# DOUBLY ROBUST ESTIMATION OF OPTIMAL INDIVIDUAL TREATMENT REGIME IN A SEMI-SUPERVISED FRAMEWORK

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**Supplementary Material**

## S1 Additional asymptotic results

### S1.1 Regularity conditions

To establish the Theorem 1, we assume the following regularity conditions. Define $\Delta(\boldsymbol{\beta}, \mathbf{x}) = E[D(\mathbf{x})I(\boldsymbol{\beta}'\mathbf{x} \geqslant 0)]$. Let $\nabla_m \Delta(\boldsymbol{\beta}, \mathbf{x})$ denote the $m$-th partial derivative operator with respect to $\boldsymbol{\beta}$, and define $|\nabla_m| \Delta(\boldsymbol{\beta}, \mathbf{x}) = \sum_{i_1,\ldots,i_m} \left| \frac{\partial^m \Delta(\boldsymbol{\beta},\mathbf{x})}{\partial \beta_{i_1} \ldots \partial \beta_{i_m}} \right|$.

C1. The propensity score $\pi(\mathbf{x})$ is known and $0 < \pi(\mathbf{x}) < 1$ for all $\mathbf{x} \in \mathcal{X}$.

C2. The estimator $\hat{\boldsymbol{\theta}}$ converges almost surely to a deterministic vector of parameters $\boldsymbol{\theta}_0$, and $n^{\frac{1}{2}} \left( \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \right) = O_p(1)$ and $\nu(\mathbf{X}, \boldsymbol{\theta})$ is continuous with respect to $\boldsymbol{\theta}$.

C3. The objective function $\Delta(\boldsymbol{\beta}, \boldsymbol{\theta}_0)$ has a unique maximizer at $\boldsymbol{\beta} = \boldsymbol{\beta}_0 = (\beta_{01}, \dots, \beta_{0p})'$ with $\|\boldsymbol{\beta}_0\| = 1$. And the parameter space $\mathcal{B}$ of $\boldsymbol{\beta}$ is compact.

C4. (a) $\mathbf{X}$ has a continuously differentiable density function $f(\cdot)$. The angular components of $\mathbf{X}$, considered as a random element of the unit sphere $\mathbb{U}$ in $\mathbb{R}^p$, has a bounded, continuous density with respect to the surface measure on $\mathbb{U}$.

(b) $E\left[D(\mathbf{X})^2\right] < \infty$ and $E\left[V(\mathbf{Z}, \boldsymbol{\theta}_0)^2\right] < \infty$.

(c) $\{\nu(\mathbf{X}, \boldsymbol{\theta}) : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ is a VC class with envelope function $C(\mathbf{X})$ and $E[C^2(\mathbf{X})] < \infty$.

C5. (a) The value function $\Delta(\boldsymbol{\beta}, \mathbf{x})$ is twice differentiable w.r.t. $\boldsymbol{\beta}$.

(b) There is an integrable function $\Upsilon(\mathbf{x})$ such that, for any $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ with $\|\boldsymbol{\beta}_1\| = \|\boldsymbol{\beta}_2\| = 1$, $\|\nabla_2 \Delta(\boldsymbol{\beta}_1, \mathbf{x}) - \nabla_2 \Delta(\boldsymbol{\beta}_2, \mathbf{x})\| < \Upsilon(\mathbf{x}) \|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\|$.

(c) $E\left\{|\nabla_1 \Delta(\boldsymbol{\beta}_0, \mathbf{X})|^2\right\} < \infty$ and $E\left\{|\nabla_2| \Delta(\boldsymbol{\beta}_0, \mathbf{X})\right\} < \infty$.

(d) $V = -\nabla_2 \Delta(\boldsymbol{\beta}_0, \mathbf{X}) = \int_{\boldsymbol{\beta}_0'\mathbf{X}=0} [f(\mathbf{X})\dot{D}(\mathbf{X}) + D(\mathbf{X})\dot{f}(\mathbf{X})]'\boldsymbol{\beta}_0 \mathbf{X}\mathbf{X}' d\sigma$ is positive definite, where $\sigma$ is the surface measure on the hyperplane $\{\mathbf{X} : \boldsymbol{\beta}_0'\mathbf{X} = 0\}$. $\dot{D}(\mathbf{X})$ and $\dot{f}(\mathbf{X})$ denote the derivatives of

$D(\mathbf{X})$ and $f(\mathbf{X})$ (the density function of $\mathbf{X}$) with respect to $\mathbf{X}$, respectively.

C6. (a) The kernel function $K(\cdot) : \mathbb{R} \to \mathbb{R}$ is a symmetric probability density function. $K(s)$ is twice continuously differentiable and Lipschitz and the second derivative satisfies the Lipschitz condition. $\int K^2(u)du < \infty, \mu_2(K) = \int u^2 K^2(u)du < \infty$.

(b) The bandwidth $h$ satisfies $h \to 0, nh \to \infty$ and $nh^4 \to 0$ as $n \to \infty$.

C7. Denote the density function of $S = \boldsymbol{\beta}'\mathbf{X}$ by $f_S(\cdot)$ with the support $\mathcal{S}$.

(a) The density function $f_S(\cdot)$ and the regression function $m(\cdot)$ are three times continuously differentiable with respect to $s$.

(b) $f_S(\cdot)$ is everywhere positive for $\boldsymbol{\beta} \in \mathcal{B}_\epsilon$, where $\mathcal{B}_\epsilon$ is a $\epsilon$-neighbourhood of $\boldsymbol{\beta}_0$, and $\inf_{s \in \mathcal{S}} f(s) \geqslant C > 0$ where C is a constant.

Condition C1 is assumed to simplify the theoretical arguments, which can be extended to the situation when the propensity score model is correctly specified. For example, by using a logistic regression, the parameters in the propensity score model can be consistently estimated from data. Condition C2 usually holds for the least squares estimator under mild conditions. Conditions 3 and 4 are assumed to establish the consistency of estimators for $\boldsymbol{\beta}$. Condition C3 is an identifiability condition for $\boldsymbol{\beta}_0$, which assumes the existence and uniqueness of population parameters that maximize the value function. Condition C4 is assumed to show the uniform convergence of $\hat{\Delta}(\boldsymbol{\beta}, \boldsymbol{\theta}_0)$ to $\Delta(\boldsymbol{\beta}, \mathbf{X})$. Condition C5 is assumed to ensure the asymptotic property of $\hat{\boldsymbol{\beta}}$. Condition C6 is a commonly used condition for kernel estimation, which requires undersmoothing of bandwidths and is standard to obtain consistency of semiparametric estimators (Tsiatis, 2006). Conditions C4, C5, and C7 are often used to establish the large sample properties of M-estimators (Sherman, 1993; Delsol and Van Keilegom, 2020).

To establish the asymptotic results in Theorem S1, we make some variations to the conditions C1 to C5 given previously as follows. Define $\Delta^{\mathrm{DR}}(\boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{x}) = E\left[\frac{\pi(\mathbf{X})}{\pi(\mathbf{X}, \boldsymbol{\alpha})} D(\mathbf{x}) I(\boldsymbol{\beta}'\mathbf{x} \geqslant 0)\right]$.

*C*1′. The true and posited propensity scores satisfy $0 < \pi(\mathbf{x}) < 1$ and $0 < \pi(\mathbf{x}, \boldsymbol{\alpha}) < 1$ for all $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\alpha}$ in a compact space.

*C*2′. The estimators $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\alpha}}$ satisfy $\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi_{\theta,i} + o_p(1)$ and $\sqrt{n}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = n^{-\frac{1}{2}} \sum_{i=1}^{n} \psi_{\alpha,i} + o_p(1)$, where $\psi_{\theta,i}$ 's and $\psi_{\alpha,i}$ 's are iid mean-zero random vectors with finite second moments.

*C*3′. The objective function $\Delta^{\mathrm{DR}}(\boldsymbol{\beta}, \boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)$ has a unique maximizer at $\boldsymbol{\beta} =$

$\boldsymbol{\beta}_0 = (\beta_{01}, \dots, \beta_{0p})'$ with $\|\boldsymbol{\beta}_0\| = 1$. And the parameter space of $\mathcal{B}$ of $\boldsymbol{\beta}$ is compact.

$C4'$. (a) $\mathbf{X}$ has a continuously differentiable density function $f(\cdot)$. The angular components of $\mathbf{X}$, considered as a random element of the unit sphere $\mathbb{U}$ in $\mathbb{R}^p$, has a bounded, continuous density with respect to the surface measure on $\mathbb{U}$.

(b) $E\{D(\mathbf{X})^2\} < \infty$ and $E[V(\mathbf{Z}, \boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)^2] < \infty$.

(c) $\{\nu(\mathbf{X}, \boldsymbol{\theta}) : \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ is a VC class with envelope function $C(\mathbf{X})$ and $E[C^2(\mathbf{X})] < \infty$.

$C5'$. (a) The function $\Delta^{\mathrm{DR}}(\boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{x})$ is twice differentiable with respect to $\boldsymbol{\beta}$.

(b) There is an integrable function $\Upsilon^{DR}(\mathbf{x})$ such that, for any $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ with $\|\boldsymbol{\beta}_1\| = \|\boldsymbol{\beta}_2\| = 1$,

$$\left\|\nabla_2 \Delta^{DR}(\boldsymbol{\beta}_1, \boldsymbol{\alpha}_0, \mathbf{x}) - \nabla_2 \Delta^{DR}(\boldsymbol{\beta}_2, \boldsymbol{\alpha}_0, \mathbf{x})\right\| < \Upsilon^{DR}(\mathbf{x}) \left\|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2\right\|.$$

(c) $E\left\{\left|\nabla_1 \Delta^{DR}(\boldsymbol{\beta}_0, \boldsymbol{\alpha}_0, \mathbf{x})\right|^2\right\} < \infty$ and $E\left\{|\nabla_2|\, \Delta^{DR}(\boldsymbol{\beta}_0, \boldsymbol{\alpha}_0, \mathbf{x})\right\} < \infty$.

(d) $V^{DR} = \int_{\boldsymbol{\beta}_0'\mathbf{X}=0} \frac{\pi(\mathbf{X})}{\pi(\mathbf{X}, \boldsymbol{\alpha}_0)} [f(\mathbf{X})\dot{D}(\mathbf{X}) + D(\mathbf{X})\dot{f}(\mathbf{X})]'\boldsymbol{\beta}_0 \mathbf{X}\mathbf{X}' d\sigma$, the second derivative matrix of $-\Delta^{DR}(\boldsymbol{\beta}_0, \boldsymbol{\alpha}_0, \mathbf{X})$ with respect to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}_0$, is positive definite, where $\sigma$ is the surface measure on the hyperplane $\{\mathbf{X} : \boldsymbol{\beta}_0'\mathbf{X} = 0\}$.

## S1.2 Asymptotic properties for doubly robust estimators

Here we will consider the asymptotic properties of the doubly robust estimators under the situation that $\pi(\mathbf{X})$ is unknown similarly.

**Theorem S1.** *Let $G^{DR}(t)$, $G_{\lambda}^{DR}(t)$ and $G_{pl}^{DR}(t)$ be the mean-zero Gaussian process with continuous sample paths. Under conditions $C1' - C5'$ and $C6 - C7$, when either $\pi(\mathbf{X}, \boldsymbol{\alpha})$ or $\nu(\mathbf{X}, \boldsymbol{\theta})$ is correctly specified, as $n, N \to \infty$, $\frac{n}{N} \to \rho$, $\lambda \in [0, 1]$ we have:*

*(a1)* $\hat{\boldsymbol{\beta}}_{sup}^{DR} \xrightarrow{p} \boldsymbol{\beta}_0$; *(a2)* $\hat{\boldsymbol{\beta}}_{pl}^{DR} \xrightarrow{p} \boldsymbol{\beta}_0$; *(a3)* $\hat{\boldsymbol{\beta}}_{\lambda}^{DR} \xrightarrow{p} \boldsymbol{\beta}_0$.

*(b1)* $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_{sup}^{DR} - \boldsymbol{\beta}_0) \xrightarrow{d} \arg\max_t Z^{DR}(t)$, *where the process* $Z^{DR}(t) = G^{DR}(t) - \frac{1}{2}t'V^{DR}t$. *Here $G^{DR}(t)$ has the covariance kernel function $Cov^{DR}(\cdot, \cdot)$ (defined in the proof of Theorem S1) and $-V^{DR}$ is the second derivative matrix of $E[V(\mathbf{Z}, \boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)]$ with respect to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}_0$.*

*(b2)* $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_{pl}^{DR} - \boldsymbol{\beta}_0) \xrightarrow{d} \arg\max_t Z_{pl}^{DR}(t)$, *where the process* $Z_{pl}^{DR}(t) = G_{pl}^{DR}(t) - \frac{1}{2}t'V^{DR}t$. *Here $G_{pl}^{DR}(t)$ has the covariance kernel function* $(\frac{\rho}{1+\rho})^2 Cov^{DR}(\cdot, \cdot)$.

*(b3)* $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_{\lambda}^{DR} - \boldsymbol{\beta}_0) \xrightarrow{d} \arg\max_t Z_{\lambda}^{DR}(t)$, *where the process* $Z_{\lambda}^{DR}(t) = G_{\lambda}^{DR}(t) - \frac{1}{2}t'V^{DR}t$. *Here $G_{\lambda}^{DR}(t)$ has the covariance kernel function* $[\lambda^2 + (1-\lambda)^2\rho^2]Cov^{DR}(\cdot, \cdot)$.

The proof of Theorem S1 follows a similar structure to that of Theorem

1, leading to comparable conclusions regarding covariance comparison.

### S1.3 Additional notes on asymptotic variance comparisons

For some constant $K > 0$, let $G_1(t)$ be a mean-zero Gaussian process with continuous sample paths and covariance kernel function $Cov(\cdot,\cdot)$, $G_2(t)$ be a mean-zero Gaussian process with continuous sample paths and covariance kernel function $KCov(\cdot,\cdot)$. Since $G_2(t)$ is a scaled version of $G_1(t)$ in covariance, we can express $G_2(t) \overset{d}{=} \sqrt{K}G_1(t)$, where $\overset{d}{=}$ denotes equality in distribution. The asymptotic distribution of $\arg\max_t G_2(t) - \frac{1}{2}t'Vt$ can be written as $\arg\max_t \sqrt{K}G_1(t) - \frac{1}{2}t'Vt$ for some positive definite matrix $V$.

Let $t = \sqrt{K}s$, we have $\sqrt{K}G_1(t) - \frac{1}{2}t'Vt = \sqrt{K}G_1(\sqrt{K}s) - \frac{K}{2}s'Vs = K\left(G_1(s) - \frac{1}{2}s'Vs\right)$. Since $K$ is a positive multiplicative constant, it does not affect the location of the maximum. Thus by $t = \sqrt{K}s$ we have $\arg\max_t \sqrt{K}G_1(t) - \frac{1}{2}t'Vt = \sqrt{K}\arg\max_s G_1(s) - \frac{1}{2}s'Vs$.

Denote random variables $W_1 \overset{d}{=} \arg\max_t G_1(t) - \frac{1}{2}t'Vt$ and $W_2 \overset{d}{=} \arg\max_t G_2(t) - \frac{1}{2}t'Vt$, then $W_2 \overset{d}{=} \sqrt{K}W_1$. Define $Var(W_1) = \Sigma$, we have $Var(W_2) = Var(\sqrt{K}W_1) = K\Sigma$.

Based on this explanation, the results presented in Theorem 1 can be directly obtained by setting $K = (\frac{\rho}{1+\rho})^2$ for $\hat{\boldsymbol{\beta}}_{pl}$, and $K = \lambda^2 + (1-\lambda)^2\rho^2$ for $\hat{\boldsymbol{\beta}}_{\lambda}$.

# S2 Theoretical proofs

## S2.1 Supporting Lemma

Let $f_S(\cdot)$ is the density of $s = \boldsymbol{\beta}'\mathbf{x}$ with $\inf_{s\in\mathcal{S}} f_S(s) \geqslant C > 0$ where $\mathcal{S}$ is the support of $s$ and C is a constant. And let $\hat{f}_n(s) = \hat{f}_n(\boldsymbol{\beta}'\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n K_h\{\boldsymbol{\beta}'(\mathbf{X}_i - \mathbf{x})\} = \frac{1}{n}\sum_{i=1}^n K_h(S_i - s)$, where $S_i = \boldsymbol{\beta}'\mathbf{X}_i$.

**Lemma S1.** *Suppose that condition C6.(a) holds. Let $E[g(\mathbf{X},U)|U=u]$ be continuous and twice differentiable at $u$ and $E|g(\mathbf{X},U)|^2 < \infty$. If the second derivative of $f(\mathbf{x})$ are continuous and bounded, then as $n \to \infty$, we have:*

$$\sup_{u\in\mathcal{U}} \left| \frac{1}{n}\sum_{i=1}^n K_h(U_i - u)(\frac{U_i - u}{h})^k g(\mathbf{X}_i, U_i) - f(u)E[g(\mathbf{X},U)|U=u]\mu_k \right| = O(c_n) \quad a.s.$$

*where $\mathcal{U}$ is the support of $U$, $\mu_k = \int u^k K(u)du$ and $c_n = h^2 + \sqrt{\frac{\log h^{-1}}{nh}}$.*

*Proof.* The proof may be constructed along the lines of Lemma A.2 in Xia and Li (1999). □

## S2.2 Proof of Lemma 1

$$
\begin{aligned}
E[\{Y^*(1) - Y^*(0)\}d(\mathbf{X})] &= E[E\{Y^*(1) - Y^*(0)|\mathbf{X}\}d(\mathbf{X})] \\
&= E[E\{Y^*(1) - Y^*(0)|\mathbf{X}, A\}d(\mathbf{X})] \\
&= E[\{E[Y|\mathbf{X}, A = 1] - E[Y|\mathbf{X}, A = 0]\}d(\mathbf{X})] \\
&= E[D(\mathbf{X})d(\mathbf{X})].
\end{aligned}
$$

Note that the first '=' holds by the law of iterated expectation, the second '=' holds by the no-unmeasured-confounders assumption and the third '=' holds by the stable unit treatment value assumption. This completes the proof.

## S2.3 Proof of Theorem 1 (a1)-(a3)

For simplicity of notation, we write $S = \boldsymbol{\beta}'\mathbf{X}$ and $S_i = \boldsymbol{\beta}'\mathbf{X}_i$. Recall that

$$
\hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) = \frac{\lambda}{n}\sum_{i=1}^{n} V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}})I(S_i \geqslant 0) + \frac{1-\lambda}{N}\sum_{i=n+1}^{M} \hat{m}(S_i, \hat{\boldsymbol{\theta}})I(S_i \geqslant 0),
$$

$$
\Delta_\lambda(\boldsymbol{\beta}, \boldsymbol{\theta}_0) = \lambda E[V(\mathbf{Z}, \boldsymbol{\theta}_0)I\,(S \geqslant 0)] + (1-\lambda)E[m(S, \boldsymbol{\theta}_0)I\,(S \geqslant 0)].
$$

Note that for M-estimate, we only need to prove $\sup_{\boldsymbol{\beta}} |\hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) - \Delta_\lambda(\boldsymbol{\beta}, \boldsymbol{\theta}_0)| \overset{P}{\to} 0$ as $n \to \infty$ to get the conclusion in Theorem 1 (a1)-(a3).

$$
\begin{aligned}
&\sup_{\boldsymbol{\beta}} |\hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) - \Delta_\lambda(\boldsymbol{\beta}, \boldsymbol{\theta}_0)| \\
\leqslant &\lambda \sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^{n} V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) I\left(S_i \geqslant 0\right) - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right| \\
&+ (1-\lambda) \sup_{\boldsymbol{\beta}} \left| \frac{1}{N} \sum_{i=n+1}^{M} \hat{m}(S_i, \hat{\boldsymbol{\theta}}) I\left(S_i \geqslant 0\right) - E[m(S, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right| \\
:= &\lambda I + (1-\lambda) II,
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^{n} V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) I\left(S_i \geqslant 0\right) - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right|, \\
II &= \sup_{\boldsymbol{\beta}} \left| \frac{1}{N} \sum_{i=n+1}^{M} \hat{m}(S_i, \hat{\boldsymbol{\theta}}) I\left(S_i \geqslant 0\right) - E[m(S, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right|.
\end{aligned}
$$

For $I$ and by condition C2, there exists a constant $\delta$ small enough, such that:

$$
\begin{aligned}
&\sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^{n} V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) I\left(S_i \geqslant 0\right) - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right| \\
\leqslant &\sup_{\boldsymbol{\beta}} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|<\delta} \left| \frac{1}{n} \sum_{i=1}^{n} (V(\mathbf{Z}_i, \boldsymbol{\theta}) - V(\mathbf{Z}_i, \boldsymbol{\theta}_0)) I\left(S_i \geqslant 0\right) - E[(V(\mathbf{Z}, \boldsymbol{\theta}) - V(\mathbf{Z}, \boldsymbol{\theta}_0)) I\left(S \geqslant 0\right)] \right| \\
&+ \sup_{\boldsymbol{\beta}} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|<\delta} \left| E[V(\mathbf{Z}, \boldsymbol{\theta}) I\left(S \geqslant 0\right)] - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right| \\
&+ \sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^{n} V(\mathbf{Z}_i, \boldsymbol{\theta}_0) I\left(S_i \geqslant 0\right) - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right| \\
:= &I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$I_1 = \sup_{\boldsymbol{\beta}} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|<\delta} \left| \frac{1}{n} \sum_{i=1}^{n} (V(\mathbf{Z}_i, \boldsymbol{\theta}) - V(\mathbf{Z}_i, \boldsymbol{\theta}_0)) I\left(S_i \geqslant 0\right) - E[(V(\mathbf{Z}, \boldsymbol{\theta}) - V(\mathbf{Z}, \boldsymbol{\theta}_0)) I\left(S \geqslant 0\right)] \right|,$$

$$I_2 = \sup_{\boldsymbol{\beta}} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|<\delta} \left| E[V(\mathbf{Z}, \boldsymbol{\theta}) I\left(S \geqslant 0\right)] - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right|,$$

$$I_3 = \sup_{\boldsymbol{\beta}} \left| \frac{1}{n} \sum_{i=1}^{n} V(\mathbf{Z}_i, \boldsymbol{\theta}_0) I\left(S_i \geqslant 0\right) - E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I\left(S \geqslant 0\right)] \right|.$$

Write

$$f(\mathbf{Z}_i, \boldsymbol{\beta}, \boldsymbol{\theta}) = \frac{A_i - \pi(\mathbf{X}_i)}{\pi(\mathbf{X}_i)\{1 - \pi(\mathbf{X}_i)\}} \nu(\mathbf{X}_i, \boldsymbol{\theta}) I(\boldsymbol{\beta}'\mathbf{X}_i \geqslant 0),$$

where $\nu(\mathbf{X}_i, \boldsymbol{\theta}) = E(Y|\mathbf{X}_i, A = 0)$. By the Conditions C4 (b) and (c)

$$E\left(\left[\frac{\{Y - \nu\left(\mathbf{X}, \boldsymbol{\theta}_0\right)\}\{A - \pi(\mathbf{X})\}}{\pi(\mathbf{X})\{1 - \pi(\mathbf{X})\}}\right]^2\right) < \infty, E[C^2(\mathbf{X})] < \infty,$$

there exists an envelope function

$$F(\mathbf{X}, A) = C(\mathbf{X}) \left| \frac{A - \pi(\mathbf{X})}{\pi(\mathbf{X})\{1 - \pi(\mathbf{X})\}} \right|,$$

with

$$E[F^2(\mathbf{X}, A)] < \infty.$$

Then the function class $\{f(\mathbf{Z}, \boldsymbol{\beta}, \boldsymbol{\theta}) : \boldsymbol{\beta} \in \mathcal{B}, \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ is an Euclidean class with a square integrable envelope $F(\mathbf{X}, A)$. In addition, $I(S \geq 0)$ also is Euclidean. By the Condition C2 and equicontinous theorem of the empirical process then $I_1 = o_p(n^{-\frac{1}{2}}) = o_p(1)$ ( See Lemma 2.17 in Pakes and Pollard (1989) ). By the continuity of $E[V(\mathbf{Z}, \boldsymbol{\theta}) I\left(S \geqslant 0\right)]$ with respect to $\boldsymbol{\theta}$ and

Condition C2, $I_2 = o_p(1)$. By the Condition C4(b), then the function class $\{V(\mathbf{Z},\boldsymbol{\theta}_0)I(\boldsymbol{\beta}'\mathbf{X}_i \geqslant 0) : \boldsymbol{\beta} \in \mathcal{B}\}$ is an Euclidean class with a square integrable envelope $V(\mathbf{Z},\boldsymbol{\theta}_0)$. Then $I_3 = O_p(n^{-\frac{1}{2}}) = o_p(1)$ by Lemma 2.8 in Pakes and Pollard (1989). Hence $I = o_p(1)$.

For $II$ we have:

$$
\begin{aligned}
&\sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}\hat{m}(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right) - E[m(S,\boldsymbol{\theta}_0)I\left(S \geqslant 0\right)]\right| \\
\leqslant &\sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}\hat{m}(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right) - \frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right)\right| \\
&+\sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right) - \frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\boldsymbol{\theta}_0)I\left(S_i \geqslant 0\right)\right| \\
&+\sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\boldsymbol{\theta}_0)I\left(S_i \geqslant 0\right) - E[m(S,\boldsymbol{\theta}_0)I\left(S \geqslant 0\right)]\right| \\
:=&II_1 + II_2 + II_3,
\end{aligned}
$$

where

$$
\begin{aligned}
II_1 &= \sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}\hat{m}(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right) - \frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right)\right|, \\
II_2 &= \sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\hat{\boldsymbol{\theta}})I\left(S_i \geqslant 0\right) - \frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\boldsymbol{\theta}_0)I\left(S_i \geqslant 0\right)\right|, \\
II_3 &= \sup_{\boldsymbol{\beta}}\left|\frac{1}{N}\sum_{i=n+1}^{M}m(S_i,\boldsymbol{\theta}_0)I\left(S_i \geqslant 0\right) - E[m(S,\boldsymbol{\theta}_0)I\left(S \geqslant 0\right)]\right|.
\end{aligned}
$$

Then we will show that $II_1$, $II_2$ and $II_3$ are all $o_p(1)$ terms respectively.

For the first term $II_1$, we have

$$\begin{aligned}
II_1 &= \sup_{\boldsymbol{\beta}} \left| \frac{1}{N} \sum_{i=n+1}^{M} \left[ \hat{m}(S_i, \hat{\boldsymbol{\theta}}) - m(S_i, \hat{\boldsymbol{\theta}}) \right] I\left(S_i \geqslant 0\right) \right| \\
&\leqslant \sup_{\boldsymbol{\beta}} \sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|<\delta} |\hat{m}(S_i, \boldsymbol{\theta}) - m(S_i, \boldsymbol{\theta})| \frac{1}{N} \sum_{i=n+1}^{M} I\left(S_i \geqslant 0\right) \\
&= O(c_n).
\end{aligned}$$

Note that the last equation is a direct conclusion of Lemma S1. And the second term $II_2$ is a $o_p(1)$ since the continuity of $m(S, \boldsymbol{\theta})I\left(S \geqslant 0\right)$ with respect to $\boldsymbol{\theta}$ and the third term $II_3$ is $o_p(1)$ simply by Lemma 2.8 in Pakes and Pollard (1989), which are similar to $I_2$ and $I_3$ respectively. Therefore, $II = o_p(1)$.

Now we get that $\sup_{\boldsymbol{\beta}} |\hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) - \Delta_\lambda(\boldsymbol{\beta}, \boldsymbol{\theta}_0)| = \lambda I + (1-\lambda)II = o_p(1)$. The proof of consistency for $\hat{\boldsymbol{\beta}}_\lambda$ is completed.

### S2.4 Proof of Theorem 1 (b1)

Denote $g(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}) = V(\mathbf{Z}, \boldsymbol{\theta})I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, $\Xi = (\boldsymbol{\beta}, \delta)$, $\Xi_0 = (\boldsymbol{\beta}_0, 0)$, where $\delta = \boldsymbol{\theta} - \boldsymbol{\theta}_0$, and $h(\cdot, \boldsymbol{\beta}, \delta) = V(\mathbf{Z}, \boldsymbol{\theta}_0 + \delta)I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0) - V(\mathbf{Z}, \boldsymbol{\theta}_0 + \delta)I(\boldsymbol{\beta}_0'\mathbf{X} \geqslant 0)$. Then $\hat{\boldsymbol{\beta}}_{sup} = \arg\max_{\boldsymbol{\beta} \in \mathcal{B}} P_n h(\cdot, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$. Note that $P_n$ denotes the empirical expectation and $Pg(\cdot, \boldsymbol{\beta}, \theta) = E[V(\mathbf{Z}, \boldsymbol{\theta})I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)] = E[D(\mathbf{X})I(\boldsymbol{\beta}_0'\mathbf{X} \geqslant 0)]$ where $D(\mathbf{X})$ is the CATE defined in Section 2.2. Thus $Ph(\cdot, \boldsymbol{\beta}, \delta) = E[D(\mathbf{X})(I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0) - I(\boldsymbol{\beta}_0'\mathbf{X} \geqslant 0))] := H(\boldsymbol{\beta})$. We notice that when

$h(\cdot, \boldsymbol{\beta}, \delta)$ is taking expectation over $\mathbf{X}$, it is unrelated to the nuisance parameter $\delta$ anymore. Next we do the Taylor expansion of $H(\boldsymbol{\beta})$ around $\boldsymbol{\beta}_0$ that

$$Ph(\cdot, \boldsymbol{\beta}, \delta) = -\frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{\beta}_0)' V (\boldsymbol{\beta} - \boldsymbol{\beta}_0) + o(\|(\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2), \tag{S2.1}$$

where $-V$ is the second derivative matrix of $H(\boldsymbol{\beta})$ at $\boldsymbol{\beta}_0$.

In order to calculate the matrix $-V$, we consider the transformation

$$T_{\boldsymbol{\beta}} = (I - \|\boldsymbol{\beta}\|^{-2}\boldsymbol{\beta}\boldsymbol{\beta}')(I - \boldsymbol{\beta}_0\boldsymbol{\beta}_0') + \|\boldsymbol{\beta}\|^{-2}\boldsymbol{\beta}\boldsymbol{\beta}_0',$$

such that $\boldsymbol{\beta}'(T_{\boldsymbol{\beta}}\mathbf{X}) = \boldsymbol{\beta}_0'\mathbf{X}$. Thus $T_{\boldsymbol{\beta}}$ maps $A(\boldsymbol{\beta}) = \{\boldsymbol{\beta}'\mathbf{X} > 0\}$ onto $A = \{\boldsymbol{\beta}_0'\mathbf{X} > 0\}$, and $\partial A(\boldsymbol{\beta}) = \{\boldsymbol{\beta}'\mathbf{X} = 0\}$ onto $\partial A = \{\boldsymbol{\beta}_0'\mathbf{X} = 0\}$. Similar to the Example 6.4 in Kim and Pollard (1990), the surface measure $\sigma_{\boldsymbol{\beta}}$ on $\partial A(\boldsymbol{\beta})$ has the constant density $\rho_{\boldsymbol{\beta}}(\mathbf{x}) = \frac{\boldsymbol{\beta}^T\boldsymbol{\beta}_0}{\|\boldsymbol{\beta}\|}$ with respect to the image of the surface measure $\sigma = \sigma_{\boldsymbol{\beta}}$ under $T_{\boldsymbol{\beta}}$. The outward pointing normal to $A(\boldsymbol{\beta})$ is the standardized vector $-\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|}$ and along $\partial A(\boldsymbol{\beta})$ the derivative $\frac{\partial T_{\boldsymbol{\beta}}\mathbf{X}}{\partial \boldsymbol{\beta}}$ reduces to $-\|\boldsymbol{\beta}\|^{-2}[\boldsymbol{\beta}\mathbf{X}^T + (\boldsymbol{\beta}^T\mathbf{X})I]$. Thus similar to the proof of Theorem 2 in Cheng and Yang (2024), we can calculate

$$V = \int_{\boldsymbol{\beta}_0'\mathbf{X}=0} [f(\mathbf{X})\dot{D}(\mathbf{X}) + D(\mathbf{X})\dot{f}(\mathbf{X})]'\boldsymbol{\beta}_0\mathbf{X}\mathbf{X}' d\sigma,$$

where $\dot{D}(\mathbf{X})$ and $\dot{f}(\mathbf{X})$ denote the derivatives of $D(\mathbf{X})$ and $f(\mathbf{X})$, the density function of $\mathbf{X}$, with respect to $\mathbf{X}$ respectively.

By the condition C4 (b) we can apply the Lemma 4.1 of Kim and Pollard (1990) that for a given constant $M$ and any $\epsilon > 0$, uniformly for $\|\Xi - \Xi_0\| \leqslant M$, we have

$$P_n h(\cdot, \boldsymbol{\beta}, \delta) \leqslant P h(\cdot, \boldsymbol{\beta}, \delta) + \epsilon(\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 + \delta^2) + O_p(n^{-\frac{2}{3}}), \qquad \text{(S2.2)}$$

where we also apply the Cauchy-Schwarz inequality that $\|\Xi - \Xi_0\|^2 \leqslant \|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 + \delta^2$. Combining (S2.1) and (S2.2) together with conditions C2 and C5 (d), we can derive that

$$P_n h(\cdot, \boldsymbol{\beta}, \delta) \leqslant -(\frac{1}{2}\lambda_{min}(V) - \epsilon)\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|^2 + \epsilon\delta^2 + O_p(n^{-\frac{2}{3}}),$$

and

$$\begin{aligned} 0 = P_n h(\cdot, \boldsymbol{\beta}_0, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) &\leqslant P_n h(\cdot, \hat{\boldsymbol{\beta}}_{sup}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \\ &\leqslant -(\frac{1}{2}\lambda_{min}(V) - \epsilon)\|\hat{\boldsymbol{\beta}}_{sup} - \boldsymbol{\beta}_0\|^2 + O_p(n^{-\frac{2}{3}}), \end{aligned}$$

where $\lambda_{min}(V)$ is the smallest eigenvalue of the positive definite matrix $V$. Then by taking $\epsilon = \frac{1}{4}\lambda_{min}(V)$ we can get the conclusion that $\|\hat{\boldsymbol{\beta}}_{sup} - \boldsymbol{\beta}_0\| = O_p(n^{-\frac{1}{3}})$. Next we will show that $P_n h(\cdot, \hat{\boldsymbol{\beta}}_{sup}, 0) \geqslant \sup_{\boldsymbol{\beta} \in \mathcal{B}} P_n h(\cdot, \boldsymbol{\beta}, 0) - o_p(n^{-\frac{2}{3}})$, which is the first condition of Theorem 1.1 in Kim and Pollard (1990). It follows from Lemma 4.6 of Kim and Pollard (1990) that uniformly in a $O_p(n^{-\frac{1}{3}})$ neighborhood of $\boldsymbol{\beta}_0$, $W_n(\cdot, n^{\frac{1}{3}}(\boldsymbol{\beta} - \boldsymbol{\beta}_0), n^{\frac{1}{3}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)) - W_n(\cdot, n^{\frac{1}{3}}(\boldsymbol{\beta} - \boldsymbol{\beta}_0), 0) = o_p(1)$ which is the stochastic equicontinuity condition (ii) of Theorem 2.3 in Kim and Pollard (1990), where the process

$W_n(\cdot, t_1, t_2) = n^{\frac{2}{3}}(P_n - P)h(\cdot, \boldsymbol{\beta}_0 + t_1 n^{-\frac{1}{3}}, t_2 n^{-\frac{1}{3}})$. By the form of Taylor expansion in (S2.1), $Ph(\cdot, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) - Ph(\cdot, \boldsymbol{\beta}, 0) = 0$ uniformly in a $O_p(n^{-\frac{1}{3}})$ neighborhood of $\boldsymbol{\beta}_0$. Denote $\hat{\boldsymbol{\beta}}_0 = \arg\max_{\boldsymbol{\beta} \in \mathcal{B}} P_n h(\cdot, \boldsymbol{\beta}, 0)$, therefore

$$
\begin{aligned}
P_n h(\cdot, \hat{\boldsymbol{\beta}}_{sup}, 0) &= P_n h(\cdot, \hat{\boldsymbol{\beta}}_{sup}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) - o_p(n^{-\frac{2}{3}}) \\
&\geqslant P_n h(\cdot, \hat{\boldsymbol{\beta}}_0, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) - o_p(n^{-\frac{2}{3}}) \\
&= P_n h(\cdot, \hat{\boldsymbol{\beta}}_0, 0) - o_p(n^{-\frac{2}{3}}) \\
&= \sup_{\boldsymbol{\beta} \in \mathcal{B}} P_n h(\cdot, \boldsymbol{\beta}, 0) - o_p(n^{-\frac{2}{3}}).
\end{aligned}
$$

In order to calculate the limiting covariance kernel function $Cov(C_1, C_2)$, we use the local coordinates as in Example 6.4 of Kim and Pollard (1990). Define $\boldsymbol{\beta}(\mathbf{u}) = \sqrt{1 - \|\mathbf{u}\|^2}\boldsymbol{\beta}_0 + \mathbf{u}$ where $\mathbf{u}$ is orthogonal to $\boldsymbol{\beta}_0$ and ranges over a neighborhood of the origin. Such a decomposition can be get similarly by taking $\mathbf{u} = \mathbf{u}(\boldsymbol{\beta}) = T_0\boldsymbol{\beta}$ with $T_0 = I - \boldsymbol{\beta}_0\boldsymbol{\beta}_0'$. Then we can write $\boldsymbol{\beta} = (\boldsymbol{\beta}_0'\boldsymbol{\beta})\boldsymbol{\beta}_0 + T_0\boldsymbol{\beta}$ such that $\boldsymbol{\beta}_0'\boldsymbol{\beta} = \sqrt{1 - \|\mathbf{u}\|^2}$ and $\boldsymbol{\beta}_0\mathbf{u} = \boldsymbol{\beta}_0' T_0 \boldsymbol{\beta} = 0$ since the parameter space is on the sphere $(\|\boldsymbol{\beta}\| = 1, \|\boldsymbol{\beta}_0\| = 1)$. Also, $\mathbf{u}(\boldsymbol{\beta}_0 + \frac{C_1}{t}) = T_0\frac{C_1}{t}$ and $\mathbf{u}(\boldsymbol{\beta}_0 + \frac{C_2}{t}) = T_0\frac{C_2}{t}$ since $T_0\boldsymbol{\beta}_0 = 0$. Decompose $\mathbf{X}$ similarly into $r\boldsymbol{\beta}_0 + \mathbf{v}$, with $\mathbf{v}$ orthogonal to $\boldsymbol{\beta}_0$, for some random variable $r$ and random vector $\mathbf{v}$. Denote $C_1^* = T_0 C_1$ and $C_2^* = T_0 C_2$, we can obtain that

$$\left(\boldsymbol{\beta}_0+\frac{C_1}{t}\right)'\mathbf{X}=\left(\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}\boldsymbol{\beta}_0+\frac{C_1^*}{t}\right)'(r\boldsymbol{\beta}_0+\mathbf{v})$$
$$=r\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}+\frac{C_1^{*\prime}\mathbf{v}}{t}.$$

Then by the identity $ab=\frac{1}{2}(a^2+b^2-(a-b)^2)$, we first calculate that

$$\left|h(\cdot,\boldsymbol{\beta}_0+\frac{C_1}{t},0)-h(\cdot,\boldsymbol{\beta}_0+\frac{C_2}{t},0)\right|^2$$
$$=V^2(\mathbf{Z},\boldsymbol{\theta}_0)\left|I\left((\boldsymbol{\beta}_0+\frac{C_1}{t})'\mathbf{X}\geqslant 0\right)-I\left((\boldsymbol{\beta}_0+\frac{C_2}{t})'\mathbf{X}\geqslant 0\right)\right|,$$

and

$$tP\left|h(\cdot,\boldsymbol{\beta}_0+\frac{C_1}{t},0)-h(\cdot,\boldsymbol{\beta}_0+\frac{C_2}{t},0)\right|^2$$
$$=tPV^2(\mathbf{Z},\boldsymbol{\theta}_0)\left|I\left((\boldsymbol{\beta}_0+\frac{C_1}{t})'\mathbf{X}\geqslant 0\right)-I\left((\boldsymbol{\beta}_0+\frac{C_2}{t})'\mathbf{X}\geqslant 0\right)\right|$$
$$=tPq(\mathbf{X})\left|I\left(r\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}+\frac{C_1^{*\prime}\mathbf{v}}{t}\geqslant 0\right)-I\left(r\sqrt{1-\frac{\|C_2^*\|^2}{t^2}}+\frac{C_2^{*\prime}\mathbf{v}}{t}\geqslant 0\right)\right|,$$

where $q(\mathbf{X})=E[V^2(\mathbf{Z},\boldsymbol{\theta}_0)|\mathbf{X}]=\pi^{-1}(\mathbf{X})E[(Y-\nu(\mathbf{X},\boldsymbol{\theta}_0))^2|\mathbf{X},A=1)]+(1-\pi(\mathbf{X}))^{-1}E[(Y-\nu(\mathbf{X},\boldsymbol{\theta}_0))^2|\mathbf{X},A=0)]$. Denote $p(r,\mathbf{v})$ be the joint density function of $(r,\mathbf{v})$. With a change of variable $w=tr$, we have

$q(\mathbf{X}) = q(r\boldsymbol{\beta}_0 + \mathbf{v}) = q(\frac{w}{t}\boldsymbol{\beta}_0 + \mathbf{v})$ and

$$tPq(\mathbf{X})\left|I\left(r\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}+\frac{C_1^{*\prime}\mathbf{v}}{t}\geqslant 0\right)-I\left(r\sqrt{1-\frac{\|C_2^*\|^2}{t^2}}+\frac{C_2^{*\prime}\mathbf{v}}{t}\geqslant 0\right)\right|$$
$$=\iint I\left(-\frac{C_1^{*\prime}\mathbf{v}}{\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}}<w\leqslant-\frac{C_2^{*\prime}\mathbf{v}}{\sqrt{1-\frac{\|C_2^*\|^2}{t^2}}}\right)q(\frac{w}{t}\boldsymbol{\beta}_0+\mathbf{v})p(\frac{w}{t},\mathbf{v})dwd\mathbf{v}$$
$$+\iint I\left(-\frac{C_2^{*\prime}\mathbf{v}}{\sqrt{1-\frac{\|C_2^*\|^2}{t^2}}}<w\leqslant-\frac{C_1^{*\prime}\mathbf{v}}{\sqrt{1-\frac{\|C_1^*\|^2}{t^2}}}\right)q(\frac{w}{t}\boldsymbol{\beta}_0+\mathbf{v})p(\frac{w}{t},\mathbf{v})dwd\mathbf{v}.$$

Integrate over $w$, then let $t \to \infty$ to get

$$\int |(C_1 - C_2)'\mathbf{v}|q(\mathbf{v})p(0, \mathbf{v})d\mathbf{v} := L(C_1 - C_2)$$

as the limit of the sum of the two terms with $L(C) \neq 0$ for $C \neq 0$. Therefore the limiting covariance kernel can now be calculated as $Cov(C_1, C_2) = \frac{1}{2}(L(C_1) + L(C_2) - L(C_1 - C_2))$.

Then the asymptotic distribution of $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_{sup} - \boldsymbol{\beta}_0)$ follows by applying the Main Theorem of Kim and Pollard (1990). This completes the proof.

## S2.5 Proof of Theorem 1 (b2)

In this part, we try to derive the asymptotic distribution of $\hat{\boldsymbol{\beta}}_{pl}$ similar to the $\hat{\boldsymbol{\beta}}_{sup}$. A direct idea is to consider $g_m(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}) = m(\boldsymbol{\beta}'\mathbf{X}, \boldsymbol{\theta})I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, $h_m(\cdot, \boldsymbol{\beta}, \delta) = g_m(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}_0 + \delta) - g_m(\cdot, \boldsymbol{\beta}_0, \boldsymbol{\theta}_0 + \delta)$ with $\delta = \boldsymbol{\theta} - \boldsymbol{\theta}_0$ and $\hat{\boldsymbol{\beta}}_m = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} P_N h_m(\cdot, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$. In implementation, however, we can not get $\hat{\boldsymbol{\beta}}_m$ directly since we need to estimate $m(\boldsymbol{\beta}'\mathbf{X}, \boldsymbol{\theta}_0)$ by its NW estimator $\hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}})$ first. Then it is sufficient to show the convergence rate of $\hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta}_0)$. By the asymptotic property of NW estimator, $\hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) = O_p((nh)^{-\frac{1}{2}})$. From the condition C6 (b) we can derive that $(nh)^{-\frac{1}{2}} = o(n^{-\frac{1}{3}})$. And from condition C2 that $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 = O_p(n^{-\frac{1}{2}})$, then by the continuous derivability of $m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$, it follows that $m(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta}_0) = O_p(n^{-\frac{1}{2}}) = o_p(n^{-\frac{1}{3}})$. Thus

$$
\begin{aligned}
&\hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta}_0)\\
=&\hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) + m(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}}) - m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta}_0)\\
=&o_p(n^{-\frac{1}{3}}).
\end{aligned}
$$

Then consider $g_{pl}(\cdot,\boldsymbol{\beta},m) = mI(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, $h_{pl}(\cdot,\boldsymbol{\beta},\delta) = g_{pl}(\cdot,\boldsymbol{\beta},m_0+\delta) - g_{pl}(\cdot,\boldsymbol{\beta}_0,m_0+\delta)$ with $\delta = m - m_0$ and $\hat{\boldsymbol{\beta}}_{pl} = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} P_M h_{pl}(\cdot,\boldsymbol{\beta},\hat{m}-m_0)$, where $m = m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta})$, $m_0 = m(\boldsymbol{\beta}'\mathbf{X},\boldsymbol{\theta}_0)$ and $\hat{m} = \hat{m}(\boldsymbol{\beta}'\mathbf{X},\hat{\boldsymbol{\theta}})$.

Similar to the proof of Lemma 4.5 in Kim and Pollard (1990), let $Z_n(t) = n^{\frac{2}{3}} P_M h_{pl}(\cdot,\boldsymbol{\beta}_0 + tn^{-\frac{1}{3}},0)$ if $\boldsymbol{\beta}_0 + tn^{-\frac{1}{3}} \in \mathcal{B}$ and zero otherwise. And the corresponding centered process is $W_n(t) = Z_n(t) - n^{\frac{2}{3}} P h_{pl}(\cdot,\boldsymbol{\beta}_0 + tn^{-\frac{1}{3}},0)$ if $\boldsymbol{\beta}_0 + tn^{-\frac{1}{3}} \in \mathcal{B}$ and zero otherwise. With fixed $t$, $\boldsymbol{\beta}_0$ is an interior point of $\mathcal{B}$ ensures that $\boldsymbol{\beta}_0 + tn^{-\frac{1}{3}} \in \mathcal{B}$ for $n$ large enough. Condition C5.(a) implies that as $n$ goes to infinity, we have

$$n^{\frac{2}{3}} P h_{pl}(\cdot, \boldsymbol{\beta}_0 + t n^{-\frac{1}{3}}, 0) \to -\frac{1}{2} t' V_{pl} t,$$

which contributes the quadratic trend to the limit process for $Z_n(t)$. Note that $-V_{pl}$ is the second derivative matrix of $E[m(\boldsymbol{\beta}'\mathbf{X}, \boldsymbol{\theta}_0) I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)]$ with respect to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}_0$, which is equal to $-V$, the second derivative matrix of $E[V(\mathbf{Z}, \boldsymbol{\theta}_0) I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)]$ with respect to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}_0$ calculating in the proof of Theorem 1 (b1) in section S2.4. Then by the Lindeberg condition,

$$\begin{aligned} W_n(t) &= n^{\frac{2}{3}} (P_M - P) h_{pl}(\cdot, \boldsymbol{\beta}_0 + t n^{-\frac{1}{3}}, 0) \\ &= \frac{\rho}{1+\rho} \sum_{i=1}^{M} n^{-\frac{1}{3}} [h_{pl}(\mathbf{X}_i, \boldsymbol{\beta}_0 + t n^{-\frac{1}{3}}, 0) - P h_{pl}(\cdot, \boldsymbol{\beta}_0 + t n^{-\frac{1}{3}}, 0)] \\ &\to G_{pl}(t). \end{aligned}$$

Here $G_{pl}(t)$ is a mean-zero Gaussian process with continuous sample paths and covariance kernel function $(\frac{\rho}{1+\rho})^2 Cov(C_1, C_2)$, where $Cov(C_1, C_2)$ was defined in section S2.4. Thus, we could drop the conclusion that

$$n^{\frac{1}{3}} (\hat{\boldsymbol{\beta}}_{pl} - \boldsymbol{\beta}_0) \xrightarrow{d} \arg\max_t Z_{pl}(t),$$

where the process $Z_{pl}(t) = G_{pl}(t) - \frac{1}{2} t' V t$.

## S2.6 Proof of Theorem 1 (b3)

Recall that $\hat{\boldsymbol{\beta}}_\lambda = \arg\max_{\boldsymbol{\beta} \in \mathcal{B}} \lambda P_n h(\cdot, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + (1-\lambda) P_{M-n} h_{pl}(\cdot, \boldsymbol{\beta}, \hat{m} - m)$, where $P_{M-n}$ represents the empirical expectation of samples in $\{n +$

$1, \ldots, M\}$. According to the previous discussions, $\hat{\boldsymbol{\beta}}_\lambda$ also has the cube root convergence rate. And applying the Theorem 6.6 in Chapter 6 of Gut (2009), we have

$$n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_\lambda - \boldsymbol{\beta}_0) \xrightarrow{d} \arg\max_t Z_\lambda(t),$$

where the process $Z_\lambda(t) = G_\lambda(t) - \frac{1}{2}t'Vt$. Here $G_\lambda(t)$ is a mean-zero Gaussian process with continuous sample paths and covariance kernel function $[\lambda^2 + (1-\lambda)^2\rho^2]Cov(C_1, C_2)$.

## S2.7 Proof of Theorem S1 (a1)-(a3)

As has discussed before that to maximize $E\{D(\mathbf{X})I(\boldsymbol{\beta}'\mathbf{X} \geqslant c)\}$ is equivalent to maximize $E\{D(\mathbf{X})g(\mathbf{X})I(\boldsymbol{\beta}'\mathbf{X} \geqslant c)\}$ for any positive function $g(\cdot)$, under the class of monotonic increasing index models for $D(\mathbf{X})$. Recall that for M-estimate, we only need to prove $\sup_{\boldsymbol{\beta}} |\hat{\Delta}_\lambda^{DR}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\alpha}}) - \Delta_\lambda^{DR}(\boldsymbol{\beta}, \boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)| \xrightarrow{P} 0$ as $n \to \infty$ to get the conclusion in Theorem S1 (a1)-(a3). Following similar arguments by Cavanagh and Sherman (1998), we can show that for any given $(\boldsymbol{\theta}, \boldsymbol{\alpha})$, $\Delta_\lambda^{DR}(\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{\alpha})$ has a unique maximizer at $\boldsymbol{\beta} = \boldsymbol{\beta}_0$. The consistency of $\hat{\boldsymbol{\beta}}_\lambda^{DR}$ can be similarly derived as for $\hat{\boldsymbol{\beta}}_\lambda$ given in the proof of Theorem 1.

### S2.8 Proof of Theorem S1 (b1)-(b3)

Similar to the proof for Theorem 1 (b1)-(b3), let $\boldsymbol{\delta} = (\boldsymbol{\theta}, \boldsymbol{\alpha})' - (\boldsymbol{\theta}_0, \boldsymbol{\alpha}_0)' = O_p(n^{-\frac{1}{2}})$ rather than $\boldsymbol{\theta} - \boldsymbol{\theta}_0$ in the proof of Theorem 1 (b1) when the propensity score is unknown, then the conclusion Theorem S1 (b1) can be dropped, where $Cov^{DR}(C_1, C_2) = \lim_{t\to\infty} th(\cdot, \boldsymbol{\beta}_0 + \frac{C_1}{t}, 0)h(\cdot, \boldsymbol{\beta}_0 + \frac{C_2}{t}, 0)$ for each $C_1$, $C_2$ in $\mathbb{R}^p$. When it comes to the proof of Theorem S1 (b2), we can also derive that $\hat{m}(\boldsymbol{\beta}'\mathbf{X}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\alpha}}) - m(\boldsymbol{\beta}'\mathbf{X}, \boldsymbol{\theta}_0, \boldsymbol{\alpha}_0) = o_p(n^{-\frac{1}{3}})$. Then (b2) holds. The conclusion of Theorem S1 (b3) obviously holds when (b1) and (b2) are both proved.

## S3 Justification for variance estimation procedure

### S3.1 Estimating procedure for $\Sigma_{sup}$

1. Generate iid perturbation $\xi_i$ from $Beta(\sqrt{2}-1, 1)$ for $i = 1, \ldots, n$.

2. Perturb the value function. Let $\hat{\boldsymbol{\theta}}^b = \arg\min_{\boldsymbol{\theta}} \frac{1}{n}\sum_{i=1}^n \xi_i(1-A_i)[Y_i - \nu(\mathbf{X}_i, \boldsymbol{\theta})]^2$, then for linear decision $d_{\boldsymbol{\beta}}(\mathbf{X}) = I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, we perturb the value function by

$$\hat{\Delta}^b_{sup}\left(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}^b\right) = \frac{1}{n}\sum_{i=1}^n \xi_i V\left(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}^b\right) d_{\boldsymbol{\beta}}(\mathbf{X}_i).$$

3. Re-estimate $\boldsymbol{\beta}$. We use the iterative algorithm derived in Remark 2 to obtain the new estimator that $\hat{\boldsymbol{\beta}}^b_{sup} = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \hat{\Delta}^b_{sup}\left(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}^b\right)$.

4. Estimate the variance. Repeat the above steps for $B$ times and compute the empirical variance matrix $\hat{\Sigma}_{sup}$ of $\{\hat{\boldsymbol{\beta}}^b_{sup}, b = 1, \ldots, B\}$ to estimate the population variance $\Sigma_{sup}$.

### S3.2 Estimating procedure for $\Sigma_{pl}$

1. Generate iid perturbation $\xi_i$ from $Beta(\sqrt{2}-1, 1)$ for $i = 1, \ldots, n+N$.

2. Perturb the value function. Let $\hat{\boldsymbol{\theta}}^b = \arg\min_{\boldsymbol{\theta}} \frac{1}{n}\sum_{i=1}^n \xi_i(1-A_i)[Y_i - \nu(\mathbf{X}_i, \boldsymbol{\theta})]^2$ and $\hat{m}^b(\boldsymbol{\beta}'\mathbf{X}_j, \boldsymbol{\theta}) = \frac{\sum_{i=1}^n \xi_i K_h(\boldsymbol{\beta}'\mathbf{X}_i - \boldsymbol{\beta}'\mathbf{X}_j)V(\mathbf{Z}_i, \boldsymbol{\theta})}{\sum_{i=1}^n \xi_i K_h(\boldsymbol{\beta}'\mathbf{X}_i - \boldsymbol{\beta}'\mathbf{X}_j)}$, then for linear decision $d_{\boldsymbol{\beta}}(\mathbf{X}) = I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, we perturb the value function by

$$\hat{\Delta}^b_{pl}\left(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}^b\right) = \frac{1}{n+N}\sum_{j=1}^{n+N} \xi_j \hat{m}^b\left(\boldsymbol{\beta}'\mathbf{X}_j, \hat{\boldsymbol{\theta}}^b\right) d_{\boldsymbol{\beta}}(\mathbf{X}_j).$$

3. Re-estimate $\boldsymbol{\beta}$. We use the iterative algorithm derived in Remark 2 to obtain the new estimator that $\hat{\boldsymbol{\beta}}^b_{pl} = \arg\max_{\boldsymbol{\beta}\in\mathcal{B}} \hat{\Delta}^b_{pl}\left(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}^b\right)$.

4. Estimate the variance. Repeat the above steps for $B$ times and compute the empirical variance matrix $\hat{\Sigma}_{pl}$ of $\{\hat{\boldsymbol{\beta}}^b_{pl}, b = 1, \ldots, B\}$ to estimate the population variance $\Sigma_{pl}$.

### S3.3 Theoretical guarantees for variance estimation procedures

As the notation in section S2.4, denote that $g(\cdot, \boldsymbol{\beta}, \boldsymbol{\theta}) = V(\mathbf{Z}, \boldsymbol{\theta})I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$, $\Xi = (\boldsymbol{\beta}, \delta)$, $\Xi_0 = (\boldsymbol{\beta}_0, 0)$, where $\delta = \boldsymbol{\theta} - \boldsymbol{\theta}_0$, and $h(\cdot, \boldsymbol{\beta}, \delta) = V(\mathbf{Z}, \boldsymbol{\theta}_0 + \delta)(I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0) - I(\boldsymbol{\beta}_0'\mathbf{X} \geqslant 0))$. Recall that $\mathbf{Z}_i = (\mathbf{X}_i, Y_i, A_i)$ for $i = 1, \ldots, n$

and

$$\hat{\Delta}_{sup}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) = \frac{1}{n}\sum_{i=1}^{n} V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) I(\boldsymbol{\beta}'\mathbf{X}_i \geqslant 0)$$

Thus, $\hat{\Delta}_{sup}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$ is a U-process of degree 1 with symmetric kernel function $g(\mathbf{Z}, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) = V(\mathbf{Z}, \hat{\boldsymbol{\theta}}) I(\boldsymbol{\beta}'\mathbf{X} \geqslant 0)$ with respect to $\mathbf{Z}$. By the definition, $\hat{\boldsymbol{\beta}}_{sup}$ is the maximizer of $\hat{\Delta}_{sup}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$. Let $\{\xi_i : i = 1, \ldots, n\}$ be $n$ i.i.d copies from the nonnegative perturbation variable $\xi$ with mean $\mu$ and variance $\mu^2$. Consider the stochastic perturbation process

$$\hat{\Delta}^b_{sup}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}}) = \frac{1}{n}\sum_{i=1}^{n} \xi_i V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) I(\boldsymbol{\beta}'\mathbf{X}_i \geqslant 0) = P_n \xi g(\mathbf{Z}; \boldsymbol{\beta}, \hat{\boldsymbol{\theta}}),$$

and let $\hat{\boldsymbol{\beta}}^b_{sup}$ be the maximizer of $\hat{\Delta}^b_{sup}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$. Following Jin et al. (2001), we assume without loss of generality that the mean and variance of $\xi$ are both 1 for simplicity, as other cases can be handled through appropriate rescaling. Then we have $\boldsymbol{\beta}_0$, the maximizer of $Pg(\mathbf{Z}, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$, is also the maximizer of $P\xi g(\mathbf{Z}, \boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$. Similar to the proof of Theorem 1 (b1), $\|\hat{\boldsymbol{\beta}}^b_{sup} - \boldsymbol{\beta}_0\| = O_p(n^{-\frac{1}{3}})$, thus $\|\hat{\boldsymbol{\beta}}^b_{sup} - \hat{\boldsymbol{\beta}}_{sup}\| = O_p(n^{-\frac{1}{3}})$ since $\|\hat{\boldsymbol{\beta}}_{sup} - \boldsymbol{\beta}_0\| = O_p(n^{-\frac{1}{3}})$. This implies that there exists a constant vector $\mathbf{C}$ such that $\hat{\boldsymbol{\beta}}_{sup} = \boldsymbol{\beta}_0 + n^{-\frac{1}{3}}\mathbf{C}$. For fixed $t_1 \in \mathbb{R}^p$ such that $\hat{\boldsymbol{\beta}}_{sup} + n^{-\frac{1}{3}}t_1$ belongs to the parameter space $\mathcal{B}$ for $n$ large enough, we denote $\tilde{t} = t_1 + \mathbf{C}$, then $n^{\frac{2}{3}} P\xi h(\cdot, \hat{\boldsymbol{\beta}}_{sup} + n^{-\frac{1}{3}}t_1, \cdot) = n^{\frac{2}{3}} P\xi h(\cdot, \boldsymbol{\beta}_0 + n^{-\frac{1}{3}}\tilde{t}, \cdot)$ converges to $-\frac{1}{2}\tilde{t}'V\tilde{t}$, where $V$ is defined the same as that in Theorem 1. Moreover, since $P|\xi h(\cdot, \Xi_1) - \xi h(\cdot, \Xi_2)| = O(\|\Xi_1 -$

$\Xi_2\|)$ for $\Xi_1$ and $\Xi_2$ near $\Xi_0$ and together with condition C4, the process $J_n(\cdot, t_1, t_2) = n^{\frac{2}{3}}(P_n - P)\xi h(\cdot, \hat{\boldsymbol{\beta}}_{sup} + t_1 n^{-\frac{1}{3}}, t_2 n^{-\frac{1}{3}}) = n^{\frac{2}{3}}(P_n - P)\xi h(\cdot, \boldsymbol{\beta}_0 + \tilde{t} n^{-\frac{1}{3}}, t_2 n^{-\frac{1}{3}})$ satisfies the stochastic equicontinuity condition (ii) in Theorem 2.3 of Kim and Pollard (1990). By Lemma 4.6 in Kim and Pollard (1990), then we have $J_n(\cdot, n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}^b_{sup} - \hat{\boldsymbol{\beta}}_{sup}), n^{\frac{1}{3}}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)) - J_n(\cdot, n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}^b_{sup} - \hat{\boldsymbol{\beta}}_{sup}), 0) = o_p(1)$. Define $\hat{\boldsymbol{\beta}}^b_0 = \arg\max_{\boldsymbol{\beta} \in \mathcal{B}} P_n \xi h(\cdot, \boldsymbol{\beta}, 0)$, thus

$$
\begin{aligned}
n^{\frac{2}{3}} P_n \xi h(\cdot, \hat{\boldsymbol{\beta}}^b_{sup}, 0) &= n^{\frac{2}{3}} P_n \xi h(\cdot, \hat{\boldsymbol{\beta}}^b_{sup}, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) - o_p(1) \\
&\geqslant n^{\frac{2}{3}} P_n \xi h(\cdot, \hat{\boldsymbol{\beta}}^b_0, \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) - o_p(1) \\
&= n^{\frac{2}{3}} P_n \xi h(\cdot, \hat{\boldsymbol{\beta}}^b_0, 0) - o_p(1) \\
&= n^{\frac{2}{3}} \sup_{\boldsymbol{\beta} \in \mathcal{B}} P_n \xi h(\cdot, \boldsymbol{\beta}, 0) - o_p(1).
\end{aligned}
$$

By Lemma 4.5 of Kim and Pollard (1990), we can derive that the limit distribution of the finite-dimensional projections of the process $Z_n(t) = n^{\frac{2}{3}} P_n \xi h(\cdot, \hat{\boldsymbol{\beta}}_{sup} + t n^{-\frac{1}{3}}, 0)$ correspond to the finite-dimensional projections of a process $Z(\tilde{t}) = G(\tilde{t}) - \frac{1}{2}\tilde{t}' V \tilde{t}$, where $G(\cdot)$ and $V$ are defined the same as that in Theorem 1. Therefore, applying Theorem 2.7 of Kim and Pollard (1990), we can derive that $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}^b_{sup} - \hat{\boldsymbol{\beta}}_{sup}) \xrightarrow{d} \arg\max_t Z(t)$. Hence $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}_{sup} - \boldsymbol{\beta}_0)$ has the same asymptotic distribution as that of $n^{\frac{1}{3}}(\hat{\boldsymbol{\beta}}^b_{sup} - \hat{\boldsymbol{\beta}}_{sup})$. And we can complete the justification of our resampling-based variance estimation procedure in a similar way to the proofs of Theorem 1(b2) and (b3).

# S4 Additional numerical results

## S4.1 Iterative algorithm

Iterative algorithm for cestimating $\hat{\boldsymbol{\beta}}_\lambda$ and similarly for $\hat{\boldsymbol{\beta}}_{pl}$.

begin

calculate $\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{n}\sum_{i=1}^n (1-A_i)[Y_i - \boldsymbol{\theta}'\mathbf{X}_i]^2$.

for i=1 to n do

calculate $V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}})$.

calculate $\hat{\boldsymbol{\beta}}_{\mathsf{sup}} = \arg\max_{\boldsymbol{\beta}} \hat{\Delta}_{\mathsf{sup}}(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$.

set t=0; $\lambda = \frac{n}{n+N}$; $\hat{\boldsymbol{\beta}}^{(t)} = \hat{\boldsymbol{\beta}}_{\mathsf{sup}}$;

for j=n+1 to n+N do

step 1

calculate $\hat{m}(\hat{\boldsymbol{\beta}}^{(t)\prime}\mathbf{X}_i, \hat{\boldsymbol{\theta}})$;

plug $\hat{m}(\hat{\boldsymbol{\beta}}^{(t)\prime}\mathbf{X}_i, \hat{\boldsymbol{\theta}})$ in $\hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$;

step 2

calculate $\hat{\boldsymbol{\beta}}^{(t+1)} = \arg\max_{\boldsymbol{\beta}} \hat{\Delta}_\lambda(\boldsymbol{\beta}, \hat{\boldsymbol{\theta}})$.

if $\|\hat{\boldsymbol{\beta}}^{(t+1)} - \hat{\boldsymbol{\beta}}^{(t)}\|_\infty > 10^{-4}$ do

t=t+1;

$\hat{\boldsymbol{\beta}}^{(t)} = \hat{\boldsymbol{\beta}}^{(t-1)}$ and iterate step 1 and step 2;

else do

$\hat{\boldsymbol{\beta}}_\lambda = \hat{\boldsymbol{\beta}}^{(t+1)}$.

end

## S4.2 Simulation description and additional results

Under all the cases, the optimal ITR that maximizes the value function and the optimal ITR that in our interested decision class $\mathcal{D}$ are the same, which is given by $d^{\text{opt}}(\mathbf{x}) = I\left(\boldsymbol{\beta}_{0,std}'\mathbf{x} \geqslant 0\right)$ with $\boldsymbol{\beta}_{0,std}$ after imposing the constraint $\left\|\boldsymbol{\beta}_{0,std}\right\| = 1$. Note that the true interception here is zero which is contained in our estimating procedure as discussion in remark 1, but what we are most interested in is the efficiency improvement of the covariate coefficients, thus we do not report the intercept term in our tables. In the simulation, we evaluate the performance of three estimation methods, namely the supervised (sup), semi-supervised (SS) with the optimal tuning weight $\lambda = \frac{\rho^2}{1+\rho^2}$, and pooled (pl) estimators respectively. We conduct 1000 simulation runs for each case. All the tables report the mean bias (Bias) of the estimators, the standard error (SE) for the estimators, the standard deviation (SD), eatimated by $\frac{1}{1000}\sum_{i=1}^{1000} SD_i$, with each $SD_i$ calculated from 200 bootstrap samples as described in section 4 in the $i$-th simulation run, using the perturbation variable $\xi \sim \text{Beta}(\sqrt{2}-1, 1)$. Additionally, we report the relative efficiency (Eff) of the proposed SS and pl estimators in terms of mean squared error (MSE) compared to the supervised (sup) estimator which is calculated by the following formula:

$$\text{Eff} = \frac{\text{MSE of sup estimator}}{\text{MSE of SS (pl) estimator}}.$$

For asymmetric distributions, directly using the 2.5-th and 97.5-th percentiles to construct a 95% confidence interval is inappropriate, as it may result in an overly wide interval. Therefore, we adopt a confidence interval based on adaptively skewness-adjusted quantiles. Specifically, for a confidence level of $\tau$, we define $q_1 = \tau/(2+2K\cdot|\text{skewness}|)$ and $q_2 = \tau - q_1$. If the skewness is positive, the confidence interval is constructed using the $q_1$-th percentile and the $(1-(\tau-q_1))$-th percentile. Conversely, if the skewness is negative, the interval is based on the $q_2$-th percentile and the $(1-(\tau-q_2))$-th percentile. Here, $K$ is a tunable positive parameter that controls the degree to which skewness influences the quantile levels. A larger $K$ increases the impact of skewness on the adjustment. In our numerical studies, we set $K = 3$. In simulation studies, we use a Gaussian kernel with bandwidth $h_n = 0.5n^{-1/3}$ which satisfies the condition C6 (b). The optimization in the proposed methods is done by the 'optim' function in R with the default method 'Nelder-Mead' for searching the maximizer.

**Remark S1.** Due to the non-smooth nature of the value function containing indicator functions, conventional gradient-based optimization methods are unsuitable for our problem. We instead employ the derivative-free

Nelder-Mead simplex algorithm (Nelder and Mead, 1965) implemented via R's ‘optim‘ function, which is particularly well-adapted to handle non-differentiable and non-convex objective functions. This method operates by iteratively evolving a simplex through reflection, expansion, contraction and shrinkage operations, systematically guiding the search toward local optima without requiring gradient information. The algorithm's robustness to discontinuous functions and its ability to perform reasonably well in avoiding local optima make it particularly suitable for our estimation problem.

When the PS is unknown, we set as follows. For correctly specified baseline treatment-free effect model for $\nu(\mathbf{X})$, we generate the true model as that in case 1 and case 2. For misspecified model for $\nu(\mathbf{X})$, we use the same setting as case 3 and case 4. For correctly specified PS model, we generate $\pi(\mathbf{X})$ by a logistic regression model, and for misspecified scenario, we set $\pi(\mathbf{X}) = s(-0.5 + X_1^2 + X_2^2)$, where $s(x) = \frac{1}{1+e^{-x}}$ is the sigmoid function.

### S4.3 Real data description and additional results

The AIDS Clinical Trials Group Protocol 175 (ACTG 175) dataset comprises clinical data from a randomized controlled trial designed to evaluate the efficacy of different antiretroviral therapies in HIV-infected patients. The study included a total of 2139 participants, who were randomly assigned to one of four treatment arms: zidovudine (ZDV) monotherapy, didanosine (ddI) monotherapy, combination therapy with ZDV and ddI, and combination therapy with ZDV and zalcitabine (ddC). Baseline characteristics $\mathbf{X}$ of participants include seven binary variables, haemophilia (0, no; 1, yes), homosexual activity (0, no; 1, yes), history of intravenous drug use (0, no; 1, yes), race (0, white; 1, non-white), gender (0, female; 1, male), antiretroviral history (0, naive; 1, experienced) and symptomatic status (0, asymptomatic; 1, symptomatic), and four continuous variables, age (years), weight (kg), CD4 T cell count at baseline and CD8 T cell count (cells per cubic millimetre) at baseline.

To investigate the validity of the MCAR assumption, we first report some summary statistics including mean and standard deviation (SD) of these covariates $\mathbf{X}$ in Table S11, along with the p-values of the Kolmogorov-Smirnov (K-S) test for continuous covariates or Chi-Square ($\chi^2$) test for discrete covariates to evaluate the equality of these distributions.

We standardize the four continuous covariates by subtracting the mean and dividing by the standard deviation before estimating procedure. The PS estimated by the logistic regression model is 0.5, which is the same as that directly set in Fan et al. (2017). The bandwidth $h = 0.28$ in kernel estimator is selected using the 'npregbw' function from the np package in R and the tuning weight $\lambda$ is chosen as the optimal value $\frac{\rho^2}{1+\rho^2} = 0.5$ derived in section 3.

To further validate the performance of our method, we conducted additional analyses using a train-test splitting approach. The dataset was randomly divided into training (70%) and testing (30%) sets, with this process repeated 50 times to account for variability in random splitting. The results consistently showed that the point estimates of optimal ITR parameters aligned closely with our original findings (Table 2 in the main text), demonstrating the robustness of our semi-supervised approach. Notably, while the supervised method continued to produce statistically insignificant confidence intervals at both 0.05 and 0.1 levels, our semi-supervised estimators achieved significant interval estimates for several covariate coefficients, attributable to their reduced asymptotic variance. These findings further reinforce the advantages of our proposed method in real-world applications, particularly in terms of estimation precision and inferential reliability. The detailed results are presented in Tables S14 and S15, which summarize the averaged outcomes across all repetitions.

We then computed the pseudo-outcome value using the doubly robust score to construct an estimate of the counterfactual difference $Y^*(1)-Y^*(0)$:

$$V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}}) = \frac{\{Y_i - \nu(\mathbf{X}_i, \hat{\boldsymbol{\theta}})\}\{A_i - \pi(\mathbf{X}_i)\}}{\pi(\mathbf{X}_i)\{1 - \pi(\mathbf{X}_i)\}},$$

for each individual in the full sample, which provides an estimate of the CATE $D(\mathbf{X}_i)$. We summarize the overall performance using the median of these individual pseudo-outcome values. In the full dataset, the median value of $V(\mathbf{Z}_i, \hat{\boldsymbol{\theta}})$ is approximately 5, indicating a substantial treatment benefit. We repeated this analysis under the 50 random sample splits. Across these repetitions, the average of the medians of the pseudo-outcome values is approximately 1.11, and the median of these 50 medians is approximately 2.51. These positive values suggest that the treatment ZDV + ddI generally provides greater benefit than the alternative ZDV + ddC for a majority of individuals in the dataset. This finding further validates the rationality of the semi-supervised treatment recommendations in the main text and aligns with the clinical literature.

Table S1: Results under case S2 with known propensity score

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | - 0.018 | 0.032 | - 0.049 | - 0.021 | - 0.065 | - 0.037 |
| | | SE | 0.121 | 0.116 | 0.117 | 0.116 | 0.163 | 0.316 |
| | | SD | 0.111 | 0.111 | 0.112 | 0.111 | 0.162 | 0.280 |
| | | CP(%) | 95.2 | 92.7 | 91.4 | 94.8 | 94.4 | 91.4 |
| SS | 200 | Bias | 0.009 | 0.021 | - 0.002 | 0.007 | - 0.009 | - 0.021 |
| | | SE | 0.066 | 0.051 | 0.059 | 0.058 | 0.056 | 0.057 |
| | | SD | 0.055 | 0.048 | 0.054 | 0.056 | 0.052 | 0.052 |
| | | CP(%) | 97.7 | 98.1 | 97.6 | 98.3 | 98.8 | 99.2 |
| | | **Eff** | **3.419** | **4.922** | **4.314** | **4.000** | **9.007** | **28.640** |
| | 500 | Bias | 0.007 | 0.020 | - 0.008 | 0.004 | - 0.012 | -0.007 |
| | | SE | 0.051 | 0.044 | 0.060 | 0.050 | 0.058 | 0.058 |
| | | SD | 0.056 | 0.047 | 0.065 | 0.056 | 0.059 | 0.062 |
| | | CP(%) | 97.4 | 98.8 | 96.5 | 98.1 | 98.0 | 97.7 |
| | | **Eff** | **5.774** | **6.536** | **4.053** | **5.531** | **8.320** | **28.831** |
| pl | 200 | Bias | 0.004 | 0.024 | - 0.016 | -0.001 | - 0.011 | -0.008 |
| | | SE | 0.044 | 0.040 | 0.065 | 0.047 | 0.053 | 0.068 |
| | | SD | 0.055 | 0.047 | 0.072 | 0.055 | 0.058 | 0.069 |
| | | CP(%) | 98.6 | 99.1 | 92.9 | 98.8 | 94.3 | 97.0 |
| | | **Eff** | **7.603** | **7.484** | **3.376** | **6.161** | **9.838** | **21.658** |
| | 500 | Bias | 0.001 | 0.018 | - 0.015 | - 0.003 | - 0.012 | -0.007 |
| | | SE | 0.040 | 0.031 | 0.062 | 0.038 | 0.047 | 0.066 |
| | | SD | 0.048 | 0.041 | 0.068 | 0.048 | 0.053 | 0.065 |
| | | CP(%) | 98.0 | 99.2 | 90.8 | 99.3 | 92.8 | 92.8 |
| | | **Eff** | **9.093** | **12.303** | **3.796** | **9.372** | **12.770** | **23.171** |

Table S2: Results under case S3 with known propensity score

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | 0.003 | 0.011 | -0.004 | 0.000 | -0.007 | -0.003 |
| | | SE | 0.050 | 0.037 | 0.047 | 0.041 | 0.050 | 0.083 |
| | | SD | 0.046 | 0.036 | 0.043 | 0.038 | 0.048 | 0.082 |
| | | CP(%) | 96.0 | 97.4 | 95.5 | 98.7 | 96.9 | 94.6 |
| SS | 200 | Bias | 0.006 | 0.017 | -0.004 | 0.006 | -0.005 | 0.008 |
| | | SE | 0.037 | 0.030 | 0.034 | 0.033 | 0.032 | 0.033 |
| | | SD | 0.032 | 0.028 | 0.029 | 0.030 | 0.030 | 0.033 |
| | | CP(%) | 96.8 | 96.5 | 97.8 | 97.9 | 99.6 | 99.3 |
| | | **Eff** | **1.816** | **1.361** | **1.912** | **1.523** | **2.387** | **6.301** |
| | 500 | Bias | 0.005 | 0.014 | -0.004 | 0.005 | -0.005 | 0.009 |
| | | SE | 0.026 | 0.020 | 0.023 | 0.022 | 0.025 | 0.029 |
| | | SD | 0.025 | 0.022 | 0.024 | 0.023 | 0.027 | 0.033 |
| | | CP(%) | 98.1 | 96.1 | 97.8 | 98.8 | 99.6 | 98.7 |
| | | **Eff** | **3.686** | **2.790** | **3.963** | **3.378** | **3.943** | **7.697** |
| pl | 200 | Bias | 0.007 | 0.014 | -0.004 | 0.005 | -0.006 | 0.011 |
| | | SE | 0.016 | 0.015 | 0.017 | 0.015 | 0.019 | 0.029 |
| | | SD | 0.020 | 0.020 | 0.022 | 0.020 | 0.023 | 0.034 |
| | | CP(%) | 98.4 | 96.6 | 96.1 | 99.0 | 99.2 | 97.8 |
| | | **Eff** | **9.420** | **4.386** | **7.139** | **7.023** | **6.926** | **7.752** |
| | 500 | Bias | 0.005 | 0.011 | -0.003 | 0.005 | -0.006 | 0.010 |
| | | SE | 0.012 | 0.011 | 0.014 | 0.012 | 0.016 | 0.026 |
| | | SD | 0.013 | 0.012 | 0.016 | 0.013 | 0.016 | 0.029 |
| | | CP(%) | 99.1 | 96.9 | 97.9 | 99.2 | 99.7 | 97.9 |
| | | **Eff** | **15.496** | **7.733** | **11.638** | **10.527** | **8.780** | **9.897** |

Table S3: Results under case S4 with known propensity score

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | -0.001 | 0.011 | -0.002 | 0.003 | -0.004 | 0.002 |
| | | SE | 0.041 | 0.038 | 0.033 | 0.031 | 0.036 | 0.062 |
| | | SD | 0.037 | 0.034 | 0.030 | 0.031 | 0.035 | 0.059 |
| | | CP(%) | 98.0 | 96.1 | 96.8 | 99.4 | 98.0 | 94.9 |
| SS | 200 | Bias | -0.002 | 0.019 | 0.001 | 0.003 | -0.004 | 0.010 |
| | | SE | 0.033 | 0.029 | 0.026 | 0.029 | 0.030 | 0.032 |
| | | SD | 0.029 | 0.027 | 0.024 | 0.026 | 0.027 | 0.030 |
| | | CP(%) | 98.3 | 92.2 | 98.4 | 98.8 | 99.6 | 99.3 |
| | | **Eff** | **1.540** | **1.400** | **1.623** | **1.140** | **1.412** | **3.556** |
| | 500 | Bias | -0.002 | 0.015 | 0.002 | 0.003 | -0.004 | 0.011 |
| | | SE | 0.022 | 0.020 | 0.018 | 0.018 | 0.022 | 0.028 |
| | | SD | 0.022 | 0.020 | 0.019 | 0.020 | 0.023 | 0.028 |
| | | CP(%) | 99.0 | 91.7 | 99.1 | 98.9 | 99.8 | 98.6 |
| | | **Eff** | **3.451** | **2.781** | **3.285** | **2.881** | **2.532** | **4.590** |
| pl | 200 | Bias | 0.000 | 0.015 | 0.001 | 0.003 | -0.003 | 0.010 |
| | | SE | 0.013 | 0.013 | 0.014 | 0.013 | 0.017 | 0.025 |
| | | SD | 0.017 | 0.017 | 0.017 | 0.017 | 0.019 | 0.029 |
| | | CP(%) | 99.0 | 92.8 | 99.3 | 99.5 | 99.5 | 98.2 |
| | | **Eff** | **9.444** | **5.495** | **5.281** | **5.828** | **4.306** | **5.773** |
| | 500 | Bias | -0.001 | 0.012 | 0.001 | 0.003 | -0.003 | 0.009 |
| | | SE | 0.011 | 0.010 | 0.011 | 0.010 | 0.013 | 0.021 |
| | | SD | 0.013 | 0.013 | 0.014 | 0.013 | 0.016 | 0.026 |
| | | CP(%) | 99.9 | 89.5 | 98.9 | 99.6 | 100.0 | 97.9 |
| | | **Eff** | **13.060** | **8.895** | **9.037** | **8.736** | **7.580** | **7.891** |

Table S4: Results under case S5 with known propensity score

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | - 0.004 | 0.032 | - 0.024 | - 0.011 | - 0.059 | - 0.025 |
| | | SE | 0.122 | 0.093 | 0.116 | 0.101 | 0.134 | 0.252 |
| | | SD | 0.107 | 0.090 | 0.105 | 0.094 | 0.128 | 0.226 |
| | | CP(%) | 95.9 | 94.6 | 92.7 | 97.8 | 93.2 | 92.0 |
| SS | 200 | Bias | 0.020 | 0.035 | - 0.003 | 0.013 | - 0.039 | 0.011 |
| | | SE | 0.078 | 0.061 | 0.074 | 0.076 | 0.072 | 0.067 |
| | | SD | 0.068 | 0.058 | 0.067 | 0.067 | 0.066 | 0.062 |
| | | CP(%) | 97.5 | 98.6 | 98.5 | 99.0 | 97.3 | 99.5 |
| | | **Eff** | **2.369** | **2.080** | **2.500** | **1.761** | **3.323** | **14.183** |
| | 500 | Bias | 0.010 | 0.039 | - 0.023 | 0.007 | - 0.039 | 0.016 |
| | | SE | 0.079 | 0.057 | 0.089 | 0.066 | 0.078 | 0.080 |
| | | SD | 0.073 | 0.060 | 0.082 | 0.071 | 0.078 | 0.079 |
| | | CP(%) | 97.8 | 97.4 | 95.5 | 98.3 | 92.2 | 96.7 |
| | | **Eff** | **2.383** | **2.282** | **1.684** | **2.335** | **2.894** | **9.868** |
| pl | 200 | Bias | 0.002 | 0.044 | - 0.030 | 0.000 | - 0.044 | 0.024 |
| | | SE | 0.073 | 0.056 | 0.089 | 0.067 | 0.083 | 0.090 |
| | | SD | 0.063 | 0.054 | 0.082 | 0.062 | 0.074 | 0.081 |
| | | CP(%) | 99.7 | 99.6 | 99.4 | 99.9 | 95.7 | 99.3 |
| | | **Eff** | **2.800** | **2.247** | **1.640** | **2.274** | **2.476** | **7.520** |
| | 500 | Bias | - 0.002 | 0.036 | - 0.031 | - 0.003 | - 0.041 | 0.025 |
| | | SE | 0.061 | 0.049 | 0.081 | 0.060 | 0.070 | 0.088 |
| | | SD | 0.054 | 0.048 | 0.077 | 0.054 | 0.067 | 0.079 |
| | | CP(%) | 99.9 | 99.6 | 99.6 | 99.9 | 95.5 | 98.6 |
| | | **Eff** | **4.054** | **3.002** | **1.976** | **2.894** | **3.426** | **7.876** |

Table S5: Results under case S6 with known propensity score

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | - 0.010 | 0.024 | - 0.022 | - 0.002 | - 0.053 | - 0.017 |
| | | SE | 0.109 | 0.100 | 0.100 | 0.095 | 0.118 | 0.237 |
| | | SD | 0.096 | 0.091 | 0.091 | 0.087 | 0.114 | 0.212 |
| | | CP(%) | 95.9 | 94.2 | 94.4 | 97.1 | 95.2 | 91.3 |
| SS | 200 | Bias | 0.007 | 0.032 | - 0.009 | 0.012 | - 0.027 | 0.014 |
| | | SE | 0.074 | 0.062 | 0.068 | 0.069 | 0.069 | 0.067 |
| | | SD | 0.067 | 0.057 | 0.063 | 0.064 | 0.061 | 0.060 |
| | | CP(%) | 98.2 | 97.9 | 99.0 | 98.9 | 98.9 | 99.0 |
| | | **Eff** | **2.149** | **2.330** | **2.247** | **1.833** | **3.015** | **12.427** |
| | 500 | Bias | -0.003 | 0.035 | - 0.020 | 0.007 | - 0.030 | 0.022 |
| | | SE | 0.075 | 0.057 | 0.077 | 0.063 | 0.072 | 0.084 |
| | | SD | 0.069 | 0.057 | 0.073 | 0.066 | 0.071 | 0.073 |
| | | CP(%) | 98.2 | 97.0 | 96.1 | 99.1 | 94.3 | 94.9 |
| | | **Eff** | **2.106** | **2.673** | **1.702** | **2.243** | **2.720** | **7.752** |
| pl | 200 | Bias | 0.001 | 0.038 | - 0.032 | 0.003 | - 0.036 | 0.029 |
| | | SE | 0.063 | 0.050 | 0.084 | 0.062 | 0.074 | 0.084 |
| | | SD | 0.058 | 0.049 | 0.074 | 0.057 | 0.067 | 0.076 |
| | | CP(%) | 99.9 | 99.7 | 99.7 | 99.9 | 97.5 | 98.1 |
| | | **Eff** | **2.990** | **3.203** | **1.381** | **2.317** | **2.481** | **7.509** |
| | 500 | Bias | - 0.003 | 0.032 | - 0.031 | - 0.003 | - 0.034 | 0.032 |
| | | SE | 0.048 | 0.043 | 0.075 | 0.049 | 0.069 | 0.084 |
| | | SD | 0.050 | 0.043 | 0.068 | 0.049 | 0.061 | 0.074 |
| | | CP(%) | 100.0 | 99.9 | 99.8 | 100.0 | 96.4 | 98.9 |
| | | **Eff** | **5.090** | **4.313** | **1.693** | **3.798** | **2.884** | **7.452** |

Table S6: Results under case S1 with both model correctly specified

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | -0.001 | 0.014 | - 0.009 | 0.000 | 0.000 | 0.001 |
| | | SE | 0.053 | 0.048 | 0.050 | 0.053 | 0.063 | 0.114 |
| | | SD | 0.051 | 0.048 | 0.049 | 0.052 | 0.061 | 0.114 |
| | | CP(%) | 97.2 | 96.6 | 96.7 | 97.9 | 95.9 | 94.3 |
| SS | 200 | Bias | 0.002 | 0.024 | - 0.006 | 0.006 | - 0.002 | 0.009 |
| | | SE | 0.045 | 0.038 | 0.038 | 0.040 | 0.041 | 0.042 |
| | | SD | 0.039 | 0.035 | 0.036 | 0.036 | 0.037 | 0.040 |
| | | CP(%) | 98.0 | 93.8 | 97.7 | 98.8 | 99.5 | 98.7 |
| | | **Eff** | **1.351** | **1.415** | **1.759** | **1.748** | **2.337** | **7.290** |
| | 500 | Bias | 0.002 | 0.020 | - 0.007 | 0.002 | - 0.003 | 0.011 |
| | | SE | 0.031 | 0.028 | 0.032 | 0.030 | 0.033 | 0.044 |
| | | SD | 0.032 | 0.029 | 0.033 | 0.030 | 0.036 | 0.041 |
| | | CP(%) | 99.2 | 95.0 | 96.1 | 99.7 | 99.3 | 96.2 |
| | | **Eff** | **2.952** | **2.432** | **2.394** | **3.192** | **3.715** | **6.577** |
| pl | 200 | Bias | 0.001 | 0.021 | - 0.006 | 0.000 | - 0.006 | 0.018 |
| | | SE | 0.024 | 0.021 | 0.026 | 0.023 | 0.027 | 0.042 |
| | | SD | 0.028 | 0.026 | 0.032 | 0.027 | 0.031 | 0.043 |
| | | CP(%) | 99.4 | 93.6 | 96.3 | 99.0 | 99.2 | 94.8 |
| | | **Eff** | **4.928** | **3.630** | **3.591** | **5.271** | **5.560** | **6.851** |
| | 500 | Bias | 0.000 | 0.018 | - 0.006 | 0.000 | - 0.006 | 0.017 |
| | | SE | 0.018 | 0.016 | 0.025 | 0.018 | 0.021 | 0.041 |
| | | SD | 0.022 | 0.020 | 0.027 | 0.022 | 0.026 | 0.039 |
| | | CP(%) | 99.6 | 91.3 | 94.6 | 99.5 | 99.4 | 92.8 |
| | | **Eff** | **8.202** | **5.567** | **4.115** | **8.229** | **8.662** | **7.259** |

Table S7: Results under case S2 with both model correctly specified

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | - 0.005 | 0.027 | - 0.032 | - 0.021 | - 0.066 | - 0.036 |
| | | SE | 0.111 | 0.108 | 0.106 | 0.116 | 0.158 | 0.285 |
| | | SD | 0.107 | 0.106 | 0.106 | 0.114 | 0.151 | 0.263 |
| | | CP(%) | 96.6 | 95.0 | 93.1 | 95.6 | 93.0 | 91.5 |
| SS | 200 | Bias | 0.006 | 0.030 | - 0.007 | 0.005 | - 0.011 | - 0.003 |
| | | SE | 0.064 | 0.058 | 0.075 | 0.068 | 0.062 | 0.057 |
| | | SD | 0.060 | 0.052 | 0.060 | 0.059 | 0.056 | 0.056 |
| | | CP(%) | 98.5 | 96.9 | 98.1 | 98.7 | 98.8 | 99.7 |
| | | **Eff** | **2.973** | **3.126** | **2.087** | **2.972** | **6.986** | **25.364** |
| | 500 | Bias | -0.004 | 0.032 | -0.021 | -0.003 | -0.012 | 0.006 |
| | | SE | 0.067 | 0.052 | 0.090 | 0.070 | 0.070 | 0.076 |
| | | SD | 0.063 | 0.052 | 0.078 | 0.062 | 0.064 | 0.068 |
| | | CP(%) | 98.4 | 97.6 | 94.2 | 99.0 | 97.2 | 98.2 |
| | | **Eff** | **2.719** | **3.723** | **1.409** | **2.769** | **5.411** | **13.991** |
| pl | 200 | Bias | -0.007 | 0.035 | -0.028 | -0.007 | -0.012 | 0.015 |
| | | SE | 0.068 | 0.054 | 0.094 | 0.060 | 0.061 | 0.077 |
| | | SD | 0.064 | 0.054 | 0.087 | 0.063 | 0.064 | 0.074 |
| | | CP(%) | 98.6 | 98.0 | 92.8 | 98.9 | 94.1 | 96.5 |
| | | **Eff** | **2.614** | **3.363** | **1.290** | **3.757** | **7.289** | **13.627** |
| | 500 | Bias | -0.009 | 0.030 | -0.028 | -0.010 | -0.015 | 0.015 |
| | | SE | 0.062 | 0.046 | 0.093 | 0.059 | 0.052 | 0.078 |
| | | SD | 0.057 | 0.048 | 0.082 | 0.057 | 0.059 | 0.071 |
| | | CP(%) | 98.1 | 98.5 | 89.2 | 99.4 | 92.2 | 96.7 |
| | | **Eff** | **3.108** | **4.693** | **1.318** | **3.915** | **9.734** | **13.104** |

Table S8: Results under case S5 with baseline treatment-free effect model misspecified

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | 0.039 | 0.011 | - 0.026 | - 0.004 | - 0.065 | - 0.034 |
| | | SE | 0.101 | 0.088 | 0.111 | 0.103 | 0.109 | 0.220 |
| | | SD | 0.095 | 0.083 | 0.101 | 0.095 | 0.107 | 0.200 |
| | | CP(%) | 94.5 | 96.3 | 95.0 | 96.7 | 93.3 | 91.2 |
| SS | 200 | Bias | 0.051 | 0.038 | - 0.024 | 0.016 | - 0.041 | - 0.026 |
| | | SE | 0.095 | 0.074 | 0.094 | 0.094 | 0.076 | 0.081 |
| | | SD | 0.077 | 0.062 | 0.078 | 0.077 | 0.069 | 0.069 |
| | | CP(%) | 89.0 | 99.3 | 99.3 | 98.1 | 98.6 | 99.9 |
| | | **Eff** | **1.058** | **1.211** | **1.399** | **1.191** | **2.080** | **7.138** |
| | 500 | Bias | 0.037 | 0.040 | -0.033 | -0.001 | -0.049 | -0.019 |
| | | SE | 0.100 | 0.072 | 0.092 | 0.087 | 0.089 | 0.094 |
| | | SD | 0.088 | 0.067 | 0.085 | 0.084 | 0.079 | 0.083 |
| | | CP(%) | 91.1 | 99.7 | 99.8 | 99.4 | 98.2 | 99.7 |
| | | **Eff** | **1.023** | **1.242** | **1.420** | **1.396** | **1.510** | **5.387** |
| pl | 200 | Bias | 0.029 | 0.046 | -0.032 | -0.010 | -0.049 | -0.015 |
| | | SE | 0.092 | 0.070 | 0.085 | 0.092 | 0.082 | 0.094 |
| | | SD | 0.093 | 0.070 | 0.086 | 0.087 | 0.079 | 0.088 |
| | | CP(%) | 92.1 | 99.8 | 99.8 | 99.7 | 96.8 | 100.0 |
| | | **Eff** | **1.223** | **1.245** | **1.663** | **1.243** | **1.766** | **5.410** |
| | 500 | Bias | 0.022 | 0.039 | -0.038 | -0.014 | -0.046 | -0.011 |
| | | SE | 0.093 | 0.066 | 0.082 | 0.084 | 0.080 | 0.094 |
| | | SD | 0.085 | 0.065 | 0.078 | 0.080 | 0.074 | 0.085 |
| | | CP(%) | 88.1 | 99.8 | 99.7 | 99.8 | 93.7 | 99.6 |
| | | **Eff** | **1.237** | **1.443** | **1.718** | **1.484** | **1.860** | **5.468** |

Table S9: Results under case S1 with propensity score model misspecified

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | 0.001 | 0.016 | - 0.009 | - 0.002 | - 0.009 | - 0.008 |
| | | SE | 0.060 | 0.061 | 0.051 | 0.054 | 0.067 | 0.121 |
| | | SD | 0.062 | 0.061 | 0.051 | 0.052 | 0.066 | 0.124 |
| | | CP(%) | 97.1 | 95.9 | 96.8 | 98.0 | 96.7 | 94.8 |
| SS | 200 | Bias | 0.007 | 0.019 | - 0.003 | 0.003 | -0.003 | 0.001 |
| | | SE | 0.038 | 0.037 | 0.036 | 0.040 | 0.038 | 0.039 |
| | | SD | 0.033 | 0.031 | 0.033 | 0.035 | 0.035 | 0.037 |
| | | CP(%) | 98.6 | 97.0 | 99.0 | 98.5 | 99.9 | 99.9 |
| | | **Eff** | **2.428** | **2.532** | **2.031** | **1.832** | **3.148** | **9.925** |
| | 500 | Bias | 0.006 | 0.016 | - 0.003 | 0.002 | -0.003 | 0.002 |
| | | SE | 0.027 | 0.024 | 0.027 | 0.028 | 0.031 | 0.038 |
| | | SD | 0.027 | 0.026 | 0.030 | 0.029 | 0.033 | 0.038 |
| | | CP(%) | 99.1 | 97.4 | 98.3 | 98.9 | 99.5 | 99.1 |
| | | **Eff** | **4.733** | **5.441** | **3.513** | **3.630** | **4.509** | **10.337** |
| pl | 200 | Bias | 0.006 | 0.017 | - 0.003 | 0.000 | - 0.005 | 0.008 |
| | | SE | 0.019 | 0.019 | 0.024 | 0.021 | 0.024 | 0.037 |
| | | SD | 0.025 | 0.024 | 0.028 | 0.025 | 0.029 | 0.040 |
| | | CP(%) | 99.5 | 97.1 | 98.3 | 99.2 | 99.7 | 99.0 |
| | | **Eff** | **9.139** | **7.840** | **4.553** | **6.504** | **7.725** | **10.569** |
| | 500 | Bias | 0.004 | 0.014 | - 0.002 | 0.000 | -0.004 | 0.006 |
| | | SE | 0.014 | 0.013 | 0.019 | 0.016 | 0.019 | 0.035 |
| | | SD | 0.017 | 0.015 | 0.021 | 0.017 | 0.021 | 0.035 |
| | | CP(%) | 99.3 | 96.8 | 97.3 | 99.7 | 100.0 | 97.4 |
| | | **Eff** | **16.621** | **14.518** | **7.150** | **11.575** | **12.283** | **11.684** |

Table S10: Results under case S2 with propensity score model misspecified

| Method | N | Statistics | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ |
|---|---|---|---|---|---|---|---|---|
| sup | | Bias | -0.036 | 0.048 | - 0.057 | - 0.030 | - 0.064 | - 0.056 |
| | | SE | 0.160 | 0.158 | 0.126 | 0.121 | 0.181 | 0.323 |
| | | SD | 0.149 | 0.148 | 0.120 | 0.118 | 0.171 | 0.293 |
| | | CP(%) | 94.7 | 92.6 | 89.2 | 94.8 | 92.6 | 91.8 |
| SS | 200 | Bias | 0.008 | 0.033 | -0.007 | 0.004 | -0.012 | -0.004 |
| | | SE | 0.076 | 0.064 | 0.062 | 0.071 | 0.066 | 0.058 |
| | | SD | 0.058 | 0.050 | 0.056 | 0.058 | 0.055 | 0.054 |
| | | CP(%) | 98.5 | 98.4 | 99.7 | 99.1 | 99.4 | 99.9 |
| | | **Eff** | **4.513** | **5.624** | **4.596** | **2.988** | **7.841** | **32.115** |
| | 500 | Bias | 0.003 | 0.029 | -0.019 | 0.000 | -0.013 | -0.006 |
| | | SE | 0.065 | 0.056 | 0.077 | 0.067 | 0.059 | 0.070 |
| | | SD | 0.057 | 0.048 | 0.066 | 0.058 | 0.060 | 0.063 |
| | | CP(%) | 98.8 | 97.9 | 97.4 | 99.0 | 97.4 | 98.7 |
| | | **Eff** | **6.289** | **7.468** | **2.855** | **3.371** | **9.796** | **21.750** |
| pl | 200 | Bias | 0.003 | 0.029 | -0.027 | -0.003 | -0.017 | 0.016 |
| | | SE | 0.053 | 0.041 | 0.084 | 0.060 | 0.055 | 0.078 |
| | | SD | 0.056 | 0.048 | 0.073 | 0.056 | 0.060 | 0.069 |
| | | CP(%) | 98.0 | 99.0 | 95.6 | 98.8 | 94.1 | 97.3 |
| | | **Eff** | **9.321** | **12.227** | **2.355** | **4.168** | **11.136** | **17.079** |
| | 500 | Bias | 0.001 | 0.025 | -0.026 | -0.003 | -0.016 | 0.015 |
| | | SE | 0.046 | 0.041 | 0.078 | 0.049 | 0.052 | 0.073 |
| | | SD | 0.049 | 0.042 | 0.068 | 0.049 | 0.054 | 0.066 |
| | | CP(%) | 98.2 | 99.0 | 94.3 | 99.3 | 93.5 | 96.4 |
| | | **Eff** | **12.599** | **13.236** | **2.751** | **6.241** | **12.259** | **19.944** |

Table S11: Testing of the MCAR assumption for ACTG 175 study

| | Labeled Data | | Unlabeled Data | | |
|---|---|---|---|---|---|
| Predictors | Mean | SD | Mean | SD | P-value of Test |
| hemo | 0.073 | 0.260 | 0.098 | 0.297 | 0.9071 |
| homo | 0.706 | 0.456 | 0.621 | 0.485 | 0.5271 |
| drugs | 0.115 | 0.319 | 0.170 | 0.376 | 1 |
| race | 0.233 | 0.432 | 0.317 | 0.466 | 1 |
| gender | 0.839 | 0.368 | 0.822 | 0.383 | 0.2671 |
| str2 | 0.598 | 0.491 | 0.589 | 0.493 | 0.2157 |
| symptom | 0.195 | 0.397 | 0.159 | 0.366 | 0.6103 |
| age | 35.59 | 8.830 | 35.08 | 8.669 | 0.5378 |
| wtkg | 74.65 | 12.51 | 75.93 | 14.16 | 0.0686 |
| cd40 | 353.9 | 119.7 | 347.6 | 126.3 | 0.0938 |
| cd80 | 986.0 | 450.9 | 1002 | 489.7 | 0.8804 |

Table S12: CI of estimated paramters for ACTG 175 study

| Method | sup | | SS | | pl | |
|---|---|---|---|---|---|---|
| CI | 95% CI | 90% CI | 95% CI | 90% CI | 95% CI | 90% CI |
| intercept | (-0.038, 0.097) | (-0.025, 0.075) | (-0.057, 0.158) | (-0.031, 0.135) | (-0.177, 0.204) | (-0.127, 0.197) |
| hemo | (-0.949, 0.323) | (-0.925, 0.101) | **(-0.854, -0.590)** | **(-0.839, -0.631)** | **(-0.859, -0.611)** | **(-0.852, -0.650)** |
| homo | (-0.658, 0.422) | (-0.586, 0.333) | **(-0.338, -0.049)** | **(-0.324, -0.082)** | **(-0.271, -0.126)** | **(-0.257, -0.133)** |
| drugs | (-0.647, 0.449) | (-0.605, 0.348) | (-0.290, 0.103) | (-0.283, 0.058) | **(-0.219, -0.082)** | **(-0.218, -0.099)** |
| race | (-0.217, 0.626) | (-0.146, 0.564) | **(0.194, 0.508)** | **(0.218, 0.463)** | **(0.240, 0.495)** | **(0.245, 0.476)** |
| gender | (-0.650, 0.505) | (-0.589, 0.474) | (-0.284, 0.082) | (-0.236, 0.049) | (-0.146, 0.065) | (-0.119, 0.058) |
| str2 | (-0.175, 0.528) | (-0.126, 0.483) | **(0.141, 0.378)** | **(0.160, 0.354)** | **(0.176, 0.304)** | **(0.184, 0.281)** |
| symptom | (-0.529, 0.284) | (-0.450, 0.217) | (-0.368, 0.022) | **(-0.330, -0.032)** | **(-0.249, -0.122)** | **(-0.242, -0.141)** |
| age | (-0.147, 0.321) | (-0.117, 0.280) | **(0.055, 0.263)** | **(0.087, 0.253)** | **(0.072, 0.209)** | **(0.074, 0.195)** |
| weight | (-0.164, 0.270) | (-0.122, 0.208) | **(0.011, 0.208)** | **(0.032, 0.187)** | **(0.027, 0.175)** | **(0.028, 0.150)** |
| cd40 | (-0.420, 0.082) | (-0.367, 0.066) | **(-0.383, -0.146)** | **(-0.364, -0.158)** | **(-0.284, -0.193)** | **(-0.273, -0.204)** |
| cd80 | (-0.135, 0.313) | (-0.084, 0.251) | **(0.072, 0.290)** | **(0.102, 0.276)** | **(0.089, 0.206)** | **(0.096, 0.193)** |

Table S13: CI length of estimated paramters for ACTG 175 study

| CI | 95% CI | | | 90% CI | | |
|---|---|---|---|---|---|---|
| Length | sup | SS | pl | sup | SS | pl |
| intercept | 0.135 | 0.214 | 0.381 | 0.100 | 0.166 | 0.324 |
| hemo | 1.272 | 0.263 | 0.247 | 1.026 | 0.208 | 0.202 |
| homo | 1.080 | 0.289 | 0.145 | 0.919 | 0.241 | 0.124 |
| drugs | 1.096 | 0.392 | 0.137 | 0.953 | 0.341 | 0.120 |
| race | 0.843 | 0.314 | 0.255 | 0.710 | 0.246 | 0.231 |
| gender | 1.154 | 0.366 | 0.211 | 1.064 | 0.285 | 0.178 |
| str2 | 0.703 | 0.237 | 0.128 | 0.609 | 0.194 | 0.097 |
| symptom | 0.813 | 0.390 | 0.127 | 0.667 | 0.298 | 0.102 |
| age | 0.468 | 0.208 | 0.137 | 0.398 | 0.166 | 0.121 |
| weight | 0.434 | 0.197 | 0.148 | 0.330 | 0.155 | 0.121 |
| cd40 | 0.502 | 0.237 | 0.091 | 0.432 | 0.206 | 0.069 |
| cd80 | 0.448 | 0.218 | 0.117 | 0.335 | 0.174 | 0.096 |

Table S14: Estimated parameters of optimal ITR with sample splitting

| Mehods | sup | | SS | | pl | |
|---|---|---|---|---|---|---|
| Predictors | Est | SD | Est | SD | Est | SD |
| intercept | 0.013 | 0.037 | 0.017 | 0.061 | 0.027 | 0.260 |
| hemo | -0.567 | 0.359 | -0.548 | 0.084 | -0.516 | 0.159 |
| homo | -0.104 | 0.265 | -0.104 | 0.066 | -0.099 | 0.132 |
| drugs | -0.130 | 0.291 | -0.103 | 0.089 | -0.105 | 0.126 |
| race | 0.269 | 0.205 | 0.302 | 0.074 | 0.308 | 0.134 |
| gender | -0.141 | 0.294 | -0.137 | 0.078 | -0.110 | 0.162 |
| str2 | 0.199 | 0.175 | 0.210 | 0.064 | 0.226 | 0.127 |
| symptom | -0.232 | 0.204 | -0.198 | 0.074 | -0.214 | 0.138 |
| age | 0.124 | 0.107 | 0.125 | 0.059 | 0.136 | 0.134 |
| weight | 0.020 | 0.103 | 0.024 | 0.057 | 0.028 | 0.120 |
| cd40 | -0.199 | 0.115 | -0.203 | 0.059 | -0.208 | 0.137 |
| cd80 | 0.121 | 0.104 | 0.121 | 0.058 | 0.123 | 0.132 |

Table S15: CI of estimated paramters with sample splitting

| Method | sup | | SS | | pl | |
|---|---|---|---|---|---|---|
| CI | 95% CI | 90% CI | 95% CI | 90% CI | 95% CI | 90% CI |
| intercept | (-0.070, 0.090) | (-0.044, 0.076) | (-0.114, 0.136) | (-0.079, 0.114) | (-0.522, 0.369) | (-0.457, 0.283) |
| hemo | (-0.882, 0.423) | (-0.778, 0.298) | **(-0.501, -0.175)** | **(-0.480, -0.212)** | **(-0.674, -0.091)** | **(-0.638, -0.156)** |
| homo | (-0.574, 0.399) | (-0.518, 0.312) | **(-0.267, -0.007)** | **(-0.245, -0.028)** | (-0.267, 0.258) | (-0.214, 0.210) |
| drugs | (-0.569, 0.538) | (-0.494, 0.441) | (-0.171, 0.178) | (-0.144, 0.136) | (-0.326, -0.183) | (-0.271, 0.138) |
| race | (-0.300, 0.507) | (-0.234, 0.439) | **(0.055, 0.355)** | **(0.080, 0.324)** | (-0.069, 0.456) | **(0.000, 0.421)** |
| gender | (-0.524, 0.568) | (-0.429, 0.498) | (-0.112, 0.192) | (-0.081, 0.171) | (-0.327, 0.304) | (-0.259, 0.242) |
| str2 | (-0.294, 0.410) | (-0.234, 0.345) | (-0.039, 0.223) | (-0.008, 0.204) | (-0.063, 0.437) | (-0.001, 0.404) |
| symptom | (-0.390, 0.415) | (-0.327, 0.348) | (-0.098, 0.203) | (-0.074, 0.171) | (-0.389, 0.151) | (-0.340, 0.100) |
| age | (-0.184, 0.254) | (-0.143, 0.208) | (-0.072, 0.179) | (-0.042, 0.155) | (-0.206, 0.326) | (-0.145, 0.286) |
| weight | (-0.177, 0.245) | (-0.138, 0.205) | (-0.049, 0.181) | (-0.023, 0.161) | (-0.231, 0.281) | (-0.170, 0.227) |
| cd40 | (-0.307, 0.153) | (-0.258, 0.119) | (-0.242, 0.003) | **(-0.215, -0.020)** | (-0.360, 0.202) | (-0.305, 0.142) |
| cd80 | (-0.168, 0.249) | (-0.135, 0.205) | (-0.044, 0.189) | (-0.016, 0.172) | (-0.205, 0.349) | (-0.140, 0.298) |

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