Frequency Domain Resampling for Gridded Spatial Data

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Supplementary Material

Abstract

These supplementary materials consist of additional numerical results as well as further mathematical proofs, which are presented in Sections S1 and S2, respectively. In particular, Section S1 contains further simulation evidence regarding the performance of spatial bootstrap methods (FDWB/HFDB) for spectral inference (Section S1.1), additional numerical details on tests of spatial isotropy (Section S1.2), as well as simulation studies regarding kernel bandwidths for bootstrap methods (Section S1.3). Section S2 provides proofs of technical lemmas needed to justify the proposed bootstrap procedure (HFDB) of the main manuscript.

Key words and phrases: Spatial Frequency Domain Bootstrap, Spectral Mean, Periodogram.

S1 Numerical Studies

To provide additional numerical evidence in conjunction with the simulation studies presented in Section 5 of the main manuscript, here we further investigate the performance of the FDWB and HFDB methods for inferring spectral means for both Gaussian and non-Gaussian spatial processes, where non-Gaussianity is introduced through transformations. This comparison aims to highlight the coverage accuracy of the methods under different settings.

S1.1 Confidence intervals for spatial covariance

Continuing with the example in Section 5.1, we consider a real-valued, mean-zero weakly stationary spatial process with a Gaussian covariance function. The parametric form of the covariance function is given by $\gamma(\mathbf{h}) = \theta^2 \exp\left(-(\|\mathbf{h}\|/\phi)^2\right)$, where θ^2 is the partial sill parameter and ϕ is the range parameter. For the simulations, we set $\theta^2 = 1$ and $\phi = 2$, with no nugget effect. We consider rectangular spatial datasets of sizes 30×30 $(n = 900)$, 50×50 $(n = 2500)$, and 70×70 $(n = 4900)$. We report results for nominal 90% confidence intervals, where bootstrap intervals have equi-tailed form $[\widehat{M}_n(\psi) - \widehat{q}_{0.95} n^{-1/2}, \widehat{M}_n(\psi) - \widehat{q}_{0.05} n^{-1/2}]$ based on quantiles $\widehat{q}_{0.05}$ and $\widehat{q}_{0.95}$ of bootstrap distributions approximated by 500 bootstrap draws for each data set.

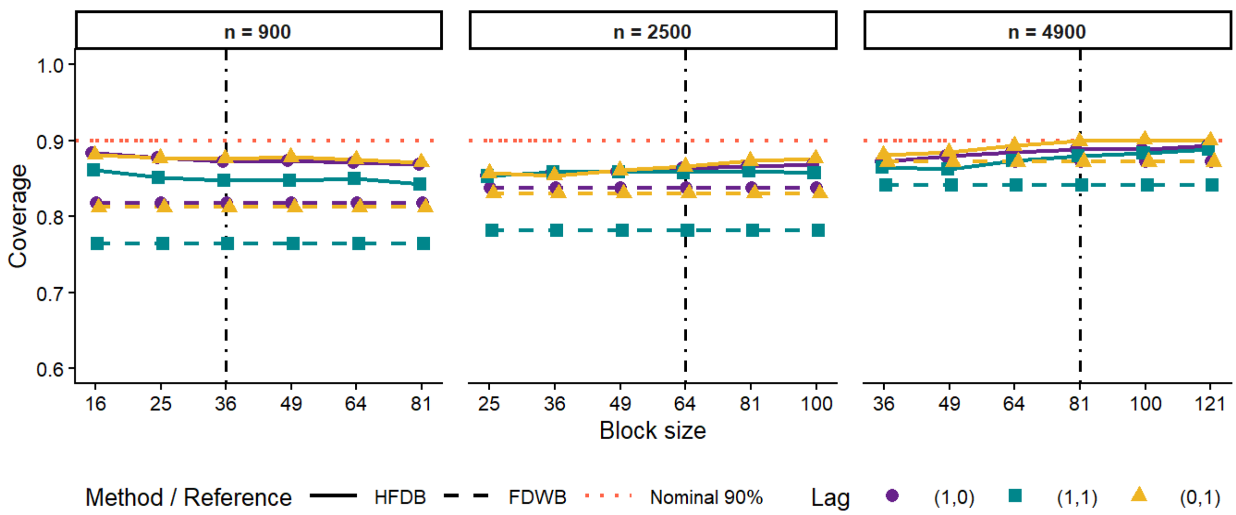

Figure 1: Empirical coverages of 90% HFDB/FDWB intervals for the covariance parameter $\gamma(\mathbf{h})$, based on different subsample sizes b_n, lags $\mathbf{h}$, and sample sizes n. The vertical line indicates a rule of thumb block choice.

Figure 1 presents the coverage accuracy of 90% two-tailed confidence intervals for the covariance parameter. HFDB performs better compared to FDWB overall. For small sample sizes, both methods show some undercoverage, especially FDWB. As the spatial sample size n increases, their coverage improves and gets closer to the nominal level. However, for $\mathbf{h} = (1,1)^\top$, FDWB still shows some undercoverage even at large sample sizes, while HFDB reaches the nominal level. Overall, HFDB and FDWB behave similarly for this Gaussian process (where $\sigma_2 = 0$ holds in the sampling distributions targeted by bootstrap), but HFDB performs slightly better. This behavior is consistent with the fact that the scaling correction in HFDB becomes most relevant only when σ_2^2 in σ^2 is non-zero. The coverage performance of HFDB is relatively stable across block sizes, and the vertical lines in Figure 1 indicate the coverages corresponding to the block rule described in Section 4.1 (with $C = 1$ for illustration), which appear reasonable.

Subsequently, we present a further comparison of HFDB and the normal approximation for confidence interval construction for the same Gaussian process using the same target parameter. As discussed in Section 3.1, we can estimate the variance of spectral mean using Eq. 3.3 via subsampling variance estimator and create confidence intervals using normal approximations. For comparison, we construct 95% confidence intervals using the normal approximation, together with a normal quantile $\mathrm{z}_{0.975} = 1.96$, yielding intervals of the form $\widehat{M}_n(\psi) \pm \mathrm{z}_{0.975} \times \widehat{\sigma}_n n^{-1/2}$.

From Figure 2, we observe that HFDB consistently outperforms normal approximation across block sizes and sample sizes, even in the case of a Gaussian process. HFDB maintains close to the nominal coverage for all lags $\mathbf{h}$ and across all sample sizes n. In contrast, the normal approximation generally undercovers, especially for smaller sample sizes, though these normal intervals improve and gradually approach the nominal level as the block size and sample size increases. The coverage performance of HFDB is relatively stable across block sizes, and the vertical lines in Figure 2 indicate the coverages corresponding to the block rule described in Section 4.1 (with $C = 1$ for illustration).

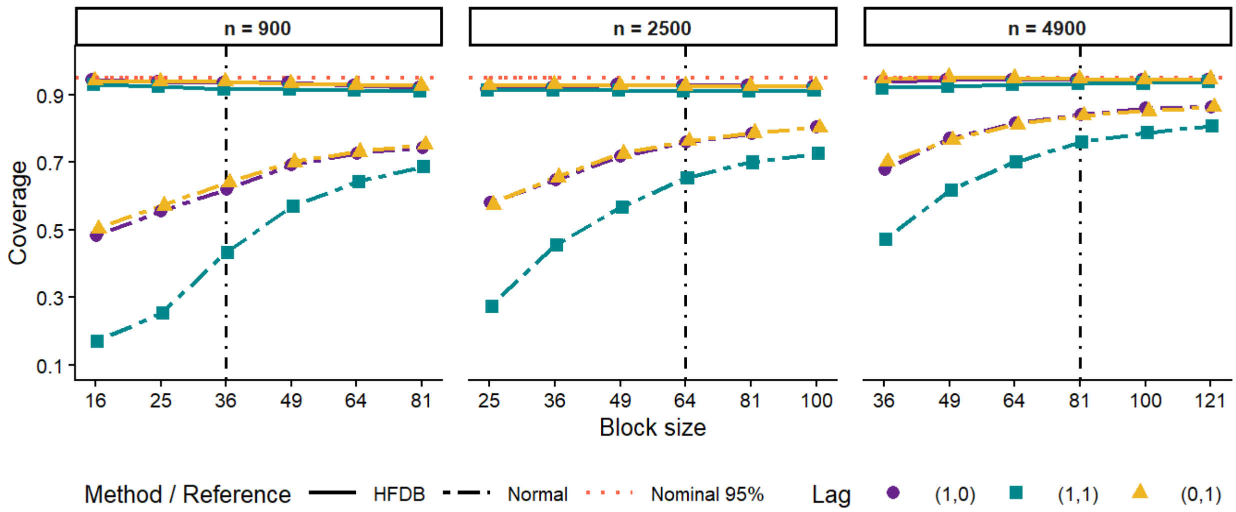

Figure 2: For covariance parameters $\gamma(\mathbf{h})$ at different lags $\mathbf{h}$, coverages of 95% HFDB/normal approximation intervals over various sample sizes n; both HFDB/normal coverages are a function of (square) block size b_n, where the vertical line indicates a rule of thumb block choice.

As for another non-Gaussian example, we consider a nonlinear transformation of a stationary Gaussian spatial field. Let $\{X(\mathbf{s}) : \mathbf{s} \in \mathbb{Z}^2\}$ be a mean-zero, variance-one stationary Gaussian process with Matérn-type covariance function (cf. Stein, 1999). We generate the non-Gaussian field $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{Z}^2\}$ according to

$$Z(\mathbf{s}) = X(\mathbf{s}) + 0.3\{X(\mathbf{s})^2 - 1\}, \quad \mathbf{s} \in \mathbb{Z}^2.$$

For the Gaussian field, we set $\nu = 1$, $\alpha = 1/7$, and $\phi = 0.007$ to ensure that the process variance is close to 1. This model is reasonable, though challenging, for our study because it preserves spatial stationarity while introducing non-Gaussian behavior through a nonlinear transformation. In particular, the process exhibits nonzero fourth-order cumulants, making the second variance component σ_2^2 relevant, when trying to provide interval estimators of covariance parameters $\gamma(\mathbf{h}) \equiv \mathrm{Cov}\{Z(\mathbf{s}), Z(\mathbf{s}+\mathbf{h})\}$ over different lags $\mathbf{h}$. Using the same simulation setup for a Gaussian process $X(\mathbf{s})$ as described in Section 5 of the main manuscript, we examine coverages of 95% FDWB and HFDB intervals over different spatial sample sizes $30 \times 30, 50 \times 50$, and 70×70. and for the three lags $\mathbf{h} = (1,0)^\top$, $(1,1)^\top$, and $(2,0)^\top$.

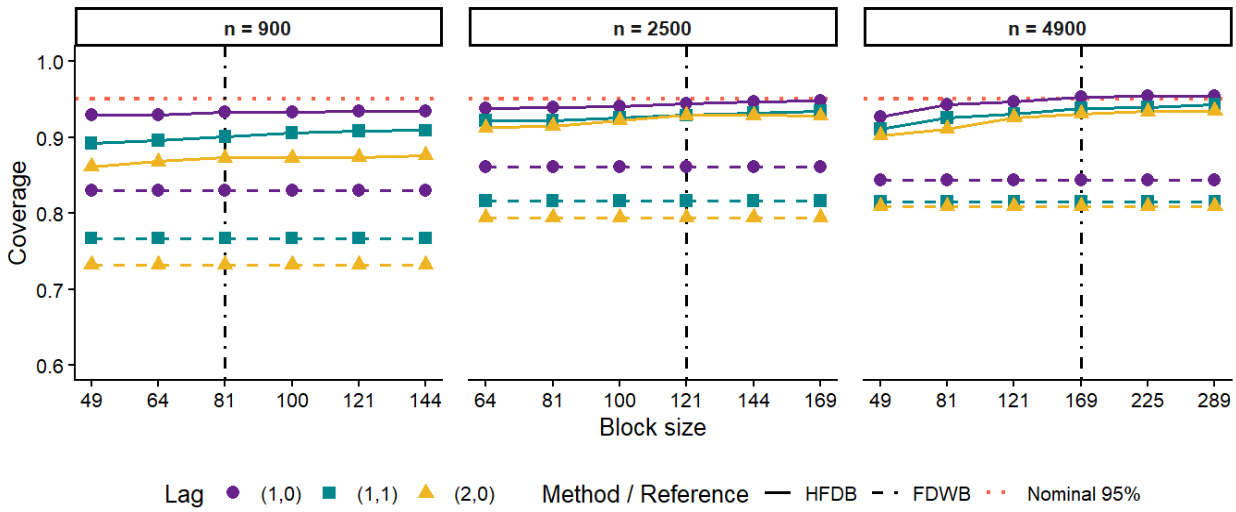

Figure 3: Empirical coverages of 95% HFDB/FDWB intervals for the covariance parameter $\gamma(\mathbf{h})$, based on different subsample sizes b_n, lags $\mathbf{h}$, and sample sizes n. The vertical line indicates a rule of thumb block choice.

We observe from the coverage results shown in Figure 3 that FDWB underperforms across all three lags, performing worse at higher lags, with only marginal improvement as the sample size increases. In contrast, HFDB achieves better coverage, with performance gradually improving toward the nominal level as the sample size increases for all three lags. The coverage performance of HFDB is relatively stable across block sizes, and the vertical lines in Figure 3 indicate the coverages corresponding to the block rule described in Section 4.1 (with $C = 1.5$ for illustration).

S1.2 Calibration of Spatial Isotropy Tests

As described in Section 5.3 of the main manuscript, the simulation results for a sample region 50×50 , using the same parameters as before for both Gaussian and non-Gaussian processes, are based on 1000 simulations with 500 replications at a 10% nominal level and are as follows.

Table 1: Rejection rates for Gaussian data.

τ_R	Sub	FDWB	HFDB
1	0.083	0.091	0.091
1.1	0.281	0.295	0.295
1.2	0.665	0.656	0.656
1.3	0.910	0.903	0.903
1.4	0.990	0.991	0.991
1.5	0.999	1	1

Table 2: Rejection rates for Non-Gaussian data.[a]

τ_R	Sub	HFDB
1	0.077	0.114
1.2	0.332	0.375
1.4	0.754	0.787
1.6	0.945	0.955
1.8	0.994	0.992
2	0.997	1

[a] For non-Gaussian data, FDWB fails to maintain the size at the 10% level, with a rejection rate of 0.174 for $\tau_R = 1$, indicating its unsuitability for such data.

Tables 1-2 indicate that both frequency domain resampling methods perform well when the underlying distribution is Gaussian, as expected. Both HFDB and FDWB perform better in terms of size and power than spatial subsampling (Sub) when the underlying process is Gaussian. However, under non-Gaussianity, FDWB fails to maintain size under isotropy, making it unsuitable for such scenarios due to inflated rejection rates and poor control of the Type I error. In contrast, both HFDB and subsampling remain effective, with HFDB outperforming spatial subsampling (Sub) in both size and power.

S1.3 Sensitivity to Bandwidth Selection

Our objective here is to examine the sensitivity of the bootstrap methods (FDWB/HFDB) to bandwidth selection in kernel spectral density estimation. Recall, from Section 4, that both FDWB/HFDB as frequency domain bootstraps require such density estimation. As outlined in Section 4.1, the HFDB approach uses a bivariate standard Gaussian kernel with a joint bandwidth (δ_1, δ_2) such that $\delta_1 = \delta_2 \in [0.05, 0.15]$ as suggested by Ng et al. (2021) for FDWB in moderate sample sizes.

As a study on the effect of bandwidth $\delta_1 = \delta_2$ choice for both bootstrap methods (FDWB/HFDB), Figure 4 shows the resulting coverage of 95% intervals for covariance parameters $\gamma(\mathbf{h})$, $\mathbf{h} = (1,0)^\top, (1,1)^\top$, and $(0,1)^\top$, with spatial data simulated from the non-Gaussian process of Section 5.1 on size $n = 2500$ sample region (i.e., 50×50). For illustration, we use a bandwidth range $\delta_1 \in [0.05, 0.25]$ that is somewhat larger than the range $[0.05, 0.15]$ proposed by Ng et al. (2021). Additionally, the HFDB method uses a 10×10 block size based on the rule of thumb block choice from Section 4.1 (e.g., $C = 1.4$).

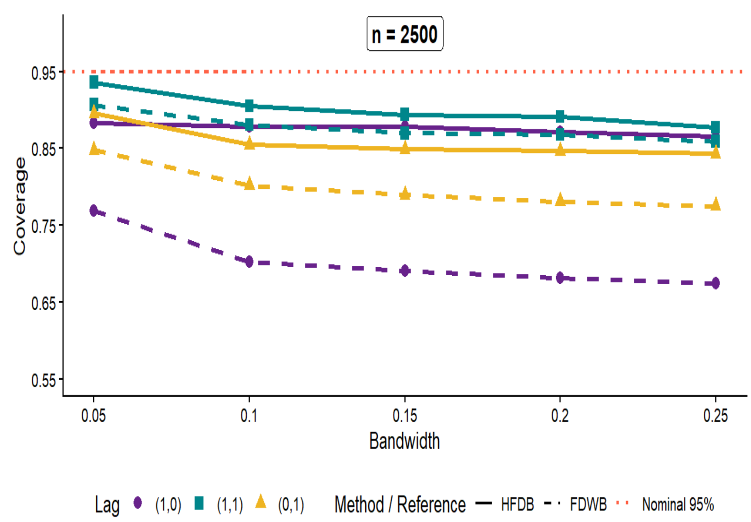

Figure 4: Coverage performance of 95% HFDB/FDWB intervals under different bandwidth choices.

From Figure 4, both bootstraps (FDWB/HFDB) exhibit coverages that similarly decrease with increasing bandwidths here, so that the shorter bandwidth range $[0.05, 0.15]$ of Ng et al. (2021) appears reasonable. While bandwidth can effect coverage, and the best coverages in Figure 4 appear to occur similarly for both bootstraps at $\delta_1 = 0.05$, the overall coverage of HFDB appears better across lags and also HFDB appears less sensitive than FDWB to bandwidth choice at some lags (e.g., $\mathbf{h} = (1, 0)^\top$).

S2 Proofs of Technical Lemmas

Here we establish the technical lemmas, mentioned in the Appendix of the main manuscript, as needed for establishing the proposed subsampling variance estimators and bootstrap distributional estimators (i.e., Theorems 3.1 and 4.1 of the main paper).

S2.1 Proof of Lemma 1

To formalize our proofs, we introduce the following: for $r = 1, 2$, and $x \in \mathbb{R}$,

$H_n \equiv H_n(\psi)$; $\quad Y \sim \mathcal{N}(0, \sigma^2(\psi))$, $\quad t_n^{(r)} \equiv E(H_n^r)$, $\quad t_{n,C}^{(r)} \equiv E\left(H_n - E(H_n)\right)^r$,

$F_n(x) \equiv P(H_n \leq x)$, $\quad F(x) \equiv P(Y \leq x)$,

$\widehat{F}_n(x) \equiv L^{-1}\sum_{\ell=1}^L \mathbb{I}[b_n^{1/2}(\widehat{M}_{\text{sub}}^{(\ell)}(\psi) - M(\psi)) \leq x]$; $\widehat{F}_{n,C}(x) \equiv L^{-1}\sum_{\ell=1}^L \mathbb{I}[b_n^{1/2}(\widehat{M}_{\text{sub}}^{(\ell)}(\psi) - \widetilde{M}_n(\psi)) \leq x]$;

$\widehat{t}_n^{(r)} \equiv L^{-1}\sum_{\ell=1}^L \left\{b_n^{1/2}\left(\widehat{M}_{\text{sub}}^{(\ell)}(\psi) - M(\psi)\right)\right\}^r$; $\widehat{t}_{n,C}^{(r)} \equiv L^{-1}\sum_{\ell=1}^L \left\{b_n^{1/2}\left(\widehat{M}_{\text{sub}}^{(\ell)}(\psi) - \widetilde{M}_n(\psi)\right)\right\}^r$.

These definitions establish key quantities used to prove the asymptotic validity of bootstrap procedures. They distinguish between population-level targets (e.g., $M(\psi)$) and sample-based estimates (e.g., $\widetilde{M}_n(\psi)$) to carefully track bias and variability in the resampling framework. We define r-th order central moments and empirical distributions centered around both the true and estimated targets.

First we prove statement 1(i). By notation defined above, we have to show $\sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F(x)| \xrightarrow{P} 0$ as $n \to \infty$. Let $\mathcal{C} \subset \mathbb{R}$ is the set of continuity points of F. It suffices to show $\widehat{F}_n(x_0) \xrightarrow{P} F(x_0)$ as $n \to \infty$ for an arbitrary fixed $x_0 \in \mathcal{C}$. From this, one may take a countable, dense(in $\mathbb{R}$) collection $\{x_i : i \geq 1\} \subset \mathcal{C}$ and, for any sub-sequence $\{n_m\}$ of $\{n\}$, extract a further sub-sequence $\{n_{m_k}\} \equiv \{n_k\}$ where, almost surely(a.s.), it holds that $\widehat{F}_{n_{m_k}}(x_i) \to F(x_i)$ as $n_{m_k} \to \infty$ for each $i \geq 1$; the latter implies that $\sup_{x\in\mathbb{R}} |\widehat{F}_{n_{m_k}}(x) - F(x)| \xrightarrow{a.s.} 0$, which is equivalent to $\sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F(x)| \xrightarrow{P} 0$ as $\{n_m\}$ was arbitrary. Then statement (i) follows from a triangle inequality.

Now, due to boundedness, $\widehat{F}_n(x_0) \xrightarrow{P} F(x_0)$ is equivalent to $E(\widehat{F}_n(x_0) - F(x_0))^2 \to 0$, and we show the latter.

To control this, fix $\lambda \in (0, 1)$ where λ sets a threshold for spatial proximity, and for $\mathbf{j} = (j_1, j_2), \mathbf{k} = (k_1, k_2)$ integer vectors in the plane (the i-th component can be $0, 1, ..., \left(n_i - n_i^{(b)}\right)$ for $i = 1, 2$) we split the double sum of covariances in the MSE into indices $\mathbf{j}, \mathbf{k}$ such that:

(a) $|j_i - k_i| < \lambda n_i$ for some $i = 1, 2$,

(b) and those for which $|j_i - k_i| \geq \lambda n_i$ for both $i = 1, 2$.

In case (a), the number of such dependent index pairs has an upper bound $n\lambda/L$.

In case (b), the assumption

$$\frac{|j_i - k_i|}{n_i^{(b)}} \to \infty \quad \text{for} \quad i = 1, 2,$$

along with $n_i^{(b)}/n_i \to 0$ (or $n_i/n_i^{(b)} \to \infty$), implies that the covariance between $\mathbb{I}(H_{\mathbf{j},b} \leq x_0)$

and $\mathbb{I}(H_{\mathbf{k},b} \le x_0)$ must disappear asymptotically. That is,

$$\begin{aligned}
&\sup_{\substack{|j_i-k_i|\ge \lambda n_i \\ \forall i=1,2}} \operatorname{Cov}\left(\mathbb{I}(H_{\mathbf{j},b} \le x_0), \mathbb{I}(H_{\mathbf{k},b} \le x_0)\right) \\
&= \sup_{\substack{|j_i-k_i|\ge \lambda n_i \\ \forall i=1,2}} |P(H_{\mathbf{j},b} \le x_0, H_{\mathbf{k},b} \le x_0) - P(H_{\mathbf{j},b} \le x_0)P(H_{\mathbf{k},b} \le x_0)| \to 0,
\end{aligned}$$

by Assumption 3 This guarantees

$$\begin{aligned}
D_{1,n}(\lambda) &:= \sup_{\substack{|j_i-k_i|\ge \lambda n_i \\ \forall i=1,2}} |P(H_{\mathbf{j},b} \le x_0, H_{\mathbf{k},b} \le x_0) - F(x_0)^2| \to 0, \\
D_{2,n}(\lambda) &:= \sup_{\substack{j_i \in \{0,1,..,(n_i-n_i^{(b)})\} \\ \forall i=1,2}} |P(H_{\mathbf{j},b} \le x_0)\left(F(x_0) - P(H_{\mathbf{j},b} \le x_0)\right)| \to 0.
\end{aligned}$$

Combining, we obtain:

$$E\left(\widehat{F}_n(x_0) - F(x_0)\right)^2 \le \frac{n\lambda}{L} + D_{1,n}(\lambda) + D_{2,n}(\lambda) \to \lambda.$$

Since $\lambda > 0$ is arbitrary, we conclude:

$$E\left(\widehat{F}_n(x_0) - F(x_0)\right)^2 \to 0,$$

completing the proof of Lemma 1(i).

Next, we prove statements (iii) - (v) together. First we would like to prove the following statements: if $\lim_{n\to\infty} E(H^2_{\mathbf{j}^{(n)},n}) = E(Y^2)$ hold for any vector of integer sequence $\mathbf{j}^{(n)}$, then

$$\widehat{t}_n^{(1)} \xrightarrow{P} t_n^{(1)}, \;\; \widehat{t}_n^{(2)} \xrightarrow{P} t_n^{(2)}, \;\; \text{and} \;\; \widehat{t}_{n,C}^{(2)} \xrightarrow{P} t_{n,C}^{(2)} \;\; \text{as} \;\; n \to \infty. \tag{S2.1}$$

We show $\widehat{t}_n^{(r)} \xrightarrow{P} t_n^{(r)}$, for $r = 1, 2$. We would also have $\widehat{t}_{n,C}^{(2)} \xrightarrow{P} t_{n,C}^{(2)}$ by Slutsky's theorem.

For any $C > 0$ such that $\pm C \in \mathcal{C}$, it follows from statement (i) that

$$\widehat{t}_n^{(r)}(C) \equiv \frac{1}{L}\sum_{\ell=1}^{L} H^r_{\mathbf{j}^{(\ell)},b}\mathbb{I}\{|H_{\mathbf{j}^{(\ell)},b}| \le C\} = \int y^r \mathbb{I}\{|y| \le C\} d\widehat{F}_n(y) \xrightarrow{P} E(Y^r \mathbb{I}\{|Y| \le C\}). \tag{S2.2}$$

That is, by statement (i), for any sub-sequence $\{n_j\}$ of $\{n\}$, there exists again further sub-sequence $\{n_k\} \subset \{n_j\}$ where $\sup_{x\in\mathbb{R}} |\widehat{F}_{n_k}(x) - F(x)| \xrightarrow{a.s.} 0$ as $\{n_k\} \to 0$. Then using

Continuous Mapping theorem, for a random variable $Y^*_{n_k} \sim \widehat{F}_{n_k}$, we have $(Y^*_{n_k})^r \mathbb{I}\{|Y^*_{n_k}| \leq C\} \xrightarrow{D} Y^r \mathbb{I}\{|Y \leq C\}$, and so the corresponding expected values (bounded) converge as $\int y^r \mathbb{I}\{|y| \leq C\} d\widehat{F}_{n_k}(y) \xrightarrow{a.s.} E(Y^r \mathbb{I}\{|Y| \leq C\})$ as $n_k \to \infty$ implying Eq. (S2.2). For given $\upsilon > 0$ and $\lambda \in (0,1)$, pick and fix C such that $E\left(Y^r \mathbb{I}\{|Y| > C\}\right) < \upsilon\lambda/3$. Then it suffices to show

$$\limsup_{n\to\infty} P\left(|\widehat{t}^{(r)}_n - \widehat{t}^{(r)}_n(C)| > \upsilon/3\right) \leq \lambda, \tag{S2.3}$$

from which

$$\begin{aligned}\limsup_{n\to\infty} P\left(|\widehat{t}^{(r)}_n - E(Y^r)| > \upsilon\right) \quad &\leq \quad \limsup_{n\to\infty} P\left(|\widehat{t}^{(r)}_n(C) - E(Y^r \mathbb{I}\{|Y| \leq C\})| > \upsilon/3\right) \\ &\quad + \limsup_{n\to\infty} P\left(|\widehat{t}^{(r)}_n - \widehat{t}^{(r)}_n(C)| > \upsilon/3\right) \leq \lambda\end{aligned}$$

follows by Eq. (S2.2)-(S2.3) and $P\left(E(Y^r \mathbb{I}\{|Y| > C\}) > \upsilon/3\right) = 0$, which gives us $\widehat{t}^{(r)}_n \xrightarrow{P} E(Y^r)$. Since we have assumed for any vector of integer sequence $\mathbf{j}^{(n)}$, $E(H^2_{\mathbf{j}^{(n)},n}) \to E(Y^2)$ as $n \to \infty$, we have $t^{(2)}_n \equiv E(H^2_n) \to E(Y^2) = \sigma^2(\psi)$ and, by dominated convergence theorem (DCT), $E(|H_n|) \to E(|H|)$ and $t^{(1)}_n \equiv E(H_n) \to E(Y) = 0$ as $n \to \infty$. Thus, we have established $\widehat{t}^{(r)}_n \xrightarrow{P} t^{(r)}_n$ for $r = 1, 2$.

By Markov's inequality, for $r = 1, 2$, we have

$$P\left(|\widehat{t}^{(r)}_n - \widehat{t}^{(r)}_n(C)| > \upsilon/3\right) \leq \frac{\upsilon}{3} \max_{1\leq \ell \leq L} E\left(|H_{\mathbf{j}^{(\ell)},b}|^r \mathbb{I}\{|H_{\mathbf{j}^{(\ell)},b}| > C\}\right),$$

so that, Eq. (S2.3) follows from $\limsup_{n\to\infty} \max_{1\leq\ell\leq L} E\left(|H_{\mathbf{j}^{(\ell)},b}|^r \mathbb{I}\{|H_{\mathbf{j}^{(\ell)},b}| > C\}\right) \leq \upsilon\lambda/3$ which is implied by the moment conditions. Note that a contrary result

$$\limsup_{n\to\infty} \max_{1\leq\ell\leq L} E\left(|H_{\mathbf{j}^{(\ell)},b}|^r \mathbb{I}\{|H_{\mathbf{j}^{(\ell)},b}| > C\}\right) > \upsilon\lambda/3$$

would imply existence of a sub-sequence $\{n_j\}$ and a positive integer m_{n_j} such that

$$E\left(|H_{\mathbf{j}^{(m_{n_j})},b_{n_j}}|^r \mathbb{I}\{|H_{\mathbf{j}^{(m_{n_j})},b_{n_j}}| > C\}\right) > \upsilon\lambda/3$$

holds for each n_j, which contradicts

$$\lim_{n_j\to\infty} E\left(|H_{\mathbf{j}^{(m_{n_j})},b_{n_j}}|^r \mathbb{I}\{|H_{\mathbf{j}^{(m_{n_j})},b_{n_j}}| > C\}\right) = E\left(|Y|^r \mathbb{I}\{|Y| > C\}\right) < \upsilon\lambda/3,$$

that follows by DCT from $H_{\mathbf{j}^{(m_{n_j})},b_{n_j}} \xrightarrow{D} Y$ along with $E(|H_{\mathbf{j}^{(m_{n_j})},b_{n_j}}|^r) \to E(|Y|^r)$ from the assumed moment conditions.

Now, the remaining is to show $\lim_{n\to\infty} E\left(H^2_{\mathbf{j}^{(n)},n}\right) = E(Y^2)$. By 4th-order stationarity we only need to prove $\lim_{\to\infty} E(H_n^2) = \sigma^2(\psi)$. By Assumption 1, Brillinger, 2001, and Fuentes, 2002 we have,

$$E(\mathcal{I}_n(\boldsymbol{\omega})) = f(\boldsymbol{\omega}) + O(n^{-1}), \quad \boldsymbol{\omega} \in \Pi^2 \tag{S2.4}$$

and for $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \Pi^2$, and $i = 1, 2$,

$$\operatorname{Cov}(\mathcal{I}_n(\boldsymbol{\omega}_1), \mathcal{I}_n(\boldsymbol{\omega}_2)) = \begin{cases} f^2(\boldsymbol{\omega}_1) + O(n^{-1}) & \text{if } |\omega_{1i}| = |\omega_{2i}| (\neq 0,\ \forall i = 1, 2) \\ \frac{(2\pi)^2}{n} f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, -\boldsymbol{\omega}_2) + O(n^{-1}) & \text{if } |\omega_{1i}| \neq |\omega_{2i}|. \end{cases} \tag{S2.5}$$

The error terms are uniform in $\boldsymbol{\omega}$. Hence, for $i = 1, 2$,

$$\begin{aligned} \operatorname{Var}(H_n(\psi)) &= n^{-1}(4\pi^2)^2 \sum_{\mathbf{j},\mathbf{k}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\psi(\boldsymbol{\omega}_{\mathbf{k},n}) \operatorname{Cov}(\mathcal{I}_n(\boldsymbol{\omega}_{\mathbf{j},n}), \mathcal{I}_n(\boldsymbol{\omega}_{\mathbf{k},n})) \\ &= n^{-1}(4\pi^2)^2 \left[\sum_{\mathbf{j}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n}) \left(\psi(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n})\right) \operatorname{Var}(\mathcal{I}_n(\boldsymbol{\omega}_{\mathbf{j},n})) \right. \\ &\qquad \left. + \sum_{\substack{\mathbf{j},\mathbf{k}\in\mathcal{J}_n \\ |\boldsymbol{\omega}_{j_i}| \neq |\boldsymbol{\omega}_{k_i}|}} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\psi(\boldsymbol{\omega}_{\mathbf{k},n}) \operatorname{Cov}(\mathcal{I}_n(\boldsymbol{\omega}_{\mathbf{j},n}), \mathcal{I}_n(\boldsymbol{\omega}_{\mathbf{k},n})) \right] \\ &=: S_1 + S_2. \end{aligned}$$

By Eq. (S2.5) and Riemann sum form, it holds that

$$\begin{aligned} S_1 &= (2\pi)^2 n^{-1} 4\pi^2 \sum_{\mathbf{j}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n}) \left(\psi(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n})\right) f^2(\boldsymbol{\omega}_{\mathbf{j},n}) + O(n^{-1}) \\ &= (2\pi)^2 \int_{\Pi^2} \psi(\boldsymbol{\omega})(\psi(\boldsymbol{\omega}) + \psi(-\boldsymbol{\omega})) f^2(\boldsymbol{\omega}) d\boldsymbol{\omega} + O(n^{-1}) = \sigma_1^2(\psi) + O(n^{-1}). \end{aligned}$$

Similarly, we have that

$$
\begin{aligned}
S_2 &= n^{-1}(4\pi^2)^2 \sum_{\substack{\mathbf{j},\mathbf{k}\in\mathcal{J}_n\\ |\boldsymbol{\omega}_{j_i}|\neq|\boldsymbol{\omega}_{k_i}|}} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\psi(\boldsymbol{\omega}_{\mathbf{k},n}) \left(\frac{(2\pi)^2}{n} f_4(\boldsymbol{\omega}_{\mathbf{j},n},\boldsymbol{\omega}_{\mathbf{k},n},-\boldsymbol{\omega}_{\mathbf{k},n}) + O(n^{-1})\right)\\
&= (2\pi)^2 n^{-2} 4\pi^2 \left[\sum_{\mathbf{j},\mathbf{k}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\psi(\boldsymbol{\omega}_{\mathbf{k},n}) f_4(\boldsymbol{\omega}_{\mathbf{j},n},\boldsymbol{\omega}_{\mathbf{k},n},-\boldsymbol{\omega}_{\mathbf{k},n})\right.\\
&\qquad \left. - \sum_{\mathbf{j}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})(\psi(\boldsymbol{\omega}_{\mathbf{j},n})+\psi(-\boldsymbol{\omega}_{\mathbf{j},n})) f_4(\boldsymbol{\omega}_{\mathbf{j},n},\boldsymbol{\omega}_{\mathbf{j},n},-\boldsymbol{\omega}_{\mathbf{j},n}) + O(n^{-1})\right]\\
&= (2\pi)^2 \int_{\Pi^2}\int_{\Pi^2} \psi(\boldsymbol{\omega}_1)\psi(\boldsymbol{\omega}_2) f_4(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,-\boldsymbol{\omega}_2)\, d\boldsymbol{\omega}_1\, d\boldsymbol{\omega}_2 + O(n^{-1}) = \sigma_2^2(\psi) + O(n^{-1}).
\end{aligned}
$$

Now, we have $\mathrm{Var}(H_n(\psi)) = \sigma_1^2(\psi) + \sigma_2^2(\psi) + O(n^{-1}) = \sigma^2(\psi) + O(n^{-1})$. By Eq. (S2.4) and bounded variation of $\psi(\cdot)$, we also have

$$
E(H_n(\psi)) = n^{1/2}\left(\frac{(2\pi)^2}{n}\sum_{\mathbf{j}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\{f(\boldsymbol{\omega}_{\mathbf{j},n}) + O(n^{-1})\} - \int_{\Pi^2}\psi(\boldsymbol{\omega})f(\omega)d\omega\right) = O(n^{-1/2}).
$$

Hence, we conclude that $E(H_n(\psi))^2 = \mathrm{Var}(H_n(\psi)) + E^2(H_n(\psi)) = \sigma^2(\psi) + O(n^{-1})$.

Finally, we prove statement (ii), i.e., we will show $\sup_{x\in\mathbb{R}}\left|\widehat{F}_{n,C}(x) - F_n(x)\right| \xrightarrow{P} 0$ as $n\to\infty$. Let Y_n^* denotes a random variable with distribution function $\widehat{F}_n$, then $(Y_n^* - \widehat{t}_n^{(1)})$ is a random variable with distribution function $\widehat{F}_{n,C}$. Because $\sup_{x\in\mathbb{R}}\left|\widehat{F}_n(x) - F(x)\right| \xrightarrow{P} 0$ and $\widehat{t}_n^{(1)} \xrightarrow{P} t_n^{(1)}$, and $t_n^{(1)} \to E(Y) = 0$ guaranteed by the moment assumption, we have, for any sub-sequence $\{n_j\}\subset\{n\}$, there exists a further sub-sequence $\{n_k\}\subset\{n_j\}$ such that $Y_{n_k}^* \xrightarrow{D} Y$ and $\widehat{t}_{n_k}^{(1)} \to 0$ hold as $n_k\to\infty$ almost surely (a.s.). By Slutsky's theorem, $(Y_{n_k}^* - \widehat{t}_{n_k}^{(1)}) \xrightarrow{D} Y$ follows a.s. which, because $\{n_j\}$ was arbitrary, implies $\sup_{x\in\mathbb{R}}\left|\widehat{F}_{n,C}(x) - F(x)\right| \xrightarrow{P} 0$. Statement (ii) follows from a triangle inequality. $\square$

S2.2 Proof of Lemma 2

Note that,

$$\begin{aligned}
&(2\pi)^2 b^{-1} \sum_{\mathbf{j}\in\mathcal{J}_b} \psi(\boldsymbol{\omega}_{\mathbf{j},b}) \frac{1}{L} \sum_{\ell=1}^{L} \left(\mathcal{I}_{\text{sub}}^{(\ell)}(\boldsymbol{\omega}_{\mathbf{j},b}) - \widetilde{\mathcal{I}}_n(\boldsymbol{\omega}_{\mathbf{j},b}) \right)^2 \\
&= (2\pi)^2 b^{-1} \left[\sum_{\ell=1}^{L} \psi(\boldsymbol{\omega}_{\mathbf{j},b}) \left(\frac{1}{L} \sum_{\ell=1}^{L} (\mathcal{I}_{\text{sub}}^{(\ell)}(\boldsymbol{\omega}_{\mathbf{j},b}))^2 - 2f^2(\boldsymbol{\omega}_{\mathbf{j},b}) \right) \right. \\
&\quad \left. + \sum_{\ell=1}^{L} \psi(\boldsymbol{\omega}_{\mathbf{j},b}) \left(f^2(\boldsymbol{\omega}_{\mathbf{j},b}) - (\widetilde{\mathcal{I}}_n(\boldsymbol{\omega}_{\mathbf{j},b}))^2 \right) + \sum_{\ell=1}^{L} \psi(\boldsymbol{\omega}_{\mathbf{j},b}) f^2(\boldsymbol{\omega}_{\mathbf{j},b}) \right] \\
&=: S_{1n} + S_{2n} + S_{3n}.
\end{aligned}$$

It is immediate from the Riemann sum expression that, $S_{3n} \to \int_{\Pi^2} \psi(\boldsymbol{\omega}) f^2(\boldsymbol{\omega})$ as $n \to \infty$. Now, proceeding in the same way as Yu, 2023, using Cauchy-Schwarz inequality and boundedness of $\psi(\cdot)$ we have $|S_{2n}| = o(1)$. Also, using the moment conditions we have $S_{1n} \xrightarrow{P} 0$, which gives us the statement of Lemma 2. $\square$

S2.3 Proof of Lemma 3

By the exponential resampling mechanism, we have $\text{Cov}_*(U^*_{\mathbf{j}}, U^*_{\mathbf{k}}) = \mathbb{I}\{\mathbf{j} = -\mathbf{k}\}$. Hence, with $\widehat{f}_n$ being an even function it follows, $\text{Cov}_*(G^*(\boldsymbol{\omega}_{\mathbf{j},n}), G^*(\boldsymbol{\omega}_{\mathbf{k},n})) = \widehat{f}_n^2(\boldsymbol{\omega}_{\mathbf{j},n})\mathbb{I}\{\mathbf{j} = -\mathbf{k}\}$ and,

$$\begin{aligned}
\text{Var}_*(Q^*_{FDWB,n}(\psi)) &= \frac{(4\pi^2)^2}{n} \sum_{\mathbf{j},\mathbf{k}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\psi(\boldsymbol{\omega}_{\mathbf{k},n})\text{Cov}_*(G^*(\boldsymbol{\omega}_{\mathbf{j},n}), G^*(\boldsymbol{\omega}_{\mathbf{k},n})) \\
&= (2\pi)^2 \sum_{\mathbf{j}\in\mathcal{J}_n} \frac{(2\pi)^2}{n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\{\psi(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(\boldsymbol{\omega}_{-\mathbf{j},n})\} f^2(\boldsymbol{\omega}_{\mathbf{j},n}) \\
&\quad + \frac{(4\pi^2)^2}{n} \sum_{\mathbf{j}\in\mathcal{J}_n} \psi(\boldsymbol{\omega}_{\mathbf{j},n})\{\psi(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(\boldsymbol{\omega}_{-\mathbf{j},n})\}(\widehat{f}_n^2(\boldsymbol{\omega}_{\mathbf{j},n}) - f^2(\boldsymbol{\omega}_{\mathbf{j},n})) \\
&=: S_1 + S_2
\end{aligned}$$

Since, S_1 is a Riemann sum, it immediately follows that $S_1 = \sigma_1^2(\psi) + O(1)$. Using $|\mathcal{J}_n| = O(n)$ and Eq. (4.7) we have $S_2 = o_P(1)$ which completes the proof of Lemma 3(i).

For notational ease, we define $\mathcal{J}_n^+ \equiv \{\mathbf{j} \equiv (j_1, j_2) \in \mathcal{J}_n : \text{either } j_1 > 0 \text{ or } j_1 = 0 \text{ and } j_2 > 0\}$. Since $\{U^*_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}_n^+\}$ are i.i.d. standard exponentially distributed and $U^*_{\mathbf{j}} \equiv U^*_{-\mathbf{j}}$ for $\mathbf{j} \in \mathcal{J}_n$ where $-\mathbf{j} \in \mathcal{J}_n^+$, then $Q^*_{FDWB,n}(\psi)$ can be written as

$$Q^*_{FDWB,n} = n^{-1/2} \sum_{\mathbf{j} \in \mathcal{J}_n^+} V^*_{\mathbf{j},n}, \tag{S2.6}$$

where,

$$V^*_{\mathbf{j},n} = (2\pi)^2 (\psi(\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(-\boldsymbol{\omega}_{\mathbf{j},n}))(U^*_{\mathbf{j}} - 1), \;\; \mathbf{j} \in \mathcal{J}_n^+.$$

Since, $|\mathcal{J}_n^+| \to \infty$, as $n \to \infty$, a conditional version of Lyapunov's CLT can be applied to Eq. (S2.6) (using the established convergence in probability of $\mathrm{Var}_*(Q^*_{FDWB,n}(\psi))$ to a positive constant from (i), i.e., we have to find a $\upsilon > 0$ such that

$$n^{-(1+\upsilon/2)} \sum_{\mathbf{j} \in \mathcal{J}_n^+} E^* \left(|V^*_{\mathbf{j},n}|^{2+\upsilon} \right) = o_P(1). \tag{S2.7}$$

Choosing $\upsilon = 1$, we have

$$\sum_{\mathbf{j} \in \mathcal{J}_n^+} |\psi(\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(-\boldsymbol{\omega}_{\mathbf{j},n})| \leq n \cdot \sup_{\boldsymbol{\omega} \in \Pi} |\psi(\boldsymbol{\omega})| \cdot \sup_{\boldsymbol{\omega} \in \Pi} |\widehat{f}_n(\boldsymbol{\omega})| = O_P(n),$$

since both ψ and f (due to absolutely summable autocovariance function) are bounded, and due to Eq. (4.7). Hence, it holds that

$$\sum_{\mathbf{j} \in \mathcal{J}_n^+} \left| \psi(\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(-\boldsymbol{\omega}_{\mathbf{j},n}) \right|^3 = O_P(n). \tag{S2.8}$$

Since, $U^*_{\mathbf{j}}$ are standard exponential we have $E^*(|U^*_{\mathbf{j}} - 1|^3) = 12e^{-1} - 2 = K(\text{Say})$ gives us

$$\sum_{\mathbf{j} \in \mathcal{J}_n^+} \mathbf{E}^* \left(|V^*_{\mathbf{j},n}|^{2+\upsilon} \right) = (4\pi^2)^3 K \sum_{\mathbf{j} \in \mathcal{J}_n^+} \left| \psi(\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(\boldsymbol{\omega}_{\mathbf{j},n}) + \psi(-\boldsymbol{\omega}_{\mathbf{j},n}) \widehat{f}_n(-\boldsymbol{\omega}_{\mathbf{j},n}) \right|^3 = O_P(n),$$

due to Eq. (S2.8). This shows Lyapunov condition holds for Eq. (S2.7) when $\upsilon = 1$. Together with Lemma 3(i), the assertion in Lemma 3(ii) follows. $\square$

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