MINIMUM ABERRATION FRACTIONAL FACTORIAL DESIGNS UNDER BASELINE PARAMETRIZATION

Xinxin Xia[1], Fasheng Sun[1] and Chunyan Wang[2]

[1]*Northeast Normal University and* [2]*Renmin University of China*

Supplementary Material

In this supplementary material, we provide the proofs of theoretical results, and list 5-level approximate BP-MA designs and their K_2 and K_3 values for runs of 25, 50, 75, 100, and 125. 5-level MA designs are not available in the literature. We use $\mathrm{OA}(25, 6, 5, 2)$ and $\mathrm{OA}(125, 31, 5, 2)$ obtained by the Rao-Hamming construction. The $\mathrm{OA}(50, 10, 5, 2)$, $\mathrm{OA}(75, 7, 5, 2)$, and $\mathrm{OA}(100, 20, 5, 2)$ are derived from Theorem 9.15 in Hedayat, Sloane and Stufken (1999), and we then select columns from these designs to obtain MA designs under OP. Next, we obtain approximate BP-MA designs for each MA design under OP by applying level permutations. For simplicity, we merely calculate the K_2 and K_3 values to identify BP-MA designs.

Corresponding author: Chunyan Wang, Center for Applied Statistics and School of Statistics, Renmin University of China, Beijing 100872, China. E-mail: chunyanwang@ruc.edu.cn.

S1. Proofs of the theoretical results

Proof of Lemma 1. P_s is an $s \times s$ column-orthogonal matrix with the first column being $\mathbf{1}_s$, and $B_s = \left(\mathbf{1}_s, (\mathbf{0}_{s-1}, I_{s-1})^T\right)$. Then, the matrices P_s, B_s^{-1}, and $B_s^{-1}P_s$ can be written as follows:

$$P_s = \begin{bmatrix} 1 & a_1^T \\ \mathbf{1}_{s-1} & A_1 \end{bmatrix}, \quad B_s^{-1} = \begin{bmatrix} 1 & \mathbf{0}_{s-1}^T \\ -\mathbf{1}_{s-1} & I_{s-1} \end{bmatrix}, \quad B_s^{-1}P_s = \begin{bmatrix} 1 & a_2^T \\ \mathbf{0}_{s-1} & A_2 \end{bmatrix},$$

where both a_1 and a_2 are $(s-1) \times 1$ column vectors, and both A_1 and A_2 are $(s-1) \times (s-1)$ matrices. Let $B_s^{-1}P_s = (u_1^T, \ldots, u_s^T)^T$, where each u_i is a $1 \times s$ row vector for $i = 1, \ldots, s$. The first element of u_1 is 1, while the first elements of $u_2, \ldots, u_s$ are 0. According to $\widetilde{\theta} = (B_s^{-1}P_s \otimes \cdots \otimes B_s^{-1}P_s)\widetilde{\beta}$, the elements of $\widetilde{\theta}$ can be expressed as a linear combination of the elements in $\widetilde{\beta}$. Thus,

$$\theta_{i_1 \cdots i_n} = (u_{j_1} \otimes \cdots \otimes u_{j_n})\widetilde{\beta} = (u_{j_1} \otimes \cdots \otimes u_{j_n})(\beta_{0 \cdots 0}, \ldots, \beta_{s-1 \cdots s-1})^T,$$

where $\{i_1, \ldots, i_n\} \subset \{0, \ldots, s-1\}$, and $\{j_1, \ldots, j_n\} \subset \{1, \ldots, s\}$. $\widetilde{\beta}$ is an $s^n \times 1$ vector with components $\beta_{0\cdots 0}, \ldots, \beta_{s-1\cdots s-1}$ in Yates order. When more than one element from $\{j_1, \ldots, j_n\}$ is in $\{2, \ldots, s-1, s\}$, we have $\lceil i_1 \ldots i_n \rceil > 1$, where $\lceil i_1 \ldots i_n \rceil$ is the number of non-zero elements in $\{i_1, \ldots, i_n\}$. It can be verified that any $\theta_{i_1 \ldots i_n}$ in $\widetilde{\theta}$, where $\lceil i_1, \ldots, i_n \rceil > 1$, can be expressed as a linear combination of elements from $\widetilde{\beta}$ with the coefficients of the main effects in $\widetilde{\beta}$ equal to 0. Thus, if only the main effects are active under OP (i.e. $\beta_{i_1 \ldots i_n} = 0$ for any $\lceil i_1 \ldots i_n \rceil > 1$), then $\theta_{i_1 \ldots i_n} = 0$ for any $\lceil i_1 \ldots i_n \rceil > 1$. Therefore, if only the main effects are active under OP, then only the main effects are active under BP.

By a similar argument, we can obtain that if only the main effects are active under BP, then only the main effects are active under OP. $\square$

To prove Theorem 1, we first introduce the following lemmas.

Lemma S1. *Let A be an $n \times n$ positive definite matrix, B be an $m \times m$ positive definite matrix, and u be an $n \times m$ matrix. Then $A \geq A - u^T Bu$, where $C_1 \geq C_2$ if $C_1 - C_2$ is a non-negative definite matrix.*

Proof. Note that $A-(A-u^T Bu) = u^T Bu$, and for all $z \in \mathbb{R}^m$, $z^T(u^T Bu)z = (uz)^T B(uz) \geq 0$. By the definition of the non-negative definite matrix, we know that $u^T Bu \geq 0$. Thus $A \geq A - u^T Bu$. $\square$

Lemma S2. *Let A and B be $n \times n$ symmetric matrices such that $A \geq B$, then* $\operatorname{tr}(A^{-1}) \leq \operatorname{tr}(B^{-1})$.

Proof. Let $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \cdots \geq \lambda_n(B)$ be the eigenvalues of A and B, respectively. Since $A \geq B$, we have $\lambda_j(A) \geq \lambda_j(B)$, and thus $\lambda_j(A^{-1}) \leq \lambda_j(B^{-1})$ for $j = 1, \ldots, n$. Therefore $\operatorname{tr}(A^{-1}) \leq \operatorname{tr}(B^{-1})$. $\square$

Proof of Theorem 1. First, we prove that orthogonal arrays are D_s-optimal among all designs. From Lemma 1, we have $W = XP^{-1}$, where $X = (\mathbf{1}_N, X_1)$, and $P = ((1, \mathbf{0}^T_{(s-1)n})^T, (p, P_1^T)^T)$. Here p is an $(s-1)n \times 1$ vector, $\mathbf{0}_{(s-1)n}$ is an $(s-1)n \times 1$ all zero vector, and $P_1 = diag(L, \ldots, L)$ with $L = A_1 - \mathbf{1}_{s-1}a_1^T$, where A_1 and a_1 are as defined in Lemma 1. Then

$$
\begin{aligned}
(W^TW)^{-1} &= [(P^{-1})^T X^T X P^{-1}]^{-1} \\
&= P(X^TX)^{-1}P^T \\
&= \begin{bmatrix} 1 & p^T \\ \mathbf{0}_{(s-1)n} & P_1 \end{bmatrix} \begin{bmatrix} \frac{1}{N} & \mathbf{1}^T_{(s-1)n}X_1 \\ X_1^T\mathbf{1}_{(s-1)n} & (X_1^TX_1)^{-1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T_{(s-1)n} \\ p & P_1^T \end{bmatrix} \\
&= \begin{bmatrix} * & * \\ * & P_1(X_1^TX_1)^{-1}P_1^T \end{bmatrix},
\end{aligned}
$$

where $*$ denotes terms that do not impact the proof. Thus, we have $(W^TW)^{-1}_{(-1,-1)} = P_1(X_1^TX_1)^{-1}P_1^T$, and

$$
|(W^TW)_{(-1,-1)}| = \frac{1}{|(W^TW)^{-1}_{(-1,-1)}|} = \frac{|X_1^TX_1|}{|P_1^TP_1|} = \frac{|X_1^TX_1|}{|L^TL|^n},
$$

where

$$
\begin{aligned}
L^TL &= (A_1 - \mathbf{1}_{s-1}a_1^T)^T(A_1 - \mathbf{1}_{s-1}a_1^T) \\
&= A_1^TA_1 - A_1^T\mathbf{1}_{s-1}a_1^T - a_1\mathbf{1}^T_{s-1}A_1 + a_1\mathbf{1}^T_{s-1}\mathbf{1}_{s-1}a_1^T \\
&= s(I + a_1a_1^T).
\end{aligned}
$$

It is worth noting that the determinant of L^TL is a constant for a given P_s. Thus, $|(W^TW)_{(-1,-1)}| = c|X_1^TX_1|$, where c is a constant. That is, a design that maximizes $|X_1^TX_1|$ will certainly maximize $|(W^TW)_{(-1,-1)}|$. From Cheng, C. S. (1980), we obtain that orthogonal arrays are D_s-optimal.

In what follows, we demonstrate that orthogonal arrays also achieve G-optimality among all designs. Let $\mathcal{Z}_A$ be the set of all possible level combinations of n s-level factors, and let $\mathcal{W}_A$ be the model matrix corresponding to $\mathcal{Z}_A$ under main-effect model under BP. For any row w of matrix $\mathcal{W}_A$, let Y_w be the response for this design point. Then

$$\begin{aligned}
E[(Y_w - w\hat{\theta})^2|Z] &= E[(Y_w - w\theta + w\theta - w\hat{\theta})^2|Z] \\
&= E[(Y_w - w\theta)^2] + E[(w\theta - w\hat{\theta})^2|Z] \\
&= \sigma^2 + w\text{var}(\hat{\theta})w^T \\
&= \sigma^2(1 + w(W^TW)^{-1}w^T).
\end{aligned}$$

According to Lemma 1, we have $\mathcal{W}_A = \mathcal{X}_AP^{-1}$ and $W = XP^{-1}$, where $\mathcal{X}_A$ and X are the orthogonal contrast matrices corresponding to the full factorial design and the fractional factorial design Z under the main effect model, respectively.

$$\begin{aligned}
\sum_{w\in\mathcal{W}_A} w(W^TW)^{-1}w^T &= \text{tr}[\mathcal{W}_A(W^TW)^{-1}\mathcal{W}_A^T] \\
&= \text{tr}[(P^{-1})^T\mathcal{X}_A^T\mathcal{X}_AP^{-1}P(X^TX)^{-1}P^T]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{tr}[\mathcal{X}_A^T\mathcal{X}(X^TX)^{-1}] \\
&= s^n \mathrm{tr}[(X^TX)^{-1}] \\
&\geq \frac{s^n(1+n(s-1))^2}{\mathrm{tr}(X^TX)} \\
&= \frac{s^n(1+n(s-1))}{N}.
\end{aligned}
$$

Then $\max_{w\in\mathcal{W}_A} E[(Y_w - w\hat{\theta})^2|Z] = \sigma^2(1+\max_{w\in\mathcal{W}_A} w(W^TW)^{-1}w^T) \geq \sigma^2(1+\{1+n(s-1)\}/N)$.

When Z is an orthogonal array of strength 2, the equality holds. Since $W = XP^{-1}$, wP is a row vector of X. The sum of squares of each row of X is $1+n(s-1)$, and we have

$$
w(W^TW)^{-1}w^T = w((P^{-1})^TX^TXP^{-1})^{-1}w^T = \frac{||wP||^2}{N} = \frac{1+n(s-1)}{N}.
$$

Therefore, when Z is an orthogonal array of strength 2, we have $E[(Y_w - w\hat{\theta})^2|Z] = \sigma^2(1+\{1+n(s-1)\}/N)$. This indicates that orthogonal arrays attain G-optimality among all designs.

Furthermore, we now establish that orthogonal arrays are also A_s-optimal within the class of balanced designs. According to main-effect model under BP, $\mathrm{var}(\hat{\theta}) = (W^TW)^{-1}\sigma^2$, where $W = (\mathbf{1}_N, Z_1)$, then

$$
W^TW = \begin{bmatrix} N & \mathbf{1}_N^TZ_1 \\ Z_1^T\mathbf{1}_N & Z_1^TZ_1 \end{bmatrix} \text{ and } (W^TW)^{-1} = \begin{bmatrix} * & * \\ * & M^{-1} \end{bmatrix},
$$

where $M = Z_1^T Z_1 - 1/N(Z_1^T \mathbf{1}_N)(\mathbf{1}_N^T Z_1) = \{\mathrm{var}(\hat{\theta}_1)\}^{-1}$ is a positive definite matrix. When Z is a balanced design, $\mathbf{1}_N^T Z_1 = N/s\mathbf{1}_{(s-1)n}^T$ and $Z_1^T Z_1 - 1/N(Z_1^T \mathbf{1}_N)(\mathbf{1}_N^T Z_1) = Z_1^T Z_1 - N/s^2 J_{(s-1)n}$. Here $J_{(s-1)n}$ is a matrix of all ones, and

$$Z_1^T Z_1 = \frac{N}{s^2}\left[\begin{array}{cccc|c} R & & & & \\ & R & & * & \\ & & \ddots & & u_1 \\ & * & & & \\ & & & R & \\ \hline & & u_1^T & & R \end{array}\right], \text{ with } R = sI_{s-1}$$

Then,

$$M = Z_1^T Z_1 - \frac{N}{s^2} J_{(s-1)n} = \frac{N}{s^2}\left[\begin{array}{cccc|c} E & & & & \\ & E & & * & \\ & & \ddots & & u_2 \\ & * & & & \\ & & & E & \\ \hline & & u_2^T & & E \end{array}\right] = \frac{N}{s^2}\begin{bmatrix} T & u_2 \\ u_2^T & E \end{bmatrix},$$

where $E = sI_{s-1} - J_{s-1}$. Since M is a positive definite matrix, then T and T^{-1} are also positive definite matrices.

Let θ^i denote the parameter vector corresponding to the main effect of the ith factor and define $M_i = \{\mathrm{var}(\hat{\theta}^i)\}^{-1}$ for $i = 1, \ldots, n$. Then, we have

$$\mathrm{tr}(\mathrm{var}(\hat{\theta}_1)) = \mathrm{tr}(M_1^{-1}) + \cdots + \mathrm{tr}(M_n^{-1}). \tag{S1.1}$$

Let M_i^{OA} be $\{\text{var}(\hat{\theta}^i)\}^{-1}$ when Z is an orthogonal array, and let M_i^{Ba} be $\{\text{var}(\hat{\theta}^i)\}^{-1}$ when Z is a balanced design. Then, according to Lemma S2, we only need to show that $M_i^{OA} \geq M_i^{Ba}$ for $i = 1, \dots, n$. When $i = n$, we have $M_n^{Ba} = N/s^2(E - u_2^T T^{-1} u_2)$. If Z is an OA$(N, n, s, 2)$, we have

$$Z_1^T Z_1 - \frac{N}{s^2} J_{(s-1)n} = \frac{N}{s^2} \text{diag}(E, \dots, E).$$

Then, from Lemma S1, we have $M_n^{OA} = (N/s^2)E \geq M_n^{Ba}$. For $i = 1, \dots, n-1$, by swapping the ith column and the nth column of design Z, we obtain $M_i^{OA} = (N/s^2)E \geq M_i^{Ba}$. Therefore, based on Lemma S2 and (S1.1), the trace of $\text{var}(\hat{\theta}_1)$ is minimized when Z is an orthogonal array within the class of balanced designs, implying that orthogonal arrays are A_s-optimal among all balanced designs. □

Proof of lemma 2. To enhance the clarity of the proof, let $\varphi_b = g_{k_1} \dots g_{k_b}$. Then, according to the definition of $\xi(\varphi_b)$, we have

$$\begin{aligned}
&\xi(\varphi_b)^T \xi(\varphi_b) \\
=&\frac{s^2}{N^2} \left[(A_c \phi(\varphi_b) - \mathbf{1}_{(s-1)n} \alpha(\varphi_b))^T (A_c \phi(\varphi_b) - \mathbf{1}_{(s-1)n} \alpha(\varphi_b)) \right] \\
=&\frac{s^2}{N^2} \left[\phi(\varphi_b)^T A_c^T A_c \phi(\varphi_b) - \phi(\varphi_b)^T A_c^T \mathbf{1}_{(s-1)n} \alpha(\varphi_b) \right. \\
&\quad \left. - \alpha(\varphi_b) \mathbf{1}_{(s-1)n}^T A_c \phi(\varphi_b) + \alpha(\varphi_b)^2 \mathbf{1}_{(s-1)n}^T \mathbf{1}_{(s-1)n} \right],
\end{aligned}$$

where A_c is an $(s-1)n \times (s-1)n$ block diagonal matrix with diagonal block

$H = I_{s-1} + J_{s-1}$. Utilizing the definition of $\phi(\varphi_b)$, we have

$$
\begin{aligned}
&\xi(\varphi_b)^T\xi(\varphi_b)\\
=&\frac{s^2}{N^2}\Bigg[b\beta(\varphi_b)^T H^T H\beta(\varphi_b) + \sum_{j\in V}\beta(j\varphi_b)^T H^T H\beta(j\varphi_b) - b\beta(\varphi_b)^T H^T \mathbf{1}_{s-1}\alpha(\varphi_b)\\
&-\sum_{j\in V}\beta(j\varphi_b)^T H^T\mathbf{1}_{s-1}\alpha(\varphi_b) - b\alpha(\varphi_b)\mathbf{1}_{s-1}^T H\beta(\varphi_b) - \sum_{j\in V}\alpha(\varphi_b)\mathbf{1}_{s-1}^T H\beta(j\varphi_b)\\
&\quad+b\alpha(\varphi_b)\mathbf{1}_{s-1}^T\mathbf{1}_{s-1}\alpha(\varphi_b) + (s-1)\sum_{j\in V}\alpha(\varphi_b)^2\Bigg]\\
=&\frac{s^2}{N^2}\left(b\left\|\beta(\varphi_b)^T H - \alpha(\varphi_b)\mathbf{1}_{s-1}^T\right\|_F^2 + \sum_{j\in V}\left\|\beta(j\varphi_b)^T H - \alpha(g_{k_1}\dots g_{k_b})\mathbf{1}_{s-1}^T\right\|_F^2\right),
\end{aligned}
$$

where $\beta(\varphi_b)$ is an $(s-1)\times 1$ vector consisting of a single element valued at $\alpha(\varphi_b)$ and the remaining elements set to be 0, and $V = \{1,\dots,n\}\backslash\{k_1,\dots,k_b\}$. $\beta(j\varphi_b) = \big(\alpha(\langle j_1\varphi_b\rangle),\dots,\alpha(\langle j_{s-1}\varphi_b\rangle)\big)^T$ and $j_l = (s-1)(j-1)+l$ for $l = 1,\dots,s-1$. □

Proof of Theorem 4. We consider (1) first. If Z is an s-level orthogonal array of strength t, then all level combinations of any $v+1$ columns of Z occur N/s^{v+1} times for $2\le v\le t-1$. Then $\alpha(g_{k_1}\dots g_{k_v}) = N/s^v$, while $\beta(g_{k_1}\dots g_{k_b})$ is an $(s-1)\times 1$ vector with one element equal to N/s^v and the remaining elements being 0. Moreover, $\beta(jg_{k_1}\dots g_{k_v}) = N/s^{v+1}\mathbf{1}_{s-1}$, and $\beta(jg_{k_1}\dots g_{k_v})^T H = N/s^v\mathbf{1}_{s-1}^T$. Thus,

$$
T_2^{\varphi_v} = \sum_{j\in V}\left\|\beta(jg_{k_1}\dots g_{k_v})^T H - \alpha(g_{k_1}\dots g_{k_v})\mathbf{1}_{s-1}^T\right\|_F^2 = 0,
$$

$$T_1^{\varphi_v} = \left\|\beta(g_{k_1}\dots g_{k_v})^T H - \alpha(g_{k_1}\dots g_{k_v})\mathbf{1}_{s-1}^T\right\|_F^2 = \frac{N^2}{s^{2v}}.$$

Hence,

$$K_v = \frac{s^2}{N^2}\sum_{\varphi_v\in\Phi_v}(vT_1^{\varphi_v} + T_2^{\varphi_v}) = \frac{v(s-1)^v}{s^{2v-2}}\binom{n}{v}.$$

We next consider (2). When $v = t$, $T_1^{\varphi_t} = N^2/s^{2t}$, then

$$K_t = \frac{t(s-1)^t}{s^{2t-2}}\binom{n}{v} + \frac{s^2}{N^2}\sum_{\varphi_t\in\Phi_t}T_2^{\varphi_t} = \frac{t(s-1)^t}{s^{2t-2}}\binom{n}{t} + s^2 J_t,$$

where $J_t = \frac{1}{N^2}\sum_{\varphi_t\in\Phi_t}T_2^{\varphi_t}$. □

To prove Theorem 5, we need the following lemma.

Lemma S3. *If $x_1,\dots,x_b,c_1,\dots,c_b \in Z_s\backslash\{0\}$, then the number of solutions for the equation $c_1x_1+\cdots+c_bx_b = 0$ $(mod\ s)$ is $\gamma_{s1}^b = (-1)^b(s-1)/s + (s-1)^b/s$, where s is a prime number or a prime power.*

Proof. When s is a prime number, let $f(b)$ represent the number of solutions for the equation $c_1x_1+\cdots+c_bx_b = 0$ $(mod\ s)$. Clearly, $f(1) = 0$, $f(2) = s-1$. When $b > 2$, let $u_{b-2} = c_1x_1+\cdots+c_{b-2}x_{b-2}$ $(mod\ s)$. If $u_{b-2} = 0$, $c_1x_1+\cdots+c_bx_b = 0$ $(mod\ s)$ is reduced to $c_{b-1}x_{b-1}+c_bx_b = 0$ $(mod\ s)$, then (x_{b-1},x_b) has $f(2)$ values. If $u_{b-2}\in Z_s\backslash\{0\}$, $c_1x_1+\cdots+c_bx_b = 0$ $(mod\ s)$ is reduced to $u_{b-2}+c_{b-1}x_{b-1}+c_bx_b = 0$ $(mod\ s)$, then x_{b-1} has $s-2$ possible values. For each value of x_{b-1}, there exists a unique value of x_b that satisfies the equation $u_{b-2}+c_{b-1}x_{b-1}+c_bx_b = 0$ $(mod\ s)$. Note that the tuple $(x_1, \ldots, x_{b-2})$ has $(s-1)^{b-2}$ possible values. Thus, $f(b) = f(b-2)f(2) + \big((s-1)^{b-2} - f(b-2)\big)(s-2) = (s-1)^{b-1} - (s-1)^{b-2} + f(b-2)$. After simplifying the expression, we obtain $f(b) = (-1)^b(s-1)/s + (s-1)^b/s$, i.e. $\gamma_{s1}^b = (-1)^b(s-1)/s + (s-1)^b/s$. When s is a prime power, the proof proceeds in a similar manner and is therefore omitted here for brevity. $\square$

Proof of Theorem 5. For an s^{n-p} design of resolution $t+1$, $\alpha(g_{k_1} \ldots g_{k_t}) = N/s^t$. Then $J_t = 1/N^2 \sum_{\varphi_t \in \Phi_t} T_2^{\varphi_t}$, where $T_2^{\varphi_t} = \sum_{j \in V} \|\beta(jg_{k_1} \ldots g_{k_t})^T H - N/s^t \mathbf{1}_{s-1}^T\|_F^2$, and $\beta(jg_{k_1} \ldots g_{k_t}) = (\alpha(\langle j_1 g_{k_1} \ldots g_{k_t} \rangle), \ldots, \alpha(\langle j_{s-1} g_{k_1} \ldots g_{k_t} \rangle))^T$. Let $G_{k_1} \ldots G_{k_t}$ be the factor combination involved in the effect corresponding to $g_{k_1} \ldots g_{k_t}$. Moreover, $G_{k_1} \ldots G_{k_t}$ corresponds to $(s-1)^t$ factorial effects. For example, when $s = 3$, we have four factors A, B, C and D. Let $g_{k_1} g_{k_2} = 1020$, the effect corresponding to $g_{k_1} g_{k_2}$ is $A_1 C_2$. Then $G_{k_1} G_{k_2} = AC$, and A_1C_1, A_1C_2, A_2C_1 and A_2C_2 are factorial effects corresponding to $G_{k_1} G_{k_2}$.

When any two words from A_{t+1} do not share t common factors, there are the following two possible cases for $\beta(jg_{k_1} \ldots g_{k_t})$.

Case I: Not all $G_{k_1} \ldots G_{k_t}$ appear in the words of length $t+1$. In this case, $\beta(jg_{k_1} \ldots g_{k_t}) = N/s^{t+1} \mathbf{1}_{s-1}$ for $j \in V$, and $T_2^{\varphi_t} = 0$.

Case II: All $G_{k_1} \ldots G_{k_t}$ appear in the words of length $t+1$. The number of $g_{k_1} \ldots g_{k_t}$ that satisfy this case is $(s-1)^t(t+1)A_{t+1}$, among these, the number of $g_{k_1} \dots g_{k_t}$ corresponding to $T_2^{\varphi_t} = N^2/s^{2t}$ is $\gamma_{s2}^t(t+1)A_{t+1}$, where $\gamma_{s2}^t = (s-1)^t - \gamma_{s1}^t$. Note that for each of these $g_{k_1} \dots g_{k_t}$, there exists a unique $j_0 \in V$ such that $\beta(j_0 g_{k_1} \dots g_{k_t}) = (\alpha(\langle j_1 g_{k_1} \dots g_{k_t}\rangle), \dots, \alpha(\langle j_{s-1} g_{k_1} \dots g_{k_t}\rangle))^T$, where there is only one element equal to N/s^t and all other elements are 0. For any $j \in V \backslash \{j_0\}$, $\beta(j g_{k_1} \dots g_{k_{t+1}}) = N/s^{t+1}\mathbf{1}_{s-1}^T$, and thus $T_2^{\varphi_t} = N^2/s^{2t}$. In addition, among these $g_{k_1} \dots g_{k_t}$, the number of $g_{k_1} \dots g_{k_t}$ corresponding to $T_2^{\varphi_t} = (s-1)N^2/s^{2t}$ is $\gamma_{s1}^t(t+1)A_{t+1}$. Note that for each of these $g_{k_1} \dots g_{k_t}$, there exists a unique $j_0 \in V$ such that $\beta(j_0 g_{k_1} \dots g_{k_t}) = \mathbf{0}_{s-1}^T$, while for any $j \in V \backslash \{j_0\}$, $\beta(j g_{k_1} \dots g_{k_{t+1}}) = N/s^{t+1}\mathbf{1}_{s-1}^T$. Then $T_2^{\varphi_t} = (s-1)N^2/s^{2t}$.

Therefore, $J_t = (t+1)/s^{2t}[(s-1)\gamma_{s1}^t + \gamma_{s2}^t]A_{t+1}$ holds when any two of A_{t+1}'s words of length $t+1$ do not share t common factors. By a similar argument, we obtain that $J_t = (t+1)/s^{2t}[(s-1)\gamma_{s1}^t + \gamma_{s2}^t]A_{t+1}$ also holds if there exists two words of length $t+1$ that share t common factors. $\square$

Proof of Theorem 6. For clarity the following proof, we now provide the definition of an effect that satisfies a word. An effect is said to satisfy the word if it meets the following two conditions: (1) the factors involved in this effect are encompassed within the word, and (2) the subscript of this effect satisfies the equation corresponding to the word. For example, for a 3^{3-1} design, there are three factors A, B, and C with $C = A + B$. Then, the effects $A_1B_1C_2$ and $A_2B_2C_1$ satisfy the word. If either of these two conditions is not satisfied, the effect is said to not satisfy the word.

From Lemma 2, we have

$$K_{t+1} = \frac{s^2(t+1)}{N^2} \sum_{\Phi_{t+1}} T_1^{\varphi_{t+1}} + \frac{s^2}{N^2} \sum_{\Phi_{t+1}} T_2^{\varphi_{t+1}}.$$

(1) First, calculate $\sum_{\Phi_{t+1}} T_1^{\varphi_{t+1}}$ for an s^{n-p} design of resolution $t+1$, with $T_1^{\varphi_{t+1}} = \|\beta(g_{k_1} \dots g_{k_{t+1}})^T H - \alpha(g_{k_1} \dots g_{k_{t+1}})\mathbf{1}_{s-1}^T\|_F^2$, where $\beta(g_{k_1} \dots g_{k_{t+1}})$ is an $(s-1) \times 1$ vector with one element being $\alpha(g_{k_1} \dots g_{k_{t+1}})$ and the remaining elements being 0. There are the following three possible cases.

Case I: Not all $G_{k_1} \dots G_{k_{t+1}}$ appear in the degenerate words of length $t+1$. Then $\alpha(g_{k_1} \dots g_{k_{t+1}}) = N/s^{t+1}$ and $T_1^{\varphi_{t+1}} = N^2/s^{2t+2}$.

Case II: All $G_{k_1} \dots G_{k_{t+1}}$ appear in the degenerate words of length $t+1$, and the effect corresponding to $g_{k_1} \dots g_{k_{t+1}}$ satisfies the words of length $t+1$. Then $\alpha(g_{k_1} \dots g_{k_{t+1}}) = N/s^t$ and $T_1^{\varphi_{t+1}} = N^2/s^{2t}$.

Case III: All $G_{k_1} \dots G_{k_{t+1}}$ appear in the degenerate words of length $t+1$, and the effect corresponding to $g_{k_1} \dots g_{k_{t+1}}$ does not satisfy the words of length $t+1$. Then $\alpha(g_{k_1} \dots g_{k_{t+1}}) = 0$ and $T_1^{\varphi_{t+1}} = 0$.

It is clear that the number of $g_{k_1} \dots g_{k_{t+1}}$ satisfying Cases I, II, and III are ξ, $\gamma_{s2}^t A_{t+1}$, and $((s-1)^{t+1} - \gamma_{s2}^t)A_{t+1}$, respectively, where $\xi = (s -$

$1)^{t+1}\big[n!/\{(t+1)!(n-t-1)!\} - A_{t+1}\big]$. Therefore, we have

$$\frac{s^2(t+1)}{N^2}\sum_{\Phi_{t+1}} T_1^{\varphi_{t+1}} = \frac{s^2(t+1)}{N^2}\left[\gamma_{s2}^t A_{t+1}\frac{N^2}{s^{2t}} + \xi\frac{N^2}{s^{2t+2}}\right]$$
$$= \frac{t+1}{s^{2t}}\left[(s^2\gamma_{s2}^t - (s-1)^{t+1})A_{t+1} + (s-1)^{t+1}\binom{n}{t+1}\right].$$

(2) Second, calculate $\sum_{\Phi_{t+1}} T_2^{\varphi_{t+1}}$ for an s^{n-p} design of resolution $t+1$ with $T_2^{\varphi_{t+1}} = \sum_{j\in V}\|\beta(jg_{k_1}\dots g_{k_{t+1}})^T H - \alpha(g_{k_1}\dots g_{k_{t+1}})\mathbf{1}_{s-1}^T\|_F^2$, where $\beta(jg_{k_1}\dots g_{k_{t+1}}) = \big(\alpha(\langle j_1 g_{k_1}\dots g_{k_{t+1}}\rangle),\dots,\alpha(\langle j_{s-1}g_{k_1}\dots g_{k_{t+1}}\rangle)\big)^T$, $j_l = (s-1)(j-1)+l$ for $l = 1,\dots,s-1$. There are following two possible cases.

Case I: All $G_{k_1}\dots G_{k_{t+1}}$ appear in the degenerate words of length $t+1$.

Case II: Not all $G_{k_1}\dots G_{k_{t+1}}$ appear in the degenerate words of length $t+1$.

Case I can be further subdivided into the following four scenarios:

(i) The effect corresponding to $g_{k_1}\dots g_{k_{t+1}}$ satisfies both the word of length $t+1$ and $t+2$. Thus, $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^t$. For some $j_0 \in V$, $\beta(j_0 g_{k_1}\dots g_{k_{t+1}})$ is an s-dimensional vector consisting of a N/s^t and $s-1$ zeros; while for any $j \in V\backslash\{j_0\}$, $\beta(jg_{k_1}\dots g_{k_{t+1}}) = N/s^{t+1}\mathbf{1}_{s-1}^T$. Hence, $T_2^{\varphi_{t+1}} = N^2/s^{2t}$.

(ii) The effect corresponding to $g_{k_1}\dots g_{k_{t+1}}$ satisfies the word of length $t+1$, but does not satisfy the word of length $t+2$, and $G_{k_1}\dots G_{k_{t+1}}$ have $t+1$ factors included in the degenerate words of length $t+2$. Thus, $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^t$. For some $j_0 \in V$, $\beta(j_0 g_{k_1}\dots g_{k_{t+1}}) = \mathbf{0}_{s-1}^T$, while for any $j \in V \backslash \{j_0\}$, $\beta(jg_{k_1} \dots g_{k_{t+1}}) = N/s^{t+1}\mathbf{1}_{s-1}^T$. Therefore, $T_2^{\varphi_{t+1}} = (s-1)N^2/s^{2t}$.

(iii) The effect corresponding to $g_{k_1} \dots g_{k_{t+1}}$ satisfies the word of length $t+1$, and $G_{k_1} \dots G_{k_{t+1}}$ have $r \leq t$ factors included in the degenerate words of length $t+2$. Thus, $\alpha(g_{k_1} \dots g_{k_{t+1}}) = N/s^t$. At this point, for any $j \in V$, $\beta(jg_{k_1} \dots g_{k_{t+1}}) = N/s^{t+1}\mathbf{1}_{s-1}^T$. Therefore, $T_2^{\varphi_{t+1}} = 0$.

(iv) The effect corresponding to $g_{k_1} \dots g_{k_{t+1}}$ does not satisfy the word of length $t+1$. Then $\alpha(g_{k_1} \dots g_{k_{t+1}}) = 0$ and $\beta(jg_{k_1} \dots g_{k_{t+1}}) = \mathbf{0}_{s-1}^T$ for $j \in V$. Therefore, $T_2^{\varphi_{t+1}} = 0$.

Let A_{t+2}^* be the number of degenerate words of length $t+2$, and A_{t+2}^1 be the number of degenerate words of length $t+2$ that share $t+1$ common factors with some degenerate word of length $t+1$. Define $A_{t+2}^2 = A_{t+2}^* - A_{t+2}^1$. For each of the A_{t+2}^1 degenerate words of length $t+2$, project them into $t+1$ factors. Then we obtain $l = (t+2)A_{t+2}^1$ degenerate words of length $t+1$ (with repetition). Let B_{t+2}^{t+1} represent the set of degenerate words of length $t+2$, and these degenerate words share $t+1$ factors with degenerate word of length $t+1$. For each the effect corresponding to $G_{k_1} \dots G_{k_{t+1}} \in B_{t+2}^{t+1}$, let c denote the number of effects among the $(s-1)^{t+1}$ effects corresponding to $G_{k_1} \dots G_{k_{t+1}}$, and these effects satisfy the words of length $t+1$ but do not satisfy the words of length $t+2$, thus, we have $c = \gamma_{s1}^t$.

In addition, the number of $g_{k_1} \dots g_{k_{t+1}}$ satisfying (i) and (ii) are $l(\gamma_{s2}^t - c)$ and lc, respectively. In (iii) and (iv), $T_2^{\varphi_{t+1}} = 0$, and therefore, we omit specifying the number of $g_{k_1} \dots g_{k_{t+1}}$ here.

Case II can be subdivided into the following four scenarios:

(i) $G_{k_1} \dots G_{k_{t+1}}$ have t factors included in the degenerate word of length $t+1$ and the degenerate word of length $t+2$ share $t+1$ common factors with $G_{k_1} \dots G_{k_{t+1}}$.

(ii) $G_{k_1} \dots G_{k_{t+1}}$ have t factors included in the degenerate word of length $t+1$ and the degenerate word of length $t+2$ share f $(f < t+1)$ common factors with $G_{k_1} \dots G_{k_{t+1}}$.

(iii) $G_{k_1} \dots G_{k_{t+1}}$ have r $(r < t)$ factors included in the degenerate word of length $t+1$ and the degenerate word of length $t+2$ share $t+1$ common factors with $G_{k_1} \dots G_{k_{t+1}}$.

(iv) $G_{k_1} \dots G_{k_{t+1}}$ have r $(r < t)$ factors included in the degenerate word of length $t+1$ and the degenerate word of length $t+2$ share f $(f < t+1)$ common factors with $G_{k_1} \dots G_{k_{t+1}}$.

Suppose that the number of $G_{k_1} \dots G_{k_{t+1}}$ satisfying (i) is a. For each $G_{k_1} \dots G_{k_{t+1}}$ satisfying (i), combine $G_{k_1} \dots G_{k_{t+1}}$ with one of the remaining $n-t-1$ factors to form $G_{k_1} \dots G_{k_{t+1}} G_{k_{t+2}}$. There are three possible cases for $G_{k_1} \dots G_{k_{t+1}} G_{k_{t+2}}$.

(i_1) If $G_{k_1}\dots G_{k_{t+1}}G_{k_{t+2}}$ contains degenerate words of length $t+1$. Among $G_{k_1}\dots G_{k_{t+1}}$ corresponds to $(s-1)^{t+1}$ factorial effects, the number of $g_{k_1}\dots g_{k_{t+1}}$ that makes $\beta(jg_{k_1}\dots g_{k_{t+1}})$ consisting of one N/s^{t+1} and $s-2$ zeros is $(s-1)\gamma_{s2}^{t}$. And all these $g_{k_1}\dots g_{k_{t+1}}$ satisfy $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^{t+1}$. Define $B(g_{k_1}\dots g_{k_{t+1}}) = \|\beta(jg_{k_1}\dots g_{k_{t+1}})^T H - \alpha(g_{k_1}\dots g_{k_{t+1}})\mathbf{1}_{s-1}^T\|_F^2$. Then, $B(g_{k_1}\dots g_{k_{t+1}}) = N^2/s^{2t+2}$. Moreover, the number of $g_{k_1}\dots g_{k_{t+1}}$ satisfying $\beta(jg_{k_1}\dots g_{k_{t+1}}) = \mathbf{0}_{s-1}^T$ is $(s-1)\gamma_{s1}^{t}$, and $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^{t+1}$, thus $B(g_{k_1}\dots g_{k_{t+1}}) = (s-1)N^2/s^{2t+2}$.

(i_2) If $G_{k_1}\dots G_{k_{t+1}}G_{k_{t+2}}$ equals to degenerate words of length $t+2$. Among $G_{k_1}\dots G_{k_{t+1}}$ corresponds to $(s-1)^{t+1}$ factorial effects, the number of $g_{k_1}\dots g_{k_{t+1}}$ that makes $\beta(jg_{k_1}\dots g_{k_{t+1}})$ consisting of one N/s^{t+1} and $s-2$ zeros is γ_{s2}^{t+1}, and $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^{t+1}$. Then, $B(g_{k_1}\dots g_{k_{t+1}}) = N^2/s^{2t+2}$. Further, the number of $g_{k_1}\dots g_{k_{t+1}}$ satisfying $\beta(jg_{k_1}\dots g_{k_{t+1}}) = \mathbf{0}_{s-1}^T$ is γ_{s1}^{t+1}, and $\alpha(g_{k_1}\dots g_{k_{t+1}}) = N/s^{t+1}$, thus $B(g_{k_1}\dots g_{k_{t+1}}) = (s-1)N^2/s^{2t+2}$.

(i_3) If $G_{k_1}\dots G_{k_{t+1}}G_{k_{t+2}}$ does not exist in the degenerate word of length $t+1$ and $t+2$, then $B(g_{k_1}\dots g_{k_{t+1}}) = 0$. Take all the three cases into account, we obtain $T_2 = [(s-1)\gamma_{s2}^{t}+\gamma_{s2}^{t+1}]N^2/s^{2t+2}+[(s-1)\gamma_{s1}^{t}+\gamma_{s1}^{t+1}](s-1)N^2/s^{2t+2}$.

Let b be the number of $G_{k_1}\dots G_{k_{t+1}}$ that satisfy case (ii), for each of $G_{k_1}\dots G_{k_{t+1}}$, $T_2 = (s-1)N^2\gamma_{s2}^{t}/s^{2t+2}+(s-1)^2N^2\gamma_{s1}^{t}/s^{2t+2}$. Then, the number of $G_{k_1}\dots G_{k_{t+1}}$ satisfying (iii) is $(t+2)A_{t+2}^2-a$, for each of $G_{k_1}\dots G_{k_{t+1}}$,

$T_2 = N^2\gamma_{s2}^{t+1}/s^{2t+2} + (s-1)N^2\gamma_{s1}^{t+1}/s^{2t+2}$. For each $G_{k_1}\dots G_{k_{t+1}}$ that satisfies case (iv), $T_2 = 0$. Then, we have

$$\begin{aligned}
\sum_{\Phi_{t+1}} T_2^{\varphi_{t+1}} =& \frac{N^2}{s^{2t+2}}\Bigg[\big[(s-1)\gamma_{s2}^t + \gamma_{s2}^{t+1} + (s-1)^2\gamma_{s1}^t + (s-1)\gamma_{s1}^{t+1}\big]a \\
&+ l(\gamma_{s2}^t - c)s^2 + lc(s-1)s^2 + \big[(s-1)\gamma_{s2}^t + (s-1)^2\gamma_{s1}^t\big]b \\
&+ \big[\gamma_{s2}^{t+1} + (s-1)\gamma_{s1}^{t+1}\big]\big((t+2)A_{t+2}^2 - a\big)\Bigg] \\
=& \frac{N^2}{s^{2t+2}}\Bigg[l(\gamma_{s2}^t - c)s^2 + lc(s-1)s^2 + \big[\gamma_{s2}^{t+1} + (s-1)\gamma_{s1}^{t+1}\big](t+2)A_{t+2}^2 \\
&+ \big[(s-1)\gamma_{s2}^t + (s-1)^2\gamma_{s1}^t\big](a+b)\Bigg].
\end{aligned}$$

Define set V as the collection of $G_{k_1}\dots G_{k_{t+1}}$ that satisfy the following two conditions: (1) $G_{k_1}\dots G_{k_{t+1}}$ does not all appear in the degenerate word of length $t+1$; (2) For a fixed word of length $t+1$, $G_{k_1}\dots G_{k_{t+1}}$ has t common factors to this word. By the definitions of a and b, it can be verified that the cardinality of the V is $a+b$. Then, $a+b = (n-t-1)(t+1)A_{t+1}$.

It is worth noting that among the $G_{k_1}\dots G_{k_{t+1}}$ satisfying (1) and (2), some possibilities in Case I are included. For any regular design, there are $t+1$ possible combinations of projecting a degenerate words of length $t+1$ into t factors. For each of these possibilities, combining them with the remaining $n-t-1$ factors generates $n-t-1$ factor combinations. Therefore, for all words of length $t+1$, there are a total of $(n-t-1)(t+1)A_{t+1}$ factor combinations. Let B_{t+1} represent the set of these $(n-t-1)(t+1)A_{t+1}$ factor combinations (including repeated factor combinations). Let u be the number of elements in set B_{t+1} that share $t+1$ factors with both the degenerate word of length $t+2$ and the degenerate word of length $t+1$. Then, $u=(t+1)(t+2)A^1_{t+2}$ represents the count of $G_{k_1}\dots G_{k_{t+1}}$ that satisfy either (i) or (ii) specified in Case I. Therefore, we have

$$
\begin{aligned}
K_{t+1} =& \frac{s^2(t+1)}{N^2}\sum_{\Phi_{t+1}} T_1^{\varphi_{t+1}} + \frac{s^2}{N^2}\sum_{\Phi_{t+1}} T_2^{\varphi_{t+1}} \\
=& \frac{t+1}{s^{2t}}\left[(s^2\gamma^t_{s2}-(s-1)^{t+1})A_{t+1}+(s-1)^{t+1}n!/\{(t+1)!(n-t-1)!\}\right] \\
& + \frac{1}{s^{2t}}\Big[l(\gamma^t_{s2}-c)s^2+lc(s-1)s^2+\left[\gamma^{t+1}_{s2}+(s-1)\gamma^{t+1}_{s1}\right](t+2)A^2_{t+2} \\
& + \left[(s-1)\gamma^t_{s2}+(s-1)^2\gamma^t_{s1}\right]\left[(n-t-1)(t+1)A_{t+1}-u\right]\Big] \\
=& \frac{1}{s^{2t}}\left[C_1(t+1)A_{t+1}+C_2(t+2)A^1_{t+2}+C_3(t+2)A^2_{t+2}+C_4\right],
\end{aligned}
$$

where $C_1=s^2\gamma^t_{s2}+\left[(s-1)\gamma^t_{s2}+(s-1)^2\gamma^t_{s1}\right](n-t-1)-(s-1)^{t+1}$, $C_2=\left[s^2-(s-1)(t+1)\right]\gamma^t_{s2}+\left[s^2(s-2)-(s-1)^2(t+1)\right]\gamma^t_{s1}$, $C_3=\gamma^{t+1}_{s2}+(s-1)\gamma^{t+1}_{s1}$, and $C_4=(s-1)^{t+1}(t+1)n!/\{(t+1)!(n-t-1)!\}$. $\square$

Proof of Corollary 2. For an s^{n-p} design, its runs are the solutions to the system of equations $A^Tx=0$, where $A=[a_1,\dots,a_p]$, and $a_1,\dots,a_p$ are linearly independent n-dimensional column vectors. Let $\lceil a_i^T\rceil$ represent the number of non-zero elements in a_i^T. Extract the row vectors from $a_1^T,\dots,a_p^T$

that satisfy $\lceil a_i^T \rceil = 3$, assuming there are q such vectors. Denote these q row vectors as $b_1, \ldots, b_q$. In this case, for $i = 1, \ldots, q$, each equation $b_i x = 0$ corresponds to a word of length 3. Since A_4^1 is the number of degenerate words of length 4 that share 3 common factors with some degenerate word of length 3. Each of these A_4^1 words is obtained from certain linear combinations of words of length 3 that share two common factors. That is, they are generated by linear combinations of u vectors from $b_1, \ldots, b_q$, where the u vectors satisfy the following condition: when these u row vectors are concatenated row-wise into a matrix, the resulting matrix can be written as $(U, cI, \mathbf{0})$ up to column permutation, where U is a $u \times 2$ matrix in which every element is non-zero, I is an $u \times u$ identity matrix, c is a constant, and 0 is an $u \times (n - u - 2)$ zero matrix. For example, when $u = 2$, assume that the two rows of the matrix are b_i and b_j, then there are $s - 1$ linear combinations of b_i and b_j, i.e. $b_i + b_j, b_i + 2b_j, \ldots, b_i + (s-1)b_j$. Among these $s-1$ words, $s-3$ words have a length of 4, and these $s-3$ words have same degenerate words. Then, any one of these A_4^1 words corresponds to $s - 3$ words that have same degenerate word. Hence, $A_4^* + (s-4)A_4^1 = A_4$, which implies $A_4^2 = A_4 - (s-3)A_4^1$. Substituting $t = 2$ and $A_4^2 = A_4 - (s-3)A_4^1$ into (5.7) in the paper, the expression for K_3 is obtained. $\square$

S2. Approximate BP-MA designs for 5-level

We present the K_2 and K_3 values for 25, 50, 75, 100, and 125 runs at 5 levels in Tables S.1 and S.2.

Table S.1: The K_2-value of the approximate BP-MA designs with n runs and m factors

n \ m	3	4	5	6	7	8	9	10
25	6.84	19.68	42.80	79.92	–	–	–	–
50	4.98	12.24	24.2	42.00	66.96	100.04	142.47	195.42
75	4.35	9.71	19.08	33.03	51.33	–	–	–
100	4.03	8.43	15.31	24.21	39.75	58.93	82.05	111.92
125	3.84	7.68	12.80	19.20	32.88	47.84	67.08	117.60

" – " represents that an $n \times m$ design cannot be obtained by selecting columns from a saturated orthogonal array with n runs.

Table S.2: The K_3-value of the approximate BP-MA designs with n runs and m factors

n \ m	3	4	5	6	7	8	9	10
25	1.56	9.28	30.80	78.48	–	–	–	–
50	0.78	4.64	15.40	38.40	82.24	154.52	266.37	430.34
75	0.51	3.13	11.00	27.35	57.02	–	–	–
100	0.39	2.34	8.29	20.70	43.00	81.22	136.40	219.33
125	0.31	1.87	6.27	15.74	34.61	64.43	109.63	193.39

The five level permutations of 5 levels are labeled as I,...,V according to the rules of Table S.3.

Table S.3: Level permutations

Label	Level permutations
I	$\{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$
II	$\{0, 1, 2, 3, 4\} \to \{1, 0, 2, 3, 4\}$
III	$\{0, 1, 2, 3, 4\} \to \{1, 2, 0, 3, 4\}$
IV	$\{0, 1, 2, 3, 4\} \to \{1, 2, 3, 0, 4\}$
V	$\{0, 1, 2, 3, 4\} \to \{1, 2, 3, 4, 0\}$

OA(25,6,5,2) and OA(125,31,5,2) are regular designs, we only present their independent columns and the definition relationship of other columns. OA(50,10,5,2), OA(75,7,5,2) and OA(100,20,5,2) are derived from Theorem 9.15 in Hedayat, Sloane and Stufken (1999). For the details of these designs, please refer to Tables S.4 to S.8. In addition, we have provided the column labels selected from OA($n, m, 5, 2$) to create MA design, along with the level permutation to obtain the MA design under BP. For specific details, please refer to Tables S.9 to S.13. It is worth noting that in Step 2 of Algorithm 1, evaluating K_r for $r = t, \ldots, n-1$ across all $5^n p$ candidate designs becomes computationally intractable for lagre $s/n/p$. For example, when $n = 7$ and $p = 1$, the total number of candidate designs reaches $5^7 = 78125$, making exhaustive evaluation infeasible. To address this combinatorial explosion, we adopt a randomized subsampling strategy when $n \geq 7$: from the $5^n p$ candidate designs, we randomly select 10,000 designs. We then compute K_r over these $10,000$ designs for $r = t, \ldots, n-1$ to approximately identify minimum aberration designs.

Table S.4: OA(25,6,5,2)

Column label	**1**	**2**	**3**	**4**	**5**	**6**
Column	A	B	AB	AB^2	AB^3	AB^4

where **Column** $A = (0,1,2,3,4)^T \otimes \mathbf{1}_5$, **Column** $B = \mathbf{1}_5 \otimes (0,1,2,3,4)^T$, and AB^u is a shortcut notation for $A + uB$ for $u \in GF(5)$.

As only a subset of the columns from $\mathrm{OA}(125, 31, 5, 2)$ be used. Thus, Table S.5 presents only the required columns. Let A, B, and C be three independent columns with **Column** $A = (0,1,2,3,4)^T \otimes \mathbf{1}_{25}$, **Column** $B = \mathbf{1}_5 \otimes (0,1,2,3,4)^T \otimes \mathbf{1}_5$, and **Column** $C = \mathbf{1}_{25} \otimes (0,1,2,3,4)^T$.

Table S.5: Some columns from OA(125,31,5,2)

Column label	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Column	A	B	C	AB	AB^2	AB^3	AC
Column label	**8**	**9**	**10**	**11**	**12**	**13**	**14**
Column	BC^2	BC^3	ABC	ABC^3	ABC^4	AB^2C	AB^2C^2
Column label	**15**	**16**	**17**	**18**	**19**	**20**	
Column	AB^2C^3	AB^3C	AB^3C^2	AB^3C^4	AB^4C^2	AB^4C^3	

Table S.6: OA(50,10,5,2)

Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	0	0	0	0	0	0	0	0	0
2	0	4	3	1	2	1	0	4	2	3
3	0	3	1	2	4	4	2	0	1	3
4	0	1	2	4	3	1	2	3	0	4
5	0	2	4	3	1	4	1	3	2	0
6	0	2	3	2	3	0	4	1	4	1
7	0	1	1	3	0	2	4	4	3	2
8	0	0	4	4	2	3	3	1	1	2
9	0	3	0	1	1	2	3	2	4	4
10	0	4	2	0	4	3	1	2	3	1
11	1	1	1	1	1	1	1	1	1	1
12	1	0	4	2	3	2	1	0	3	4
13	1	4	2	3	0	0	3	1	2	4
14	1	2	3	0	4	2	3	4	1	0
15	1	3	0	4	2	0	2	4	3	1
16	1	3	4	3	4	1	0	2	0	2
17	1	2	2	4	1	3	0	0	4	3
18	1	1	0	0	3	4	4	2	2	3
19	1	4	1	2	2	3	4	3	0	0
20	1	0	3	1	0	4	2	3	4	2
21	2	2	2	2	2	2	2	2	2	2
22	2	1	0	3	4	3	2	1	4	0
23	2	0	3	4	1	1	4	2	3	0
24	2	3	4	1	0	3	4	0	2	1
25	2	4	1	0	3	1	3	0	4	2
26	2	4	0	4	0	2	1	3	1	3
27	2	3	3	0	2	4	1	1	0	4
28	2	2	1	1	4	0	0	3	3	4
29	2	0	2	3	3	4	0	4	1	1
30	2	1	4	2	1	0	3	4	0	3
31	3	3	3	3	3	3	3	3	3	3
32	3	2	1	4	0	4	3	2	0	1
33	3	1	4	0	2	2	0	3	4	1
34	3	4	0	2	1	4	0	1	3	2
35	3	0	2	1	4	2	4	1	0	3
36	3	0	1	0	1	3	2	4	2	4
37	3	4	4	1	3	0	2	2	1	0
38	3	3	2	2	0	1	1	4	4	0
39	3	1	3	4	4	0	1	0	2	2
40	3	2	0	3	2	1	4	0	1	4
41	4	4	4	4	4	4	4	4	4	4
42	4	3	2	0	1	0	4	3	1	2
43	4	2	0	1	3	3	1	4	0	2
44	4	0	1	3	2	0	1	2	4	3
45	4	1	3	2	0	3	0	2	1	4
46	4	1	2	1	2	4	3	0	3	0
47	4	0	0	2	4	1	3	3	2	1
48	4	4	3	3	1	2	2	0	0	1
49	4	2	4	0	0	1	2	1	3	3
50	4	3	1	4	3	2	0	1	2	0

Table S.7: OA(75,7,5,2)

Run	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	1	1	0	2	4	2
3	1	1	0	2	4	2	0
4	1	0	2	4	2	0	1
5	0	2	4	2	0	1	1
6	2	4	2	0	1	1	0
7	4	2	0	1	1	0	2
8	2	0	1	1	0	2	4
9	0	2	2	0	4	3	4
10	2	2	0	4	3	4	0
11	2	0	4	3	4	0	2
12	0	4	3	4	0	2	2
13	4	3	4	0	2	2	0
14	3	4	0	2	2	0	4
15	4	0	2	2	0	4	3
16	1	1	1	1	1	1	1
17	1	2	2	1	3	0	3
18	2	2	1	3	0	3	1
19	2	1	3	0	3	1	2
20	1	3	0	3	1	2	2
21	3	0	3	1	2	2	1
22	0	3	1	2	2	1	3
23	3	1	2	2	1	3	0
24	1	3	3	1	0	4	0
25	3	3	1	0	4	0	1
26	3	1	0	4	0	1	3
27	1	0	4	0	1	3	3
28	0	4	0	1	3	3	1
29	4	0	1	3	3	1	0
30	0	1	3	3	1	0	4
31	2	2	2	2	2	2	2
32	2	3	3	2	4	1	4
33	3	3	2	4	1	4	2
34	3	2	4	1	4	2	3
35	2	4	1	4	2	3	3
36	4	1	4	2	3	3	2
37	1	4	2	3	3	2	4
38	4	2	3	3	2	4	1
39	0	4	4	2	1	0	1
40	4	4	2	1	0	1	2
41	4	2	1	0	1	2	4
42	2	1	0	1	2	4	4
43	1	0	1	2	4	4	2
44	0	1	2	4	4	2	1
45	1	2	4	4	2	1	0
46	3	3	3	3	3	3	3
47	3	4	4	3	0	2	0
48	4	4	3	0	2	0	3
49	4	3	0	2	0	3	4
50	3	0	2	0	3	4	4
51	0	2	0	3	4	4	3
52	2	0	3	4	4	3	0
53	0	3	4	4	3	0	2
54	3	0	0	3	2	1	2
55	0	0	3	2	1	2	3
56	0	3	2	1	2	3	0
57	3	2	1	2	3	0	0
58	2	1	2	3	0	0	3
59	1	2	3	0	0	3	2
60	2	3	0	0	3	2	1
61	4	4	4	4	4	4	4
62	4	0	0	4	1	3	1
63	0	0	4	1	3	1	4
64	0	4	1	3	1	4	0
65	4	1	3	1	4	0	0
66	1	3	1	4	0	0	4
67	3	1	4	0	0	4	1
68	1	4	0	0	4	1	3
69	4	1	1	4	3	2	3
70	1	1	4	3	2	3	4
71	1	4	3	2	3	4	1
72	4	3	2	3	4	1	1
73	3	2	3	4	1	1	4
74	2	3	4	1	1	4	3
75	3	4	1	1	4	3	2

Table S.8: OA(100,20,5,2)

Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
3	0	0	0	0	2	2	2	2	4	4	4	4	1	1	1	1	3	3	3	3
4	0	0	0	0	3	3	3	3	1	1	1	1	4	4	4	4	2	2	2	2
5	0	1	2	3	0	2	4	1	0	3	1	4	0	4	3	2	2	4	1	3
6	0	1	2	3	1	4	2	0	1	0	4	3	3	0	2	4	3	1	4	2
7	0	1	2	3	2	1	0	4	4	1	3	0	4	2	0	3	4	3	2	1
8	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
9	0	2	4	1	0	3	1	4	2	3	4	0	4	3	2	1	3	2	1	0
10	0	2	4	1	1	0	4	3	4	2	0	3	1	2	3	4	2	0	3	1
11	0	2	4	1	3	4	0	1	0	4	3	2	3	1	4	2	1	3	0	2
12	0	2	4	1	4	1	3	0	3	0	2	4	2	4	1	3	0	1	2	3
13	0	3	1	4	0	4	3	2	1	2	3	4	1	3	0	2	0	2	4	1
14	0	3	1	4	2	3	4	0	2	4	1	3	3	2	1	0	1	4	2	0
15	0	3	1	4	3	0	2	4	4	3	2	1	0	1	2	3	2	1	0	4
16	0	3	1	4	4	2	0	3	3	1	4	2	2	0	3	1	4	0	1	2
17	0	4	3	2	1	2	3	4	2	0	3	1	0	2	4	1	1	0	4	3
18	0	4	3	2	2	4	1	3	0	1	2	3	1	4	2	0	0	3	1	4
19	0	4	3	2	3	1	4	2	1	3	0	2	4	0	1	2	4	1	3	0
20	0	4	3	2	4	3	2	1	3	2	1	0	2	1	0	4	3	4	0	1
21	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
22	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	0	0	0	0
23	1	1	1	1	3	3	3	3	0	0	0	0	2	2	2	2	4	4	4	4
24	1	1	1	1	4	4	4	4	2	2	2	2	0	0	0	0	3	3	3	3
25	1	2	3	4	1	3	0	2	1	4	2	0	1	0	4	3	3	0	2	4
26	1	2	3	4	2	0	3	1	2	1	0	4	4	1	3	0	4	2	0	3
27	1	2	3	4	3	2	1	0	0	2	4	1	0	3	1	4	0	4	3	2
28	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0
29	1	3	0	2	1	4	2	0	3	4	0	1	0	4	3	2	4	3	2	1
30	1	3	0	2	2	1	0	4	0	3	1	4	2	3	4	0	3	1	4	2
31	1	3	0	2	4	0	1	2	1	0	4	3	4	2	0	3	2	4	1	3
32	1	3	0	2	0	2	4	1	4	1	3	0	3	0	2	4	1	2	3	4
33	1	4	2	0	1	0	4	3	2	3	4	0	2	4	1	3	1	3	0	2
34	1	4	2	0	3	4	0	1	3	0	2	4	4	3	2	1	2	0	3	1
35	1	4	2	0	4	1	3	0	0	4	3	2	1	2	3	4	3	2	1	0
36	1	4	2	0	0	3	1	4	4	2	0	3	3	1	4	2	0	1	2	3
37	1	0	4	3	2	3	4	0	3	1	4	2	1	3	0	2	2	1	0	4
38	1	0	4	3	3	0	2	4	1	2	3	4	2	0	3	1	1	4	2	0
39	1	0	4	3	4	2	0	3	2	4	1	3	0	1	2	3	0	2	4	1
40	1	0	4	3	0	4	3	2	4	3	2	1	3	2	1	0	4	0	1	2
41	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
42	2	2	2	2	3	3	3	3	4	4	4	4	0	0	0	0	1	1	1	1
43	2	2	2	2	4	4	4	4	1	1	1	1	3	3	3	3	0	0	0	0
44	2	2	2	2	0	0	0	0	3	3	3	3	1	1	1	1	4	4	4	4
45	2	3	4	0	2	4	1	3	2	0	3	1	2	1	0	4	4	1	3	0
46	2	3	4	0	3	1	4	2	3	2	1	0	0	2	4	1	0	3	1	4
47	2	3	4	0	4	3	2	1	1	3	0	2	1	4	2	0	1	0	4	3
48	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1
49	2	4	1	3	2	0	3	1	4	0	1	2	1	0	4	3	0	4	3	2
50	2	4	1	3	3	2	1	0	1	4	2	0	3	4	0	1	4	2	0	3

Table S.8 (continued): OA(100,20,5,2)

Run	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
51	2	4	1	3	0	1	2	3	2	1	0	4	0	3	1	4	3	0	2	4
52	2	4	1	3	1	3	0	2	0	2	4	1	4	1	3	0	2	3	4	0
53	2	0	3	1	2	1	0	4	3	4	0	1	3	0	2	4	2	4	1	3
54	2	0	3	1	4	0	1	2	4	1	3	0	0	4	3	2	3	1	4	2
55	2	0	3	1	0	2	4	1	1	0	4	3	2	3	4	0	4	3	2	1
56	2	0	3	1	1	4	2	0	0	3	1	4	4	2	0	3	1	2	3	4
57	2	1	0	4	3	4	0	1	4	2	0	3	2	4	1	3	3	2	1	0
58	2	1	0	4	4	1	3	0	2	3	4	0	3	1	4	2	2	0	3	1
59	2	1	0	4	0	3	1	4	3	0	2	4	1	2	3	4	1	3	0	2
60	2	1	0	4	1	0	4	3	0	4	3	2	4	3	2	1	0	1	2	3
61	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
62	3	3	3	3	4	4	4	4	0	0	0	0	1	1	1	1	2	2	2	2
63	3	3	3	3	0	0	0	0	2	2	2	2	4	4	4	4	1	1	1	1
64	3	3	3	3	1	1	1	1	4	4	4	4	2	2	2	2	0	0	0	0
65	3	4	0	1	3	0	2	4	3	1	4	2	3	2	1	0	0	2	4	1
66	3	4	0	1	4	2	0	3	4	3	2	1	1	3	0	2	1	4	2	0
67	3	4	0	1	0	4	3	2	2	4	1	3	2	0	3	1	2	1	0	4
68	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2
69	3	0	2	4	3	1	4	2	0	1	2	3	2	1	0	4	1	0	4	3
70	3	0	2	4	4	3	2	1	2	0	3	1	4	0	1	2	0	3	1	4
71	3	0	2	4	1	2	3	4	3	2	1	0	1	4	2	0	4	1	3	0
72	3	0	2	4	2	4	1	3	1	3	0	2	0	2	4	1	3	4	0	1
73	3	1	4	2	3	2	1	0	4	0	1	2	4	1	3	0	3	0	2	4
74	3	1	4	2	0	1	2	3	0	2	4	1	1	0	4	3	4	2	0	3
75	3	1	4	2	1	3	0	2	2	1	0	4	3	4	0	1	0	4	3	2
76	3	1	4	2	2	0	3	1	1	4	2	0	0	3	1	4	2	3	4	0
77	3	2	1	0	4	0	1	2	0	3	1	4	3	0	2	4	4	3	2	1
78	3	2	1	0	0	2	4	1	3	4	0	1	4	2	0	3	3	1	4	2
79	3	2	1	0	1	4	2	0	4	1	3	0	2	3	4	0	2	4	1	3
80	3	2	1	0	2	1	0	4	1	0	4	3	0	4	3	2	1	2	3	4
81	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
82	4	4	4	4	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
83	4	4	4	4	1	1	1	1	3	3	3	3	0	0	0	0	2	2	2	2
84	4	4	4	4	2	2	2	2	0	0	0	0	3	3	3	3	1	1	1	1
85	4	0	1	2	4	1	3	0	4	2	0	3	4	3	2	1	1	3	0	2
86	4	0	1	2	0	3	1	4	0	4	3	2	2	4	1	3	2	0	3	1
87	4	0	1	2	1	0	4	3	3	0	2	4	3	1	4	2	3	2	1	0
88	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3
89	4	1	3	0	4	2	0	3	1	2	3	4	3	2	1	0	2	1	0	4
90	4	1	3	0	0	4	3	2	3	1	4	2	0	1	2	3	1	4	2	0
91	4	1	3	0	2	3	4	0	4	3	2	1	2	0	3	1	0	2	4	1
92	4	1	3	0	3	0	2	4	2	4	1	3	1	3	0	2	4	0	1	2
93	4	2	0	3	4	3	2	1	0	1	2	3	0	2	4	1	4	1	3	0
94	4	2	0	3	1	2	3	4	1	3	0	2	2	1	0	4	0	3	1	4
95	4	2	0	3	2	4	1	3	3	2	1	0	4	0	1	2	1	0	4	3
96	4	2	0	3	3	1	4	2	2	0	3	1	1	4	2	0	3	4	0	1
97	4	3	2	1	0	1	2	3	1	4	2	0	4	1	3	0	0	4	3	2
98	4	3	2	1	1	3	0	2	4	0	1	2	0	3	1	4	4	2	0	3
99	4	3	2	1	2	0	3	1	0	2	4	1	3	4	0	1	3	0	2	4
100	4	3	2	1	3	2	1	0	2	1	0	4	1	0	4	3	2	3	4	0

Table S.9: Selected columns and Level permutation for OA(25,6,5,2)

Design	25 × 3	25 × 4	25 × 5	25 × 6
Sel-col	(**1**,**2**,**3**)	(**1**,**2**,**3**,**4**)	(**1**,**2**,**3**,**4**,**5**)	(**1**,**2**,**3**,**4**,**5**,**6**)
Lel-per	(III,I,I)	(II,II,I,I)	(III,III,III,I,I)	(V,II,II,I,I,I)

[a] Sel-col represents the column labels selected from the orthogonal array.

[b] Lel-per represents the level permutation applied to the selected columns.

Table S.10: Selected columns and Level permutation for OA(50,10,5,2)

Design	50 × 3	50 × 4	50 × 5	50 × 6
Sel-col	(**1**,**2**,**3**)	(**1**,**2**,**3**,**4**)	(**1**,**2**,**3**,**4**,**5**)	(**1**,**2**,**3**,**4**,**5**,**6**)
Lel-per	(III,I,I)	(III,III,I,I)	(II,IV,II,I,I)	(I,V,II,V,II,I)
Design	50 × 7		50 × 8	
Sel-col	(**1**,**2**,**3**,**4**,**5**,**6**,**7**)		(**1**,**2**,**3**,**4**,**5**,**6**,**7**,**8**)	
Lel-per	(II,I,III,I,III,II,I)		(II,III,II,IV,IV,III,III,IV)	
Design	50 × 9		50 × 10	
Sel-col	(**1**,**2**,**3**,**4**,**5**,**6**,**7**,**8**,**9**)		(**1**,**2**,**3**,**4**,**5**,**6**,**7**,**8**,**9**,**10**)	
Lel-per	(IV,V,IV,I,I,IV,V,II,II)		(II,V,I,II,V,II,III,I,IV,I)	

Table S.11: Selected columns and Level permutation for OA(75,7,5,2)

Design	75 × 3	75 × 4	75 × 5
Sel-col	(**1**,**2**,**3**)	(**1**,**2**,**3**,**4**)	(**1**,**2**,**3**,**4**,**5**)
Lel-per	(I,I,I)	(I,I,I,I)	(V,II,IV,I,I)
Design	75 × 6	75 × 7	
Sel-col	(**1**,**2**,**3**,**4**,**5**,**6**)	(**1**,**2**,**3**,**4**,**5**,**6**,**7**)	
Lel-per	(IV,II,V,V,III,I)	(I,III,III,V,II,II,IV)	

Table S.12: Selected columns and Level permutation for OA(100,20,5,2)

Design	100 × 3	100 × 4	100 × 5	100 × 6
Sel-col	(**1**,**2**,**5**)	(**1**,**2**,5,**6**)	(**1**,**4**,**5**,**8**,**20**)	(**1**,**4**,**5**,**8**,**17**,**20**)
Lel-per	(I,I,I)	(I,I,I,I)	(I,I,I,I,I)	(I,I,I,I,I,I)
Design	100 × 7		100 × 8	
Sel-col	(**1**,**4**,**5**,**6**,**8**,**17**,**20**)		(**1**,**2**,**6**,**7**,**9**,**10**,**13**,**15**)	
Lel-per	(V,V,V,III,V,IV,IV)		(I,II,V,I,IV,V,V,III)	
Design	100 × 9		100 × 10	
Sel-col	(**9**,**10**,**11**,**13**,**14**,**15**,**17**,**18**,**19**)		(**1**,**2**,**4**,**6**,**7**,**9**,**10**,**13**,**15**,**16**)	
Lel-per	(II,V,V,III,V,V,I,II,I)		(V,IV,I,I,III,V,II,IV,II,V)	

Table S.13: Selected columns and Level permutation for OA(125,31,5,2)

Design	125 × 3	125 × 4	125 × 5	125 × 6
Sel-col	(**1**,**2**,**3**)	(**1**,**2**,**3**,**10**)	(**1**,**2**,**3**,**10**,**15**)	(**1**,**2**,**3**,**10**,**15**,**18**)
Lel-per	(I,I,I)	(I,I,I,I)	(I,I,I,I,I)	(I,I,I,I,I,I)
Design	125 × 7		125 × 8	
Sel-col	(**1**,**2**,**3**,**4**,**7**,**18**,**20**)		(**1**,**2**,**3**,**4**,**12**,**13**,**14**,**16**)	
Lel-per	(II,III,III,V,V,II,II)		(IV,II,IV,II,V,II,IV,V)	
Design	125 × 9		125 × 10	
Sel-col	(**1**,**2**,**3**,**4**,**7**,**8**,**12**,**15**,**19**)		(**1**,**2**,**3**,**4**,**5**,**6**,**10**,**14**,**15**,**17**)	
Lel-per	(IV,II,I,II,I,I,III,III,II)		(V,IV,IV,IV,IV,III,V,IV,V,III)	

References

Cheng, C. S. (1980). Orthogonal arrays with variable numbers of symbols. *Ann. Statist.* **8**, 447–453.

Hedayat, A. S., Sloane, N. J. and Stufken, J. (1999). *Orthogonal Arrays: Theory and Applications*. Springer Series in Statistics. New York: Springer-Verlag.