\mathcal{R}}_i - \mathcal{R}_i) + (\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h) (\widehat{\mathcal{R}}_i - \mathcal{R}_i)
\end{aligned}$$

$$\begin{aligned}
& +(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i + (\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h)\mathcal{R}_i \\
& +\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))(\widehat{\mathcal{R}}_i - \mathcal{R}_i) + \mathcal{R}_i^\top \Sigma_Y(h)(\widehat{\mathcal{R}}_i - \mathcal{R}_i) \\
& +\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i \\
& = \widehat{\Gamma}_{1i}(h) + \mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i,
\end{aligned} \tag{B.6}$$

where

$$\begin{aligned}
\widehat{\Gamma}_{1i}(h) & = (\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))(\widehat{\mathcal{R}}_i - \mathcal{R}_i) + (\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h)(\widehat{\mathcal{R}}_i - \mathcal{R}_i) \\
& +(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i + \mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))(\widehat{\mathcal{R}}_i - \mathcal{R}_i) \\
& +(\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)(\widehat{\mathcal{R}}_i - \mathcal{R}_i) + \mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i.
\end{aligned}$$

By Lemma 2 and Lemma 4, we have $\sigma_1\left((\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))(\widehat{\mathcal{R}}_i - \mathcal{R}_i)\right) = \mathbf{O}_p(p^{1+2\delta}q^{1+2\omega}n^{-3/2})$; $\sigma_1\left((\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h)(\widehat{\mathcal{R}}_i - \mathcal{R}_i)\right) = \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1})$; $\sigma_1\left((\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i\right) = \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1})$; $\sigma_1\left(\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))(\widehat{\mathcal{R}}_i - \mathcal{R}_i)\right) = \mathbf{O}_p(p^{1+\delta}q^{1+\omega}n^{-1})$; $\sigma_1\left((\widehat{\mathcal{R}}_i - \mathcal{R}_i)^\top \Sigma_Y(h)\mathcal{R}_i\right) = \mathbf{O}_p(pqn^{-1/2})$; $\sigma_1\left(\mathcal{R}_i^\top \Sigma_Y(h)(\widehat{\mathcal{R}}_i - \mathcal{R}_i)\right) = \mathbf{O}_p(pqn^{-1/2})$; $\sigma_1\left(\mathcal{R}_i^\top (\widehat{\Sigma}_Y(h) - \Sigma_Y(h))\mathcal{R}_i\right) = \mathbf{O}_p(pqn^{-1/2})$. Then, it is obtained that

$$\sigma_1(\widehat{\Gamma}_{1i}(h)) = \mathbf{O}_p(pqn^{-1/2}) \quad \text{and} \quad \sigma_1(\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i) = \mathbf{O}(p^{1-\delta}q^{1-\omega}).$$

Thus, we have

$$\begin{aligned}
\widehat{G}_{i,n} - G_{i,n} & = \sum_{h=1}^K \text{Tr}\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \sum_{h=1}^K \text{Tr}\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \\
& = \sum_{h=1}^K \sum_{j=1}^J \left\{ \lambda_j\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \lambda_j\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \right\} \\
& \leq \sum_{h=1}^K \sum_{j=1}^J \lambda_1\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h) - \Gamma_i^\top(h)\Gamma_i(h)\right) \\
& \leq J \sum_{h=1}^K \left\{ \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + 2\sigma_1\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i\right) \right\} \\
& \leq J \sum_{h=1}^K \left\{ \sigma_1^2\left(\widehat{\Gamma}_{1i}(h)\right) + 2\sigma_1\left(\widehat{\Gamma}_{1i}^\top(h)\right)\sigma_1\left(\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i\right) \right\}
\end{aligned}$$

$$= \mathbf{O}_p(p^{2-\delta}q^{2-\omega}n^{-1/2}),$$

where the second-to-last inequality is obtained by (B.6); and

$$\begin{aligned} \widehat{G}_{i,n} - G_{i,n} &= \sum_{h=1}^K \text{Tr}\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \sum_{h=1}^K \text{Tr}\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \\ &= \sum_{h=1}^K \sum_{j=1}^J \left\{ \lambda_j\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \lambda_j\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \right\} \\ &= \sum_{h=1}^K \sum_{j=1}^J \lambda_j\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h) - \Gamma_i^\top(h)\Gamma_i(h)\right) \\ &\geq \sum_{h=1}^K \sum_{j=1}^J \left\{ \lambda_j\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + \lambda_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \\ &\geq \sum_{h=1}^K \left\{ \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + J\lambda_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \\ &= \sum_{h=1}^K \left\{ \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + J\sigma_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \\ &\geq \sum_{h=1}^K \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) = \sum_{h=1}^K \sigma_1^2\left(\widehat{\Gamma}_{1i}(h)\right) = \mathbf{O}_p(pqn^{-1/2}), \end{aligned}$$

the fourth-equality follows from the condition $\lambda_{p-i+1} = \sigma_{p-i+1}$. If $\lambda_{p-i+1} = -\sigma_{p-i+1}$, we have

$$\begin{aligned} \widehat{G}_{i,n} - G_{i,n} &= \sum_{h=1}^K \text{Tr}\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \sum_{h=1}^K \text{Tr}\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \\ &= \sum_{h=1}^K \sum_{j=1}^J \left\{ \lambda_j\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h)\right) - \lambda_j\left(\Gamma_i^\top(h)\Gamma_i(h)\right) \right\} \\ &= \sum_{h=1}^K \sum_{j=1}^J \lambda_j\left(\widehat{\Gamma}_i^\top(h)\widehat{\Gamma}_i(h) - \Gamma_i^\top(h)\Gamma_i(h)\right) \\ &\geq \sum_{h=1}^K \sum_{j=1}^J \left\{ \lambda_j\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + \lambda_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \\ &\geq \sum_{h=1}^K \left\{ \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) + J\lambda_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \\ &= \sum_{h=1}^K \left\{ \lambda_1\left(\widehat{\Gamma}_{1i}^\top(h)\widehat{\Gamma}_{1i}(h)\right) - J\sigma_{p-i+1}\left(\widehat{\Gamma}_{1i}^\top(h)\mathcal{R}_i^\top \Sigma_Y(h)\mathcal{R}_i + \mathcal{R}_i^\top \Sigma_Y(h)^\top \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h)^\top\right) \right\} \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{h=1}^K \left\{ \lambda_1 \left(\widehat{\Gamma}_{1i}^\top(h) \widehat{\Gamma}_{1i}(h) \right) - J \sigma_{p-i+1} \left(\widehat{\Gamma}_{1i}^\top(h) \mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i \right) - J \sigma_1 \left(\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i \widehat{\Gamma}_{1i}^\top(h) \right) \right\} \\
&\geq \sum_{h=1}^K \left\{ \lambda_1 \left(\widehat{\Gamma}_{1i}^\top(h) \widehat{\Gamma}_{1i}(h) \right) - J \sigma_1 \left(\widehat{\Gamma}_{1i}^\top(h) \right) \sigma_1 \left(\mathcal{R}_i^\top \Sigma_Y(h) \mathcal{R}_i \right) \right\} \\
&= \mathbf{O}_p(p^{2-\delta} q^{2-\omega} n^{-1/2}),
\end{aligned}$$

the last inequality holds because $\mathfrak{R}(\Sigma_Y(h)) \leq r$. Thus, it holds that $|\widehat{G}_{i,n} - G_{i,n}| = \mathbf{O}_p(p^{2-\delta} q^{2-\omega} n^{-1/2})$.

By the result of first part, we can obtain

$$\widehat{\Gamma}_{1i}^\top(h) \asymp G_{i,n} \asymp p^{2-2\delta} q^{2-2\omega}.$$

As $i = r + 1, \dots, p$, by $\sigma_1(\widehat{\Gamma}_i(h)) = \sigma_1(\widehat{\Gamma}_{1i}(h) + \mathcal{R}_i^\top \mathbb{E}(\mathbf{E}_t \mathbf{E}_{t+h}^\top) \mathcal{R}_i) \approx \sigma_1(\widehat{\Gamma}_{1i}(h)) = \mathbf{O}_p(pqn^{-1/2})$, it holds that

$$\widehat{G}_{i,n} = \sum_{h=1}^K \text{Tr} \left(\widehat{\Gamma}_i^\top(h) \widehat{\Gamma}_i(h) \right) = \sum_{h=1}^K \sum_{j=1}^J \sigma_j^2 \left(\widehat{\Gamma}_i(h) \right) = \mathbf{O}_p(p^2 q^2 n^{-1}).$$

Put these results together, Lemma 6 is proved. □

Proof of Theorem 2.

By Lemmas 6, we have $\widehat{G}_{i,n} - \widehat{G}_{i+1,n} \asymp p^{2-2\delta} q^{2-2\omega}$ for $i = 1, \dots, r$, and $\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n} = \mathbf{O}_p(p^2 q^2 n^{-1})$. This implies

$$\begin{aligned}
\frac{\widehat{G}_{i+1,n} - \widehat{G}_{i+2,n}}{\widehat{G}_{i,n} - \widehat{G}_{i+1,n}} &\asymp 1 \quad \text{for } i = 1, \dots, r-1, \\
\frac{\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n}}{\widehat{G}_{r,n} - \widehat{G}_{r+1,n}} &= \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}).
\end{aligned}$$

□

Lemma 7. *Under Assumption 1, it holds that*

$$\|\widehat{\Omega}_x(h) - \Omega_x(h)\|_2 = \mathbf{O}_p(pq^\omega n^{-1/2}), \quad \|\Omega_x(h)\|_2 = \mathbf{O}(p^{1-\delta}).$$

Proof of Lemma 7. Let

$$\tilde{\Omega}_x(h) = \frac{1}{(n-h)q^{2-2\omega}} \sum_{t=1}^{n-h} \mathbf{Y}_t \tilde{\mathbf{C}} \tilde{\mathbf{C}}^\top \mathbf{Y}_{t+h}^\top.$$

We only show that

$$\|\hat{\Omega}_x(h) - \tilde{\Omega}_x(h)\|_2 = \mathbf{O}_p(pq^\omega n^{-1/2}), \quad \|\tilde{\Omega}_x(h) - \Omega_x(h)\|_2 = \mathbf{O}_p(pq^\omega n^{-1/2}).$$

By Lemma 4, we have

$$\|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2 = \mathbf{O}_p(q^{\frac{1-\omega}{2}} p^\delta q^\omega n^{-1/2}).$$

By Assumption 1-(C4), we have $\sum_{h=1}^{h_0} \frac{1}{n-h} \sum_{t=1}^{n-h} \|\mathbf{F}_{t+h}^\top \mathbf{F}_t\|_2 = \mathbf{O}_p(1)$. By Assumption 1-(C5) and Lemma 1, we have $\sum_{h=1}^{h_0} \frac{1}{n-h} \sum_{t=1}^{n-h} \|\mathbf{E}_t^\top \otimes \mathbf{F}_{t+h}^\top\|_2 = \mathbf{O}_p(1)$. By Lemma 1, we have $\sum_{h=1}^{h_0} \frac{1}{n-h} \sum_{t=1}^{n-h} \|\mathbf{F}_t^\top \otimes \mathbf{E}_{t+h}^\top\|_2 = \mathbf{O}_p(p^{1/2} q^{1/2} n^{-1/2})$. By Assumption 1-(C1) and Lemma 1, we have $\sum_{h=1}^{h_0} \frac{1}{n-h} \sum_{t=1}^{n-h} \|\mathbf{E}_{t+h}^\top \mathbf{E}_t\|_2 = \mathbf{O}_p(pqn^{-1/2})$. By Assumption 1-(C2), $\|\text{vec}(\mathbf{R})\|_2 = p^{\frac{1-\delta}{2}}$. Then, it holds that

$$\begin{aligned} & \|\hat{\Omega}_x(h) - \tilde{\Omega}_x(h)\|_2 \\ &= \sum_{h=1}^{h_0} \frac{1}{(n-h)q^{2-2\omega}} \sum_{t=1}^{n-h} \left\| (\mathbf{R}\mathbf{F}_t\mathbf{C}^\top + \mathbf{E}_t) (\hat{\tilde{\mathbf{C}}}\hat{\tilde{\mathbf{C}}}^\top - \tilde{\mathbf{C}}\tilde{\mathbf{C}}^\top) (\mathbf{R}\mathbf{F}_{t+h}\mathbf{C}^\top + \mathbf{E}_{t+h})^\top \right\|_2 \\ &\leq \sum_{h=1}^{h_0} \sum_{t=1}^{n-h} \left(\|\mathbf{R}\mathbf{F}_t\mathbf{C}^\top (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}})^\top \mathbf{C}\mathbf{F}_{t+h}^\top \mathbf{R}^\top\|_2 + 2\|\mathbf{R}\mathbf{F}_t\mathbf{C}^\top (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) \tilde{\mathbf{C}}^\top \mathbf{C}\mathbf{F}_{t+h}^\top \mathbf{R}^\top\|_2 \right. \\ &\quad + \|\mathbf{R}\mathbf{F}_t\mathbf{C}^\top (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}})^\top \mathbf{E}_{t+h}^\top\|_2 + 2\|\mathbf{R}\mathbf{F}_t\mathbf{C}^\top \tilde{\mathbf{C}} (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}})^\top \mathbf{E}_{t+h}^\top\|_2 \\ &\quad + \|\mathbf{E}_t (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}})^\top \mathbf{C}\mathbf{F}_{t+h}^\top \mathbf{R}^\top\|_2 + 2\|\mathbf{E}_t (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) \tilde{\mathbf{C}}^\top \mathbf{C}\mathbf{F}_{t+h}^\top \mathbf{R}^\top\|_2 \\ &\quad \left. + \|\mathbf{E}_t (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}})^\top \mathbf{E}_{t+h}^\top\|_2 + 2\|\mathbf{E}_t (\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}) \tilde{\mathbf{C}}^\top \mathbf{E}_{t+h}^\top\|_2 \right) / ((n-h)q^{2-2\omega}) \\ &\leq \sum_{h=1}^{h_0} \sum_{t=1}^{n-h} \left(\|\mathbf{R}\|_2^2 \|\mathbf{C}\|_2^2 \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2^2 \|\mathbf{F}_{t+h}^\top \mathbf{F}_t\|_2 + 2q^{1-\omega} \|\mathbf{R}\|_2^2 \|\mathbf{C}\|_2 \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2 \|\mathbf{F}_{t+h}^\top \mathbf{F}_t\|_2 \right. \\ &\quad + \|\mathbf{C}\|_2 \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2^2 \|\mathbf{F}_t^\top \otimes \mathbf{E}_{t+h}^\top\|_2 \|\text{vec}(\mathbf{R})\|_2 + 2q^{1-\omega} \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2 \|\mathbf{F}_t^\top \otimes \mathbf{E}_{t+h}^\top\|_2 \|\text{vec}(\mathbf{R})\|_2 \\ &\quad + \|\mathbf{C}\|_2 \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2^2 \|\mathbf{E}_t^\top \otimes \mathbf{F}_{t+h}^\top\|_2 \|\text{vec}(\mathbf{R}^\top)\|_2 + 2q^{1-\omega} \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2 \|\mathbf{E}_t^\top \otimes \mathbf{F}_{t+h}^\top\|_2 \|\text{vec}(\mathbf{R}^\top)\|_2 \\ &\quad \left. + \|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2^2 \|\mathbf{E}_{t+h}^\top \mathbf{E}_t\|_2 + 2\|\hat{\tilde{\mathbf{C}}} - \tilde{\mathbf{C}}\|_2 \|\mathbf{E}_{t+h}^\top \mathbf{E}_t\|_2 \right) / ((n-h)q^{2-2\omega}) \\ &= \mathbf{O}_p(p^{1+\delta} q^{2\omega} n^{-1}) + \mathbf{O}_p(pq^\omega n^{-1/2}) + \mathbf{O}_p(p^{\frac{2+3\delta}{2}} q^{\frac{5\omega}{2}} n^{-3/2}) + \mathbf{O}_p(p^{\frac{2+\delta}{2}} q^{\frac{3\omega}{2}} n^{-1}) \end{aligned}$$

$$\begin{aligned}
& + \mathbf{O}_p(p^{\frac{1+3\delta}{2}} q^{-\frac{1-5\omega}{2}} n^{-1}) + \mathbf{O}_p(p^{\frac{1+\delta}{2}} q^{-\frac{1-3\omega}{2}} n^{-1/2}) + \mathbf{O}_p(p^{1+2\delta} q^{-1+3\omega} n^{-3/2}) \\
& + \mathbf{O}_p(p^{1+\delta} q^{2\omega} n^{-1}) = \mathbf{O}_p(pq^\omega n^{-1/2}).
\end{aligned}$$

and

$$\begin{aligned}
& \|\tilde{\Omega}_x(h) - \Omega_x(h)\|_2 \\
= & \left\| \sum_{h=1}^{h_0} \frac{1}{(n-h)q^{2-2\omega}} \sum_{t=1}^{n-h} \mathbf{Y}_t \tilde{\mathbf{C}} \tilde{\mathbf{C}}^\top \mathbf{Y}_{t+h}^\top - \mathbb{E}(\mathbf{Y}_t \tilde{\mathbf{C}} \tilde{\mathbf{C}}^\top \mathbf{Y}_{t+h}^\top) \right\|_2 \\
= & \sum_{h=1}^{h_0} \left\| \mathbf{R} \left(\frac{1}{n-h} \sum_{t=1}^{n-h} \mathbf{F}_t \mathbf{F}_{t+h}^\top - \Omega_f(h) \right) \mathbf{R}^\top \right\|_2 + \sum_{h=1}^{h_0} \left\| \mathbf{R} \frac{1}{(n-h)q^{1-\omega}} \sum_{t=1}^{n-h} \mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top \right\|_2 \\
& + \sum_{h=1}^{h_0} \left\| \left(\frac{1}{(n-h)q^{1-\omega}} \sum_{t=1}^{n-h} \mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top - \Omega_{\xi_f}(h) \right) \mathbf{R}^\top \right\|_2 + \sum_{h=1}^{h_0} \left\| \frac{1}{(n-h)q^{2-2\omega}} \sum_{t=1}^{n-h} \mathbf{E}_t \tilde{\mathbf{C}} \tilde{\mathbf{C}}^\top \mathbf{E}_{t+h}^\top \right\|_2 \\
\leq & \sum_{h=1}^{h_0} \sum_{t=1}^{n-h} \left(\|\mathbf{R}\|_2^2 \|\text{tr}_{sr}(\text{vec}(\mathbf{F}_t) \text{vec}^\top(\mathbf{F}_{t+h})) - \text{tr}_{sr}(\Sigma_f(h))\|_2 + \|\mathbf{R}\|_2 \|\text{vec}(\mathbf{F}_t \mathbf{C}^\top \mathbf{E}_{t+h}^\top)\|_2 / q^{1-\omega} \right. \\
& \left. + \|\text{vec}(\mathbf{E}_t \mathbf{C} \mathbf{F}_{t+h}^\top - \Omega_{\xi_f}(h))\|_2 \|\mathbf{R}^\top\|_2 / q^{1-\omega} + \|\mathbf{E}_t \tilde{\mathbf{C}} \tilde{\mathbf{C}}^\top \mathbf{E}_{t+h}^\top\|_2 / q^{2-2\omega} \right) / (n-h) \\
\leq & \sum_{h=1}^{h_0} \sum_{t=1}^{n-h} \left(\|\mathbf{R}\|_2^2 \|\text{tr}_{sr}(\text{vec}(\mathbf{F}_t) \text{vec}^\top(\mathbf{F}_{t+h})) - \text{tr}_{sr}(\Sigma_f(h))\|_2 + \|\mathbf{R}\|_2 \|\mathbf{E}_{t+h} \otimes \mathbf{F}_t\|_2 \|\text{vec}(\mathbf{C}^\top)\|_2 / q^{1-\omega} \right. \\
& \left. + \|(\mathbf{F}_{t+h} \otimes \mathbf{E}_t) - \Sigma_{\xi_f}(h)\|_2 \|\text{vec}(\mathbf{C})\|_2 \|\mathbf{R}^\top\|_2 / q^{1-\omega} + \|\mathbf{E}_{t+h}^\top \mathbf{E}_t\|_2 \|\tilde{\mathbf{C}}\|_2^2 / q^{2-2\omega} \right) / (n-h) \\
= & \mathbf{O}_p(p^{1-\delta} n^{-1/2}) + \mathbf{O}_p(p^{\frac{2-\delta}{2}} q^{\frac{\omega}{2}} n^{-1/2}) + \mathbf{O}_p(p^{\frac{2-\delta}{2}} q^{\frac{\omega}{2}} n^{-1/2}) + \mathbf{O}_p(pq^\omega n^{-1/2}) = \mathbf{O}_p(pq^\omega n^{-1/2}).
\end{aligned}$$

Then we have

$$\|\hat{\Omega}_x(h) - \Omega_x(h)\|_2 \leq \|\hat{\Omega}_x(h) - \tilde{\Omega}_x(h)\|_2 + \|\tilde{\Omega}_x(h) - \Omega_x(h)\|_2 = \mathbf{O}_p(pq^\omega n^{-1/2}).$$

Further,

$$\begin{aligned}
\|\Omega_x(h)\|_2 & = \|\mathbf{R} \Omega_f(h) \mathbf{R}^\top + \Omega_{\xi_f}(h) \mathbf{R}^\top\|_2 \leq \|\mathbf{R}\|_2^2 \|\Omega_f(h)\|_2 + \|\Omega_{\xi_f}(h) \mathbf{R}^\top\|_2 \\
& = \|\mathbf{R}\|_2^2 \|\Omega_f(h)\|_2 + \|\text{tr}_p((\mathbf{C}^\top \otimes \mathbf{I}_p) \Sigma_{ef}(h) (\mathbf{I}_c \otimes \mathbf{R}^\top))\|_2 / q^{1-\omega} = \mathbf{O}(p^{1-\delta}).
\end{aligned}$$

□

Lemma 8. *Under Assumption 1, for $h \geq 1$ it holds that*

$$\sigma_i(\Omega_x(h)) \asymp p^{1-\delta}, \quad i = 1, \dots, r, \quad \sigma_i(\Omega_x(h)) = o(p^{1-\delta}), \quad i = r+1, \dots, p,$$

where $\sigma_i(\Omega_x(h))$ denotes the i -th largest singular value of $\Omega_x(h)$.

Proof of Lemma 8. Similar to the proof of Lemma 3, the proof of Lemma 8 can be proved. □

Lemma 9. Let Assumption 1 and $p^\delta q^\omega n^{-1/2} = o(1)$ hold, then

$$\|\widehat{\mathbf{O}} - \mathbf{O}\|_2 = \mathcal{O}_p(p^\delta q^\omega n^{-1/2}), \quad \|\widehat{\mathbf{O}}^\dagger - \mathbf{O}^\dagger\|_2 = \mathcal{O}_p(p^\delta q^\omega n^{-1/2}).$$

Proof of Lemma 9. Similar arguments for proving Lemma 6 can be used, we then omit the detailed proof of Lemma 9. □

Lemma 10. $\check{T}_{i,n}$ and $\check{G}_{i,n}$ defined in (3.8) and (3.9), respectively, and their sample versions $\widehat{T}_{i,n}$ and $\widehat{G}_{i,n}$ are monotonically decreasing with respect to i .

Proof of Lemma 10.

Since $\mathfrak{R}_i = (O_i, \dots, O_p)$, for $i = 1, \dots, p$, we have $\mathfrak{R}_i = (O_i, \mathfrak{R}_{i+1})$.

$$\begin{aligned} \check{T}_{i,n} &= \max_{1 \leq h \leq K} n^{1/2} |\mathfrak{R}_i^\top \Sigma_X(h) \mathfrak{R}_i|_\infty = \max_{1 \leq h \leq K} n^{1/2} \left| \begin{pmatrix} O_i^\top \Sigma_X(h) O_i & O_i^\top \Sigma_X(h) \mathfrak{R}_{i+1} \\ \mathfrak{R}_{i+1}^\top \Sigma_X(h) O_i & \mathfrak{R}_{i+1}^\top \Sigma_X(h) \mathfrak{R}_{i+1} \end{pmatrix} \right|_\infty \\ &> \max_{1 \leq h \leq K} n^{1/2} |\mathfrak{R}_{i+1}^\top \Sigma_X(h) \mathfrak{R}_{i+1}|_\infty = \check{T}_{i+1,n}, \\ \check{G}_{i,n} &= \sum_{1 \leq h \leq K} \sum_j \check{\sigma}_j^2 (\mathfrak{R}_i^\top \Sigma_X(h) \mathfrak{R}_i) = \sum_{1 \leq h \leq K} \sum_j \check{\sigma}_j^2 \left\{ \begin{pmatrix} O_i^\top \Sigma_X(h) O_i & O_i^\top \Sigma_X(h) \mathfrak{R}_{i+1} \\ \mathfrak{R}_{i+1}^\top \Sigma_X(h) O_i & \mathfrak{R}_{i+1}^\top \Sigma_X(h) \mathfrak{R}_{i+1} \end{pmatrix} \right\} \\ &> \sum_{1 \leq h \leq K} \sum_j \check{\sigma}_j^2 (\mathfrak{R}_{i+1}^\top \Sigma_X(h) \mathfrak{R}_{i+1}) = \check{G}_{i+1,n}. \end{aligned}$$

Similarly, $\widehat{T}_{i,n}$ and $\widehat{G}_{i,n}$ can also be shown to be monotonically decreasing with respect to i . □

Lemma 11. *Under Assumption 1, it holds that*

$$\widehat{T}_{i,n} \asymp p^{1-\delta} n^{1/2}, \quad i = 1, \dots, r; \quad \widehat{T}_{i,n} = \mathbf{O}_p(p^{1+\delta} q^{2\omega} n^{-1/2}), \quad i = r+1, \dots, p.$$

Proof of Lemma 11. Similar to the proof of Lemma 5, the proof of Lemma 11 is omitted. □

Lemma 12. *Under Assumption 1, it holds that*

$$\widehat{G}_{i,n} \asymp p^{2-2\delta}, \quad i = 1, \dots, r; \quad \widehat{G}_{i,n} = \mathbf{O}_p(p^2 q^{2\omega} n^{-1}), \quad i = r+1, \dots, p.$$

Proof of Lemma 12. Similar to the proof of Lemma 6, the proof of Lemma 12 is omitted. □

Proof of Theorems 1 and 2. By Lemmas 11 and 12, the proof of Theorems 1 and 2 is obtained. □

Proof of Theorem 3. As the other results are parallel to $P(\widehat{r}_{\text{SR}} = r) \rightarrow 1$ in Theorem 3, the other proof will follow the spirit of $P(\widehat{r}_{\text{SR}} = r) \rightarrow 1$. Thus, we will only give the proof for $P(\widehat{r}_{\text{SR}} = r) \rightarrow 1$. □

On the one hand, in probability, for any $i < r$, as $|\widehat{G}_{i,n} - G_{i,n}| = \mathbf{O}_p(p^{2-\delta} q^{2-\omega} n^{-1/2})$, we have $\widehat{G}_{i,n} \asymp G_{i,n} \asymp p^{2-2\delta} q^{2-2\omega}$. By $\widehat{G}_{r+1,n} = \mathbf{O}_p(p^2 q^2 n^{-1})$, we have $\widehat{G}_{i,n} - \widehat{G}_{i+1,n} \asymp p^{2-2\delta} q^{2-2\omega}$ for any $i \leq r$ and $\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n} = \mathbf{O}_p(p^2 q^2 n^{-1})$. It implies that $\frac{\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n}}{\widehat{G}_{r,n} - \widehat{G}_{r+1,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1})$. Thus, for large n , as $i < r$, we have

$$\frac{\widehat{G}_{i+1,n} - \widehat{G}_{i+2,n}}{\widehat{G}_{i,n} - \widehat{G}_{i+1,n}} \asymp 1 > \frac{\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n}}{\widehat{G}_{r,n} - \widehat{G}_{r+1,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1}) \rightarrow 0$$

in probability. On the other hand, for any j with $r < j$, as $\widehat{G}_{j,n} > 0$, thus, for large n

$$\frac{\widehat{G}_{j+1,n} - \widehat{G}_{j+2,n}}{\widehat{G}_{j,n} - \widehat{G}_{j+1,n}} > 0 \leftarrow \frac{\widehat{G}_{r+1,n} - \widehat{G}_{r+2,n}}{\widehat{G}_{r,n} - \widehat{G}_{r+1,n}} = \mathbf{O}_p(p^{2\delta} q^{2\omega} n^{-1})$$

in probability uniformly over all $r < j$. Hence, putting these results together, we have shown that $\widehat{r}_{\text{SR}} \rightarrow r$ with probability one.

□

3 Simulation Results

References

- Lam, C., Yao, Q. and Bathia, N. (2011). Estimation for latent factor models for high-dimensional time series, *Biometrika* **98**: 901–918.
- Wang, D., Liu, X. and Chen, R. (2019). Factor models for matrix-valued high-dimensional time series, *Journal of Econometrics* **208**: 231–248.

Table 1: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (1) and Σ_1 and Σ_2 with case (1)

(δ, ω)	(0.5, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.435(111 2)	0.705(57 2)	0.410(115 3)	0.420(114 2)	0.710(57 1)	0.460(106 2)
	800	0.370(125 1)	0.700(58 2)	0.400(119 1)	0.400(120 0)	0.640(69 3)	0.410(116 2)
(20,40)	200	0.345(131 0)	0.520(96 0)	0.385(123 0)	0.520(96 0)	0.915(17 0)	0.525(95 0)
	800	0.355(129 0)	0.450(99 1)	0.400(119 1)	0.460(107 1)	0.920(15 1)	0.550(89 1)
(40,40)	200	0.450(109 1)	0.770(46 0)	0.540(92 0)	0.495(111 0)	0.730(54 0)	0.510(98 0)
	800	0.385(123 0)	0.770(46 0)	0.560(87 1)	0.435(113 0)	0.770(46 0)	0.495(101 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.480(100 4)	0.860(23 5)	0.580(83 1)	0.495(91 10)	0.890(19 3)	0.655(68 1)
	800	0.480(99 5)	0.880(21 3)	0.555(87 2)	0.505(99 0)	0.885(23 0)	0.590(81 1)
(20,40)	200	0.510(97 1)	0.800(40 0)	0.570(86 0)	0.585(82 1)	0.970(4 2)	0.735(53 0)
	800	0.480(103 1)	0.720(56 0)	0.530(93 1)	0.505(98 1)	0.975(3 2)	0.730(52 2)
(40,40)	200	0.585(83 0)	0.910(17 1)	0.715(56 1)	0.520(96 0)	0.945(11 0)	0.675(65 0)
	800	0.500(100 0)	0.955(9 0)	0.765(45 2)	0.565(86 1)	0.970(6 0)	0.710(58 0)
(δ, ω)	(0, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.755(49 0)	0.965(6 1)	0.855(27 2)	0.740(52 0)	0.960(8 0)	0.880(24 0)
	800	0.800(40 0)	0.975(5 0)	0.870(25 1)	0.720(55 1)	0.970(5 1)	0.865(25 1)
(20,40)	200	0.770(46 0)	0.910(18 0)	0.865(27 0)	0.820(36 0)	0.990(2 0)	0.875(25 0)
	800	0.800(40 0)	0.940(12 0)	0.845(31 0)	0.785(42 1)	1.000(1 0)	0.900(20 0)
(40,40)	200	0.845(31 0)	0.990(2 0)	0.925(15 0)	0.885(23 0)	0.985(3 0)	0.930(14 0)
	800	0.850(30 0)	0.990(2 0)	0.930(13 0)	0.825(35 0)	0.995(1 0)	0.945(10 1)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.770(44 2)	0.990(2 0)	0.900(17 3)	0.780(42 2)	0.995(1 0)	0.900(19 1)
	800	0.795(36 5)	0.990(1 1)	0.900(17 3)	0.725(50 5)	0.980(3 1)	0.885(22 1)
(20,40)	200	0.800(40 0)	0.965(7 0)	0.905(18 1)	0.810(36 2)	0.995(1 0)	0.870(26 0)
	800	0.810(38 0)	0.985(3 0)	0.855(28 1)	0.790(40 2)	1.000(0 0)	0.890(21 1)
(40,40)	200	0.850(30 0)	1.000(0 0)	0.940(12 0)	0.890(22 0)	1.000(0 0)	0.925(14 1)
	800	0.855(28 1)	0.990(2 0)	0.920(15 1)	0.840(32 0)	0.995(1 0)	0.940(9 3)
(δ, ω)	(0, 0)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.905(19 0)	0.995(1 0)	0.980(4 0)	0.895(21 0)	0.990(2 0)	0.960(8 0)
	800	0.880(23 1)	0.995(1 0)	0.965(7 0)	0.900(20 0)	1.000(0 0)	0.980(4 0)
(20,40)	200	0.965(7 0)	0.990(2 0)	0.975(4 1)	0.935(13 0)	1.000(0 0)	0.930(2 2)
	800	0.940(11 1)	0.985(3 0)	0.980(4 0)	0.910(17 1)	1.000(1 0)	0.940(9 0)
(40,40)	200	0.940(10 2)	1.000(0 0)	0.990(2 0)	0.930(14 0)	0.995(1 0)	0.970(4 2)
	800	0.950(7 0)	1.000(0 0)	0.990(2 0)	0.950(10 0)	1.000(0 0)	0.990(2 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.890(16 6)	0.995(1 0)	0.980(4 0)	0.875(21 4)	0.995(1 0)	0.955(8 1)
	800	0.875(21 4)	1.000(0 0)	0.950(8 2)	0.895(19 2)	1.000(1 0)	0.960(4 0)
(20,40)	200	0.960(7 1)	0.995(1 0)	0.975(4 1)	0.925(12 3)	1.000(0 0)	0.970(4 2)
	800	0.950(10 0)	1.000(0 0)	0.990(2 0)	0.930(14 0)	1.000(0 0)	0.965(4 3)
(40,40)	200	0.950(10 0)	1.000(0 0)	0.990(2 0)	0.930(14 0)	1.000(0 0)	0.965(4 3)
	800	0.965(7 0)	1.000(0 0)	0.990(2 0)	0.950(10 0)	1.000(0 0)	0.990(2 0)

Table 2: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (3) and Σ_1 and Σ_2 with case (1)

(δ, ω)		$(0.5, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.520(95 1)	0.790(41 1)	0.545(90 1)	0.530(94 0)	0.790(42 0)	0.575(85 0)
	800	0.500(99 1)	0.775(45 0)	0.525(95 0)	0.460(106 2)	0.775(44 1)	0.515(96 1)
(20,40)	200	0.495(100 1)	0.680(64 0)	0.555(89 0)	0.525(95 0)	0.945(11 0)	0.720(56 0)
	800	0.535(92 1)	0.710(58 0)	0.550(89 1)	0.525(90 5)	0.955(9 0)	0.775(44 1)
(40,40)	200	0.585(83 0)	0.845(31 0)	0.665(67 0)	0.555(89 0)	0.830(34 0)	0.640(72 0)
	800	0.530(94 0)	0.905(10 0)	0.650(70 0)	0.545(91 0)	0.885(23 0)	0.620(76 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.585(79 4)	0.905(17 2)	0.685(63 0)	0.565(83 4)	0.895(20 1)	0.700(60 0)
	800	0.560(82 6)	0.920(13 3)	0.630(72 2)	0.530(92 2)	0.905(16 3)	0.690(59 3)
(20,40)	200	0.600(79 1)	0.850(30 0)	0.690(62 0)	0.565(87 0)	0.975(4 1)	0.815(37 0)
	800	0.625(73 2)	0.900(20 0)	0.710(57 1)	0.675(62 3)	0.970(6 0)	0.850(29 1)
(40,40)	200	0.655(69 0)	0.955(8 1)	0.780(44 0)	0.605(79 0)	0.925(15 0)	0.785(42 1)
	800	0.590(82 0)	0.975(4 1)	0.810(38 0)	0.615(77 0)	0.965(7 0)	0.740(51 1)
(δ, ω)		$(0, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.850(30 0)	0.965(7 0)	0.890(22 0)	0.830(33 1)	0.985(3 0)	0.885(22 1)
	800	0.800(40 0)	0.975(5 0)	0.820(26 0)	0.815(37 0)	0.990(2 0)	0.935(13 0)
(20,40)	200	0.855(28 1)	0.960(8 0)	0.900(20 0)	0.815(37 0)	1.000(0 0)	0.920(12 0)
	800	0.830(34 0)	0.920(16 0)	0.925(15 0)	0.815(37 0)	1.000(0 0)	0.920(12 0)
(40,40)	200	0.815(37 0)	0.995(1 0)	0.940(12 0)	0.840(32 0)	1.000(0 0)	0.940(12 0)
	800	0.845(31 0)	0.995(1 0)	0.950(10 0)	0.875(25 0)	0.985(3 0)	0.945(11 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.850(26 4)	0.995(1 0)	0.925(15 0)	0.835(31 2)	1.000(0 0)	0.900(18 2)
	800	0.810(36 2)	0.985(3 0)	0.895(20 1)	0.825(34 1)	0.995(0 1)	0.940(11 1)
(20,40)	200	0.865(26 1)	0.980(3 1)	0.925(15 0)	0.825(34 2)	1.000(0 0)	0.910(12 2)
	800	0.830(34 0)	0.980(4 0)	0.955(9 0)	0.825(32 3)	1.000(0 0)	0.915(12 1)
(40,40)	200	0.840(32 0)	0.995(1 0)	0.945(10 1)	0.850(30 0)	1.000(0 0)	0.950(9 1)
	800	0.850(30 0)	1.000(0 0)	0.945(11 0)	0.880(24 0)	1.000(0 0)	0.945(11 0)
(δ, ω)		$(0, 0)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.930(12 0)	1.000(0 0)	0.990(2 0)	0.910(16 2)	1.000(0 0)	0.965(5 2)
	800	0.930(12 2)	1.000(0 0)	0.990(2 0)	0.935(13 0)	1.000(0 0)	0.975(5 0)
(20,40)	200	0.975(5 0)	1.000(0 0)	1.000(0 0)	0.945(11 0)	1.000(0 0)	0.975(5 0)
	800	0.970(6 0)	1.000(0 0)	1.000(0 0)	0.940(10 0)	1.000(0 0)	0.805(39 0)
(40,40)	200	0.960(8 0)	1.000(0 0)	1.000(0 0)	0.980(3 1)	1.000(0 0)	0.995(10)
	800	0.970(6 0)	0.995(1 0)	0.770(46 0)	0.965(7 0)	1.000(0 0)	0.980(2 2)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.930(10 2)	1.000(0 0)	0.995(0 1)	0.905(12 7)	1.000(0 0)	0.970(4 2)
	800	0.930(10 4)	1.000(0 0)	0.990(2 0)	0.930(12 2)	1.000(0 0)	0.975(5 0)
(20,40)	200	0.975(5 0)	1.000(0 0)	1.000(0 0)	0.940(11 1)	1.000(0 0)	0.975(4 1)
	800	0.970(6 0)	1.000(0 0)	1.000(0 0)	0.945(9 2)	1.000(0 0)	0.980(1 3)
(40,40)	200	0.960(8 0)	1.000(0 0)	1.000(0 0)	0.980(3 1)	1.000(0 0)	0.990(0 2)
	800	0.970(6 0)	1.000(0 0)	1.000(0 0)	0.965(6 1)	1.000(0 0)	0.985(1 2)

Table 3: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (4) and Σ_1 and Σ_2 with case (1)

(δ, ω)		$(0.5, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.825(35 0)	0.960(8 0)	0.915(17 0)	0.795(41 0)	0.960(8 0)	0.925(15 0)
	800	0.820(36 0)	0.985(3 0)	0.960(8 0)	0.850(30 0)	0.995(1 0)	0.935(13 0)
(20,40)	200	0.840(32 0)	0.965(7 0)	0.950(10 0)	0.800(40 0)	0.990(2 0)	0.935(13 0)
	800	0.840(32 0)	0.975(5 0)	0.955(9 0)	0.805(39 0)	0.990(2 0)	0.945(11 0)
(40,40)	200	0.810(38 0)	0.980(4 0)	0.970(6 0)	0.855(29 0)	0.975(5 0)	0.940(12 05)
	800	0.855(29 0)	0.990(2 0)	0.965(7 0)	0.805(39 0)	0.990(2 0)	0.935(13 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.825(32 3)	1.000(0 0)	0.940(11 1)	0.810(37 1)	0.980(4 0)	0.955(9 0)
	800	0.835(33 0)	1.000(3 0)	0.975(5 0)	0.860(26 2)	0.995(1 0)	0.960(8 0)
(20,40)	200	0.840(32 0)	0.985(3 0)	0.970(6 0)	0.800(39 1)	0.990(2 0)	0.930(13 1)
	800	0.845(31 0)	0.995(1 0)	0.975(5 0)	0.810(38 0)	0.995(0 1)	0.940(12 0)
(40,40)	200	0.825(35 0)	0.985(3 0)	0.975(5 0)	0.865(27 0)	0.990(2 0)	0.940(12 05)
	800	0.870(26 0)	1.000(0 0)	0.965(6 1)	0.825(34 1)	0.990(2 0)	0.935(12 1)
(δ, ω)		$(0, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.940(12 0)	1.000(0 0)	0.985(3 0)	0.915(17 0)	1.000(0 0)	0.995(1 0)
	800	0.960(8 0)	0.990(2 0)	0.980(3 1)	0.940(12 0)	1.000(0 0)	0.990(2 0)
(20,40)	200	0.955(9 0)	0.985(3 0)	0.970(6 0)	0.915(17 0)	1.000(0 0)	0.970(4 2)
	800	0.955(9 0)	0.985(3 0)	0.990(2 0)	0.950(10 0)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.985(3 0)	1.000(0 0)	0.995(1 0)	0.970(6 0)	1.000(0 0)	0.985(1 2)
	800	0.970(6 0)	1.000(0 0)	0.990(1 1)	0.955(9 0)	1.000(0 0)	0.990(2 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.935(11 2)	1.000(0 0)	0.985(2 1)	0.915(16 1)	1.000(0 0)	0.990(1 1)
	800	0.950(7 3)	0.995(1 0)	0.980(2 2)	0.930(11 3)	1.000(0 0)	0.990(2 0)
(20,40)	200	0.960(7 1)	0.990(3 0)	0.975(5 0)	0.910(16 2)	1.000(0 0)	0.965(5 2)
	800	0.965(7 0)	0.990(2 0)	0.990(1 1)	0.950(10 0)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.985(3 0)	1.000(0 0)	0.995(1 0)	0.970(6 0)	1.000(0 0)	0.980(3 15)
	800	0.975(5 0)	1.000(0 0)	0.995(0 0)	0.965(7 0)	1.000(0 0)	0.985(1 2)
(δ, ω)		$(0, 0)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.985(2 1)	1.000(0 0)	1.000(1 0)	0.980(4 0)	0.995(1 0)	0.985(2 1)
	800	0.990(2 0)	0.995(1 0)	0.995(1 0)	0.990(1 3)	1.000(0 0)	1.000(0 0)
(20,40)	200	0.985(2 1)	1.000(0 0)	0.995(1 0)	0.980(4 0)	1.000(0 0)	1.000(0 0)
	800	0.990(2 0)	1.000(1 0)	0.995(1 0)	0.990(2 0)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.980(3 1)	1.000(0 0)	1.000(0 0)	0.990(2 0)	1.000(0 0)	1.000(0 0)
	800	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.980(2 2)	1.000(0 0)	1.000(0 0)	0.980(4 0)	0.995(1 0)	0.985(2 1)
	800	0.985(2 3)	0.995(1 0)	0.995(1 0)	0.980(1 3)	1.000(0 0)	1.000(0 0)
(20,40)	200	0.985(2 1)	1.000(0 0)	1.000(0 0)	0.970(4 2)	1.000(0 0)	1.000(0 0)
	800	0.985(1 2)	1.000(0 0)	0.995(1 0)	0.985(2 1)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.975(3 2)	0.975(5 0)	0.790(42 0)	0.990(1 1)	1.000(0 0)	1.000(0 0)
	800	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)	1.000(0 0)

Table 4: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (1) and Σ_1 and Σ_2 with case (2)

(δ, ω)	(0.5, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.360(128 0)	0.640(70 2)	0.415(114 3)	0.340(130 2)	0.640(70 2)	0.370(124 2)
	800	0.445(110 1)	0.665(84 3)	0.475(103 2)	0.465(106 1)	0.685(62 1)	0.445(111 0)
(20,40)	200	0.355(170 1)	0.480(103 1)	0.345(131 0)	0.470(106 0)	0.910(18 0)	0.580(83 1)
	800	0.420(114 2)	0.515(97 0)	0.380(123 1)	0.465(107 0)	0.860(26 2)	0.570(86 0)
(40,40)	200	0.430(113 1)	0.770(46 0)	0.535(92 1)	0.390(122 0)	0.790(42 0)	0.565(87 0)
	800	0.485(103 0)	0.760(48 0)	0.520(96 0)	0.415(117 0)	0.755(49 0)	0.500(100 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.445(109 2)	0.885(23 0)	0.570(85 1)	0.410(112 6)	0.800(35 5)	0.520(94 2)
	800	0.545(88 3)	0.875(21 4)	0.560(84 4)	0.495(96 5)	0.895(19 2)	0.630(71 2)
(20,40)	200	0.555(87 2)	0.825(34 1)	0.570(86 0)	0.570(86 0)	0.970(6 0)	0.745(50 1)
	800	0.545(91 0)	0.855(28 1)	0.595(81 0)	0.545(88 3)	0.955(5 4)	0.715(56 1)
(40,40)	200	0.520(95 1)	0.935(13 0)	0.720(56 0)	0.510(98 0)	0.940(11 1)	0.690(61 15)
	800	0.575(85 0)	0.970(5 1)	0.705(59 0)	0.485(102 1)	0.965(7 0)	0.690(62 0)
(δ, ω)	(0, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.750(48 2)	0.950(9 0)	0.845(29 2)	0.795(40 1)	0.980(4 0)	0.890(22 0)
	800	0.770(44 2)	0.965(7 0)	0.875(25 0)	0.805(38 1)	0.985(3 0)	0.875(24 1)
(20,40)	200	0.770(46 0)	0.925(15 0)	0.895(21 0)	0.720(56 0)	1.000(0 0)	0.885(23 0)
	800	0.750(50 0)	0.920(16 0)	0.870(26 0)	0.735(53 0)	1.000(0 0)	0.915(17 0)
(40,40)	200	0.830(34 0)	0.990(2 0)	0.920(16 0)	0.820(36 0)	0.995(1 0)	0.930(14 05)
	800	0.840(32 0)	0.990(2 0)	0.935(13 0)	0.820(36 0)	0.990(2 0)	0.920(16 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.755(45 4)	0.980(2 2)	0.895(17 4)	0.800(39 1)	1.000(0 0)	0.910(16 2)
	800	0.775(43 2)	0.990(2 0)	0.900(19 1)	0.815(35 2)	0.990(2 0)	0.900(19 1)
(20,40)	200	0.790(42 0)	0.980(4 0)	0.890(22 0)	0.715(57 0)	1.000(0 0)	0.905(17 2)
	800	0.755(49 0)	0.970(6 0)	0.895(21 0)	0.755(46 3)	1.000(0 0)	0.910(18 0)
(40,40)	200	0.830(34 0)	1.000(0 0)	0.960(16 0)	0.825(35 0)	1.000(0 0)	0.930(13 1)
	800	0.855(29 0)	1.000(0 0)	0.940(12 0)	0.840(32 0)	0.995(1 0)	0.925(15 0)
(δ, ω)	(0, 0)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.910(18 0)	1.000(0 0)	0.985(3 0)	0.910(18 0)	1.000(0 0)	0.980(2 0)
	800	0.900(19 1)	0.990(2 0)	0.970(6 0)	0.905(19 0)	0.985(3 0)	0.975(5 0)
(20,40)	200	0.965(7 0)	1.000(0 0)	0.985(3 0)	0.910(18 0)	1.000(0 0)	0.970(6 0)
	800	0.940(12 0)	1.000(0 0)	0.985(3 0)	0.915(16 1)	1.000(0 0)	0.965(5 2)
(40,40)	200	0.925(15 0)	1.000(0 0)	0.965(6 1)	0.955(9 0)	1.000(0 0)	0.985(3 0)
	800	0.960(8 0)	1.000(0 0)	0.985(3 0)	0.935(13 0)	1.000(0 0)	0.970(4 2)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.910(13 5)	1.000(0 0)	0.990(2 0)	0.910(17 1)	1.000(0 0)	0.980(2 0)
	800	0.900(18 1)	0.990(2 0)	0.970(5 1)	0.900(18 2)	0.995(1 0)	0.975(5 0)
(20,40)	200	0.965(6 1)	1.000(0 0)	0.990(2 0)	0.910(17 1)	1.000(0 0)	0.965(5 2)
	800	0.945(10 1)	1.000(0 0)	0.995(1 0)	0.915(15 2)	1.000(0 0)	0.955(5 4)
(40,40)	200	0.925(15 0)	1.000(0 0)	0.965(6 1)	0.955(9 0)	1.000(0 0)	0.990(2 0)
	800	0.960(7 1)	1.000(0 0)	0.995(1 0)	0.935(13 0)	1.000(0 0)	0.965(3 4)

Table 5: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (3) and Σ_1 and Σ_2 with case (2)

(δ, ω)	(0.5, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.500(97 3)	0.670(66 0)	0.535(91 2)	0.400(118 2)	0.680(63 1)	0.540(91 1)
	800	0.525(95 0)	0.690(55 7)	0.500(97 3)	0.515(96 1)	0.725(54 1)	0.510(97 1)
(20,40)	200	0.415(114 3)	0.540(90 2)	0.510(98 0)	0.510(98 0)	0.830(31 3)	0.630(73 1)
	800	0.470(106 0)	0.535(92 1)	0.485(103 0)	0.550(89 1)	0.770(39 7)	0.570(84 2)
(40,40)	200	0.490(102 0)	0.490(96 6)	0.460(105 3)	0.480(104 0)	0.545(88 3)	0.555(87 2)
	800	0.500(100 0)	0.565(78 9)	0.585(81 2)	0.450(110 0)	0.485(91 12)	0.500(98 2)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.555(85 4)	0.870(26 0)	0.620(76 0)	0.535(91 2)	0.905(18 1)	0.635(69 4)
	800	0.530(91 3)	0.840(29 3)	0.675(63 2)	0.585(81 2)	0.880(22 2)	0.650(67 3)
(20,40)	200	0.500(99 0)	0.750(49 1)	0.615(76 1)	0.620(74 2)	0.985(1 2)	0.790(41 1)
	800	0.610(78 0)	0.795(41 0)	0.610(78 0)	0.675(65 0)	0.960(0 1)	0.755(47 2)
(40,40)	200	0.550(90 0)	0.830(31 3)	0.660(68 0)	0.575(85 0)	0.840(30 2)	0.705(58 1)
	800	0.655(69 0)	0.905(19 0)	0.735(51 2)	0.565(86 1)	0.875(23 2)	0.670(65 1)
(δ, ω)	(0, 0.5)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.790(41 1)	0.975(5 0)	0.895(21 0)	0.815(37 0)	0.985(3 0)	0.915(17 0)
	800	0.795(41 0)	0.970(5 1)	0.890(21 1)	0.840(30 0)	0.950(10 0)	0.910(18 0)
(20,40)	200	0.880(24 0)	0.940(12 0)	0.895(21 0)	0.785(42 1)	0.995(0 1)	0.915(15 2)
	800	0.830(34 0)	0.955(9 0)	0.915(17 0)	0.790(42 0)	0.990(2 0)	0.895(21 0)
(40,40)	200	0.855(29 0)	1.000(0 0)	0.995(1 0)	0.850(30 0)	0.990(2 0)	0.910(18 0)
	800	0.890(22 0)	0.980(4 0)	0.945(11 0)	0.855(29 0)	1.000(0 0)	0.925(15 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.785(42 1)	0.995(1 0)	0.900(18 2)	0.830(34 0)	0.995(1 0)	0.930(13 1)
	800	0.795(41 0)	0.985(3 0)	0.910(17 1)	0.845(28 3)	0.975(5 0)	0.925(15 0)
(20,40)	200	0.885(23 0)	0.970(6 0)	0.910(18 0)	0.780(43 1)	0.995(0 1)	0.925(13 2)
	800	0.850(30 0)	0.975(5 0)	0.935(13 0)	0.810(37 1)	0.995(1 0)	0.910(18 0)
(40,40)	200	0.855(28 1)	1.000(0 0)	0.995(1 0)	0.850(30 0)	0.995(1 0)	0.925(15 0)
	800	0.900(20 0)	1.000(0 0)	0.950(10 0)	0.870(26 0)	1.000(0 0)	0.920(14 2)
(δ, ω)	(0, 0)						
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.935(13 0)	1.000(0 0)	0.970(5 1)	0.920(16 0)	0.990(2 0)	0.970(5 1)
	800	0.905(18 1)	1.000(0 0)	0.985(2 1)	0.950(8 2)	0.990(2 0)	0.980(4 0)
(20,40)	200	0.970(6 0)	1.000(0 0)	0.995(1 0)	0.950(8 2)	1.000(0 0)	0.990(1 1)
	800	0.975(5 0)	0.995(1 0)	0.995(1 0)	0.950(9 1)	1.000(0 0)	0.990(2 0)
(40,40)	200	0.955(9 0)	1.000(0 0)	0.980(2 0)	0.970(6 0)	1.000(0 0)	0.995(1 0)
	800	0.965(6 0)	1.000(0 0)	1.000(0 0)	0.960(8 0)	1.000(0 0)	0.990(1 1)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.930(11 3)	1.000(0 0)	0.970(5 1)	0.920(14 2)	1.000(0 0)	0.970(5 1)
	800	0.905(17 2)	1.000(0 0)	0.985(2 1)	0.950(7 3)	1.000(0 0)	0.980(4 0)
(20,40)	200	0.970(5 1)	1.000(0 0)	0.995(1 0)	0.950(8 2)	1.000(0 0)	0.995(0 1)
	800	0.975(5 0)	0.995(1 0)	0.995(1 0)	0.950(8 2)	1.000(0 0)	0.990(2 0)
(40,40)	200	0.950(9 0)	1.000(0 0)	0.985(1 2)	0.970(6 0)	1.000(0 0)	0.995(1 0)
	800	0.965(6 0)	1.000(0 0)	1.000(0 0)	0.960(8 0)	1.000(0 0)	0.990(1 1)

Table 6: Relative frequency estimates for $P(\hat{r} = 3)$ and $P(\hat{c} = 3)$ with 200 replicate samples for the AR coefficients with case (4) and Σ_1 and Σ_2 with case (2)

(δ, ω)		$(0.5, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.820(35 1)	0.975(5 0)	0.940(12 0)	0.785(42 1)	0.965(7 0)	0.930(14 0)
	800	0.845(31 0)	0.990(12 0)	0.935(13 0)	0.770(46 0)	0.950(10 0)	0.935(13 0)
(20,40)	200	0.810(38 0)	0.925(15 0)	0.925(15 0)	0.825(35 0)	0.990(2 0)	0.905(19 0)
	800	0.850(30 0)	0.950(10 0)	0.925(15 0)	0.825(35 0)	0.975(5 0)	0.930(14 0)
(40,40)	200	0.845(31 0)	0.970(6 0)	0.895(21 0)	0.825(35 0)	0.975(5 0)	0.915(16 1)
	800	0.825(35 0)	0.985(3 0)	0.905(19 0)	0.825(35 0)	0.995(1 0)	0.935(13 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.820(33 3)	0.990(2 0)	0.950(8 2)	0.810(37 1)	0.985(2 1)	0.930(14 0)
	800	0.850(30 0)	0.995(1 0)	0.950(10 0)	0.785(43 0)	0.975(5 0)	0.940(12 0)
(20,40)	200	0.820(36 0)	0.965(7 0)	0.950(10 0)	0.835(32 1)	0.995(1 0)	0.910(16 2)
	800	0.855(29 0)	0.975(5 0)	0.955(9 0)	0.830(32 2)	1.000(0 0)	0.930(14 0)
(40,40)	200	0.845(31 0)	0.985(3 0)	0.910(18 0)	0.840(32 0)	0.995(1 0)	0.945(10 1)
	800	0.835(33 0)	1.000(3 0)	0.920(16 0)	0.860(26 2)	0.995(1 0)	0.940(11 1)
(δ, ω)		$(0, 0.5)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.935(12 1)	0.995(1 0)	0.990(2 0)	0.940(12 0)	1.000(0 0)	0.990(2 0)
	800	0.970(5 1)	1.000(0 0)	0.985(3 0)	0.945(11 0)	0.990(1 1)	0.985(2 1)
(20,40)	200	0.940(12 0)	0.990(2 0)	0.985(3 0)	0.900(19 1)	1.000(0 0)	0.975(4 1)
	800	0.960(8 0)	0.995(1 0)	0.990(2 0)	0.950(10 0)	1.000(0 0)	0.980(3 1)
(40,40)	200	0.940(11 1)	0.995(1 0)	0.985(2 1)	0.950(10 0)	0.990(2 0)	0.985(3 0)
	800	0.955(9 0)	0.995(0 0)	0.990(2 0)	0.955(9 0)	0.995(1 0)	0.990(2 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.930(12 2)	0.995(1 0)	0.990(2 0)	0.940(9 3)	1.000(0 0)	0.990(2 0)
	800	0.975(4 1)	1.000(0 0)	0.985(3 0)	0.940(10 2)	0.995(0 1)	0.985(2 1)
(20,40)	200	0.945(11 0)	0.995(1 0)	0.990(1 0)	0.910(16 2)	1.000(0 0)	0.970(3 3)
	800	0.965(7 0)	1.000(0 0)	0.990(1 1)	0.950(10 0)	1.000(0 0)	0.980(2 2)
(40,40)	200	0.940(11 1)	1.000(0 0)	0.985(2 1)	0.950(10 0)	1.000(0 0)	0.980(2 2)
	800	0.975(5 0)	1.000(0 0)	0.990(2 0)	0.955(9 0)	0.995(1 0)	0.990(2 0)
(δ, ω)		$(0, 0)$					
(p, q)	n	\hat{r}_{ER_o}	\hat{r}_{SR_o}	\hat{r}_{MR_o}	\hat{c}_{ER_o}	\hat{c}_{SR_o}	\hat{c}_{MR_o}
(20,20)	200	0.985(2 1)	0.995(1 0)	0.990(2 0)	0.980(4 0)	1.000(0 0)	1.000(0 0)
	800	0.970(5 1)	0.995(1 0)	0.995(1 0)	0.970(5 1)	0.995(1 0)	0.995(1 0)
(20,40)	200	0.985(2 1)	1.000(0 0)	1.000(0 0)	0.985(2 1)	1.000(0 0)	0.990(1 1)
	800	0.990(2 0)	1.000(1 0)	0.995(1 0)	0.975(1 2)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.995(1 0)	1.000(0 0)	1.000(0 0)	0.990(2 0)	1.000(0 0)	0.995(0 1)
	800	0.990(2 0)	1.000(0 0)	1.000(0 0)	0.995(0 0)	1.000(0 0)	1.000(0 0)
(p, q)	n	\hat{r}_{ER}	\hat{r}_{SR}	\hat{r}_{MR}	\hat{c}_{ER}	\hat{c}_{SR}	\hat{c}_{MR}
(20,20)	200	0.985(1 2)	0.995(1 0)	0.990(2 0)	0.980(4 0)	1.000(0 0)	1.000(0 0)
	800	0.975(1 2)	1.000(0 0)	1.000(0 0)	0.970(4 2)	0.995(1 0)	0.995(1 0)
(20,40)	200	0.985(2 1)	1.000(0 0)	1.000(0 0)	0.985(2 3)	1.000(0 0)	0.995(0 1)
	800	0.990(1 1)	1.000(0 0)	0.995(1 0)	0.970(1 5)	1.000(0 0)	1.000(0 0)
(40,40)	200	0.995(1 0)	1.000(0 0)	1.000(0 0)	0.990(2 0)	1.000(0 0)	1.000(0 0)
	800	0.995(1 0)	1.000(0 0)	1.000(0 0)	0.995(1 0)	1.000(0 0)	1.000(0 0)