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Supplement to "A VARIATION-RATIO TEST FOR VOLATILITY JUMPS USING NOISY HIGH FREQUENCY DATA"

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Supplementary Material

This document supplements the paper entitled "A Variation-Ratio Test for Volatility Jumps Using Noisy High Frequency Data". Section S1 introduces an auxiliary lemma used in establishing the main theoretical results. Section S2 provides the proofs of Propositions 1–3, while Section S3 contains the proofs of Theorems 1 and 2.

In this supplementary material, K denotes a generic constant that may change from line to line. We shall from time to time use K_p instead of K to emphasize that it depends on an additional parameter p. By a standard localization procedure of, e.g., Aït-Sahalia and Jacod (2014), the local boundedness of the processes $b, \widetilde{b}, \overline{b}, \sigma, \widetilde{\sigma}$ and $\overline{\sigma}$ can be replaced by the boundedness (uniformly in (ω, t)) condition: $b, \widetilde{b}, \overline{b}, \sigma, \widetilde{\sigma}$ and $\overline{\sigma}$ are bounded. Following Jacod and Protter (2012), we also assume that X is bounded.

Note that the following notation are adapted from that of Li, Liu, and Zhang (2022). To further simplify our notation, we define

$$\delta_n = 1/n \quad \text{and} \quad \Delta_n = l_n \delta_n = l_n/n.$$

In what follows, we shall use E_i^n to denote the conditional expectation with respect to the σ-field $\mathcal{F}_{i\delta_n}$. For each n, we define the function

$$g_n(s) = \sum_{j=1}^{k_n-1} g_j^n 1_{((j-1)\delta_n, j\delta_n]}(s),$$

which is bounded uniformly in n and equals zero for $s > (k_n - 1)\delta_n$ and $s \leq 0$. Define

$$X(n,s)_t = \int_0^{s+t} b_u g_n(u-s)du + \int_0^{s+t} \sigma_u g_n(u-s)dW_u,$$
$$C(n,s)_t = \int_0^{s+t} \sigma_u^2 g_n(u-s)^2 du, \quad C_t = \int_0^t \sigma_s^2 ds, \quad c_i^n = \sum_{j=1}^{k_n-1} (g_j^n)^2 \Delta_{i+j}^n C.$$

Note that the processes $X(n,s)_t$ and $C(n,s)_t$ vanish for $t \leq 0$ and are constant for $t \geq (k_n - 1)\delta_n$. It is also readily seen that $\overline{X}_i^n = X(n, i\delta_n)_{k_n\delta_n}$. Additionally, we define

$$B(n,s)_t = 2\int_0^{s+t} X(n,s)_u b_u g_n(u-s)du \quad \text{and} \quad M(n,s)_t = 2\int_0^{s+t} X(n,s)_u \sigma_u g_n(u-s)dW_u.$$

By Itô's formula, we have

$$X(n,s)^2 = B(n,s) + C(n,s) + M(n,s). \tag{S0.1}$$

S1 A Lemma

We need a lemma to prove Proposition 1 of the main text. To this end, we make the following notations for decomposing $\widehat{\nu}_{(i+l_n)\delta_n} - \widehat{\nu}_{i\delta_n}$:

$$\begin{aligned}
\eta(1,l_n)^n_i &= \frac{1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [c^n_{i+l_n+j} - c^n_{i+j}], \\
\eta(2,l_n)^n_i &= \frac{1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [(\overline{X^c}^n_{i+l_n+j})^2 - c^n_{i+l_n+j} - (\overline{X^c}^n_{i+j})^2 + c^n_{i+j}], \\
\eta(3,l_n)^n_i &= \frac{2}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [\overline{X^c}^n_{i+l_n+j}\overline{\epsilon}^n_{i+l_n+j} - \overline{X^c}^n_{i+j}\overline{\epsilon}^n_{i+j}], \\
\eta(4,l_n)^n_i &= \frac{1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [(\overline{\epsilon}^n_{i+l_n+j})^2 - (\overline{\epsilon}^n_{i+j})^2], \\
\eta(5,l_n)^n_i &= -\frac{1}{2\psi_2 k_n^2 l_n \delta_n} \sum_{j=0}^{l_n} [(\Delta^n_{i+l_n+j}Y)^2 - (\Delta^n_{i+j}Y)^2], \\
\eta(6,l_n)^n_i &= \frac{-1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [(\overline{X^c+\epsilon}^n_{i+l_n+j})^2 1_{|\overline{Y}^n_{i+l_n+j}|\geq u_{1,n}} - (\overline{X^c+\epsilon}^n_{i+j})^2 1_{|\overline{Y}^n_{i+j}|\geq u_{1,n}}], \\
\eta(7,l_n)^n_i &= \frac{1}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [(\overline{J}^n_{i+l_n+j})^2 1_{|\overline{Y}^n_{i+l_n+j}|< u_{1,n}} - (\overline{J}^n_{i+j})^2 1_{|\overline{Y}^n_{i+j}|< u_{1,n}}], \\
\eta(8,l_n)^n_i &= \frac{2}{\psi_2 k_n l_n \delta_n} \sum_{j=0}^{l_n-k_n-1} [\overline{X^c+\epsilon}^n_{i+l_n+j}\overline{J}^n_{i+l_n+j} 1_{|\overline{Y}^n_{i+l_n+j}|< u_{1,n}} - \overline{X^c+\epsilon}^n_{i+j}\overline{J}^n_{i+j} 1_{|\overline{Y}^n_{i+j}|< u_{1,n}}].
\end{aligned}$$

With the above notations, we can rewrite the increment $\widehat{\nu}_{(i+l_n)\delta_n} - \widehat{\nu}_{i\delta_n}$ as follows,

$$\widehat{\nu}_{(i+l_n)\delta_n} - \widehat{\nu}_{i\delta_n} = \sum_{j=1}^{8} \eta(j, l_n)_i^n. \tag{S1.2}$$

We further define:

$$\begin{aligned}
\breve{\eta}(1, l_n)_{i,m}^n &= \frac{\widetilde{\sigma}_{m\delta_n}}{\Delta_n} \int_0^{\Delta_n} (\widetilde{W}_{(i+l_n)\delta_n+s} - \widetilde{W}_{i\delta_n+s}) ds, \\
\breve{\eta}(2, l_n)_{i,m}^n &= \frac{2\sigma_{m\delta_n}^2}{\psi_2 k_n \Delta_n} \sum_{j=0}^{l_n-k_n-1} (\breve{\eta}(2)_{i+l_n+j}^n - \breve{\eta}(2)_{i+j}^n), \\
\breve{\eta}(3, l_n)_{i,m}^n &= \frac{2\sigma_{m\delta_n}}{\psi_2 k_n \Delta_n} \sum_{j=0}^{l_n-k_n-1} (\overline{W}_{i+l_n+j}^n \overline{\epsilon}_{i+l_n+j}^n - \overline{W}_{i+j}^n \overline{\epsilon}_{i+j}^n),
\end{aligned}$$

where

$$\breve{\eta}(2)_i^n = \int_{i\delta_n}^{(i+k_n)\delta_n} \int_{i\delta_n}^{u} g_n(s - i\delta_n) dW_s g_n(u - i\delta_n) dW_u.$$

When $m = i$, we simply write $\breve{\eta}(j, l_n)_i^n$ instead of $\breve{\eta}(j, l_n)_{i,i}^n$ for $j = 1, 2, 3$, which are estimates of $\{\eta(j, l_n)_i^n\}_{1\le j\le 3}$. The following lemma, which provides estimates for the bounds of relevant approximation errors and quantities, is adapted from Lemma 1 of Li, Liu, and Zhang (2022) and contains additional new results. We provide it and its proof below for completeness.

Lemma 1. *Suppose that Assumption 1 with $\vartheta > r$, and conditions $r > 0$, $\frac{1}{2} < \theta_1 < 1$, and $0 < \theta_2 < \frac{1}{4}$ hold. Then,*

$$\left.\begin{aligned}
E_i^n|\eta(2, l_n)_i^n - \breve{\eta}(2, l_n)_i^n|^r &\le K_r (k_n\delta_n)^{\frac{r}{2}}, \\
E_i^n|\eta(3, l_n)_i^n - \breve{\eta}(3, l_n)_i^n|^r &\le K_r (k_n^{-1})^{\frac{r}{2}},
\end{aligned}\right\} \tag{S1.3}$$

and

$$\left.\begin{array}{ll} E_i^n|\eta(1,l_n)_i^n|^r \leq K_r(l_n\delta_n)^{\frac{r}{2}}, & E_i^n|\eta(2,l_n)_i^n|^r \leq K_r(k_n/l_n)^{\frac{r}{2}}, \\ E_i^n|\eta(3,l_n)_i^n|^r \leq K_r(k_nl_n\delta_n)^{-\frac{r}{2}}, & E_i^n|\eta(4,l_n)_i^n|^r \leq K_r(k_n^3l_n\delta_n^2)^{-\frac{r}{2}}, \\ E_i^n|\eta(5,l_n)_i^n|^r \leq K_r(k_n^4l_n\delta_n^2)^{-\frac{r}{2}}, & \end{array}\right\} \tag{S1.4}$$

and for a positive integer q with $q \leq r$,

$$E_i^n|\eta(6,l_n)_i^n|^q \quad \leq \quad K\delta_n^{(\frac{1}{2}-\beta_1\theta_2)q}, \tag{S1.5}$$

$$E_i^n|\eta(j,l_n)_i^n|^q \quad \leq \quad K\left[u_{1,n}^{(2-\beta_1)q} + \left(\frac{n}{l_nk_n}\right)^{q-1} u_{1,n}^{2q-\beta_1}\right], \quad j = 7,8, \tag{S1.6}$$

where inequalities (S1.4) also hold when $\eta(j,l_n)_i^n$ is replaced by $\breve{\eta}(j,l_n)_i^n$ for $j = 1,2,3$. For $r \leq 1$, then

$$E_i^n|\eta(6,l_n)_i^n|^r \quad \leq \quad K\delta_n^{(\frac{1}{2}-\beta_1\theta_2)+(r-1)\theta_1}, \tag{S1.7}$$

$$E_i^n|\eta(j,l_n)_i^n|^r \quad \leq \quad \begin{cases} K\delta_n^{(\theta_1-\frac{1}{2})r-\theta_1}(k_n\delta_n)^{\frac{2r}{\beta_1}}, & r \leq \beta_1/2, \\ K\delta_n^{(\theta_1-\frac{1}{2})r-\theta_1}(k_n\delta_n)u_{1,n}^{2r-\beta_1}, & r \geq \beta_1/2, \end{cases} \quad j = 7,8. \tag{S1.8}$$

Proof. When $\widetilde{J} \equiv 0$, a straightforward adaptation of the proof of Lemma 1 in Li, Liu, and Zhang (2022) yield the results in (S1.3) and (S1.4) under the condition $\vartheta > r$. It also readily follows from the proof of Lemma 1 in Li, Liu, and Zhang (2022) that the results in (S1.3) and (S1.4) hold when $\widetilde{J} \neq 0$.

We next prove the result in (S1.6). By equation (2.1.47) of Jacod and Protter (2012), similarly to the procedure used for obtaining (A.4) in Jing, Liu, and Kong (2014), we have

$$E_i^n||\overline{J}_i^n| \wedge u_{1,n}|^r \le \begin{cases} K(k_n\delta_n)^{r/\beta_1}, & \text{for } r \le \beta_1, \\ K(k_n\delta_n)u_{1,n}^{r-\beta_1}, & \text{for } r \ge \beta_1. \end{cases} \tag{S1.9}$$

By noting that $1_{|\overline{Y}_i^n| \le u_{1,n}} \le 1_{|\overline{J}_i^n| \le 2u_{1,n}} + 1_{|\overline{(X^c+\epsilon)}_i^n| > u_{1,n}}$, we obtain

$$E_i^n|1_{|\overline{(X^c+\epsilon)}_i^n| > u_{1,n}}| \le E_i^n \frac{|\overline{(X^c+\epsilon)}_i^n|^q}{u_{1,n}^q} \le K\frac{(k_n\delta_n)^{q/2}}{u_{1,n}^p} \le Kn^{(\theta_2-\frac{1}{4})q} \tag{S1.10}$$

for any $q > 0$. Then, set

$$\breve{\eta}(7, l_n)_i^n = \frac{1}{k_n l_n \delta_n}\sum_{j=0}^{l_n}(\overline{J}_{i+j}^n)^2 1_{|\overline{J}_{i+j}^n| < 2u_{1,n}}. \tag{S1.11}$$

To obtain the result in (S1.6) for $j = 7$, by Hölder's inequality, the arbitrariness of q in (S1.10), and the condition $\theta_2 < 1/4$, it suffices to show the result in (S1.6) with $\eta(7, l_n)_i^n$ being replaced by $\breve{\eta}(7, l_n)_i^n$. For $1 \le j_1 < j_2$, by (S1.9), we have

$$\begin{aligned} &E_i^n(\overline{J}_{i+j_1}^n)^2 1_{|\overline{J}_{i+j_1}^n| < 2u_{1,n}}(\overline{J}_{i+j_2}^n)^2 1_{|\overline{J}_{i+j_2}^n| < 2u_{1,n}} \\ &= E_i^n[(\overline{J}_{i+j_1}^n)^2 1_{|\overline{J}_{i+j_1}^n| < 2u_{1,n}} E_{i+j_2-1}^n(\overline{J}_{i+j_2}^n)^2 1_{|\overline{J}_{i+j_2}^n| < 2u_{1,n}}] \le K(k_n\delta_n u_{1,n}^{2-\beta_1})^2. \end{aligned} \tag{S1.12}$$

Similarly, for any positive integer q, we have

$$\begin{aligned} & E_i^n|\breve{\eta}(7, k_n)_i^n|^q \\ \le\ & K\frac{n^q}{k_n^q l_n^q}\left[C_{l_n}^q(k_n\delta_n u_{1,n}^{2-\beta_1})^q + C_{l_n}^{q-1}(k_n\delta_n u_{1,n}^{2-\beta_1})^{q-1}u_{1,n}^2 + \cdots + C_{l_n}^1(k_n\delta_n u_{1,n}^{2-\beta_1})u_{1,n}^{2(q-1)}\right] \\ \le\ & K\left[u_{1,n}^{(2-\beta_1)q} + \left(\frac{n}{l_n k_n}\right)^{q-1} u_{1,n}^{2q-\beta_1}\right]. \end{aligned}$$

Hence, we have proved the result in (S1.6) for $j = 7$. The result in (S1.6) for $j = 8$ follows similarly.

We now turn to the result in (S1.5). It suffices to prove that the result in (S1.5) holds with $\eta(6, l_n)_i^n$ being replaced with

$$\frac{1}{k_n l_n \delta_n} \sum_{j=0}^{l_n} (\overline{X^c + \epsilon}_{i+j}^n)^2 1_{|\overline{Y}_{i+j}^n| \geq u_{1,n}}.$$

For any positive integer q, by $1_{|\overline{Y}_{i+j}^n| \geq u_{1,n}} \leq 1_{|\overline{(X^c+\epsilon)}_{i+j}^n| \geq u_{1,n}/2} + 1_{|\overline{J}_{i+j}^n| \geq u_{1,n}/2}$ and Hölder's inequality, we have

$$\begin{aligned}
& E_i^n \left(\frac{1}{k_n l_n \delta_n} \sum_{j=0}^{l_n} (\overline{X^c + \epsilon}_{i+j}^n)^2 1_{|\overline{Y}_{i+j}^n| \geq u_{1,n}} \right)^q \\
= & \frac{1}{(k_n l_n \delta_n)^q} \sum_{j_1, \cdots, j_q = 1}^{l_n} E_i^n (\overline{X^c + \epsilon}_{i+j_1}^n)^2 1_{|\overline{Y}_{i+j_1}^n| \geq u_{1,n}} \cdots (\overline{X^c + \epsilon}_{i+j_q}^n)^2 1_{|\overline{Y}_{i+j_q}^n| \geq u_{1,n}} \\
\leq & K \frac{1}{(l_n)^q} \sum_{j_1, \cdots, j_q = 1}^{l_n} E_i^n 1_{|\overline{Y}_{i+j_1}^n| \geq u_{1,n}} \cdots 1_{|\overline{Y}_{i+j_q}^n| \geq u_{1,n}} \\
= & K E_i^n \left(\frac{1}{l_n} \sum_{j=1}^{l_n} 1_{|\overline{Y}_{i+j}^n| \geq u_{1,n}} \right)^q \\
\leq & K E_i^n \left(\frac{1}{l_n} \sum_{j=1}^{l_n} 1_{|\overline{J}_{i+j}^n| \geq u_{1,n}/2} \right)^q + K E_i^n \left(\frac{1}{l_n} \sum_{j=1}^{l_n} 1_{|\overline{(X^c+\epsilon)}_{i+j}^n| \geq u_{1,n}/2} \right)^q .
\end{aligned}$$

By (S1.10), it suffices to show that the result in (S1.5) holds with $\eta(6, l_n)_i^n$ being replaced with

$$\breve{\eta}(6, l_n)_i^n = \frac{1}{l_n} \sum_{j=1}^{l_n} 1_{|\overline{J}_{i+j}^n| \geq u_{1,n}}. \tag{S1.13}$$

Letting $\alpha_i^n = 1_{|\overline{J}_i^n| \geq u_{1,n}}$, we have

$$E_i^n|\alpha_i^n| \quad \leq \quad K\frac{E_i^n||\overline{J}_i^n| \wedge u_{1,n}|}{u_{1,n}} \leq Kk_n\delta_n u_{1,n}^{-\beta_1} \leq K\delta_n^{(1/2-\beta_1\theta_2)}. \tag{S1.14}$$

Hence, for $0 \leq j_1 < j_2$, we obtain

$$\begin{aligned} & E_i^n|\alpha_{i+j_1}^n\alpha_{i+j_2}^n| = E_i^n(|\alpha_{i+j_1}^n|E_{i+j_2}^n|\alpha_{i+j_2}^n|) \\ & \leq KE_i^n(|\alpha_{i+j_1}^n|\delta_n^{(1/2-\beta_1\theta_2)}) \leq K\delta_n^{2(1/2-\beta_1\theta_2)}. \end{aligned} \tag{S1.15}$$

Similarly, for any positive integer q, by (S1.14), it follows easily that

$$E_i^n|\alpha_{i+j_1}^n\alpha_{i+j_2}^n\cdots\alpha_{i+j_q}^n| \leq K\delta_n^{n_q(1/2-\beta_1\theta_2)}, \tag{S1.16}$$

where n_q is the number of distinct elements in the set $I_q = \{j_1, j_2, \cdots, j_q\}$. Thus, by (S1.16) and (S1.13), we have

$$\begin{aligned} E_i^n|\breve{\eta}(6,k_n)_i^n|^q \quad & \leq \quad K\frac{1}{l_n^q}\left[l_n^q\delta_n^{(1/2-\beta_1\theta_2)q} + l_n^{q-1}\delta_n^{(1/2-\beta_1\theta_2)(q-1)} + \cdots + l_n^1\delta_n^{(1/2-\beta_1\theta_2)}\right] \\ & \leq \quad K\left[\delta_n^{(1/2-\beta_1\theta_2)q} + \delta_n^{(1/2-\beta_1\theta_2)+(q-1)\theta_1)}\right] \leq K\delta_n^{(1/2-\beta_1\theta_2)q}, \end{aligned} \tag{S1.17}$$

where we use the fact $\frac{1}{2} - \beta_1\theta_2 - \theta_1 < 0$ in the last step. Hence, we complete the proof of the result in (S1.5). □

S2 Proof of Propositions 1, 2 and 3

Proof of Proposition 1. We only prove the result in (3.9) of the main text for $\theta_1 = 3/4$. The proofs of the rest results are quite similar and hence omitted.

We prove the result in (3.9) for $\theta_1 = 3/4$ when $\widetilde{J} \equiv 0$. Recalling (S1.2) and by triangle inequality (for $p \leq 1$) and Minkowski inequality (for $p > 1$), one readily sees that it suffices to show

$$n^{\frac{p}{8}-1} c_1^{-\frac{p}{2}} \sum_{i=0}^{[nt]-2l_n} \Big|\sum_{j=1}^{4} \eta(j,l_n)_i^n\Big|^p \xrightarrow{P} \mu_p \int_0^t \left(\frac{2}{3}\widetilde{\sigma}_s^2 + \frac{2}{c_1^2}\gamma_s^2\right)^{\frac{p}{2}} ds, \tag{S2.18}$$

$$n^{\frac{p}{8}-1} c_1^{-\frac{p}{2}} \sum_{i=0}^{[nt]-2l_n} \Big|\sum_{j=5}^{8} \eta(j,l_n)_i^n\Big|^p \xrightarrow{P} 0. \tag{S2.19}$$

We prove the result in (S2.18) first. Using the notations listed right before Lemma 1 and by simple calculations, we obtain

$$\left.\begin{aligned}
E_i^n(\breve{\eta}(1,l_n)_i^n)^2 &= \frac{2}{3}\widetilde{\sigma}_{i\delta_n}^2 \Delta_n, \\
E_i^n(\breve{\eta}(2,l_n)_i^n)^2 &= \frac{8\Phi_{22}}{\psi_2^2}\sigma_{i\delta_n}^4 \frac{k_n}{l_n} + O_P\left(\frac{k_n^2}{l_n^2}\right), \\
E_i^n(\breve{\eta}(3,l_n)_i^n)^2 &= \frac{16\Phi_{12}}{\psi_2^2}\sigma_{i\delta_n}^2 \sigma_\epsilon^2 \frac{1}{k_n\Delta_n} + O_P\left(\frac{1}{l_n\Delta_n}\right), \\
E_i^n(\eta(4,l_n)_i^n)^2 &= \frac{8\Phi_{11}}{\psi_2^2}\sigma_\epsilon^4 \frac{1}{k_n^3 l_n \delta_n^2} + O_P\left(\frac{1}{k_n^2 l_n^2 \delta_n}\right), \\
E_i^n(\eta(5,l_n)_i^n)^2 &= \frac{2}{\psi_2^2}\sigma_\epsilon^4 \frac{1}{k_n^4 l_n \delta_n^2} + O_P\left(\frac{1}{k_n^4 l_n \delta_n}\right).
\end{aligned}\right\} \tag{S2.20}$$

For notational convenience, we let $\breve{\eta}(j,l_n)_i^n \equiv \eta(j,l_n)_i^n$ for $j = 4$. Then, by Lemma 1, we have

$$\begin{aligned}
E_i^n\left(\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n\right)^2 \quad &= \quad \frac{2}{3}\widetilde{\sigma}_{i\delta_n}^2 l_n\delta_n + 2\gamma_{i\delta_n}^2 \frac{n^{\frac{1}{2}}}{l_n} + O_P\left((l_n\delta_n)^{\frac{3}{2}}\right) + O_P\left(\left(\frac{n^{\frac{1}{2}}}{l_n}\right)^{\frac{3}{2}}\right) \\
&\quad + O_P\left(l_n^{-\frac{1}{2}}\right) + O_P\left(l_n^{\frac{1}{2}} n^{-\frac{3}{4}}\right).
\end{aligned}$$

Note that each $(l_n\delta_n)^{-1/2}\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n$ has (conditionally on $\mathcal{F}_{i\delta_n}$) mean zero and a variance of the form $\frac{2}{3}\widetilde{\sigma}_{i\delta_n}^2 + 2\gamma_{i\delta_n}^2/c_1^2 + o_P(1)$ and is (conditionally on $\mathcal{F}_{i\delta_n}$) asymptotically normal.

Then,

$$E_i^n(l_n\delta_n)^{-\frac{p}{2}}\left|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n\right|^p \quad = \quad \mu_p\left(\frac{2}{3}\widetilde{\sigma}_{i\delta_n}^2+\frac{2\gamma_{i\delta_n}^2}{c_1^2}\right)^{\frac{p}{2}}+o_P\left(1\right).$$

Therefore, we obtain, for $\theta_1=3/4$,

$$\begin{aligned}\delta_n\sum_{i=0}^{[nt]-2l_n}E_i^n(l_n\delta_n)^{-\frac{p}{2}}\left|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n\right|^p &= n^{\frac{p}{8}-1}c_1^{-\frac{p}{2}}\sum_{i=0}^{[nt]-2l_n}E_i^n\left|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n\right|^p\\ &\xrightarrow{P}\mu_p\int_0^t\left(\frac{2}{3}\widetilde{\sigma}_s^2+\frac{2}{c_1^2}\gamma_s^2\right)^{\frac{p}{2}}ds. \end{aligned}\tag{S2.21}$$

Now in order to obtain (S2.18), it remains to show

$$n^{\frac{p}{8}-1}\sum_{i=0}^{[nt]-2l_n}\left||\sum_{j=1}^{4}\eta(j,l_n)_i^n|^p-|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n|^p\right|\xrightarrow{P}0,\tag{S2.22}$$

$$n^{\frac{p}{8}-1}\sum_{i=0}^{[nt]-2l_n}\left(|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n|^p-E_i^n|\sum_{j=1}^{4}\breve{\eta}(j,l_n)_i^n|^p\right)\xrightarrow{P}0.\tag{S2.23}$$

For (S2.22), it suffices to prove $n^{\frac{p}{8}-1}\sum_{i=0}^{[nt]-2l_n}|\eta(j,l_n)_i^n-\breve{\eta}(j,l_n)_i^n|^p\xrightarrow{P}0$ for $j=1,\cdots,4$, which are direct consequences of Lemma 1. As to (S2.23), by Lemma 2.2.11 of Jacod and Protter (2012), it is sufficient to prove

$$n^{\frac{p}{4}-2}\sum_{i=0}^{[nt]-2l_n}E_i^n|\breve{\eta}(j,l_n)_i^n|^{2p}\xrightarrow{P}0,\quad j=1,2,\cdots,4,$$

which are again direct consequences of Lemma 1.

We now turn to deal with (S2.19). To this end, it suffices to prove

$$n^{\frac{p}{8}-1}\sum_{i=0}^{[nt]-2l_n}|\eta(j,l_n)_i^n|^p\xrightarrow{P}0,\quad j=5,\cdots,8.\tag{S2.24}$$

It is easily seen that the result in (S2.24) with $j = 5$ is a direct consequence of Lemma 1. Let $\lceil p \rceil$ be the ceiling function in p, i.e., the minimum integer greater than or equal to p. Hence, $0 < \frac{p}{\lceil p \rceil} \leq 1$ for $p > 0$. In order to prove (S2.24) with $j = 6$, by (S1.5), we have, for $p > 0$,

$$\begin{aligned} n^{\frac{p}{8}-1} \sum_{i=0}^{[nt]-2l_n} E_i^n \, |\eta(6, l_n)_i^n|^p &\leq n^{\frac{p}{8}-1} \sum_{i=0}^{[nt]-2l_n} (E_i^n \, |\eta(6, l_n)_i^n|^{\lceil p \rceil})^{\frac{p}{\lceil p \rceil}} \\ &\leq K n^{\frac{p}{8}} \delta_n^{(1/2-\beta_1\theta_2)p} = K \delta_n^{(\frac{1}{2}-\beta_1\theta_2-\frac{1}{8})p}. \end{aligned}$$

Therefore, by $\theta_2 < 1/4$ and $\beta_1 < 1$, we obtain the result in (S2.24) with $j = 6$ for $p > 0$.

We now prove the result in (S2.24) with $j = 7$. When $p \geq 1$, by Lemma 1, we have

$$\begin{aligned} & n^{\frac{p}{8}} E_i^n |\eta(7, l_n)_i^n|^p \leq n^{\frac{p}{8}} (E_i^n |\eta(7, l_n)_i^n|^{\lceil p \rceil})^{\frac{p}{\lceil p \rceil}} \\ & \leq K n^{\frac{p}{8}} \left(n^{-\theta_2(2-\beta_1)p} + n^{(1-\theta_1-\frac{1}{2})(p-\frac{p}{\lceil p \rceil})-\theta_2(2p-\frac{p}{\lceil p \rceil}\beta_1)} \right) \\ & \leq K \left(n^{p(-\theta_2(2-\beta_1)+\frac{1}{8})} + n^{(\frac{1}{2}-\theta_1-2\theta_2+\frac{1}{8})p+(\theta_2\beta_1-\frac{1}{2}+\theta_1)\frac{p}{\lceil p \rceil}} \right) \\ & \leq K \left(n^{p(-\theta_2(2-\beta_1)+\frac{1}{8})} + n^{(\frac{1}{2}-\theta_1-2\theta_2+\frac{1}{8})p+(\theta_2\beta_1-\frac{1}{2}+\theta_1)} \right) \\ & \leq K \left(n^{p(-\theta_2(2-\beta_1)+\frac{1}{8})} + n^{((\frac{5}{8}-\theta_1)p-\frac{1}{2}+\theta_1)-\theta_2(2p-\beta_1)} \right), \end{aligned}$$

where we use the fact that $\theta_2\beta_1 - \frac{1}{2} + \theta_1 > 0$. Hence, we obtain the result in (S2.24) with $j = 7$ for $p \geq 1$ provided that the condition $\theta_2 > \frac{1}{8(2-\beta_1)} \vee \frac{(5-8\theta_1)p-4+8\theta_1}{8(2p-\beta_1)}$ holds. When $\theta_1 = 3/4$ and $(5-8\theta_1)p-4+8\theta_1 = 2-p < 1$ for $p \geq 1$ hold, the condition $\theta_2 > \frac{1}{8(2-\beta_1)}$ readily implies that (S2.24) holds with $j = 7$ for $p \geq 1$. When $p < 1$, we have

$$n^{\frac{p}{8}} E_i^n |\eta(7, l_n)_i^n|^p \leq (E_i^n |n^{\frac{1}{8}} \eta(7, l_n)_i^n|)^p \leq K n^{p(-\theta_2(2-\beta_1)+\frac{1}{8})}.$$

Hence, the condition $\theta_2 > \frac{1}{8(2-\beta_1)}$ readily implies that (S2.24) holds with $j = 7$ for $p \leq 1$. Therefore, under the condition $\theta_2 > \frac{1}{8(2-\beta_1)}$, we have that (S2.24) holds with $j = 7$ for $p > 0$. The result in (S2.24) with $j = 8$ follows similarly. Therefore, we have proved the results in (S2.24) and (S2.19). $\square$

Proof of Proposition 2. We prove the result in (3.11) when $\widetilde{J} \neq 0$. Setting $a_n = \sum_{j=1}^{k_n-1}(g_j^n)^2$, we have

$$\begin{aligned}
&\sum_{j_1=0}^{l_n-k_n+1} c_{i+j_1}^n = \sum_{j_1=0}^{l_n-k_n+1}\sum_{l=j_1+1}^{j_1+k_n-1}(g_{l-j_1}^n)^2\Delta_{l+i}^n C = \sum_{l=1}^{l_n}\Delta_{l+i}^n C\sum_{j=1\vee(l+k_n-1-l_n)}^{l\wedge(k_n-1)}(g_j^n)^2\\
= \;& a_n\sum_{l=k_n-1}^{l_n-k_n+2}\Delta_{i+l}^n C + \sum_{l=1}^{k_n-2}\sum_{j=1}^{l}(g_j^n)^2\Delta_{l+i}^n C + \sum_{l=l_n-k_n+3}^{l_n}\sum_{j=l+k_n-1-l_n}^{k_n-1}(g_j^n)^2\Delta_{l+i}^n C\\
= \;& a_n\int_{(k_n-2)\delta_n}^{(l_n-k_n+2)\delta_n}\sigma_{s+i\delta_n}^2 ds + \sum_{l=1}^{k_n-2}\sum_{j=1}^{l}(g_j^n)^2\Delta_{l+i}^n C\\
&+ \sum_{l=l_n-k_n+3}^{l_n}\sum_{j=l+k_n-1-l_n}^{k_n-1}(g_j^n)^2\Delta_{l+i}^n C.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\eta(1,l_n)_i^n \;=\; & \frac{1}{\psi_2 k_n\Delta_n}\Bigg[a_n\int_{(k_n-2)\delta_n}^{(l_n-k_n+2)\delta_n}(\sigma_{(i+l_n)\delta_n+s}^2 - \sigma_{i\delta_n+s}^2)ds\\
&+ \sum_{l=1}^{k_n-2}\sum_{j=1}^{l}(g_j^n)^2(\Delta_{l+i+l_n}^n C - \Delta_{l+i}^n C)\\
&+ \sum_{l=l_n-k_n+3}^{l_n}\sum_{j=l+k_n-1-l_n}^{k_n-1}(g_j^n)^2(\Delta_{l+i+l_n}^n C - \Delta_{l+i}^n C)\Bigg]. \qquad \text{(S2.25)}
\end{aligned}$$

Noting that

$$\begin{aligned}E_i^n\left|\sum_{l=1}^{k_n-2}\sum_{j=1}^{l}(g_j^n)^2(\Delta_{l+i+l_n}^n C-\Delta_{l+i}^n C)\right| &\le Kk_nE_i^n\sum_{l=1}^{k_n}\left|\Delta_{l+i+l_n}^n C-\Delta_{l+i}^n C\right|\\ \le Kk_n\sum_{l=1}^{k_n}E_i^n\int_{(i+l-1)\delta_n}^{(i+l)\delta_n}\left|\sigma^2_{l_n\delta_n+s}-\sigma_s^2\right|ds &\le Kk_nk_n\delta_n(l_n\delta_n)^{\frac{1}{2}}\le K(l_n\delta_n)^{\frac{1}{2}},\end{aligned}$$

and $a_n=\sum_{j=1}^{k_n-1}(g_j^n)^2=\psi_2k_n+O(1)$, we obtain that

$$\begin{aligned}\eta(1,l_n)_i^n &= \frac{1}{\Delta_n}\int_{(k_n-2)\delta_n}^{(l_n-k_n+2)\delta_n}(\sigma^2_{(i+l_n)\delta_n+s}-\sigma^2_{i\delta_n+s})ds+O_P\left(\frac{(l_n\delta_n)^{\frac{1}{2}}}{k_n\Delta_n}\right)\\ &= \frac{1}{\Delta_n}\int_{0}^{\Delta_n}(\sigma^2_{(i+l_n)\delta_n+s}-\sigma^2_{i\delta_n+s})ds+O_P\left(\frac{(l_n\delta_n)^{\frac{1}{2}}}{k_n\Delta_n}\right).\end{aligned}\tag{S2.26}$$

Furthermore,

$$\begin{aligned}\eta(1,l_n)_i^n &= \breve{\eta}(1,l_n)_i^n+\frac{1}{\Delta_n}\int_0^{\Delta_n}(\widetilde{J}_{(i+l_n)\delta_n+s}-\widetilde{J}_{i\delta_n+s})ds\\ &\quad+\frac{1}{\Delta_n}\int_0^{\Delta_n}\int_{i\delta_n+s}^{(i+l_n)\delta_n+s}\widetilde{b}_u du ds+\frac{1}{\Delta_n}\int_0^{\Delta_n}\int_{i\delta_n+s}^{(i+l_n)\delta_n+s}(\widetilde{\sigma}_u-\widetilde{\sigma}_{i\delta_n})d\widetilde{W}_u ds\\ &\quad+O_P\left(\frac{(l_n\delta_n)^{\frac{1}{2}}}{k_n\Delta_n}\right)\\ &= \breve{\eta}(1,l_n)_i^n+\overline{\eta}(1,l_n)_i^n+o_P((l_n\delta_n)^{\frac{1}{2}})+O_P\left(\frac{(l_n\delta_n)^{\frac{1}{2}}}{k_n\Delta_n}\right),\end{aligned}\tag{S2.27}$$

where

$$\overline{\eta}(1,l_n)_i^n=\frac{1}{\Delta_n}\int_0^{\Delta_n}(\widetilde{J}_{(i+l_n)\delta_n+s}-\widetilde{J}_{i\delta_n+s})ds.\tag{S2.28}$$

Then, we have, for $p>\beta_2$,

$$c_1^{-1}n^{-\theta_1}\sum_{i=0}^{[nt]-2l_n}|\overline{\eta}(1,l_n)_i^n|^p=\frac{1}{l_n}\sum_{i=0}^{[nt]-2l_n}|\overline{\eta}(1,l_n)_i^n|^p\xrightarrow{P}\sum_{0\le s\le t}|\Delta\widetilde{J}_s|^p.\tag{S2.29}$$

Hence, to obtain the result in (3.11), it suffices to show

$$c_1^{-1}n^{-\theta_1}\sum_{i=0}^{[nt]-2l_n}|\breve{\eta}(1,l_n)_i^n|^p \xrightarrow{P} 0, \tag{S2.30}$$

$$c_1^{-1}n^{-\theta_1}\sum_{i=0}^{[nt]-2l_n}|\eta(j,l_n)_i^n|^p \xrightarrow{P} 0, \quad j=2,\cdots,8. \tag{S2.31}$$

By Lemma 1 and $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1-1}\}$, we obtain (S2.30) and results in (S2.31) with $j = 2, 3, 4, 5$. Similarly to the proof of the result in (S2.24) with $j = 6$, by the condition $p > \frac{2(1-\theta_1)}{1-2\beta_1\theta_2}$, we obtain the result in (S2.31) with $j = 6$. However, the condition $p > \max\{2, \frac{4(1-\theta_1)}{2\theta_1-1}\}$ suggests $p > \frac{2(1-\theta_1)}{1-2\beta_1\theta_2}$ since $1/2 < \theta_1 < 1, \theta_2 < 1/4$, and $\beta_1 < 1$. By the same arguments as that used in the proof of the result in (S2.24) with $j = 7$, we have that

$$\begin{aligned} E_i^n n^{-\theta_1}\sum_{i=0}^{[nt]-2l_n}|\eta(j,l_n)_i^n|^p &\le Kn^{-\theta_1+1}\times\left(n^{-p\theta_2(2-\beta_1)}+n^{(\frac{1}{2}-\theta_1-2\theta_2)p+(\theta_2\beta_1-\frac{1}{2}+\theta_1)}\right) \\ &\le K\left(n^{1-\theta_1-p\theta_2(2-\beta_1)}+n^{(\frac{1}{2}-\theta_1)p+\frac{1}{2}-\theta_2(2p-\beta_1)}\right). \end{aligned}$$

Hence, the condition $\theta_2 > \frac{1-\theta_1}{p(2-\beta_1)} \vee \frac{(1-2\theta_1)p+1}{2(2p-\beta_1)}$ readily implies the result in (S2.31) with $j = 7$. Similarly to the proof of the result in (S2.24) with $j = 8$, we obtain the result in (S2.31) with $j = 8$. Therefore, we complete the proof of the result in (3.11) of the main text. □

Proof of Proposition 3. We only prove the result in (3.9) for $TV_p(\nu; c_1, \theta_1)_t^n$ when $\widetilde{J} \neq 0$. The proofs of the other results are similar and hence omitted.

By the proof of Proposition 1, (S1.2), and (S2.27), we obtain

$$n^{\frac{p-8}{8}}c_1^{-\frac{p}{2}}\sum_{i=1}^{[nt]-2l_n}\left|\sum_{j=1}^{8}\eta(j,l_n)_i^n-\overline{\eta}(1,l_n)_i^n\right|^p \xrightarrow{P} \mu_p\int_0^t\left(\frac{2}{3}\widetilde{\sigma}_s^2+\frac{2}{c_1^2}\gamma_s^2\right)^{\frac{p}{2}}ds. \tag{S2.32}$$

Then, it remains to prove

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} \left| \left| \sum_{j=1}^{8} \eta(j,l_n)_i^n \right|^p 1_{\left|\sum_{j=1}^{8} \eta(j,l_n)_i^n\right| < u_{2,n}} - \left| \sum_{j=1}^{8} \eta(j,l_n)_i^n - \overline{\eta}(1,l_n)_i^n \right|^p \right| \xrightarrow{P} 0.$$

Note that we have, for any $p > 0$ and $\gamma > 0$, the following inequality,

$$\left| |x+y|^p 1_{|x+y|<\gamma} - |x|^p \right| \leq K_p \left[\frac{|x|^{p+r_1}}{\gamma^{r_1}} + \frac{|x|^p |y|^{r_2}}{\gamma^{r_2}} + (|y| \wedge \gamma)^p + 1_{p>1} |x|^{p-1} (|y| \wedge \gamma) \right], \tag{S2.33}$$

where r_1 and r_2 are two arbitrary positive constants. By (S2.27), setting $x = \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n$, $y = \overline{\eta}(1,l_n)_i^n$ and $\gamma = u_{2,n}$ in (S2.33), one readily sees that it suffices to prove the following,

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} \left| \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n \right|^p \left| \frac{\eta(j,l_n)_i^n}{u_{2,n}} \right|^{r_1} \xrightarrow{P} 0, \ j = 2, \cdots, 8 \tag{S2.34}$$

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} \left| \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n \right|^p \left| \frac{\breve{\eta}(1,l_n)_i^n}{u_{2,n}} \right|^{r_1} \xrightarrow{P} 0, \tag{S2.35}$$

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} \left| \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n \right|^p \left| \frac{\overline{\eta}(1,l_n)_i^n}{u_{2,n}} \right|^{r_2} \xrightarrow{P} 0, \tag{S2.36}$$

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} (|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n})^p \xrightarrow{P} 0, \tag{S2.37}$$

$$n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} 1_{p>1} \left| \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n \right|^{p-1} (|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n}) \xrightarrow{P} 0. \tag{S2.38}$$

We deal with the above statements one by one. For the result in (S2.35), by Hölder's inequality, we have

$$\begin{aligned} & n^{\frac{p-8}{8}} \sum_{i=1}^{[nt]-2l_n} \left| \sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n \right|^p \left| \frac{\breve{\eta}(1,l_n)_i^n}{u_{2,n}} \right|^{r_1} \\ \le & \left(\frac{1}{n} \sum_{i=1}^{[nt]-2l_n} \left| n^{\frac{1}{8}} (\sum_{j=2}^{8} \eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n) \right|^{ap} \right)^{\frac{1}{a}} \left(\frac{1}{n} \sum_{i=1}^{[nt]-2l_n} \left| \frac{\breve{\eta}(1,l_n)_i^n}{u_{2,n}} \right|^{br_1} \right)^{\frac{1}{b}} \end{aligned} \tag{S2.39}$$

where $a, b > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. By (S2.32) and letting $a \to 1$, we obtain that the first term in (S2.39) is bounded in probability. Then, by the arbitrariness of r_1, it is enough to prove

$$E_i^n \left| \frac{\breve{\eta}(1,l_n)_i^n}{u_{2,n}} \right|^r \le K n^{-\gamma}, \quad i = 1, 2, \cdots, [nt] - 2l_n, \tag{S2.40}$$

for some $r > 0$ and $\gamma > 0$. This is a direct consequence of $E|\breve{\eta}(1,l_n)_i^n| = O_P((l_n\delta_n)^{\frac{1}{2}}) = O_P(n^{\frac{\theta_1-1}{2}})$ and $\theta_4 < \frac{1-\theta_1}{2}$. Hence, we have proved the result in (S2.35).

Similarly, by Lemma 1, one readily sees that the condition $\theta_4 < \frac{1-\theta_1}{2}$ implies the results in (S2.34) with $j = 2, 3, 4, 5, 6$. As to the results in (S2.34) with $j = 7$ and $j = 8$, due to Lemma 1, it suffices to deal with the case of $j = 7$. To this end, by (S1.6) and $\beta_1 < 1$, we obtain that

$$E_i^n \left| \frac{\eta(7,l_n)_i^n}{u_{2,n}} \right| \le K \delta_n^{(2-\beta_1)\theta_2 - \theta_4}.$$

Hence, the condition $\theta_4 < (2 - \beta_1)\theta_2$ leads to the result in (S2.34) with $j = 7$.

We next turn to (S2.36). Similarly to the proof of the result in (S2.35), it suffices to prove the result in (S2.40) with $\breve{\eta}(1,l_n)_i^n$ being replaced by $\overline{\eta}(1,l_n)_i^n$. By (S2.28), we have $E_i^n|\overline{\eta}(1,l_n)_i^n| \le K(l_n\delta_n) \le K n^{\theta_1-1}$. Then, the condition $\theta_4 < \frac{1-\theta_1}{2}$ implies (S2.36).

As to (S2.37), similarly to how we obtain result in (S1.9), we obtain that, for $r > 0$,

$$E_i^n||\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n}|^r \leq \begin{cases} K(l_n\delta_n)^{r/\beta_2}, & \text{for } r < \beta_2, \\ K(l_n\delta_n)u_{2,n}^{r-\beta_2}, & \text{for } r \geq \beta_2. \end{cases} \tag{S2.41}$$

When $p < \beta_2$, we have

$$E\left|n^{\frac{p-8}{8}}\sum_{i=1}^{[nt]-2l_n}(|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n})^p\right| \leq Kn^{\frac{p-8}{8}}n(l_n\delta_n)^{p/\beta_2} \leq Kn^{\frac{p(\beta_2+8\theta_1-8)}{8\beta_2}},$$

where note that $\beta_2 + 8\theta_1 - 8 < 0$ because $\theta_1 = 3/4$ and $\beta_2 < 1$. Hence, we obtain the result in (S2.37) when $p < \beta_2$. When $p \geq \beta_2$, we have

$$E\left|n^{\frac{p-8}{8}}\sum_{i=1}^{[nt]-2l_n}(|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n})^p\right| \leq Kn^{\frac{p-8}{8}}n(l_n\delta_n)u_{2,n}^{p-\beta_2} \leq Kn^{\frac{p}{8}+(\theta_1-1)-\theta_4(p-\beta_2)}. \tag{S2.42}$$

Hence, the condition $\theta_4 > \frac{p+8\theta_1-8}{8(p-\beta_2)}$ implies the result in (S2.37) when $p \geq \beta_2$.

We finally deal with (S2.38). By Hölder's inequality and $p > 1$, we have

$$\begin{aligned} &n^{\frac{p-8}{8}}\sum_{i=1}^{[nt]-2l_n}1_{p>1}\left|\sum_{j=2}^{8}\eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n\right|^{p-1}(|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n}) \\ &\leq \left(\frac{1}{n}\sum_{i=1}^{[nt]-2l_n}\left|n^{\frac{1}{8}}(\sum_{j=2}^{8}\eta(j,l_n)_i^n + \breve{\eta}(1,l_n)_i^n)\right|^p\right)^{\frac{p-1}{p}}\left(\frac{1}{n}\sum_{i=1}^{[nt]-2l_n}\left|n^{\frac{1}{8}}(|\overline{\eta}(1,l_n)_i^n| \wedge u_{2,n})\right|^p\right)^{\frac{1}{p}} \end{aligned} \tag{S2.43}$$

By (S2.32), we obtain that the first term in (S2.43) is bounded in probability. Then, we obtain the result in (S2.38) by (S2.42) and the condition $\theta_4 > \frac{p+8\theta_1-8}{8(p-\beta_2)}$. □

S3 Proofs of Theorems 1 and 2

Proof of Theorem 1. By delta method, (3.12) and (3.8) of the main text, it suffices to prove

$$\begin{pmatrix} U^n_{1,t} \\ \\ U^n_{2,t} \end{pmatrix} := n^{\frac{1-\theta_1}{2}} \begin{pmatrix} n^{(\theta_1-\frac{1}{2})\frac{p}{2}-1} c_1^{\frac{p}{2}} V_p(\nu; c_1, \theta_1)^n_t - \mu_p 2^{\frac{p}{2}} \int_0^t \gamma_s^p ds \\ \\ n^{(\theta_1-\frac{1}{2})\frac{p}{2}-1} (2c_1)^{\frac{p}{2}} V_p(\nu; 2c_1, \theta_1)^n_t - \mu_p 2^{\frac{p}{2}} \int_0^t \gamma_s^p ds \end{pmatrix} \xrightarrow{L-s} \int_0^t \overline{\Sigma}_s d\mathbb{B}_s, \qquad \text{(S3.44)}$$

where

$$\overline{\Sigma}_t^T \overline{\Sigma}_t = \begin{pmatrix} c_1 2^{p+1} \int_0^2 h_p(\rho_1(s)) ds \gamma_t^{2p} & c_1 2^p \int_{-4}^2 h_p(\rho_2(s)) ds \gamma_t^{2p} \\ c_1 2^p \int_{-4}^2 h_p(\rho_2(s)) ds \gamma_t^{2p} & c_1 2^{p+2} \int_0^2 h_p(\rho_1(s)) ds \gamma_t^{2p} \end{pmatrix} =: \begin{pmatrix} \Sigma^2_{11,t} & \Sigma^2_{12,t} \\ \\ \Sigma^2_{21,t} & \Sigma^2_{22,t} \end{pmatrix},$$

and $\mathbb{B}$ is a standard two-dimensional Brownian motion. We need some additional notations. Let q be an arbitrary positive integer, which is allowed to go to infinity. Let $J_n(q) = J_{n,t}(q) := [([nt]-4l_n)/(2(q+2)l_n)]$ denote the number of big blocks with common time span $2(q+2)\Delta_n$. Define

$$a_m(q) := (m-1)2(q+2)l_n, \quad b_m(q) := a_m(q) + 2ql_n, \quad s_m := a_m(q),$$

for $m = 1, 2, \cdots, J_n(q)$. Define the set of integers:

$$A_m(q) = \{i : a_m(q) \le i < b_m(q)\}, \quad B_m(q) = \{i : b_m(q) \le i < a_{m+1}(q)\}.$$

For $i \in A_m(q) \bigcup B_m(q), j = 1, 2$, define

$$\xi(j)^n_{i,m} \;:=\; (jc_1)^{\frac{1}{2}} n^{\frac{2\theta_1-1}{4}} \left(\breve{\eta}(2, jl_n)^n_{i,a_m(q)} + \breve{\eta}(3, jl_n)^n_{i,a_m(q)} + \eta(4, jl_n)^n_i \right).$$

Note that, conditional on $\mathcal{F}_{s_m\delta_n}$, $\xi(j)^n_{i,m}$ is asymptotically normal with mean zero and variance of the form $2\gamma^2_{s_m\delta_n} + o_P(1)$. Moreover, we have

$$E^n_{s_m}\left|\xi(j)^n_{i,m}\right|^p \;=\; \mu_p 2^{\frac{p}{2}}\gamma^p_{s_m\delta_n} + o_P(1). \tag{S3.45}$$

For $j = 1, 2$, by (S1.2), we can now decompose $U^n_{j,t}$ in (S3.44) as follows:

$$U^n_{j,t} \equiv n^{\frac{1-\theta_1}{2}}\left(n^{(\theta_1-\frac{1}{2})\frac{p}{2}-1}(jc_1)^{\frac{p}{2}}V_p(\nu; jc_1, \theta_1)^n_t - \mu_p 2^{\frac{p}{2}}\int_0^t \gamma^p_s ds\right) = M^{n,q}_{j,t} + \sum_{j_1=1}^{5} R(j_1)^n_{j,t},$$

where

$$\begin{aligned}
M^{n,q}_{j,t} &= n^{\frac{1-\theta_1}{2}}\sum_{m=1}^{J_n(q)}\left(\sum_{i=a_m(q)}^{b_m(q)}\left(|\xi(j)^n_{i,m}|^p - E^n_{s_m}|\xi(j)^n_{i,m}|^p\right)\right)\delta_n =: n^{\frac{1-\theta_1}{2}}\sum_{m=1}^{J_n(q)} D^n_{j,m}\\
R(1)^n_{j,t} &= n^{\frac{1-\theta_1}{2}}\sum_{i=0}^{[nt]-2jl_n}\left|(jc_1)^{\frac{1}{2}}n^{\frac{2\theta_1-1}{4}}\right|^p\left(\left|\sum_{j_1=1}^{8}\eta(j_1, jl_n)^n_i\right|^p - \left|\sum_{j_1=2}^{4}\eta(j_1, jl_n)^n_i\right|^p\right)\delta_n\\
R(2)^n_{j,t} &= n^{\frac{1-\theta_1}{2}}\sum_{i=0}^{[nt]-2jl_n}\left(\left|(jc_1)^{\frac{1}{2}}n^{\frac{2\theta_1-1}{4}}\sum_{j_1=2}^{4}\eta(j_1, jl_n)^n_i\right|^p - |\xi(j)^n_{i,m}|^p\right)\delta_n\\
R(3)^n_{j,t} &= n^{\frac{1-\theta_1}{2}}\sum_{m=1}^{J_n(q)}\left(\sum_{i=b_m(q)+1}^{a_{m+1}(q)-1}\left(|\xi(j)^n_{i,m}|^p - E^n_{s_m}|\xi(j)^n_{i,m}|^p\right)\right)\delta_n\\
R(4)^n_{j,t} &= n^{\frac{1-\theta_1}{2}}\sum_{i=J_n(q)2(q+2)l_n+1}^{[nt]-2jl_n}|\xi(j)^n_{i,m}|^p\delta_n\\
R(5)^n_{j,t} &= n^{\frac{1-\theta_1}{2}}\left(\sum_{i=0}^{J_n(q)2(q+2)l_n}E^n_{s_m}|\xi(j)^n_{i,m}|^p\delta_n - \mu_p 2^{\frac{p}{2}}\int_0^t \gamma^p_s ds\right),
\end{aligned}$$

and

$$D^n_{j,m} := \left(\sum_{i=a_m(q)}^{b_m(q)}\left(|\xi(j)^n_{i,m}|^p - E^n_{s_m}|\xi(j)^n_{i,m}|^p\right)\right)\delta_n.$$

Hence, it suffices to prove

$$\begin{pmatrix} M_{1,t}^{n,q} \\ \\ M_{2,t}^{n,q} \end{pmatrix} \equiv n^{\frac{1-\theta_1}{2}} \sum_{m=1}^{J_n(q)} \begin{pmatrix} D_{1,m}^{n} \\ \\ D_{2,m}^{n} \end{pmatrix} \xrightarrow{L-s} \int_0^t \overline{\Sigma}_s d\mathbb{B}_s, \tag{S3.46}$$

as $q, n \to \infty$; and, for any $\gamma > 0$,

$$\lim_{q\to\infty} \limsup_{n\to\infty} P(|R(j_1)_{j,t}^n| > \gamma) = 0, \quad j = 1, 2,\ j_1 = 1, \cdots, 5. \tag{S3.47}$$

For (S3.47), we only prove the case of $j = 1$. The case of $j = 2$ can be treated similarly. By (S3.45), it follows from the property of Riemann integral that (S3.47) holds for $j_1 = 5$. It is also straightforward that (S3.47) holds for $j_1 = 3, 4$. For the case of $j_1 = 1$, it suffices to show

$$n^{(p-1)\frac{\theta_1}{2}-(\frac{1}{2}+\frac{p}{4})} \sum_{i=0}^{[nt]-2l_n} |\eta(j_2, l_n)_i^n|^p \xrightarrow{P} 0, \qquad j_2 = 1, 5, \cdots, 8, \tag{S3.48}$$

as $n \to \infty$. The result in (S3.48) with $j_2 = 1$ follows straightforwardly from (S1.4) and $1/2 < \theta_1 < 3/4$. The result in (S3.48) with $j_2 = 5$ is a direct consequence of (S1.4). Similarly to how we obtain the result in (S2.24) with $j = 6$, the conditions $p > 1,\ \beta_1 < 1$, and $\theta_2 < 1/4$ imply $p > \frac{2(1-\theta_1)}{3-2\theta_1-4\beta_1\theta_2}$ and hence the result in (S3.48) with $j_2 = 6$. By similar arguments to that used in proving (S2.24) with $j = 7$, we have

$$\begin{aligned}
&n^{(p-1)\frac{\theta_1}{2}-(\frac{1}{2}+\frac{p}{4})} \sum_{i=0}^{[nt]-2l_n} E_i^n\, |\eta(7, l_n)_i^n|^p \\
&\leq K n^{(p-1)\frac{\theta_1}{2}-(\frac{1}{2}+\frac{p}{4})+1} \times \left(n^{-p\theta_2(2-\beta_1)} + n^{(\frac{1}{2}-\theta_1-2\theta_2)p+(\theta_2\beta_1-\frac{1}{2}+\theta_1)}\right) \\
&\leq K \left(n^{(p-1)\frac{\theta_1}{2}+(\frac{1}{2}-\frac{p}{4})-p\theta_2(2-\beta_1)} + n^{-(p-1)\frac{\theta_1}{2}+\frac{p}{4}-\theta_2(2p-\beta_1)}\right).
\end{aligned}$$

Hence, the condition $\theta_2 > \frac{2\theta_1(p-1)+2-p}{4p(2-\beta_1)} \vee \frac{p-2\theta_1(p-1)}{4(2p-\beta_1)}$ leads to the result in (S3.48) with $j_2 = 7$.

Likewise the similar arguments to that of obtaining (S2.24) with $j = 8$ leads to the result in (S3.48) with $j_2 = 8$. We finally turn to proving (S3.47) with $j_1 = 2$. By Minkowski inequality, it suffices to prove, for any fixed q,

$$n^{(p-1)\frac{\theta_1}{2}-(\frac{1}{2}+\frac{p}{4})} \sum_{i=0}^{[nt]-2l_n} \left|\eta(j_2, l_n)_i^n - \breve{\eta}(j_2, l_n)_{i,a_m(q)}^n\right|^p \xrightarrow{P} 0, \qquad j_2 = 2, 3, \tag{S3.49}$$

as $n \to \infty$. This is a direct consequence of the result in (S1.3) and $p > 1$. Hence, we have proved the result in (S3.47). It remains to deal with (S3.46).

We now prove the result in (S3.46). To proceed, we need to introduce sequences (indexed by n and q) of continuous-time filtrations $(\mathscr{H}_t^{n,q})_{t\geq 0}$ defined by

$$\mathscr{H}_t^{n,q} := \begin{cases} \mathcal{F}_0, \text{ for } t \in [0, 2ql_n + 8l_n\delta_n); \\ \\ \mathcal{F}_{s_m\delta_n}, \text{ for } t \in [a_m(q)\delta_n + 4l_n\delta_n, a_{m+1}(q)\delta_n + 4l_n\delta_n),\ m = 2, \ldots \end{cases} \tag{S3.50}$$

Note that $J_n(q) = J_{n,t}(q) = m - 1$ for $t \in [a_m(q)\delta_n + 4l_n\delta_n, a_{m+1}(q)\delta_n + 4l_n\delta_n),\ m = 2, \ldots$ It is easily verified that, for fixed n and q, $(M_{j,t}^{n,q})_{t\geq 0}, j = 1, 2$, are martingales with respect to the filtration $(\mathscr{H}_t^{n,q})_{t\geq 0}$ defined in (S3.50). Hence, by Theorem 2.2.15 in Jacod and Protter (2012), it suffices to show that

$$n^{1-\theta_1} \sum_{m=1}^{J_n(q)} E_{s_m}^n \left(D_{j_1,m}^n D_{j_2,m}^n\right) \xrightarrow{P} \int_0^t \Sigma_{j_1 j_2,s}^2 ds, \quad j_1, j_2 = 1, 2, \tag{S3.51}$$

$$n^{2(1-\theta_1)} \sum_{m=1}^{J_n(q)} E_{s_m}^n \left(D_{j,m}^n\right)^4 \xrightarrow{P} 0, \quad j = 1, 2, \tag{S3.52}$$

$$n^{\frac{1-\theta_1}{2}} \sum_{m=1}^{J_n(q)} E_{s_m}^n \left(D_{j,m}^n (N_{b_m(q)\delta_n} - N_{a_m(q)\delta_n})\right) \xrightarrow{P} 0, \quad j = 1, 2, \tag{S3.53}$$

as n and q go to infinity, where N is either one of the entries of $(W, \widetilde{W})$ or any other bounded martingale orthogonal to both W and $\widetilde{W}$.

Recall that, conditional on $\mathcal{F}_{s_m\delta_n}$, $\xi(j)^n_{i,m}$ is asymptotically normal with mean zero and variance of the form $2\gamma^2_{s_m\delta_n} + o_P(1)$. Additionally, it is easily verified that the sequences $|\xi(j)^n_{i,m}|^p$ are uniformly integrable under the condition $\vartheta > 2p$. To illustrate, straightforward calculations yield that, for $a_m(q) \leq i_1, i_2 \leq b_m(q)$,

$$E^n_{s_m}\left(\xi(1)^n_{i_1,m}\xi(1)^n_{i_2,m}\right) = \begin{cases} 2\gamma^2_{s_m\delta_n}\dfrac{l_n - |i_1 - i_2|}{l_n} + O_P\left(\dfrac{k_n^2}{l_n^2}\right), \quad 0 \leq |i_1 - i_2| < l_n, \\ -\gamma^2_{s_m\delta_n}\dfrac{2l_n - |i_1 - i_2|}{2l_n} + O_P\left(\dfrac{k_n^2}{l_n^2}\right), \quad l_n \leq |i_1 - i_2| \leq 2l_n, \\ O_P\left(\dfrac{k_n^2}{l_n^2}\right), \quad \text{otherwise}; \end{cases}$$

$$E^n_{s_m}\left(\xi(2)^n_{i_1,m}\xi(1)^n_{i_2,m}\right) =$$

$$O_P\left(\frac{k_n^2}{l_n^2}\right) + \frac{\gamma^2_{s_m\delta_n}}{2^{\frac{1}{2}}}\frac{k_n}{l_n^2} \times \begin{cases} -2l_n + (i_2 - i_1), \quad l_n \leq i_2 - i_1 \leq 2l_n, \\ -(i_2 - i_1), \quad 0 \leq i_2 - i_1 \leq l_n, \\ -2(i_2 - i_1), \quad -l_n \leq i_2 - i_1 \leq 0, \\ 4l_n - 2(i_2 - i_1), \quad -2l_n \leq i_2 - i_1 \leq -l_n, \\ 2l_n + (i_2 - i_1), \quad -3l_n \leq i_2 - i_1 \leq -2l_n, \\ -4l_n - (i_2 - i_1), \quad -4l_n \leq i_2 - i_1 \leq -3l_n, \\ 0, \quad \text{otherwise}. \end{cases}$$

Therefore, the result in (S3.51) is an immediate consequence of the classical result that convergence in distribution, supplemented by uniform integrability, entails convergence in moments.

By Lemma 1 and Hölder's inequality, we have $E^n_{s_m}\left(D^n_{j,m}\right)^4 \leq Kn^{4(\theta_1-1)}$. Hence, the result in (S3.52) follows. Similarly to how (A.12) is proved in Vetter (2015), the result in (S3.53) is an easy consequence of an application of Itô's lemma and properties of normal distributions. We are done. $\square$

Proof of Theorem 2. The reasoning is largely analogous to the proof of Theorem 6 in Aït-Sahalia and Jacod (2009). Nevertheless, for the sake of completeness, we present the proof in full detail below.

We first prove the result $\lim_{n\to\infty} P(T_n < z_\alpha|\Omega_{T,0}) = \alpha$ under the null hypothesis. It follows from Theorem 1 that, when σ is continuous, T_n converges stably to $N(0,1)$. This further implies that for any $\mathcal{F}$-measurable subset B of Ω,

$$P\left(\{T_n < z_\alpha\} \cap B\right) \longrightarrow \alpha P(B). \tag{S3.54}$$

Nonetheless, we are primarily concerned with the case where σ is discontinuous but the observed path is continuous on $[0,T]$. To this end, we introduce the process

$$\nu^{(c)}_t = \nu_0 + \int_0^t \widetilde{b}_s\, ds + \int_0^t \widetilde{\sigma}_s\, d\widetilde{W}_s.$$

By changing ν to $\nu^{(c)}$ in the price-volatility model (2.1), we obtain a price process $X^{(c)}$ with continuous volatility. Put an additional superscript (c) for the variables defined on the basis of $X^{(c)}$, writing, for example, $\breve{T}^{(c)}_n$, $\widehat{\int_0^T \Sigma^2_s ds}^{(c)}$, and $T^{(c)}_n$. Then, by applying (S3.54) to the price process $X^{(c)}$, we have $P\left(\{T_n^{(c)} < z_\alpha\} \cap B\right) \longrightarrow \alpha P(B)$ for any $B \in \mathcal{F}$. However, on the set $\Omega_{T,0}$ where $X_t^{(c)} = X_t$ for all $t \in [0, T]$, we have $T_n = T_n^{(c)}$, and hence $\{T_n^{(c)} < z_\alpha\} \cap \Omega_{T,0} = \{T_n < z_\alpha\} \cap \Omega_{T,0}$. Therefore, $P\left(\{T_n < z_\alpha\} \cap \Omega_{T,0}\right) \longrightarrow \alpha P(\Omega_{T,0})$, completing the proof of the result under the null hypothesis as long as $P(\Omega_{T,0}) > 0$.

We next turn to the result under the alternative hypothesis. It follows from Proposition 3 and the construction of the estimator for the asymptotic variance in (3.16) that, irrespective of whether the volatility process σ is continuous,

$$\widehat{\int_0^T \Sigma_s^2 ds} \xrightarrow{P} \int_0^T \Sigma_s^2 ds$$

under the conditions $\frac{1}{2} < \theta_1 < \frac{3}{4}$, $\frac{2\theta_1 - 1}{4(2-\beta_1)} < \theta_2 < \frac{1}{4}$, $\beta_2 < \frac{1-\theta_1}{2\theta_1 - 1}$ and $\frac{p(2\theta_1-1)+4(\theta_1-1)}{4(p-\beta_2)} < \theta_4 < \frac{2\theta_1-1}{4}$. By the consistency of the asymptotic variance estimator, Proposition 3, and the results in (3.13), we readily obtain that for any $\varepsilon > 0$,

$$P\left(\left|\frac{\left(\breve{T}_n - 2^{-1}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}}\right| > \varepsilon, \Omega_{T,1}\right) \to 0, \tag{S3.55}$$

or equivalently, $\left(\breve{T}_n - 2^{-1}\right)/(\widehat{\int_0^T \Sigma_s^2 ds})^{1/2} = o_P(1)$, conditional on $\Omega_{T,1}$ and provided that $P(\Omega_{T,1}) > 0$. Additionally, we have that, for any $p > 2$,

$$\frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} - \frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\int_0^T \Sigma_s^2 ds}} = o_P(1). \tag{S3.56}$$

Therefore, conditional on $\Omega_{T,1}$ and provided that $P(\Omega_{T,1}) > 0$, we obtain

$$n^{-\frac{1-\theta_1}{2}} T_n = \frac{\left(\breve{T}_n - 2^{-1}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + \frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} = O_P(1),$$

completing the proof of $T_n = O_P(n^{\frac{1-\theta_1}{2}})$.

For any fixed $p > 2$ and $\varepsilon > 0$, we introduce the following positive constant

$$K = -\frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\int_0^T \Sigma_s^2 ds}},$$

and events

$$\mathcal{E}_{1,n,\varepsilon} := \left\{ \left| \frac{\left(\breve{T}_n - 2^{-1}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} \right| > \varepsilon \right\} \text{ and } \mathcal{E}_{2,n,\varepsilon} := \left\{ \left| \frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + K \right| > \varepsilon \right\}.$$

It follows directly from (S3.55) and (S3.56) that

$$P(\mathcal{E}_{1,n,\varepsilon}, \Omega_{T,1}) \to 0 \text{ and } P(\mathcal{E}_{2,n,\varepsilon}) \to 0 \tag{S3.57}$$

for any $\varepsilon > 0$. Let $\mathcal{E}_{1,n,\varepsilon}^c$ and $\mathcal{E}_{2,n,\varepsilon}^c$ denote the complements of $\mathcal{E}_{1,n,\varepsilon}$ and $\mathcal{E}_{2,n,\varepsilon}$. Now for any $M > 0$, we have that, under the conditions $p > 2$ and $\frac{1}{2} < \theta_1 < \frac{3}{4}$,

$$\begin{aligned}
&P\left(T_n \geq -M, \Omega_{T,1}\right) \\
&= P\left(n^{\frac{1-\theta_1}{2}} \left[\frac{\left(\breve{T}_n - 2^{-1}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + \left(\frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + K \right) - K \right] \geq -M, \mathcal{E}_{1,n,K/3}^c, \mathcal{E}_{2,n,K/3}^c, \Omega_{T,1}\right) \\
&\quad + P\left(n^{\frac{1-\theta_1}{2}} \left[\frac{\left(\breve{T}_n - 2^{-1}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + \left(\frac{\left(2^{-1} - 2^{\frac{p}{2}}\right)}{\sqrt{\widehat{\int_0^T \Sigma_s^2 ds}}} + K \right) - K \right] \geq -M, \mathcal{E}_{1,n,K/3} \cup \mathcal{E}_{2,n,K/3}, \Omega_{T,1}\right) \\
&\leq P\left(-n^{\frac{1-\theta_1}{2}} K/3 \geq -M\right) + P(\mathcal{E}_{1,n,K/3}, \Omega_{T,1}) + P(\mathcal{E}_{2,n,K/3}) \\
&\to 0,
\end{aligned}$$

where the last convergence follows from (S3.57) and the fact that $-n^{\frac{1-\theta_1}{2}} K/3 < -M$

for sufficiently large n. This completes the proof of $T_n \overset{P}{\to} -\infty$ conditional on $\Omega_{T,1}$ and $\lim_{n\to\infty} P(T_n < z_\alpha|\Omega_{T,1}) = 1$ as long as $P(\Omega_{T,1}) > 0$.

$\square$

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