Testing Conditional Tail Independence

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Supplementary Material

In the supplementary material, we provide the proofs of Proposition 1 and Theorem 1.

Let $\mathbb{P}_{|\mathbf{Z}}$ and $\mathbb{E}_{|\mathbf{Z}}$ denote the conditional probability and conditional expectation given $\mathbf{Z}$, respectively. Also, let $\mathbb{P}$ and $\mathbb{E}$ denote the unconditional probability and expectation.

S1 Proof of Proposition 1

First, we derive (1) $\Rightarrow$ (2). Suppose that (1) holds, which means

$$\lim_{\tau\to 1}\mathbb{P}_{|\mathbf{Z}}\big(X > Q_X(\tau \mid \mathbf{Z}) \mid Y > Q_Y(\tau \mid \mathbf{Z})\big) = 0, \quad \text{for } \mathbf{Z} \in \mathcal{Z}, a.s.$$

Taking $\tau = 1 - x/n$ and by the definition of conditional probability, then for all $0 < x, y < \infty$, we have

$$n\Big(\mathbb{P}_{|\mathbf{Z}}\big(X > Q_X(1 - x/n \mid \mathbf{Z}), Y > Q_Y(1 - y/n \mid \mathbf{Z})\big)\Big) = o_{\mathbb{P}}(1),$$

uniformly for $\mathbf{Z} \in \mathcal{Z}$, a.s., and hence (2) holds.

Second, we derive (2) $\Rightarrow$ (3). Under Assumption 1, for a given random sample $U_1, \ldots, U_n$ from $F_X(\cdot \mid \mathbf{Z})$ and a given random sample $V_1, \ldots, V_n$ from $F_Y(\cdot \mid \mathbf{Z})$, there exist constants $a_n^1 > 0$, $a_n^2 > 0$, $b_n^1 \in R$ and $b_n^2 \in R$ such that

$$\mathbb{P}\left(\frac{\max_{1\le i\le n} U_i - b_n^1}{a_n^1} \le z\right) \to G_{\gamma_1}(x) = \exp\left\{-(1+\gamma_1 x)^{-1/\gamma_1}\right\},$$

as $n \to \infty$, for $1 + \gamma_1 x \ge 0$ and that

$$\mathbb{P}\left(\frac{\max_{1\le i\le n} V_i - b_n^2}{a_n^2} \le z\right) \to G_{\gamma_2}(y) = \exp\left\{-(1+\gamma_2 y)^{-1/\gamma_2}\right\},$$

as $n \to \infty$, for $1 + \gamma_2 y \ge 0$. By Theorem 6.3.2 in de Haan and Ferreira (2006), we have

$$\mathbb{P}\left(\frac{\max_{1\le i\le n} U_i - b_n^1}{a_n^1} \le x, \frac{\max_{1\le i\le n} V_i - b_n^2}{a_n^2} \le y\right) \to G_{\gamma_1}(x)G_{\gamma_2}(y),$$

which implies that $F_{XY}(\cdot, \cdot \mid \mathbf{Z})$ belongs to the maximum domain of attraction of the bivariate extreme value distribution $G_{\gamma_1}(\cdot)G_{\gamma_2}(\cdot)$, for $\mathbf{Z} \in \mathcal{Z}$, a.s.

Finally we derive (3) $\Rightarrow$ (1). Under the bivariate maximum domain of attraction condition, we have

$$\mathbb{P}^n \left(X \le Q_X(1-x/n \mid \mathbf{Z}), Y \le Q_Y(1-y/n \mid \mathbf{Z})\right) \to \exp(-x^{-1}-y^{-1}),$$

uniformly for $\mathbf{Z} \in \mathcal{Z}$. Taking logarithms to the both sides, we have

$$\lim_{n\to\infty} n \log \mathbb{P}\left(X \le Q_X(1-x/n \mid \mathbf{Z}), Y \le Q_Y(1-y/n \mid \mathbf{Z})\right) = -x^{-1}-y^{-1},$$

which implies that

$$\frac{-\log \mathbb{P}\left(X \le Q_X(1-x/n \mid \mathbf{Z}), Y \le Q_Y(1-y/n \mid \mathbf{Z})\right)}{1-\mathbb{P}\left(X \le Q_X(1-x/n \mid \mathbf{Z}), Y \le Q_Y(1-y/n \mid \mathbf{Z})\right)} \to 1,$$

Thus, (1) follows.

S2 Proof of Theorem 1

We begin by introducing several lemmas that are required for the proof of Theorem 1.

Lemma 1. *Suppose that model (1) and Assumptions 1-6 hold. Then for any $\varepsilon > 0$ and $i = 1, \cdots, n$,*

$$\mathbb{P}\left(\left|\frac{\widehat{Q}_X\left(1-c/n \mid \mathbf{Z}_i\right)}{Q_X(1-c/n \mid \mathbf{Z}_i)} - 1\right| > \varepsilon\right) = o(1), \quad \mathbb{P}\left(\left|\frac{\widehat{Q}_Y\left(1-c/n \mid \mathbf{Z}_i\right)}{Q_Y(1-c/n \mid \mathbf{Z}_i)} - 1\right| > \varepsilon\right) = o(1).$$

Proof. Let $\epsilon_{n1}(\mathbf{Z}) = \frac{\widehat{Q}_X(1-c/n|\mathbf{Z})}{Q_X(1-c/n|\mathbf{Z})} - 1, \epsilon_{n2}(\mathbf{Z}) = \frac{\widehat{Q}_Y(1-c/n|\mathbf{Z})}{Q_Y(1-c/n|\mathbf{Z})} - 1$. By Theorem 4 in Wang et al. (2012) we have

$$\epsilon_{n1}(\mathbf{Z}) = \frac{\log(k/c)}{\sqrt{k}}\left[\gamma_1 \mathbf{Z}^T H_1^{-1} W_{n1}(1)\{K_1(\mathbf{Z})\}^{-\gamma_1} + o_{\mathbb{P}}(1)\right],$$

$$\epsilon_{n2}(\mathbf{Z}) = \frac{\log(k/c)}{\sqrt{k}}\left[\gamma_2 \mathbf{Z}^T H_2^{-1} W_{n2}(1)\{K_2(\mathbf{Z})\}^{-\gamma_2} + o_{\mathbb{P}}(1)\right],$$

uniformly for $\mathbf{Z} \in \mathcal{Z}$ a.s., where $H_1 = \mathbb{E}\left[\{K_1(\mathbf{Z})\}^{-\gamma}\mathbf{Z}\mathbf{Z}^T\right]$ and $H_2 = \mathbb{E}\left[\{K_2(\mathbf{Z})\}^{-\gamma}\mathbf{Z}\mathbf{Z}^T\right]$ are bounded, $W_{1n}(1) = \lim_{\tau\to 1} W_{1n}(\tau)$ and $W_{2n}(1) = \lim_{\tau\to 1} W_{2n}(\tau)$ are both normal distributed with mean zero and variance $\mathbb{E}(\mathbf{Z}\mathbf{Z}^T)$. Here,

$$W_{1n}(\tau) = \{n(1-\tau)\}^{-1/2}\sum_{j=1}^{n}\mathbf{Z}_j\left[\tau - I\left\{X_j \le \mathbf{Z}_j^T\boldsymbol{\beta}_1(\tau)\right\}\right]$$

and

$$W_{2n}(\tau) = \{n(1-\tau)\}^{-1/2}\sum_{j=1}^{n}\mathbf{Z}_j\left[\tau - I\left\{Y_j \le \mathbf{Z}_j^T\boldsymbol{\beta}_2(\tau)\right\}\right].$$

Recall that both K_1 and K_2 are bounded, which means that for any $\varepsilon > 0$, and fr fixed $\mathbf{Z} \in \mathcal{Z}$ a.s.,

$$\begin{aligned}
&\mathbb{P}_{|\mathbf{Z}}\left(\left|\frac{\widehat{Q}_X(1-c/n \mid \mathbf{Z})}{Q_X(1-c/n \mid \mathbf{Z})} - 1\right| > \varepsilon\right)\\
=&\mathbb{P}_{|\mathbf{Z}}\left(\left|\frac{\log(k/c)}{\sqrt{k}}\left[\gamma_1\mathbf{Z}^T H_1^{-1} W_{n1}(1)\{K_1(\mathbf{Z})\}^{-\gamma_1} + o_{\mathbb{P}}(1)\right]\right| > \varepsilon\right) = o_{\mathbb{P}}(1).
\end{aligned}$$

Since the item $o_{\mathbb{P}}(1)$ above is bounded away from 1, by taking expection with respect to $\mathbf{Z}$, we have

$$\mathbb{P}\left(\left|\frac{\widehat{Q}_X(1-c/n \mid \mathbf{Z})}{Q_X(1-c/n \mid \mathbf{Z})} - 1\right| > \varepsilon\right) = o(1).$$

Similarly, we also have

$$\mathbb{P}\left(\left|\frac{\widehat{Q}_Y(1-c/n \mid \mathbf{Z})}{Q_Y(1-c/n \mid \mathbf{Z})} - 1\right| > \varepsilon\right) = o(1).$$

$\square$

Lemma 2. *Suppose Assumptions 1-6 hold. Under the conditional tail independence, it follows that for all* $1 < x, y < \infty$ *and constant* $c > 0$,

$$\mathbb{P}\left(X \leq xQ_X(1-c/n \mid \mathbf{Z}), Y \leq yQ_Y(1-c/n \mid \mathbf{Z})\right) = 1 - \frac{c}{n}\left(x^{-1/\gamma_1} + y^{-1/\gamma_2} + o(1)\right).$$

Proof. Let $\widetilde{d_1} = d_1^* I\left(\varrho_1^* \geq \gamma_1\right) - c_1^{-1}\{K_1(\mathbf{Z})\}^{-\gamma_1}\mathbf{Z}^T\boldsymbol{\beta}_{r_1} I\left(\varrho_1^* \leq \gamma_1\right)$, where $\varrho_1^* = \max(\varrho_1, -\delta_1)$, $d_1^* = \{K_1(\mathbf{Z})\}^{-\delta_1}\left\{d_1 I(\varrho_1 \geq -\delta_1) - \delta_1 \widetilde{K}_1(\mathbf{Z}) I(\varrho_1 \leq -\delta_1)\right\}$, c_1 is a positive constant, and denote $\widetilde{\varrho}_1 = \max(\varrho_1^*, -\gamma_1)$,. By Lemma 2 in Wang et al. (2012), for fixed $\mathbf{Z} \in \mathcal{Z}$ a.s.,

$$Q_X(1-1/t \mid \mathbf{Z}) = c_1\{K_1(\mathbf{Z})\}^{\gamma_1} t^{\gamma_1}\left[1 + \frac{\gamma_1 \widetilde{d}_1 t^{\widetilde{\varrho}_1}}{\widetilde{\varrho}_1}\{1 + o_{\mathbb{P}}(1)\}\right],$$

as $t \to \infty$, which implies that for all fixed $\mathbf{Z} \in \mathcal{Z}$,

$$\mathbb{P}_{|\mathbf{Z}}(X > t) = K_1(\mathbf{Z})\left(\frac{t}{c_1}\right)^{-1/\gamma_1}\left[1 + \frac{\widetilde{d}_1}{\widetilde{\varrho}_1}\left(\frac{t}{c_1}\right)^{\widetilde{\varrho}_1/\gamma_1}\{K_1(\mathbf{Z})\}^{-\widetilde{\varrho_1}}\{1 + o_{\mathbb{P}}(1)\}\right],$$

as $t \to \infty$. For any $1 < x < \infty$, define

$$L(x; t; \mathbf{Z}) := x^{1/\gamma_1}\frac{\mathbb{P}_{|\mathbf{Z}}(X > xt)}{\mathbb{P}_{|\mathbf{Z}}(X > t)}.$$

Thus,

$$\begin{aligned}
L(x; t; \mathbf{Z}) =& \frac{1 + \frac{\widetilde{d}_1}{\widetilde{\varrho}_1}\left(\frac{xt}{c_1}\right)^{\widetilde{\varrho}_1/\gamma_1}\{K_1(\mathbf{Z})\}^{-\widetilde{\varrho_1}}\{1 + o_{\mathbb{P}}(1)\}}{1 + \frac{\widetilde{d}_1}{\widetilde{\varrho}_1}\left(\frac{t}{c_1}\right)^{\widetilde{\varrho}_1/\gamma_1}\{K_1(\mathbf{Z})\}^{-\widetilde{\varrho_1}}\{1 + o_{\mathbb{P}}(1)\}} \\
=& \frac{1 + o_{\mathbb{P}}(1)}{1 + o_{\mathbb{P}}(1)} \\
=& 1 + o_{\mathbb{P}}(1),
\end{aligned}$$

uniformly for $\mathbf{Z} \in \mathcal{Z}$ a.s. Notice that $L(x;t;\mathbf{Z})$ is bounded away from x^{1/γ_1}.

So $\mathbb{E}\left[L(x;t;\mathbf{Z})\right] = 1 + o(1)$, as $t \to \infty$, which leads to

$$\begin{aligned}
&\mathbb{P}\left(X > xQ_X(1-c/n \mid \mathbf{Z})\right) \\
=&\mathbb{E}\left[x^{1/\gamma_1}\mathbb{P}_{|\mathbf{Z}}\left(X > Q_X(1-c/n \mid \mathbf{Z})L(x;Q_X(1-c/n \mid \mathbf{Z});\mathbf{Z})\right)\right] \\
=&x^{1/\gamma_1}c/n\mathbb{E}\left[L(x;Q_X(1-c/n \mid \mathbf{Z});\mathbf{Z})\right] \\
=&x^{1/\gamma_1}\frac{c}{n}(1+o(1)).
\end{aligned}$$

Similarly, for $1 < y < \infty$, we have

$$\mathbb{P}\left(Y > yQ_Y(1-c/n \mid \mathbf{Z})\right) = y^{1/\gamma_2}\frac{c}{n}(1+o(1)).$$

Observe that

$$\begin{aligned}
&\mathbb{P}\left(X \le xQ_X(1-c/n \mid \mathbf{Z}), Y \le yQ_Y(1-c/n \mid \mathbf{Z})\right) \\
=&1 - \mathbb{P}\left(X > xQ_X(1-c/n \mid \mathbf{Z})\right) - \mathbb{P}\left(Y \le yQ_Y(1-c/n \mid \mathbf{Z})\right) \\
&+ \mathbb{P}\left(X > xQ_X(1-c/n \mid \mathbf{Z}), Y > yQ_Y(1-c/n \mid \mathbf{Z})\right),
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{P}\left(X > xQ_X(1-c/n \mid \mathbf{Z}), Y > yQ_Y(1-c/n \mid \mathbf{Z})\right) \\
\le&\mathbb{P}\left(X > Q_X(1-c/n \mid \mathbf{Z}), Y > Q_Y(1-c/n \mid \mathbf{Z})\right) \\
=&\mathbb{E}\left[\lambda_{1-\frac{c}{n}}(\mathbf{Z})\mathbb{P}_{|\mathbf{Z}}\left(Y > Q_Y(1-c/n \mid \mathbf{Z})\right)\right] \\
=&\frac{c}{n}\mathbb{E}\left[\lambda_{1-\frac{c}{n}}(\mathbf{Z})\right] \\
=&o(n^{-1}).
\end{aligned}$$

Taking them together, we get

$$\mathbb{P}\left(X \le xQ_X(1-c/n \mid \mathbf{Z}), Y \le yQ_Y(1-c/n \mid \mathbf{Z})\right) = 1-\frac{c}{n}\left(x^{-1/\gamma_1} + y^{-1/\gamma_2} + o(1)\right).$$

□

Proof of Theorem 1. For any $\varepsilon > 0$, uniformly for $i = 1, \ldots, n$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)} > 1+\varepsilon\right) \\
=&\mathbb{P}\left(X_i > u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
&+\mathbb{P}\left(X_i > u_n(\mathbf{Z}_i), X_i < Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{X_i} > 1+\varepsilon\right) \\
&+\mathbb{P}\left(X_i < u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{X_i}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
&+\mathbb{P}\left(X_i < u_n(\mathbf{Z}_i), X_i < Q_X(1-c/n \mid \mathbf{Z}_i), 1 > 1+\varepsilon\right) \\
=&\mathbb{P}\left(X_i > u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
&+\mathbb{P}\left(X_i < u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{X_i}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
\le&\mathbb{P}\left(X_i > u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
&+\mathbb{P}\left(X_i < u_n(\mathbf{Z}_i), X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
=&\mathbb{P}\left(X_i > Q_X(1-c/n \mid \mathbf{Z}_i), \frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
\le&\mathbb{P}\left(\frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n \mid \mathbf{Z}_i)} > 1+\varepsilon\right) \\
=&o(1),
\end{aligned}$$

where the last equation holds by Lemma 1. Similarly,

$$
\begin{aligned}
&\mathbb{P}\left(\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)},1\right)}<1-\varepsilon\right)\\
=&\mathbb{P}\left(X_i>u_n(\mathbf{Z}_i),X_i>Q_X(1-c/n\mid\mathbf{Z}_i),\frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n\mid\mathbf{Z}_i)}<1-\varepsilon\right)\\
&+\mathbb{P}\left(X_i>u_n(\mathbf{Z}_i),X_i<Q_X(1-c/n\mid\mathbf{Z}_i),\frac{u_n(\mathbf{Z}_i)}{X_i}<1-\varepsilon\right)\\
&+\mathbb{P}\left(X_i<u_n(\mathbf{Z}_i),X_i>Q_X(1-c/n\mid\mathbf{Z}_i),\frac{X_i}{Q_X(1-c/n\mid\mathbf{Z}_i)}<1-\varepsilon\right)\\
&+\mathbb{P}\left(X_i<u_n(\mathbf{Z}_i),X_i<Q_X(1-c/n\mid\mathbf{Z}_i),1<1-\varepsilon\right)\\
\le&\mathbb{P}\left(X_i>Q_X(1-c/n\mid\mathbf{Z}_i),\frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n\mid\mathbf{Z}_i)}<1-\varepsilon\right)\\
&+\mathbb{P}\left(X_i<Q_X(1-c/n\mid\mathbf{Z}_i),\frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n\mid\mathbf{Z}_i)}<1-\varepsilon\right)\\
\le&\mathbb{P}\left(\frac{u_n(\mathbf{Z}_i)}{Q_X(1-c/n\mid\mathbf{Z}_i)}<1+\varepsilon\right)\\
=&o(1),
\end{aligned}
$$

uniformly for $i=1,\ldots,n$. Taken together,

$$
\lim_{n\to\infty}\mathbb{P}\left(1-\varepsilon<\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)},1\right)}<1+\varepsilon,\quad i=1,\ldots,n\right)=1.
$$

By Lemma 1 in Zhang et al. (2017), there exist two sequences of positive random variables $\xi_n^{(1)}$ and $\xi_n^{(2)}$ such that $\xi_n^{(1)}\xrightarrow{\mathbb{P}}1,\xi_n^{(2)}\xrightarrow{\mathbb{P}}1$, and for $i=1,\ldots,n$,

$$
\xi_n^{(1)}\max\left(\frac{X_i}{Q_X(1-c/n\mid\mathbf{Z}_i)},1\right)\le\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)},1\right)\le\xi_n^{(2)}\max\left(\frac{X_i}{Q_X(1-c/n\mid\mathbf{Z}_i)},1\right).
$$

Similarly, there exist two sequences of positive random variables $\eta_n^{(1)}$ and $\eta_n^{(2)}$ such that $\eta_n^{(1)} \xrightarrow{\mathbb{P}} 1, \eta_n^{(2)} \xrightarrow{\mathbb{P}} 1$, and for $i = 1, \ldots, n$,

$$\eta_n^{(1)} \max\left(\frac{Y_i}{Q_Y(1-c/n \mid \mathbf{Z}_i)}, 1\right) \leq \max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right) \leq \eta_n^{(2)} \max\left(\frac{Y_i}{Q_Y(1-c/n \mid \mathbf{Z}_i)}, 1\right).$$

Then, we have

$$\frac{\xi_n^{(1)}}{\eta_n^{(2)}} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq \frac{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)} \leq \frac{\xi_n^{(2)}}{\eta_n^{(1)}} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)},$$

$$\frac{\eta_n^{(1)}}{\xi_n^{(2)}} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq \frac{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)} \leq \frac{\eta_n^{(2)}}{\xi_n^{(1)}} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}.$$

For any $x > 0, y > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq i\leq n} \frac{\xi_n^{(1)}}{\eta_n^{(2)}} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq x, \max_{1\leq i\leq n} \frac{\eta_n^{(1)}}{\xi_n^{(2)}} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq y\right) \\
\geq &\mathbb{P}\left(\max_{1\leq i\leq n} \frac{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)} \leq x, \max_{1\leq i\leq n} \frac{\max\left(\frac{Y_i}{v_n(\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{u_n(\mathbf{Z}_i)}, 1\right)} \leq y\right) \\
\geq &\mathbb{P}\left(\max_{1\leq i\leq n} \frac{\xi_n^{(2)}}{\eta_n^{(1)}} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq x, \max_{1\leq i\leq n} \frac{\eta_n^{(2)}}{\xi_n^{(1)}} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq y\right).
\end{aligned}$$

Since for all $j, k = 1, 2$, $\eta_n^{(j)}/\xi_n^{(k)} \xrightarrow{\mathbb{P}} 1, \eta_n^{(j)}/\xi_n^{(k)} \xrightarrow{\mathbb{P}} 1$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(\Delta_n \leq x, \Theta_n \leq y\right) \\
= &\lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq i\leq n} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq x, \max_{1\leq i\leq n} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq y\right).
\end{aligned}$$

Note that for all $x > 0$, $y > 0$,

$$\mathbb{P}\left(\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)},1\right)} \le x, \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)} \le y\right)$$

$$=\mathbb{P}\left(X_i \le Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \le Q_Y(1-c/n \mid \mathbf{Z}_i), x \ge 1, y \ge 1\right)$$

$$+\mathbb{P}\left(X_i \le Q_X(1-c/n \mid \mathbf{Z}_i), Y_i > Q_Y(1-c/n \mid \mathbf{Z}_i), x^{-1}Q_Y(1-c/n \mid \mathbf{Z}_i) \le Y_i \le yQ_Y(1-c/n \mid \mathbf{Z}_i)\right)$$

$$+\mathbb{P}\left(X_i > Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \le Q_Y(1-c/n \mid \mathbf{Z}_i), y^{-1}Q_X(1-c/n \mid \mathbf{Z}_i) \le X_i \le xQ_X(1-c/n \mid \mathbf{Z}_i)\right)$$

$$+\mathbb{P}\left(X_i > Q_X(1-c/n \mid \mathbf{Z}_i), Y_i > Q_Y(1-c/n \mid \mathbf{Z}_i), \frac{Q_X(1-c/n \mid \mathbf{Z}_i)}{yQ_Y(1-c/n \mid \mathbf{Z}_i)} \le \frac{X_i}{Y_i} \le \frac{xQ_X(1-c/n \mid \mathbf{Z}_i)}{Q_Y(1-c/n \mid \mathbf{Z}_i)}\right).$$

That is to say,

$$\mathbb{P}\left(\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)},1\right)} \le x, \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)},1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)},1\right)} \le y\right)$$
$$=\begin{cases} A_1 + A_2 - A_3 + A_4, & \text{if } xy \ge 1, x \ge 1, y \ge 1, \\ A_1 - A_5 + A_4, & \text{if } xy \ge 1, 0 < x < 1, y \ge 1, \\ A_2 - A_6 + A_4, & \text{if } xy \ge 1, x \ge 1, 0 < y < 1, \\ 0, & \text{if } xy < 1, x > 0, y > 0. \end{cases}$$

where

$$A_1 = \mathbb{P}\left(X_i \le Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \le yQ_Y(1-c/n \mid \mathbf{Z}_i)\right),$$

$$A_2 = \mathbb{P}\left(X_i \le xQ_X(1-c/n \mid \mathbf{Z}_i), Y_i \le Q_Y(1-c/n \mid \mathbf{Z}_i)\right),$$

$$A_3 = \mathbb{P}\left(X_i \le Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \le Q_Y(1-c/n \mid \mathbf{Z}_i)\right),$$

$$A_4 = \mathbb{P}\left(X_i > Q_X(1-c/n \mid \mathbf{Z}_i), Y_i > Q_Y(1-c/n \mid \mathbf{Z}_i), \frac{Q_X(1-c/n \mid \mathbf{Z}_i)}{yQ_Y(1-c/n \mid \mathbf{Z}_i)} \le \frac{X_i}{Y_i} \le \frac{xQ_X(1-c/n \mid \mathbf{Z}_i)}{Q_Y(1-c/n \mid \mathbf{Z}_i)}\right),$$

$$A_5 = \mathbb{P}\left(X_i \leq Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \leq x^{-1}Q_Y(1-c/n \mid \mathbf{Z}_i)\right),$$

$$A_6 = \mathbb{P}\left(X_i \leq y^{-1}Q_X(1-c/n \mid \mathbf{Z}_i), Y_i \leq Q_Y(1-c/n \mid \mathbf{Z}_i)\right).$$

Plug in Lemma 2, we have

$$A_1 = 1 - \frac{c}{n}\left(1 + y^{-\frac{1}{\gamma_2}} + o(1)\right), A_2 = 1 - \frac{c}{n}\left(1 + x^{-\frac{1}{\gamma_1}} + o(1)\right), A_3 = 1 - \frac{c}{n}\left(2 + o(1)\right),$$
$$A_5 = 1 - \frac{c}{n}\left(1 + x^{-\frac{1}{\gamma_2}} + o(1)\right), A_6 = 1 - \frac{c}{n}\left(1 + y^{-\frac{1}{\gamma_1}} + o(1)\right).$$

On the other hand,

$$A_4 \leq \mathbb{P}\left(X_i > Q_X(1-c/n \mid \mathbf{Z}_i), Y_i > Q_Y(1-c/n \mid \mathbf{Z}_i)\right) = \frac{c}{n}E\left(\lambda_{1-c/n}(\mathbf{Z})\right) = o(n^{-1}).$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq i\leq n} \frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq x, \max_{1\leq i\leq n} \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq y\right) \\
=&\mathbb{P}^n\left(\frac{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq x, \frac{\max\left(\frac{Y_i}{Q_Y(1-c/n|\mathbf{Z}_i)}, 1\right)}{\max\left(\frac{X_i}{Q_X(1-c/n|\mathbf{Z}_i)}, 1\right)} \leq y\right) \\
=&\begin{cases}
\left(1 - \frac{c}{n}\left(x^{-\frac{1}{\gamma_1}} + y^{-\frac{1}{\gamma_2}} + o(1)\right)\right)^n, & \text{if } xy \geq 1, x \geq 1, y \geq 1, \\
\left(\frac{c}{n}\left(x^{-\frac{1}{\gamma_2}} - y^{-\frac{1}{\gamma_2}} + o(1)\right)\right)^n, & \text{if } xy \geq 1, 0 < x < 1, y \geq 1, \\
\left(\frac{c}{n}\left(y^{-\frac{1}{\gamma_1}} - x^{-\frac{1}{\gamma_1}} + o(1)\right)\right)^n, & \text{if } xy \geq 1, x \geq 1, 0 < y < 1, \\
0, & \text{if } xy < 1, x > 0, y > 0.
\end{cases}
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \mathbb{P}(\Delta_n \leq x, \Theta_n \leq y) = \begin{cases}
\exp(-cx^{-\frac{1}{\gamma_1}} - cy^{-\frac{1}{\gamma_2}}), & \text{if } x \geq 1, y \geq 1, \\
0, & \text{otherwise.}
\end{cases}$$

Together with continuous mapping theorem, we obtain

$$\lim_{n\to\infty} \mathbb{P}(q_n \leq x) = \mathbb{P}(f(\max(U_1, 1), \max(U_2, 1)) < x)$$

where $U_1 \sim$ Fréchet $(\gamma_1^{-1}, c^{\gamma_1})$, $U_2 \sim$ Fréchet $(\gamma_2^{-1}, c^{\gamma_2})$ and U_1 and U_2 are independent. That is to say, q_n converges in distribution to $f(\max(U_1, 1), \max(U_2, 1))$.

□

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