Appendix

Proof of Corollary 2

Letting $\xi = \mu/(1-\mu)$, where $\mu(x) = \mathbb{P}[Y = 1 \mid X = x, R_1 = R_2 = 1]$, denoting $dP_{r_1 r_2}(x) = d\mathbb{P}(x \mid R_1 = r_1, R_2 = r)$, and omitting function arguments for simplicity, note that

$$\begin{aligned}\int \varphi(\overline{\mathbb{P}})\, d\mathbb{P} &= \int \left\{ \frac{\rho}{(\rho+\overline{\xi})^2}\Big(1+\overline{\xi}\Big)^2 \frac{R_1 R_2 \overline{\varpi}}{\overline{\mathbb{E}}(R_1 R_2)}\Big(Y - \overline{\mu}\Big) + \frac{(1-R_1)R_2}{\overline{\mathbb{E}}((1-R_1)R_2)}\left(\frac{\overline{\xi}}{\rho+\overline{\xi}} - \overline{\theta}\right)\right\} \, d\mathbb{P} \\ &= \frac{\mathbb{E}(R_1R_2)}{\overline{\mathbb{E}}(R_1R_2)} \int \frac{\rho}{(\rho+\overline{\xi})^2}\Big(1+\overline{\xi}\Big)^2 \Big(\mu - \overline{\mu}\Big)\overline{\varpi}\, dP_{11} + \frac{\mathbb{E}((1-R_1)R_2)}{\overline{\mathbb{E}}((1-R_1)R_2)} \int \left(\frac{\overline{\xi}}{\rho+\overline{\xi}} - \overline{\theta}\right) \, dP_{01}.\end{aligned}$$

Therefore rearranging shows that the remainder $R_\theta(\overline{\mathbb{P}}; \mathbb{P})$ equals

$$\frac{\mathbb{E}(R_1R_2)}{\overline{\mathbb{E}}(R_1R_2)} \int \frac{\rho}{(\rho+\overline{\xi})^2}\Big(1+\overline{\xi}\Big)^2 \Big(\mu - \overline{\mu}\Big)\overline{\varpi}\, dP_{11} + \frac{\mathbb{E}((1-R_1)R_2)}{\overline{\mathbb{E}}((1-R_1)R_2)} \int \left(\frac{\overline{\xi}}{\rho+\overline{\xi}} - \theta\right) \, dP_{01} \tag{4}$$

$$+ \left\{ 1 - \frac{\mathbb{E}((1-R_1)R_2)}{\overline{\mathbb{E}}((1-R_1)R_2)} \right\} \Big(\overline{\theta} - \theta\Big). \tag{5}$$

The term (5) above is second-order. Therefore consider term (4).

By the mean value theorem we have, for some $\overline{\xi}'$ between ξ and $\overline{\xi}$, that

$$\begin{aligned}(\mu - \overline{\mu})(1+\overline{\xi})^2 &= (\mu - \overline{\mu})(1+\overline{\xi}')^2 + (\mu - \overline{\mu})\Big\{(1+\overline{\xi})^2 - (1+\overline{\xi}')^2\Big\} \\ &= (\xi - \overline{\xi}) + (\mu - \overline{\mu})\Big\{(1+\overline{\xi})^2 - (1+\overline{\xi}')^2\Big\}\end{aligned}$$

since for $\xi = f(\mu) = \frac{\mu}{1-\mu}$ we have $f'(\mu) = 1/(1-\mu)^2 = (1+\xi)^2$. Call the second term in the last line above $S_{2f}(x)$. Similarly we have for some other $\overline{\xi}''$ between ξ and $\overline{\xi}$, that

$$\begin{aligned}(\xi - \overline{\xi})\frac{\rho}{(\rho+\overline{\xi})^2} &= (\xi - \overline{\xi})\frac{\rho}{(\rho+\overline{\xi}'')^2} + (\xi - \overline{\xi})\left\{\frac{\rho}{(\rho+\overline{\xi})^2} - \frac{\rho}{(\rho+\overline{\xi}'')^2}\right\} \\ &= \left(\frac{\xi}{\rho+\xi} - \frac{\overline{\xi}}{\rho+\overline{\xi}}\right) + (\xi - \overline{\xi})\left\{\frac{\rho}{(\rho+\overline{\xi})^2} - \frac{\rho}{(\rho+\overline{\xi}'')^2}\right\}\end{aligned}$$

since for $g(\xi) = \frac{\xi}{\rho+\xi}$ we have $g'(\xi) = \frac{\rho}{(\rho+\xi)^2}$. Call the second term in the last line above $S_{2g}(x)$.

Therefore for the first term in (4) we have

$$\begin{aligned}\int \frac{\rho}{(\rho+\overline{\xi})^2}\Big(1+\overline{\xi}\Big)^2\Big(\mu - \overline{\mu}\Big)\overline{\varpi}\, dP_{11} &= \int \frac{\rho}{(\rho+\overline{\xi})^2}\Big(\xi - \overline{\xi} + S_{2f}\Big)\overline{\varpi}\, dP_{11} \\ &= \int \left\{ \left(\frac{\xi}{\rho+\xi} - \frac{\overline{\xi}}{\rho+\overline{\xi}}\right) + \left(S_{2g} + \frac{S_{2f}\rho}{(\rho+\overline{\xi})^2}\right)\right\}\overline{\varpi}\, dP_{11}.\end{aligned}$$

The second term involving (S_{2g}, S_{2f}) is second-order. For the first term, note

$$\int \left(\frac{\xi}{\rho+\xi} - \frac{\overline{\xi}}{\rho+\overline{\xi}}\right)\overline{\varpi}\, dP_{11} = \int \left(\frac{\xi}{\rho+\xi} - \frac{\overline{\xi}}{\rho+\overline{\xi}}\right)\Big(\overline{\varpi} - \varpi\Big)\, dP_{11} - \int \left(\frac{\overline{\xi}}{\rho+\overline{\xi}} - \frac{\xi}{\rho+\xi}\right)\, dP_{01}.$$

Putting all of the above together gives the result.