Supplementary Material

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This supplement contains technical proofs for the manuscript "Simple Inferential Heritability Analyses of Big GWAS Data", referred to as MS in the supplement.

Overview

In this supplement, we present the proofs of Theorem 3.1, Theorem 4.1, Lemma 5.1 and Theorem 5.1. Theorem 3.2 and Theorem 4.2 then follow immediately as we explained in Sections 3 and 4 of MS, and all of the remaining theorems and corollaries can be derived fairly straightforwardly, given the established results.

Throughout this supplement, c denotes a generic constant, whose value may be different at different places. The notation δ is sometimes used to denote an arbitrary positive constant, following the mathematical tradition; this should have no confusion with the error vector $\delta = (\delta_i)_{1 \le i \le n-q}$ in (3.17) of MS. The spectral norm of a matrix, A, is defined as $\|A\| = \sqrt{\lambda_{\max}(A'A)}$, where $\lambda_{\max}$ denotes the largest eigenvalue; the Euclidean norm of A is defined as $\|A\|_2 = \sqrt{\mathrm{tr}(A'A)}$.

1 Proof of Theorem 3.1

Part (I): For $\hat{\sigma}^2$, the denominator can be expressed as $\mathrm{tr}(S_z^2) - \sum_{i=1}^n s_{z,i,i}^2$. By the RMT (e.g., Corollary 16.2 of Jiang 2022), we have

$$\frac{1}{n}\mathrm{tr}(S_z^2) \xrightarrow{\text{a.s.}} \nu_2 = 1 + \gamma. \tag{1}$$

Next, we argue that

$$\frac{1}{n}\sum_{i=1}^n s_{z,i,i}^2 \xrightarrow{\text{P}} 1. \tag{2}$$

It follows from (1), (2) that the denominator of $\hat{\sigma}^2$, divided by n, converges in probability to γ. To show (2), note that, for any $\delta > 0$, by Marcinkiewicz–Zygmund inequality [e.g., (5.71) of Jiang (2022)], we have

$$\begin{aligned} \mathrm{P}(|s_{z,i,i} - 1| > \delta) &= \mathrm{P}\left[\left|\sum_{j=1}^{p}(z_{ij}^2 - 1)\right| > p\delta\right] \\ &\le \frac{1}{(p\delta)^3}\mathrm{E}\left\{\left|\sum_{j=1}^{p}(z_{ij}^2 - 1)\right|^3\right\} \\ &\le \frac{c}{\delta^3 p^3}\mathrm{E}\left\{\sum_{j=1}^{p}(z_{ij}^2 - 1)^2\right\}^{3/2} \\ &\le \frac{c}{p^{3/2}}\left\{\frac{1}{p}\sum_{j=1}^{p}\mathrm{E}(|z_{ij}^2 - 1|^3)\right\} \le \frac{c}{p^{3/2}}, \end{aligned}$$

applying Jensen's inequality to the second-to-last step. It follows that

$$\mathrm{P}\left(\max_{1\le i\le n}|s_{z,i,i} - 1| > \delta\right) \le \sum_{i=1}^{n}\mathrm{P}(|s_{z,i,i} - 1| > \delta) \le \frac{c}{\sqrt{p}},$$

using (2.14) of MS. Conclusion (2) thus follows by straightforward arguments.

For any symmetric matrix A, which depends only on Z and satisfies

$$\mathrm{tr}(A) = 0, \tag{3}$$

we have $\mathrm{E}(y'Ay|Z) = \mathrm{E}\{\mathrm{tr}(Ayy')|Z\} = \mathrm{tr}\{A\mathrm{E}(yy'|Z)\} = \omega\sigma^2\mathrm{tr}(AS_z)$. Note that $\mathrm{E}(\alpha_j) = 0$ and $\mathrm{var}(\alpha_j) = \omega\sigma^2$, hence

$$\mathrm{E}(yy'|Z) = \tau^2 I_n + \omega\sigma^2 S_z \equiv V_{y|Z}. \tag{4}$$

Because $A = S_{z,\mathrm{o}}$ satisfies (3), we have

$$\mathrm{E}(\hat{\sigma}^2|Z) = \frac{\mathrm{E}(y'S_{z,\mathrm{o}}y|Z)}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} = \omega\sigma^2\frac{\mathrm{tr}(S_{z,\mathrm{o}}S_z)}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} = \omega\sigma^2, \tag{5}$$

because $\mathrm{tr}(S_{z,\mathrm{o}}S_z) = \mathrm{tr}(S_{z,\mathrm{o}}^2)$. On the other hand, let $b = (b_j)_{1\le j\le p}$ and note that $y|Z, b \sim N(0, \Sigma)$, where $\Sigma = \tau^2 I_n + \sigma^2\tilde{Z}D_b\tilde{Z}'$ and $D_b = \mathrm{diag}(b_j, 1 \le j \le p)$. Thus, we have (e.g., Jiang 2022, sec. A.3.1)

$$\begin{aligned}\mathrm{var}(y'Ay|Z) &= \mathrm{var}\{\mathrm{E}(y'Ay|Z,b)|Z\} + \mathrm{E}\{\mathrm{var}(y'Ay|Z,b)|Z\} \\ &= \mathrm{var}\{\mathrm{tr}(A\Sigma)|Z\} + 2\mathrm{E}\{\mathrm{tr}(A\Sigma A\Sigma)|Z\} \\ &= 3\omega(1-\omega)\left(\frac{\sigma^4}{p^2}\right)\sum_{j=1}^{p}(Z_j'AZ_j)^2 + 2\{\tau^4\mathrm{tr}(A^2) \\ &\quad +2\tau^2\omega\sigma^2\mathrm{tr}(AS_zA) + (\omega\sigma^2)^2\mathrm{tr}(AS_zAS_z)\}, \qquad (6)\end{aligned}$$

where Z_j is the jth column of Z. By (1), (2), we have

$$\frac{1}{n}\mathrm{tr}(A^2) = \frac{1}{n}\mathrm{tr}(S_z^2) - \frac{1}{n}\sum_{i=1}^{n} s_{z,i,i}^2 \xrightarrow{\mathrm{P}} \gamma,$$

hence $\mathrm{tr}(A^2) = n\gamma\{1 + o_{\mathrm{P}}(1)\}$. Also, we have $\|A\| \leq \|S_z\| + \|dS_z\| \leq 2\|S_z\|$, hence $\lambda_{\max}(A^2) = \|A\|^2 \leq 4\|S_z\|^2 = 4\lambda_{\max}(S_z^2) \xrightarrow{\mathrm{a.s.}} 4(1+\sqrt{\gamma})^4$ by the RMT (e.g., Theorem 16.5 of Jiang 2022). Therefore, we have

$$\mathrm{tr}(AS_zA) = \mathrm{tr}(S_z^{1/2}A^2S_z^{1/2}) \leq \lambda_{\max}(A^2)\mathrm{tr}(S_z) = nO_{\mathrm{P}}(1)$$

(e.g., Corollary 16.2 of Jiang 2022). Similarly, we have [e.g., (5.40) of Jiang 2022]

$$|\mathrm{tr}(AS_zAS_z)| \leq \lambda_{\max}(S_z)\|A\|_2\|AS_z\|_2; \quad \lambda_{\max}(S_z) \xrightarrow{\mathrm{a.s.}} (1+\sqrt{\gamma})^2,$$

$\|A\|_2^2 = \mathrm{tr}(A^2) \leq \lambda_{\max}(A^2)n = nO_{\mathrm{P}}(1)$; $\|AS_z\|_2^2 = \mathrm{tr}(S_zA^2S_z) \leq \lambda_{\max}(A^2)\mathrm{tr}(S_z^2) = nO_{\mathrm{P}}(1)$. Thus, we have $\mathrm{tr}(AS_zAS_z) = nO_{\mathrm{P}}(1)$. Finally, we have

$$\sum_{j=1}^{p}(Z_j'AZ_j)^2 \leq \|A\|^2\sum_{j=1}^{p}(Z_j'Z_j)^2 = O_{\mathrm{P}}(1)\sum_{j=1}^{p}(Z_j'Z_j)^2,$$

and $\sum_{j=1}^{p}(Z_j'Z_j)^2 = O_{\mathrm{P}}(1)n^2p$, because

$$\begin{aligned}\sum_{j=1}^{p}\mathrm{E}\{Z_j'Z_j)^2\} &= n^2\sum_{j=1}^{p}\mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n}z_{ij}^2\right)^2 \\ &\leq n^2\sum_{j=1}^{p}\mathrm{E}\left(\frac{1}{n}\sum_{i=1}^{n}z_{ij}^4\right) = 3n^2p.\end{aligned}$$

It follows that $p^{-2}\sum_{j=1}^{p}(Z_j'AZ_j)^2 = O_{\mathrm{P}}(1)n$. Therefore, combining (6) and the above results, we have,

$$\mathrm{var}(\hat{\sigma}^2|Z) = \frac{\mathrm{var}(y'Ay|Z)}{\{\mathrm{tr}(A^2)\}^2} = \frac{O_{\mathrm{P}}(1)}{n}. \qquad (7)$$

For any $\delta > 0$, by (5) and (7), we have

$$\mathrm{P}(|\hat{\sigma}^2 - \omega\sigma^2| > \delta|Z) \leq \delta^{-2}\mathrm{var}(\hat{\sigma}^2|Z) = \delta^{-2}O_{\mathrm{P}}(n^{-1}) \xrightarrow{\mathrm{P}} 0.$$

Thus, by the dominated convergence theorem, we have

$$\mathrm{P}(|\hat{\sigma}^2 - \omega\sigma^2| > \delta) = \mathrm{E}\{\mathrm{P}(|\hat{\sigma}^2 - \omega\sigma^2| > \delta|Z)\} \to 0.$$

It follows, by the arbitrariness of δ, that

$$\hat{\sigma}^2 \quad \xrightarrow{\mathrm{P}} \quad \omega\sigma^2. \tag{8}$$

For $\hat{\tau}^2$, we have, by (1.2) of MS, (4), (8), we have

$$\begin{aligned} \hat{\tau}^2 &= n^{-1}\mathrm{E}(y'y|Z) - \hat{\sigma}^2 n^{-1}\mathrm{tr}(S_z) + \Delta \\ &= n^{-1}\mathrm{tr}(V_{y|Z}) - \hat{\sigma}^2 n^{-1}\mathrm{tr}(S_z) + \Delta \\ &= \tau^2 + (\omega\sigma^2 - \hat{\sigma}^2)n^{-1}\mathrm{tr}(S_z) + \Delta \\ &= \tau^2 + o_{\mathrm{P}}(1)\{1 + o_{\mathrm{P}}(1)\} + \Delta, \end{aligned} \tag{9}$$

where $\Delta = n^{-1}\{y'y - \mathrm{E}(y'y|Z)\}$. Furthermore, applying (6) to $A = I_n$, noting that $n^{-1}\mathrm{tr}(A^2) = 1 = \|A\|$, we have, by similar arguments, that

$$\mathrm{var}(y'y|Z) = nO_{\mathrm{P}}(1).$$

It follows that $\mathrm{var}(n^{-1}y'y|Z) = n^{-1}O_{\mathrm{P}}(1)$; hence, by similar arguments as above, we have $\Delta = o_{\mathrm{P}}(1)$. It follows by (9) that

$$\hat{\tau}^2 \quad \xrightarrow{\mathrm{P}} \quad \tau^2. \tag{10}$$

Part (II): For any constants a_0, a_1, it can be shown that

$$a_0\sqrt{n}(\hat{\tau}^2 - \tau^2) + a_1\sqrt{n}(\hat{\sigma}^2 - \omega\sigma^2) = \sum_{k=1}^{n+p} \eta_{nk}, \tag{11}$$

where $\eta_{nk} = \sqrt{n}[v_{kk}\{\xi_k^2 - \mathrm{E}(\xi_k^2)\} + 2(\sum_{l<k} v_{kl}\xi_l)\xi_k]$,

$$V = (v_{kl})_{1\leq k,l\leq n+p} = \begin{pmatrix} I_n \\ \tilde{Z}' \end{pmatrix} A(I_n\ \tilde{Z}), \quad \xi = \begin{pmatrix} \epsilon \\ \alpha \end{pmatrix},$$

$\tilde{Z} = p^{-1/2}Z$, $A = a_0A_0 + a_1A_1$ with $A_0 = n^{-1}[I_n - \{\mathrm{tr}(S_z)/\mathrm{tr}(S_{z,\mathrm{o}}^2)\}S_{z,o}]$, $A_1 = S_{z,\mathrm{o}}/\mathrm{tr}(S_{z,\mathrm{o}}^2)$. It follows that $\mathrm{E}(\xi_k^2) = \tau^2$, $1 \le k \le n$ and $\mathrm{E}(\xi_k^2) = \omega\sigma^2$, $n+1 \le k \le n+p$. Write $A = B + \Delta$, where

$$B = \frac{a_0}{n}I_n + \frac{a_1 - a_0}{\gamma n}S_{z,\mathrm{o}},$$
$$\Delta = \left[a_1\left\{\frac{n}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} - \frac{1}{\gamma}\right\} - a_0\left\{\frac{\mathrm{tr}(S_z)}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} - \frac{1}{\gamma}\right\}\right]\frac{S_{z,\mathrm{o}}}{n},$$

and let W, D be V with A replaced by B, Δ, respectively. Then, we have $\eta_{nk} = \zeta_{nk} + \delta_{nk}$, where ζ_{nk}, δ_{nk} are η_{nk} with V replaced by W, D, respectively.

By the RMT (e.g., Jiang 2022, sec. 16.2), it can be show that $\|\Delta\| = n^{-1}o_{\mathrm{P}}(1)$. Furthermore, it can be shown that $\mathrm{tr}(D^2) \le (1 + \|S_z\|)\|\Delta\|^2\{n + \mathrm{tr}(S_z)\} = n^{-1}o_{\mathrm{P}}(1)$. It follows that $\mathrm{E}\{(\sum_{k=1}^{n+p}\delta_{nk})^2|Z\} = \sum_{k=1}^{n+p}\mathrm{E}(\delta_{nk}^2|Z) =$

$$n\sum_{k=1}^{n+p}\left[d_{kk}^2\mathrm{var}(\xi_k^2) + 2\left\{\sum_{l<k}d_{kl}^2\mathrm{E}(\xi_l^2)\right\}\mathrm{E}(\xi_k^2)\right] \le cn\mathrm{tr}(D^2) \le o_{\mathrm{P}}(1).$$

Thus, by similar arguments as before, it can be shown that $\sum_{k=1}^{n+p}\delta_{nk} = o_{\mathrm{P}}(1)$. Therefore, we can focus on $\sum_{k=1}^{n+p}\zeta_{nk}$.

It is seen that $\zeta_{nk}, \mathcal{F}_{nk} = \sigma(Z, \xi_1, \dots, \xi_k), 1 \le k \le n + p$ is a triangular array of martingale differences. By the martingale central limit theorem (Hall & Heyde 1980, p. 58), we need to verify the following three conditions:

$$\max_{1\le k\le n+p}|\zeta_{nk}| \xrightarrow{\mathrm{P}} 0, \tag{12}$$
$$\sum_{k=1}^{n+p}\zeta_{nk}^2 \xrightarrow{\mathrm{P}} (a_0, a_1)\Sigma(a_0, a_1)', \tag{13}$$
$$\mathrm{E}\left(\max_{1\le k\le n+p}\zeta_{nk}^2\right) \text{ is bounded}, \tag{14}$$

and determine the constant 2×2 matrix Σ in (13).

(12): For any $K \ge 0$, we have

$$\begin{aligned}|w_{kk}\{\xi_k^2 - \mathrm{E}(\xi_k^2)\}| \;\le\; & \{K^2 \vee \tau^2 \vee (\omega\sigma^2)\}|w_{kk}| \\ & +|w_{kk}||\xi_k^2 - \mathrm{E}(\xi_k^2)|1_{(|\xi_k|>K)}\end{aligned}$$

$$\leq \{K^2 \vee \tau^2 \vee (\omega\sigma^2)\}\|W\| + \sqrt{\sum_{k=1}^{n+p} w_{kk}^2 |\xi_k^2 - \mathrm{E}(\xi_k^2)|^2 1_{(|\xi_k|>K)}}. \tag{15}$$

It can be shown that $\|W\| \leq 2(\|B\| \vee \|B\tilde{Z}\| \vee \|\tilde{Z}'B\tilde{Z}\|)$, and $\|B\| = n^{-1}O_{\mathrm{P}}(1)$ by RMT. It can then be shown that $\|W\| = n^{-1}O_{\mathrm{P}}(1)$. Also, we have

$$\sum_{k=1}^{n+p} w_{kk}^2 \{\xi_k^2 - \mathrm{E}(\xi_k^2)\} 1_{(|\xi_k|>K)} \leq \|W\|^2 \sum_{k=1}^{n+p} |\xi_k^2 - \mathrm{E}(\xi_k^2)|^2 1_{(|\xi_k|>K)}.$$

Combining the above results [note that (15) holds for every k], we have

$$\max_{1\leq k\leq n+p} \left[\sqrt{n}|w_{kk}\{\xi_k^2 - \mathrm{E}(\xi_k^2)\}|\right] \leq n\|W\| \left[\frac{K^2 \vee \tau^2 \vee (\omega\sigma^2)}{\sqrt{n}} + \sqrt{\frac{1}{n}\sum_{k=1}^{n+p} |\xi_k^2 - \mathrm{E}(\xi_k^2)|^2 1_{(|\xi_k|>K)}}\right]. \tag{16}$$

If we choose $K = n^{1/8}$, and denote the term under the square root in (16), with this choice of K, by u_n, then the right side of (16) is bounded by $O_{\mathrm{P}}(1)(n^{-1/4} + u_n^{1/2})$, when n is large. Furthermore, we have

$$\mathrm{E}(u_n) = \mathrm{E}\{|\epsilon_1^2 - \tau^2|^2 1_{(|\epsilon_1|>n^{1/8})}\} + \frac{p}{n}\mathrm{E}\{|\alpha_1 - \omega\sigma^2|^2 1_{(|\alpha_1|>n^{1/8})}\} = o(1),$$

using (1.4) of MS. Thus, the left side of (16) is bounded by $O_{\mathrm{P}}(1)\{n^{-1/4} + o_{\mathrm{P}}(1)\}$.

Next, by the Chebyshev, Marcinkiewicz–Zygmund, and Cauchy-Schwarz inequalities, we have, for any $\delta > 0$,

$$\begin{aligned}
&\mathrm{P}\left[\sqrt{n}\max_{1\leq k\leq n+p}\left|\left(\sum_{l<k} w_{kl}\xi_l\right)\xi_k\right| > \delta\middle| Z\right] \\
&\leq \sum_{k=1}^{n+p} \mathrm{P}\left[\left|\left(\sum_{l<k} w_{kl}\xi_l\right)\xi_k\right| > \frac{\delta}{\sqrt{n}}\middle| Z\right] \\
&\leq \frac{n^2}{\delta^4}\sum_{k=1}^{n+p} \mathrm{E}\left(\left|\sum_{l<k} w_{kl}\xi_l\right|^4\middle| Z\right)\mathrm{E}(\xi_k^4) \\
&\leq \frac{cn^2}{\delta^4}\sum_{k=1}^{n+p} \mathrm{E}\left\{\left(\sum_{l<k} w_{kl}^2\xi_l^2\right)^2\middle| Z\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{cn^2}{\delta^4}\sum_{k=1}^{n+p}\left(\sum_{l<k} w_{kl}^2\right)^2 \\
&\leq \frac{cn^2}{\delta^4}(n+p)\|W\|^4 = \frac{O_{\rm P}(1)}{n}. \qquad (17)
\end{aligned}
$$

Here in the last step we have used the fact that $\sum_{l<k} w_{kl}^2 \leq \sum_{l=1}^{n+p} w_{kl}^2 =$ the kth diagonal element of W^2, hence is bounded by $\|W\|^2 = n^{-2}O_{\rm P}(1)$, according to an earlier result, and (1.4) of MS. It follows, by similar arguments, that $\max_{1\leq k\leq n+p}\sqrt{n}|(\sum_{l<k} w_{kl}\xi_l)\xi_k| = o_{\rm P}(1)$. (12) has been verified.

(13): We can write

$$
\begin{aligned}
\sum_{k=1}^{n+p}\zeta_{nk}^2 &= n\sum_{k=1}^{n}\left\{w_{kk}(\epsilon_k^2-\tau^2) + 2\left(\sum_{l<k} w_{kl}\epsilon_l\right)\epsilon_k\right\}^2 \\
&\quad + n\sum_{k=n+1}^{n+p}\left\{w_{kk}(\xi_k^2-\omega\sigma^2) + 2\left(\sum_{l<k} w_{kl}\xi_l\right)\xi_k\right\}^2 \\
&= I_1 + I_2 \qquad (18)
\end{aligned}
$$

with I_1, I_2 defined in obvious ways. It can be shown that $\mathrm{E}(I_1|Z) = 2n\tau^4\|B\|_2^2$ and, by an earlier result, we have

$$
n\|B\|_2^2 = a_0^2 + (a_1-a_0)^2\frac{\mathrm{tr}(S_{z,\rm o}^2)}{\gamma^2 n} \xrightarrow{\rm P} a_0^2 + \frac{(a_1-a_0)^2}{\gamma}.
$$

Next, it can be shown that $\mathrm{E}(I_2|Z) =$

$$
n\left\{3\left(\frac{1}{\omega}-1\right)(\omega\sigma^2)^2\sum_{k=n+1}^{n+p} w_{kk}^2 + 4\omega\sigma^2\tau^2\|B\tilde{Z}\|_2^2 + 2(\omega\sigma^2)^2\|\tilde{Z}'B\tilde{Z}\|_2^2\right\}.
$$

Furthermore, it can be shown that $n\|B\tilde{Z}\|_2^2 =$

$$
a_0^2\frac{\mathrm{tr}(S_z)}{n} + 2\frac{a_0(a_1-a_0)}{\gamma}\cdot\frac{\mathrm{tr}(S_{z,\rm o}^2)}{n} + \frac{(a_1-a_0)^2}{\gamma^2}\cdot\frac{\mathrm{tr}(S_{z,\rm o}S_zS_{z,\rm o})}{n}.
$$

We have $n^{-1}\mathrm{tr}(S_z) \xrightarrow{\rm P} 1$, $n^{-1}\mathrm{tr}(S_{z,\rm o}^2) \xrightarrow{\rm P} \gamma$ by earlier results, and it can be shown that $n^{-1}\mathrm{tr}(S_{z,\rm o}S_zS_{z,\rm o}) \xrightarrow{\rm P} \gamma(\gamma+1)$. It follows that

$$
n\|B\tilde{Z}\|_2^2 \xrightarrow{\rm P} a_0^2 + 2a_0(a_1-a_0) + \frac{\gamma+1}{\gamma}(a_1-a_0)^2 = a_1^2 + \frac{(a_1-a_0)^2}{\gamma}.
$$

Similarly, it can be shown that $n\|\tilde{Z}'B\tilde{Z}\|_2^2 =$

$$
\begin{aligned}
& a_0^2\frac{\mathrm{tr}(S_z^2)}{n} + 2\frac{a_0(a_1-a_0)}{\gamma}\cdot\frac{\mathrm{tr}(S_zS_{z,\mathrm{o}}S_z)}{n} \\
& +\frac{(a_1-a_0)^2}{\gamma^2}\cdot\frac{\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}}S_z)}{n} \\
\xrightarrow{\mathrm{P}} \quad & a_0^2(1+\gamma)+2a_0(a_1-a_0)(\gamma+2)+(a_1-a_0)^2(\gamma+4+\gamma^{-2}) \\
= \quad & \gamma a_1^2+(2a_1-a_0)^2+\frac{(a_1-a_0)^2}{\gamma}.
\end{aligned}
$$

Finally, it can be shown that $n\sum_{k=n+1}^{n+p} w_{kk}^2 = np^{-2}\sum_{j=1}^p(Z_j'BZ_j)^2$, where Z_j is the jth column of Z. Furthermore, it can be shown that

$$
Z_j'BZ_j = \frac{d}{n}Z_j'Z_j + \frac{a_1-a_0}{\gamma n}\left\{\mathrm{tr}(S_z)+\frac{(Z_j'Z_j)^2-Z_j'Z_j}{p}\right\}+\Delta_j = Q_j+\Delta_j
$$

with Q_j defined in an obvious way, where $d = a_0-\gamma^{-1}(a_1-a_0)$ and

$$
\Delta_j = \frac{1}{n}Z_j'(D_z-dI_n)Z_j + \frac{a_1-a_0}{\gamma n}\{Z_j'S_{z,-j}Z_j-\mathrm{tr}(S_{z,-j})\}
$$

with $D_z = a_0I_n-\gamma^{-1}(a_1-a_0)dS_z$ and $S_{z,-j} = S_z-p^{-1}Z_jZ_j'$. According to earlier results, we have

$$
\|D_z-dI_n\| = \frac{|a_1-a_0|}{\gamma}\max_{1\le i\le n}|s_{z,i,i}-1| = o_{\mathrm{P}}(1).
$$

Also, by the Hanson-Wright inequality (e.g., Jiang 2022, Lemma 5.4), we have, for $t_j = K^2\max\{\sqrt{2\log p/c}\|S_{z,-j}\|_2, (2\log p/c)\|S_{z,-j}\|\}$,

$$
\begin{aligned}
& \mathrm{P}[|Z_j'S_{z,-j}Z_j-\mathrm{tr}(S_{z,-j})|>t_j|Z_{-j}] \\
& \le 2\exp\left\{-c\min\left(\frac{t_j^2}{K^4\|S_{z,-j}\|_2^2},\frac{t_j}{K^2\|S_{z,-j}\|}\right)\right\}\le\frac{2}{p^2},
\end{aligned}
$$

$1\le j\le p$. It follows that

$$
\begin{aligned}
& \mathrm{P}\left[\max_{1\le j\le p}t_j^{-1}|Z_j'S_{z,-j}Z_j-\mathrm{tr}(S_{z,-j})|>1\right] \\
& \le\sum_{j=1}^p\mathrm{P}[|Z_j'S_{z,-j}Z_j-\mathrm{tr}(S_{z,-j})|>t_j] \\
& =\sum_{j=1}^p\mathrm{E}(\mathrm{P}[|Z_j'S_{z,-j}Z_j-\mathrm{tr}(S_{z,-j})|>t_j|Z_{-j}])\le\frac{2}{p}.
\end{aligned}
$$

Thus, we have $|Z_j' S_{z,-j} Z_j - \mathrm{tr}(S_{z,-j})| = O_{\mathrm{P}}(1) t_j$, where the $O_{\mathrm{P}}(1)$ does not depend on j. Furthermore, it can be shown that

$$t_j \leq \sqrt{n \log p} O_{\mathrm{P}}(1) + O(\log p/p) Z_j' Z_j,$$

$\leq j \leq p$. Therefore, we have

$$\begin{aligned} \frac{n}{p^2} \sum_{j=1}^{p} \{(Z_j' B Z_j)^2 - Q_j^2\} &= \frac{n}{p^2} \sum_{j=1}^{p} (2 Q_j \Delta_j + \Delta_j^2) \\ &= \frac{2n}{p^2} \sum_{j=1}^{p} Q_j \Delta_j + \frac{n}{p^2} \sum_{j=1}^{p} \Delta_j^2, \end{aligned}$$

and $|\Delta_j| \leq n^{-1} \{o_{\mathrm{P}}(1) Z_j' Z_j + O_{\mathrm{P}}(1) t_j\}$. It follows that

$$\begin{aligned} \frac{n}{p^2} \sum_{j=1}^{p} \Delta_j^2 &\leq \frac{o_{\mathrm{P}}(1)}{np^2} \sum_{j=1}^{p} (Z_j' Z_j)^2 \\ &\quad + \frac{O_{\mathrm{P}}(1)}{np^2} \sum_{j=1}^{p} \left\{ O_{\mathrm{P}}(1) n \log p + O(1) (\log p/p)^2 (Z_j' Z_j)^2 \right\} \\ &= \frac{o_{\mathrm{P}}(1)}{np^2} \sum_{j=1}^{p} (Z_j' Z_j)^2 + \frac{\log p}{p} O_{\mathrm{P}}(1). \end{aligned}$$

Furthermore, it is easy to show that

$$\mathrm{E} \left\{ \sum_{j=1}^{p} (Z_j' Z_j)^2 \right\} \leq 3 n^2 p.$$

It follows that $np^{-2} \sum_{j=1}^{p} \Delta_j^2 = o_{\mathrm{P}}(1)$. Also, we have

$$\begin{aligned} \frac{n}{p^2} \sum_{j=1}^{p} |Q_j \Delta_j| &\leq \left(\frac{n}{p^2} \sum_{j=1}^{p} Q_j^2 \right)^{1/2} \left(\frac{n}{p^2} \sum_{j=1}^{p} \Delta_j^2 \right)^{1/2} \\ &= O_{\mathrm{P}}(1) o_{\mathrm{P}}(1), \end{aligned}$$

where the $O_{\mathrm{P}}(1)$ follows from the result below. Write

$$U_j = (Z_j' Z_j / n) \{ d + (a_1 - a_0) (\gamma p)^{-1} (Z_j' Z_j - 1) \}.$$

Then, we have

$$\begin{aligned} \frac{n}{p^2} \sum_{j=1}^{p} Q_j^2 &= \left(\frac{a_1 - a_0}{\gamma} \right)^2 \left\{ \frac{\mathrm{tr}(S_z)}{n} \right\}^2 \left(\frac{n}{p} \right) \\ &\quad + 2 \left(\frac{a_1 - a_0}{\gamma} \right) \left\{ \frac{\mathrm{tr}(S_z)}{n} \right\} \left(\frac{n}{p} \right) \bar{U} + \left(\frac{n}{p} \right) \overline{U^2}, \end{aligned} \qquad (19)$$

where $\bar{U} = p^{-1}\sum_{j=1}^{p} U_j$ and $\overline{U^2} = p^{-2}\sum_{j=1}^{p} U_j^2$. By the RMT, and (1.4) of MS, the first term on the right side of (19) converges in probability to $(a_1 - a_0)^2/\gamma$. Also, it is easy to see that $U_1, \dots, U_p$ are i.i.d. with $\mathrm{E}(U_1^4)$ bounded. It is then easy to show that $\bar{U} - \mathrm{E}(U_1) \xrightarrow{\mathrm{P}} 0$ and $\overline{U^2} - \mathrm{E}(U_1^2) \xrightarrow{\mathrm{P}} 0$. Finally, it can be shown that $\mathrm{E}(U_1^s) \to (d + a_1 - a_0)^s, s = 1, 2$. It then follows that

$$\begin{aligned} \frac{n}{p^2}\sum_{j=1}^{p} Q_j^2 \quad &\xrightarrow{\mathrm{P}} \quad \frac{(a_1 - a_0)^2}{\gamma} \\ &\quad +2(a_1 - a_0)(d + a_1 - a_0) + \gamma(d + a_1 - a_0)^2 \\ &= \quad \gamma a_1^2. \end{aligned}$$

Combining the above results and after some algebraic manipulations, we get

$$\begin{aligned} \sum_{k=1}^{n+p} \mathrm{E}(\zeta_{nk}^2|Z) \quad &\xrightarrow{\mathrm{P}} \quad 2\tau^4\left\{a_0^2 + \frac{(a_1 - a_0)^2}{\gamma}\right\} \\ &\quad +3\left(\frac{1}{\omega} - 1\right)(\omega\sigma^2)^2\gamma a_1^2 \\ &\quad +4\omega\sigma^2\tau^2\left\{a_1^2 + \frac{(a_1 - a_0)^2}{\gamma}\right\} \\ &\quad +2(\omega\sigma^2)^2\left\{\gamma a_1^2 + (2a_1 - a_0)^2 + \frac{(a_1 - a_0)^2}{\gamma}\right\} \\ &= \quad (a_0, a_1)\Sigma(a_0, a_1)', \end{aligned} \tag{20}$$

where the Σ is the same as in (2.13) of MS.

Next, we can write $\sum_{k=1}^{n+p} \zeta_{nk}^2 - \sum_{k=1}^{n+p} \mathrm{E}(\zeta_{nk}^2|Z) =$

$$\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\} + \sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\}.$$

Also, we have $\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) = n\{\mathrm{var}(\xi_k^2)w_{kk}^2 + 4\mathrm{E}(\xi_k^2)(\sum_{l<k} w_{kl}\xi_l)^2\}$. Thus, we have

$$\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z) = 4n\mathrm{E}(\xi_k^2)\left[\left(\sum_{l<k} w_{kl}\xi_l\right)^2 - \mathrm{E}\left\{\left(\sum_{l<k} w_{kl}\xi_l\right)^2 \middle| Z\right\}\right].$$

It can then be shown that $\sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\} = \xi' A\xi - \mathrm{tr}(AD)$, where $A = 4nW_{\mathrm{L}}'DW_{\mathrm{L}}$ and $D = \mathrm{diag}(\mathrm{E}(\xi_k^2), 1 \le k \le n+p)$. By Jiang *et al.* (2023; Lemma

1 in the supplement), we have the following result:

$$\xi' A\xi - \mathrm{tr}(AD) \xrightarrow{\mathrm{P}} 0 \ \text{ provided that } \mathrm{tr}(ADAD) \xrightarrow{\mathrm{P}} 0.$$

Note that $\mathrm{tr}(ADAD) = 16n^2\mathrm{tr}(W_{\mathrm{L}}'DW_{\mathrm{L}}DW_{\mathrm{L}}'DW_{\mathrm{L}}D) \leq 16\lambda_{\max}^4(D)\|W_{\mathrm{L}}'W_{\mathrm{L}}\|_2^2$. We now apply the following matrix inequality (see Lemma 5.3 of Jiang 2022):

$$\|W_{\mathrm{L}}'W_{\mathrm{L}}\|_2^2 \leq 2\|W_{\mathrm{L}}' + W_{\mathrm{L}}\|^2\|W_{\mathrm{L}}\|_2^2.$$

Note that $\|W_{\mathrm{L}}' + W_{\mathrm{L}}\| = \|W - dW\| \leq 2\|W\|$ and, by an earlier result, $\|W\| = n^{-1}O_{\mathrm{P}}(1)$. Also, again by an earlier result, it can be shown that $\|W_{\mathrm{L}}\|_2^2 \leq (1/2)\|W\|_2^2 = n^{-1}O_{\mathrm{P}}(1)$. It follows that $\mathrm{tr}(ADAD) \leq n^{-1}O_{\mathrm{P}}(1)$, hence $\sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\} \xrightarrow{\mathrm{P}} 0$. On the other hand, we have

$$\begin{aligned}
&\mathrm{E}\left(\left[\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\}\right]^2 \middle| Z\right) = \sum_{k=1}^{n+p}\mathrm{E}[\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\}^2|Z] \\
&= \sum_{k=1}^{n+p}\mathrm{E}\{\mathrm{var}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})|Z\} \leq \sum_{k=1}^{n+p}(\zeta_{nk}^4|Z) \leq cn^2\sum_{k=1}^{n+p}\left\{w_{kk}^4 + \left(\sum_{l<k}w_{kl}^2\right)^2\right\} \\
&\leq cn^2\|W\|^2\|W\|_2^2 = n^{-1}O_{\mathrm{P}}(1),
\end{aligned}$$

by the earlier results. It can then be shown, by the dominated convergence theorem that

$$\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\} \xrightarrow{\mathrm{P}} 0. \tag{21}$$

Combining (20) and (21), (13) has been verified with the limit given by the right side of (20), and (2.14)–(2.16) of MS.

(14): According to earlier results, we have

$$\mathrm{E}\left(\max_{1\leq k\leq n+p}\zeta_{nk}^2\right) \leq \sum_{k=1}^{n+p}\mathrm{E}(\zeta_{nk}^2) \leq cn\mathrm{E}(\|W\|_2^2), \tag{22}$$

and $\|W\|_2^2 = \mathrm{tr}[B(I_n + S_z)B(I_n + S_z)] = \mathrm{tr}(B^2) + 2\mathrm{tr}(BS_zB) + \mathrm{tr}(BS_zBS_z)$. First, we have $\mathrm{tr}(B^2) = a_0^2n^{-1} + \{(a_1 - a_0)/\gamma\}^2n^{-2}\mathrm{tr}(S_{z,\mathrm{o}}^2)$. It is easy to show that $\mathrm{E}(s_{z,i_1,i_2}^2) = p^{-1}$ if $i_1 \neq i_2$, and $\mathrm{E}(s_{z,i,i}^2) \leq 3$. Thus, we have $\mathrm{E}\{\mathrm{tr}(S_{z,\mathrm{o}}^2)\} = \sum_{i_1\neq i_2}\mathrm{E}(s_{z,i_1,i_2}^2) = n(n-1)/p$. It follows that $\mathrm{E}\{\mathrm{tr}(B^2)\} = O(n^{-1})$.

Next, using the above result, it can be shown that

$$\begin{aligned}\mathrm{E}\{\mathrm{tr}(BS_zB)\} &= \frac{a_0^2}{n} + 2a_0\left(\frac{a_1-a_0}{\gamma}\right)\frac{n-1}{np} \\ &\quad + \frac{(a_1-a_0)^2}{\gamma^2 n^2}\mathrm{E}\{\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}})\}.\end{aligned}$$

Furthermore, the following expression can be derived:

$$\mathrm{E}\{\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}})\} = \sum_{i_2,i_3}\sum_{i_1\notin\{i_2,i_3\}}\mathrm{E}(s_{z,i_1,i_2}s_{z,i_2,i_3}s_{z,i_3,i_1}).$$

If $i_1 \notin \{i_2, i_3\}$, we have

$$\mathrm{E}(s_{z,i_1,i_2}s_{z,i_2,i_3}s_{z,i_3,i_1}) = \mathrm{E}\{s_{z,i_2,i_3}\mathrm{E}(s_{z,i_1,i_2}s_{z,i_1,i_3}|z_{i_2}, z_{i_3})\},$$

where z_i denotes the ith row of Z. Furthermore, we have

$$\mathrm{E}(s_{z,i_1,i_2}s_{z,i_1,i_3}|z_{i_2}, z_{i_3}) = \frac{1}{p^2}\sum_{j=1}^{p} z_{i_2j}z_{i_3j} = \frac{s_{z,i_2,i_3}}{p}.$$

It follows, by the results mentioned above, that

$$\begin{aligned}\mathrm{E}\{\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}})\} &= \frac{1}{p}\sum_{i_2,i_3}\sum_{i_1\notin\{i_2,i_3\}}\mathrm{E}(s_{z,i_2,i_3}^2) \\ &\leq \frac{1}{p}\left(\sum_{i_2\neq i_3}\sum_{i_1\notin\{i_2,i_3\}}\frac{1}{p} + \sum_{i_2}\sum_{i_1\neq i_2} 3\right) \\ &= \frac{n(n-1)(n-2)}{p^2} + \frac{3n(n-1)}{p}.\end{aligned}$$

Thus, combining the above results, we have $\mathrm{E}\{\mathrm{tr}(BS_zB))\} = O(n^{-1})$.

Finally, we have the following expression:

$$\begin{aligned}\mathrm{tr}(BS_zBS_z) &= \frac{a_0^2}{n^2}\mathrm{tr}(S_z^2) + 2\frac{a_0(a_1-a_0)}{\gamma n^2}\mathrm{tr}(S_zS_{z,\mathrm{o}}S_z) \\ &\quad + \frac{(a_1-a_0)^2}{\gamma^2n^2}\mathrm{tr}(S_zS_{z,\mathrm{o}}S_zS_{z,\mathrm{o}}).\end{aligned}$$

Using the above results, and similar arguments, it can be shown that

$$\begin{aligned}&\mathrm{E}\{\mathrm{tr}(S_z^2)\} \leq n(n-1)p^{-1} + 3n, \\ &\mathrm{E}\{\mathrm{tr}(S_zS_{z,\mathrm{o}}S_z)\} = \frac{n^2(n-1)}{p^2}, \\ &\mathrm{E}\{\mathrm{tr}(S_zS_{z,\mathrm{o}}S_zS_{z,\mathrm{o}})\} \leq c\left(\frac{n^4}{p^3} + \frac{n^3}{p^2}\right)\end{aligned}$$

for some constant c. It follows that $\mathrm{E}\{\mathrm{tr}(BS_zBS_z)\} = O(n^{-1})$.

Combining the above results, (14) has been verified.

2 Proof of Theorem 4.1

The main task is dealing with M. The conditional mean of M given Z is given by (3.24) of MS. Also, note that we can write

$$Z_j'y = Z_j'(\tilde{Z}\alpha + \epsilon) = Z_j'W\xi,$$

where $G = (I_n \ \tilde{Z})$ and ξ is as below (11). Define $W = (w_{jk})_{1\leq j\leq p, 1\leq k\leq n+p}$ as the matrix whose jth row is $W_j' = Z_j'G$, $1 \leq j \leq p$. Then, we have the expression $Z_j'y = W_j'\xi = \sum_{k=1}^{n+p} w_{jk}\xi_k$; hence, we can write

$$\begin{aligned} M &= \frac{1}{3n^2p}\sum_{j=1}^{p}(Z_j'y)^4 \\ &= \frac{1}{3n^2p}\sum_{k_1,k_2,k_3,k_4=1}^{n+p}\left(\sum_{j=1}^{p} w_{jk_1}w_{jk_2}w_{jk_3}w_{jk_3}\right)\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}. \end{aligned} \quad (23)$$

Using the notation $K = (k_1, k_2, k_2, k_4)$,$L = (l_1, l_2, l_3, l_4)$, $w_{jK} = w_{jk_1}w_{jk_2}w_{jk_3}w_{jk_4}$, $w_K = \sum_{j=1}^{p} w_{jK}$, and $\xi_K = \xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}$, we can write

$$\mathrm{var}(M|Z) = \frac{1}{9n^4p^2}\sum_{K,L} w_K w_L c_{KL}, \quad (24)$$

where $c_{KL} = \mathrm{cov}(\xi_K, \xi_L|Z)$. Furthermore, for fixed K, L, one can write $\xi_L\xi_L = ABC$, where A, B, C note the products of the ξ's that appear in both ξ_K and ξ_L, in ξ_K only, and in ξ_L only, respectively. We can further write

$$A = \xi_{a_1}^{r_1}\cdots\xi_{a_f}^{r_f}, \ \ B = \xi_{b_1}^{s_1}\cdots\xi_{b_g}^{s_g}, \ \ C = \xi_{c_1}^{t_1}\cdots\xi_{c_h}^{t_h},$$

where $a_1, \dots, a_f$, $b_1, \dots, b_g$, $c_1, \dots, c_h$ are distinct indexes; as long as f, g, h are positive, $r_1, \dots, r_f$, $s_1, \dots, s_g$, $t_1, \dots, t_h$ are positive integers; in case f, g, or h is 0, it means that the corresponding factor, A, B, or C, does not exist.

First note that for c_{KL} to be non-zero, all of the $r_1, \dots, t_h$ must be even numbers. The reason is simple, suppose that one of those numbers, denoted by d, is odd, which corresponds to the factor ξ_k^d. Then, we have

$$\mathrm{E}(\xi_K \xi_L) = \mathrm{E}(\xi_k^d)\mathrm{E}(\cdots) = 0.$$

On the other hand, if ξ_k^d belongs entirely to ξ_K, or ξ_L, then for a similar reason we have $\mathrm{E}(\xi_K)\mathrm{E}(\xi_L) = 0$. If ξ_k^d is split into $\xi_k^{d_1}\xi_k^{d_2}$ such that $d_1 + d_2 = d$, and $\xi_k^{d_1}, \xi_k^{d_2}$ belongs to ξ_K, ξ_L, respectively, then at least one of the d_1, d_2 is odd. Thus, for a similar reason, we have $\mathrm{E}(\xi_K)\mathrm{E}(\xi_L) = 0$. It follows that

$$c_{KL} = \mathrm{E}(\xi_K \xi_L) - \mathrm{E}(\xi_K)\mathrm{E}(\xi_L) = 0.$$

Below we check all the cases and sub-cases for the zero status of c_{KL}.

(0) If $f = 0$, that is, no overlap between ξ_K and ξ_L, we have $c_{KL} = 0$.

(1) If $f = 1$, there are 4 cases: (I) $r_1 = 2$. Then, we have $\mathrm{E}(\xi_K) = \mathrm{E}(\xi_L) = 0$, hence $c_{KL} = \mathrm{E}(\xi_K \xi_L) = \mathrm{E}(\xi_{a_1}^2)\mathrm{E}(B)\mathrm{E}(C) = 0$, because B is the product of $\xi_{k_1}\xi_{k_2}\xi_{k_3}$, whose expectation is zero no matter what k_1, k_2, k_3; similarly, $\mathrm{E}(C) = 0$. (II) $r_1 = 4$. Then, there are three ways to split the 4 between ξ_K and ξ_L: 1-3, 2-2, 3-1. If 1-3, we have $\mathrm{E}(\xi_K) = 0$, hence $c_{KL} = \mathrm{E}(\xi_{a_1}^4)\mathrm{E}(B)\mathrm{E}(C) = 0$, because $\mathrm{E}(B) = 0$ for the reason above. Similarly, we have $c_{KL} = 0$ if 3-1. If 2-2, the only non-zero scenario is $g = h = 1$, $s_1 = t_1 = 2$. These are cases such as $k_1 = k_2 = l_1 = l_2 = a_1$, $k_3 = k_4 = b_1$, and $l_3 = l_4 = c_1$. (III) $r_1 = 6$. Similarly, the only non-zero scenarios are 2-4 and 4-2; the former corresponds to cases like $k_1 = k_2 = l_1 = l_2 = l_3 = l_4 = a_1$ and $k_3 = k_4 = b_1$; the latter corresponds to cases like $k_1 = k_2 = k_3 = k_4 = l_1 = l_2 = a_1$ and $l_3 = l_4 = b_1$. (IV) $r_1 = 8$. This corresponds to the case $k_1 = \cdots = l_4 = a_1$.

(2) If $f = 2$, the only non-zero scenarios are (I) $\max(r_1, r_2) = 2$, which corresponds to cases like $k_1 = l_1 = a_1$, $k_2 = l_2 = a_2$, $k_3 = k_4 = b_1$, and $l_3 = l_4 = c_1$. (II) $\max(r_1, r_2) = 4$, which corresponds to cases like $k_1 = k_2 = l_1 = l_2 = a_1$ and $k_3 = k_4 = l_3 = l_4 = a_2$.

(3) If $f = 3$, the only non-zero scenario is (I) $\max(r_1, r_2, r_3) = 4$, corresponding to cases like $k_1 = l_1 = a_1$, $k_2 = l_2 = a_2$, and $k_3 = k_4 = l_3 = l_4 = a_3$.

(4) If $f = 4$, the only non-zero scenario is (I) $r_1 = r_2 = r_3 = r_4 = 2$, corresponding to cases like $k_1 = l_1 = a_1$, $k_2 = l_2 = a_2$, $a_3 = l_3 = a_3$, and $k_4 = l_4 = a_4$.

Thus, focusing on the nonzero cases, the summation on the right side of (24) is bounded by the sum of $w_K w_L c_{KL}$ over the seven cases below.

Let c denote a generic constant, whose value may be different at different places.

Case 1: 1-II-2-2. It can be shown that

$$\begin{aligned}
&\sum_{1-\mathrm{II}-2-2} w_K w_L c_{KL} \\
&\leq c \sum_{a_1,b_1,c_1=1}^{n+p} \left(\sum_{j=1}^{p} w_{ja_1}^2 w_{jb_1}^2 \right) \left(\sum_{j=1}^{p} w_{ja_1}^2 w_{jc_1}^2 \right) \\
&= \sum_{j_1,j_2=1}^{p} \left(\sum_{a_1=1}^{n+p} w_{j_1a_1}^2 w_{j_2a_1}^2 \right) \left(\sum_{b_1=1}^{n+p} w_{j_1b_1}^2 \right) \left(\sum_{c_1=1}^{n+p} w_{j_2c_1}^2 \right) \\
&\leq \|W\|^4 \sum_{a_1=1}^{n+p} \left(\sum_{j_1=1}^{p} w_{j_1a_1}^2 \right) \left(\sum_{j_2=1}^{p} w_{j_2a_1}^2 \right) \\
&\leq \|W\|^8 (n+p).
\end{aligned}$$

It can be shown, by the RMT, that $\|W\|^2 = pO_{\mathrm{P}}(1)$. It follows that

$$\sum_{1-\mathrm{II}-2-2} w_K w_L c_{KL} \leq p^4 (n+p) O_{\mathrm{P}}(1). \tag{25}$$

Case 2: 1-III-2-4. By similar arguments, it can be shown that

$$\sum_{1-\mathrm{III}-2-4} w_K w_L c_{KL} \leq p^5 O_{\mathrm{P}}(1). \tag{26}$$

Case 3: 1-III-4-2: Similarly, we have

$$\sum_{1-\mathrm{III}-4-2} w_K w_L c_{KL} \leq p^5 O_{\mathrm{P}}(1). \tag{27}$$

Case 4: 2-I. It can be shown that

$$\sum_{2-\mathrm{I}} w_K w_L c_{KL}$$

$$\leq c \sum_{a_1,a_2,b_1,c_1=1}^{n+p} \left(\sum_{j=1}^{p} w_{ja_1} w_{ja_2} w_{jb_1}^2 \right) \left(\sum_{j=1}^{p} w_{ja_1} w_{ja_2} w_{jc_1}^2 \right)$$
$$= c \sum_{j_1,j_2=1}^{p} (W_{j_1}' W_{j_2})^2 \left(\sum_{b_1=1}^{n+p} w_{j_1 b_1}^2 \right) \left(\sum_{c_1=1}^{n+p} w_{j_2 c_1}^2 \right)$$
$$\leq c \|W\|^4 \sum_{j_1,j_2=1}^{p} (W_{j_1}' W_{j_2})^2.$$

Furthermore, it can be shown that $\mathrm{E}(W_{j_1}' W_{j_2})^2 \leq cn$ if $j_1 \neq j_2$. Thus, we have

$$\mathrm{E} \left\{ \sum_{j_1 \neq j_2} (W_{j_1}' W_{j_2})^2 \right\} \leq cnp(p-1),$$

hence $\sum_{j_1 \neq j_2} (W_{j_1}' W_{j_2})^2 = np^2 O_{\mathrm{P}}(1)$. Also, we have $\sum_{j=1}^{p} (W_j' W_j)^2 \leq \|W\|^4 p = p^3 O_{\mathrm{P}}(1)$. It follows that

$$\sum_{2-\mathrm{I}} w_K w_L c_{KL} \leq np^4 O_{\mathrm{P}}(1) + p^5 O_{\mathrm{P}}(1). \tag{28}$$

Case 5: 2-II-4-4. It can be shown that $\sum_{2-\mathrm{II}-4-4} w_K w_L c_{KL} \leq$

$$c \sum_{a_1,a_2=1}^{n+p} \left(\sum_{j=1}^{p} w_{ja_1}^2 w_{ja_2}^2 \right)^2 = c \sum_{j_1,j_2=1}^{p} \left\{ \sum_{k=1}^{n+p} (w_{j_1 k} w_{j_2 k})^2 \right\}^2.$$

First assume that $j_1 \neq j_2$. It can be seen that $w_{ji} = z_{ij}, 1 \leq i \leq n$ and $w_{j\underline{n+k}} = Z_j' Z_k / \sqrt{p}, 1 \leq k \leq p$. It follows that

$$\sum_{k=1}^{n+p} (w_{j_1 k} w_{j_2 k})^2 = \sum_{i=1}^{n} z_{ij_1}^2 z_{ij_2}^2 + p^{-1} \sum_{k=1}^{p} (Z_{j_1}' Z_k)^2 (Z_{j_2}' Z_k)^2 = I_1 + I_2,$$

I_1, I_2 defined in obvious ways. It can be shown that $\mathrm{E}(I_1^2) \leq cn^2$. As for I_2, define $u_{jk} = Z_j' Z_k / \sqrt{p}$. It can be shown that $\mathrm{E}(\sum_{k \notin \{j_1, j_2} u_{j_1 k}^2 u_{j_2 k}^2)^2 \leq cp^2$, $\mathrm{E}(u_{j_1 j_1}^4 u_{j_2 j_1}^4) \leq cn^2$, and $\mathrm{E}(u_{j_1 j_2}^4 u_{j_2 j_2}^4) \leq cn^2$. It follows that $\mathrm{E}\{\sum_{k=1}^{n+p} (w_{j_1 k} w_{j_2 k})^2\}^2 \leq c(n^2 + p^2)$. It then follows that $\sum_{j_1 \neq j_2} \{\sum_{k=1}^{n+p} (w_{j_1 k} w_{j_2 k})^2\}^2 = n^2 p^2 O_{\mathrm{P}}(1) + p^4 O_{\mathrm{P}}(1)$. On the other hand, by similar arguments as in, say, Case 1, it can be shown that $\sum_{j=1}^{p} (\sum_{k=1}^{n+p} w_{jk}^4)^2 \leq \|W\|^8 p = p^5 O_{\mathrm{P}}(1)$. It follows that

$$\sum_{2-\mathrm{II}-4-4} w_K w_L c_{KL} \leq p^5 O_{\mathrm{P}}(1). \tag{29}$$

Case 6: 3-I. It can be shown that $\sum_{3-\mathrm{I}} w_K w_L c_{KL} \leq$

$$c \sum_{a_1,a_2,a_3=1}^{n+p} \left(\sum_{j=1}^{p} w_{ja_1} w_{ja_2} w_{ja_3}^2 \right)^2 = c \sum_{j_1,j_2=1}^{p} (W_{j_1}' W_{j_2})^2 \sum_{k=1}^{n+p} w_{j_1 k}^2 w_{j_2 k}^2.$$

It can be shown that, for $j_1 \neq j_2$, we have $\mathrm{E}(W_{j_1}' W_{j_2})^4 \leq cn^3$. Thus, by the Cauchy-Schwarz inequality and a previous result, we have

$$\mathrm{E}\left\{ (W_{j_1}' W_{j_2})^2 \sum_{k=1}^{n+p} w_{j_1 k}^2 w_{j_2 k}^2 \right\} \leq (cn^3)^{1/2} \{c(n^2+p^2)\}^{1/2} \leq cn^{3/2}(n+p).$$

On the other hand, by earlier results, we have

$$\sum_{j=1}^{p} (W_j' W_j)^2 \sum_{k=1}^{n+p} w_{jk}^4 \leq O_{\mathrm{P}}(1) \|W\|^4 \sum_{j=1}^{p} (Z_j' Z_j)^2 = n^2 p^3 O_{\mathrm{P}}(1),$$

because $\mathrm{E}\{\sum_{j=1}^{p} (Z_j' Z_j)^2\} = n(n+2)p$. It follows that

$$\sum_{3-\mathrm{I}} w_K w_L c_{KL} \leq n^{3/2} p^2 (n+p) O_{\mathrm{P}}(1) + n^2 p^3 O_{\mathrm{P}}(1). \tag{30}$$

Case 7: 4-I. It can be shown that $\sum_{4-\mathrm{I}} w_K w_L c_{KL} \leq$

$$c \sum_{a_1,a_2,a_3,a_4=1}^{n+p} \left(\sum_{j=1}^{p} w_{ja_1} w_{ja_2} w_{ja_3} w_{ja_4} \right)^2 = c \sum_{j_1,j_2=1}^{p} (W_{j_1}' W_{j_4})^4.$$

By a result mentioned above, we have $\mathrm{E}\{\sum_{j_1 \neq j_2} (W_{j_1}' W_{j_2})^4\} \leq cn^3 p^2$. Also, we have $\sum_{j=1}^{p} (W_j' W_j)^4 \leq O_{\mathrm{P}}(1) \sum_{j=1}^{p} (Z_j' Z_j)^4 = n^4 p O_{\mathrm{P}}(1)$, using the fact that $Z_j' Z_j \sim \chi_n^2$, hence $\mathrm{E}(Z_j' Z_j)^4 = n(n+2)(n+4)(n+6)$. It follows that

$$\sum_{4-\mathrm{I}} w_K w_L c_{KL} \leq n^3 p^2 O_{\mathrm{P}}(1) + n^4 p O_{\mathrm{P}}(1). \tag{31}$$

Combining (24) and the subsequent arguments (up to the paragraph above Case 1), and (25)–(31), we have proved the following:

$$\mathrm{var}(M|Z) \;\; = \;\; n^{-1} O_{\mathrm{P}}(1). \tag{32}$$

Now consider $\eta = \mathrm{E}(M|Z)$, with the expression (3.24) of MS. First, we have $\mathrm{E}(T_1) = 1 + 2/n$, and, by independence and the fact that $Z_j' Z_j \sim \chi_n^2$, we have

$$\mathrm{var}(T_1) = \frac{1}{n^4 p^2} \sum_{j=1}^{p} \mathrm{var}\{(Z_j' Z_j)^2\} = \frac{8}{n^3}(n+2)(n+3). \tag{33}$$

It follows that $T_1 \stackrel{\mathrm{P}}{\longrightarrow} 1$. Next, we can write $T_2 =$

$$\frac{1}{np^2}\sum_{j=1}^{p} Z_j' Z Z' Z_j + \frac{1}{n^2p^2}(Z_j'Z_j - n)Z_j'ZZ'Z_j = \frac{1}{n}\mathrm{tr}(S_z^2) + R_2,$$

where $R_2 = (np^2)^{-1}\sum_{j=1}^{p}\Delta_j Z_j'ZZ'Z_j$ with $\Delta_j = n^{-1}Z_j'Z_j - 1$. By the RMT (e.g., Corollary 16.2 of Jiang 2022), we have $n^{-1}\mathrm{tr}(S_z^2) \stackrel{\mathrm{P}}{\longrightarrow} \gamma + 1$. On the other hand, we have $|R_2| \le (\max_{1\le j\le p}|\Delta_j|)n^{-1}\mathrm{tr}(S_z^2)$. By similar arguments as, for example, those showing (2), we have $\max_{1\le j\le p}|\Delta_j| = o_{\mathrm{P}}(1)$. It follows that $T_2 \stackrel{\mathrm{P}}{\longrightarrow} \gamma + 1$. Next, we can write

$$T_4 = \frac{1}{n^2p^3}\sum_{j=1}^{p}(Z_j'Z_j)^4 + \frac{1}{n^2p^3}\sum_{j\ne k}(Z_j'Z_k)^4 = I_1 + I_2,$$

with I_1, I_2 defined in obvious ways. We have

$$\mathrm{E}(I_1) = (n+2)(n+4)(n+6)/(np^2) \longrightarrow \gamma^2,$$
$$\mathrm{var}(I_1) = \frac{1}{n^4p^6}\sum_{j=1}^{p}\mathrm{var}\{(Z_j'Z_j)^4\} \le \frac{cn^7p}{n^4p^6} = \frac{cn^3}{p^5}.$$

Thus, we have $I_1 \stackrel{\mathrm{P}}{\longrightarrow} \gamma^2$. On the other hand, for $j \ne k$, we have $\mathrm{E}(Z_j'Z_k)^4 \le cn^2$. It follows that $I_2 = p^{-1}O_{\mathrm{P}}(1)$. Therefore, we have $T_4 \stackrel{\mathrm{P}}{\longrightarrow} \gamma^2$. Finally, can write $T_3 = I_3 - T_4$, where $I_3 = (n^2p^3)^{-1}\sum_{j=1}^{p}\{\sum_{k=1}^{p}(Z_j'Z_k)^2\}^2$. We can write $\sum_{k=1}^{p}(Z_j'Z_k)^2 = v_j + \delta_j$, where $v_j = Z_j'Z_j(Z_j'Z_j + p - 1)$ and $\delta_j = \sum_{k\ne j}\delta_{jk}$ with $\delta_{jk} = (Z_j'Z_k)^2 - Z_j'Z_j$. Then, we have

$$I_3 = \frac{1}{n^2p^3}\sum_{j=1}^{p}v_j^2 + \frac{2}{n^2p^3}\sum_{j=1}^{p}v_j\delta_j + \frac{1}{n^2p^3}\sum_{j=1}^{p}\delta_j^2. \tag{34}$$

We have $\mathrm{E}(\delta_j^2) = \mathrm{E}[\mathrm{E}\{\sum_{k\ne j}\delta_{jk})^2|Z_j\}] = \sum_{k\ne j}\mathrm{E}\{\mathrm{E}(\delta_{jk}^2|Z_j)\}$, using the fact that, conditional on Z_j, $\delta_{jk}, k \ne j$ are independent with mean 0. Furthermore, it can be shown that $\mathrm{E}(\delta_{jk}^2|Z_j) \le cn\sum_{i=1}^{n} z_{ij}^4$. It follows that $\mathrm{E}(\delta_j^2) \le cn^2p$. Thus, we have

$$\frac{1}{n^2p^3}\sum_{j=1}^{p}\mathrm{E}(\delta_j^2) \le \frac{c}{p}.$$

Next, we have

$$\mathrm{E}(v_j^2) = \mathrm{E}(Z_j'Z_j)^4 + 2(p-1)\mathrm{E}(Z_j'Z_j)^3 + (p-1)^2\mathrm{E}(Z_j'Z_j)^2$$

$$
\begin{aligned}
&= n(n+2)(n+4)(n+6) + 2(p-1)n(n+2)(n+4) \\
&\quad +(p-1)^2 n(n+2) \\
&= n(n+2)\{(n+4)(n+6) + 2(n+4)(p-1) + (p-1)^2\}. \qquad (35)
\end{aligned}
$$

Thus, in particular, we have $\mathrm{E}(v_j^2) \le cn^2p^2$, hence

$$\mathrm{E}(|v_j\delta_j|) \le \{\mathrm{E}(v_j^2)\}^{1/2}\{\mathrm{E}(\delta_j^2)\}^{1/2} \le cn^2p^{3/2}.$$

It follows that the expected absolute value of the second term on the right side of (34) is bounded by $c/\sqrt{p}$. Therefore, we have $I_3 = J_3 + o_{\mathrm{P}}(1)$, where J_3 is the first term on the right side of (34). Now, applying (35), it is easy to show that $\mathrm{E}(J_3) \to (\gamma+1)^2$. Furthermore, we have

$$\mathrm{var}(J_3) = \frac{1}{n^4p^6}\sum_{j=1}^{p}\mathrm{var}(v_j^2) \le \frac{cn^4p^5}{n^4p^6} = \frac{c}{p},$$

using the fact that $\mathrm{var}(v_j^2) \le \mathrm{E}(v_j^4) \le c\{\mathrm{E}(Z_j'Z_j)^8 + (p-1)^4\mathrm{E}(Z_j'Z_j)^4)\} \le cn^4p^4$. It follows that $J_3 \xrightarrow{\mathrm{P}} (\gamma+1)^2$, hence $T_3 \xrightarrow{\mathrm{P}} (\gamma+1)^2 - \gamma^2 = 2\gamma+1$. Combining the above results, we have established the following:

$$
\begin{aligned}
\mathrm{E}(S|Z) \;&\xrightarrow{\mathrm{P}}\; \tau^4 + 2\tau^2\omega\sigma^2(\gamma+1) + (\omega\sigma^2)^2(2\gamma+1) \\
&\quad +(\omega\sigma^2)^2\gamma^2\psi. \qquad (36)
\end{aligned}
$$

It follows, by (32), (36), and the dominated convergence theorem, that M converges in probability to the same limit as the right side of (36).

3 Proof of Lemma 5.1

We consider the case that ρ is finite. From the proof, it can be seen that the result also holds when $\rho = \infty$.

Let $D = R - I_p$ and $\Delta = \Gamma D \Gamma'$. We have $ZZ' = \Gamma R \Gamma' = \Gamma\Gamma' + \Delta$, hence

$$\mathrm{tr}(ZZ') = \mathrm{tr}(\Gamma\Gamma') + \mathrm{tr}(\Delta) = \mathrm{tr}(\Gamma\Gamma') + \sum_{i=1}^{n}\gamma_i' D\gamma_i = \mathrm{tr}(\Gamma\Gamma') + \delta_1,$$

δ_1 defined in an obvious way. By the random matrix theory [RMT; e.g., Jiang 2022, Corollay 16.2 (ii)], we have $(np)^{-1}\mathrm{tr}(\Gamma\Gamma') \stackrel{\mathrm{P}}{\longrightarrow} 1$. Also, we have $\mathrm{E}(\gamma_i' D\gamma_i) = \mathrm{tr}(D) = 0$; hence, by independence, we have $\mathrm{E}(\delta_1^2) = \sum_{i=1}^n \mathrm{E}(\gamma_i' D\gamma_i)^2$. By a martingale expression of a quadratic form (e.g., Jiang 2022, Example 8.3), we have

$$\gamma_i' D\gamma_i = 2\sum_{j=1}^p \left(\sum_{k<j} d_{jk}\gamma_{ik}\right)\gamma_{ij},$$

where d_{jk} is the (j,k) element of D. Thus, we have

$$\mathrm{E}(\gamma_i' D\gamma_i)^2 = 4\sum_{j=1}^p \mathrm{E}\left(\sum_{k<j} d_{jk}\gamma_{ik}\right)^2 \mathrm{E}(\gamma_{ij}^2) = 4\sum_{j=1}^p\sum_{k<j} d_{jk}^2 \leq 2p(p-1),$$

because the $d_{jk} = r_{jk} - 1_{(j=k)}$, and $|r_{jk}| \leq 1$, hence $|d_{jk}| \leq 1$. It follows that $\mathrm{E}(\delta_1^2) \leq 2np(p-1)$, hence $\mathrm{E}\{(np)^{-1}\delta_1\}^2 \leq 2/n \to 0$, implying $(np)^{-1}\delta_1 = o_{\mathrm{P}}(1)$. Combining the above results, we have

$$(np)^{-1}\mathrm{tr}(ZZ') = (np)^{-1}\mathrm{tr}(\Gamma\Gamma') + (np)^{-1}\delta_1 = 1 + o_{\mathrm{P}}(1).$$

Similarly, we have

$$\mathrm{tr}((ZZ')^2) = \mathrm{tr}((\Gamma\Gamma')^2) + 2\mathrm{tr}(\Gamma'\Delta\Gamma) + \mathrm{tr}(\Delta^2). \tag{37}$$

By the RMT (see the above reference), we have $(np^2)^{-1}\mathrm{tr}((\Gamma\Gamma')^2) \stackrel{\mathrm{P}}{\longrightarrow} 1+\gamma$. Next, denote the jth column of Γ by Γ_j, $1 \leq j \leq p$. It is easy to show

$$\begin{aligned}
\mathrm{tr}(\Gamma'\Delta\Gamma) &= \sum_{j=1}^p \sum_{1\leq k\neq l\leq p} d_{kl}\Gamma_k'\Gamma_j\Gamma_j'\Gamma_l \\
&= p\sum_{k\neq l} d_{kl}\Gamma_k'\Gamma_l + \sum_{j=1}^p\sum_{k\neq l} d_{kl}\Gamma_k' D_j\Gamma_l \\
&= \delta_2 + \delta_3,
\end{aligned}$$

where $D_j = \Gamma_j\Gamma_j' - I_n$, and δ_2, δ_3 are defined in obvious ways. We have

$$\sum_{k\neq l} d_{kl}\Gamma_k'\Gamma_l = 2\sum_{k>l} d_{kl}\Gamma_l'\Gamma_k = 2\sum_{k=1}^p \xi_k, \tag{38}$$

where $\xi_k = (\sum_{l<k} d_{kl}\Gamma_l)'\Gamma_k, \mathcal{F}_k = \sigma(\Gamma_1, \ldots, \Gamma_k)$, $1 \le k \le p$ is a sequence of margingale differences. It follows that

$$\mathrm{E}\left(\sum_{k\neq l} d_{kl}\Gamma_k'\Gamma_l\right)^2 = 4\sum_{k=1}^{p}\mathrm{E}(\xi_k^2).$$

Furthermore, we have

$$\begin{aligned}
\mathrm{E}(\xi_k^2) &= \mathrm{E}\left\{\Gamma_k'\left(\sum_{l<k} d_{kl}\Gamma_l\right)\left(\sum_{l<k} d_{kl}\Gamma_l\right)'\Gamma_k\right\} \\
&= \mathrm{E}\left\{\mathrm{tr}\left(\left(\sum_{l<k} d_{kl}\Gamma_l\right)\left(\sum_{l<k} d_{kl}\Gamma_l\right)'\Gamma_k\Gamma_k'\right)\right\} \\
&= \mathrm{tr}\left(\mathrm{E}\left\{\left(\sum_{l<k} d_{kl}\Gamma_l\right)\left(\sum_{l<k} d_{kl}\Gamma_l\right)'\right\}\mathrm{E}(\Gamma_k\Gamma_k')\right) \\
&= n\sum_{l<k} d_{kl}^2.
\end{aligned}$$

It follows that $\mathrm{E}(\sum_{k\le l} d_{kl}\Gamma_k'\Gamma_l)^2 = 2n\sum_{k\neq l} d_{kl}^2 \le 2np(p-1)$, implying

$$\sum_{k\neq l} d_{kl}\Gamma_k'\Gamma_l = \sqrt{n}pO_{\mathrm{P}}(1);\ \ \text{hence}\ \ (np^2)^{-1}\delta_2 = O_{\mathrm{P}}(1)/\sqrt{n} = o_{\mathrm{P}}(1).$$

Next, we have $\delta_3 = 2\sum_{j=1}^{p}\sum_{k>l} d_{kl}\Gamma_l' D_j\Gamma_k = 2\sum_{k>l} d_{kl}\Gamma_l'(\sum_{j=1}^{p} D_j)\Gamma_k$. Furthermore, for $k > l$, we have $\sum_{j=1}^{p} D_j = D_l + D_k + \sum_{j\notin\{k,l\}} D_j$. It can then be shown that $\delta_3 = \sum_{s=1}^{3}\delta_{3,s}$, where

$$\begin{aligned}
\delta_{3,1} &= \sum_{k=1}^{p}\left(\sum_{l<k} d_{kl}\Gamma_l' D_l\right)\Gamma_k = \sum_{k=1}^{p}\eta_{1,k}, \\
\delta_{3,2} &= \sum_{l=1}^{p}\Gamma_l'\left(\sum_{k>l} d_{kl}D_k\Gamma_k\right) = \sum_{l=1}^{p}\eta_{2,l}, \\
\delta_{3,3} &= \sum_{k>l, j\notin\{k,l\}} d_{kl}\Gamma_l' D_j\Gamma_k,
\end{aligned}$$

with $\eta_{1,k}, \eta_{2,l}$ defined in obvious ways. Using a similar martingale arguments, we have $\mathrm{E}(\delta_{3,1}^2) = \sum_{k=1}^{p}\mathrm{E}(\eta_{1,k}^2)$. Furthermore, we have

$$\begin{aligned}
\mathrm{E}(\eta_{1,k}^2) &= \mathrm{tr}\left(\mathrm{E}\left\{\left(\sum_{l<k} d_{kl}D_l\Gamma_l\right)\left(\sum_{l<k} d_{kl}\Gamma_l' D_l\right)\right\}\right) \\
&= \sum_{l_1,l_2<k} d_{kl_1}d_{kl_2}\mathrm{tr}(\mathrm{E}(D_{l_1}\Gamma_{l_1}\Gamma_{l_2}' D_{l_2})).
\end{aligned}$$

If $l_1 \neq l_2$, we have $\mathrm{E}(D_{l_1}\Gamma_{l_1}\Gamma'_{l_2}D_{l_2}) = \mathrm{E}(D_{l_1}\Gamma_{l_1})\mathrm{E}(\Gamma'_{l_2}D_{l_2})$. Furthermore, it can be shown that $\mathrm{E}(D_l\Gamma_l) = \mu_3 1_n$ with $\mu_s = \mathrm{E}(\gamma_{11}^s)$ for positive integer s. Thus, for $l_1 \neq l_2$, we have $\mathrm{E}(D_{l_1}\Gamma_{l_1}\Gamma'_{l_2}D_{l_2}) = \mu_3^2 1_n 1'_n$, hence $\mathrm{tr}(\mathrm{E}(D_{l_1}\Gamma_{l_1}\Gamma'_{l_2}D_{l_2})) = \mu_3^2 n$. Similarly, the following expression can be derived:

$$\mathrm{tr}(\mathrm{E}(D_l\Gamma_l\Gamma'_lD_l)) = n^3 - 5n^2 + 5n + (3n^2 - 5n)\mu_4 + n\mu_6.$$

It follows that $\mathrm{E}(\eta_{1,k}^2) \leq cn^3(k-1) + \mu_3^2 n(k-1)(k-2)$; therefore, we have

$$\mathrm{E}(\delta_{3,1}^2) \leq cn^3 \sum_{k=1}^{p-1} k + \mu_3^2 n \sum_{k=1}^{p-1} k(k-1) \leq c(n^3p^2 + np^3).$$

It follows that $\delta_{3,1} = \sqrt{n^3p^2 + np^3}O_{\mathrm{P}}(1)$; thus, we have

$$(np^2)^{-1}\delta_{3,1} = \sqrt{\frac{n}{p^2} + \frac{1}{np}}O_{\mathrm{P}}(1) = o_{\mathrm{P}}(1).$$

Next, it can be shown that $\mathrm{E}(\delta_{3,2}^2) = \sum_{l=1}^{p} \mathrm{E}(\eta_{2,l}^2)$, and

$$\mathrm{E}(\eta_{2,l}^2) = \sum_{k_1,k_2>l} d_{k_1l}d_{k_2l}\mathrm{tr}(\mathrm{E}(D_{k_1}\Gamma_{k_1}\Gamma'_{k_2}D_{k_2})).$$

Thus, by similar arguments, we have

$$\mathrm{E}(\delta_{3,2}^2) \leq c(n^3p^2 + np^3),$$

hence $\delta_{3,2} = \sqrt{n^3p^2 + np^3}O_{\mathrm{P}}(1)$ and

$$(np^2)^{-1}\delta_{3,2} = \sqrt{\frac{n}{p^2} + \frac{1}{np}}O_{\mathrm{P}}(1) = o_{\mathrm{P}}(1).$$

Finally, note that $k > l$ and $j \notin \{k, l\}$ imply not only that $, k, l$ are distinct but also there are three cases: I. $j > k > l$; II. $k > j > l$; and III. $k > l > j$. Thus, we have $\delta_{3,3} = \sum_{S=I,II,III} \delta_{3,3,S}$ with $\delta_{3,3,S} = \sum_{j,k,l \in S} d_{kl}\Gamma'_lD_j\Gamma_k$.

Write $S_{3,3,I} = \sum_{j=3}^{p} \xi_j$ with $\xi_j = \sum_{k<j}\sum_{l<k} d_{kl}\Gamma'_lD_j\Gamma_k$. Let $\mathcal{F}_j$ be defined as below (38). Clearly, we have $\xi_j \in \mathcal{F}_j$ (i.e., ξ_j is $\mathcal{F}_j$ measurable),

$$\mathrm{E}(\xi_j|\mathcal{F}_{j-1}) = \sum_{k<j}\sum_{l<k} d_{kl}\Gamma'_l\mathrm{E}(D_j|\mathcal{F}_{j-1})\Gamma_k$$

and $\mathrm{E}(D_j|\mathcal{F}_{j-1}) = \mathrm{E}(D_j) = 0$. Thus, $\xi_j, \mathcal{F}_j, 3 \le j \le p$ is a sequence of martingale differences. Thus, we have $\mathrm{E}(S_{3,3,I}^2) = \sum_{j=3}^p \mathrm{E}(\xi_j^2)$.

Furthermore, write $\mathrm{E}(\xi_j^2) = \mathrm{E}\{\mathrm{E}(\xi_j^2|\Gamma_j)\}$. For a fix $n \times 1$ vector g, define $\xi_{jk,g} = (\sum_{l<k} d_{kl}\Gamma_l')(gg' - I_n)\Gamma_k$, and $\xi_{j,g} = \sum_{k=1}^{j-1} \xi_{jk,g}$. Then, we have

$$\mathrm{E}(\xi_j^2|\Gamma_j = g) = \mathrm{E}(\xi_{j,g}^2|\Gamma_j = g) = \mathrm{E}(\xi_{j,g}^2),$$

due to the independence. Note that $\xi_{jk,g} \in \mathcal{F}_k$ and

$$\mathrm{E}(\xi_{jk,g}|\mathcal{F}_{k-1}) = \left(\sum_{l<k} d_{kl}\Gamma_l'\right)(gg' - I_n)\mathrm{E}(\Gamma_k|\mathcal{F}_{k-1})$$

and $\mathrm{E}(\Gamma_k|\mathcal{F}_{k-1}) = \mathrm{E}(\Gamma_k) = 0$. It follows that $\xi_{jk,g}, \mathcal{F}_k, 1 \le k \le j-1$ is a sequence of martingale differences; hence $\mathrm{E}(\xi_{j,g}^2) = \sum_{k=1}^{j-1} \mathrm{E}(\xi_{jk,g}^2)$. It can also be shown that $\mathrm{E}(\xi_{jk,g}^2) = (\sum_{l<k} d_{kl}^2)\{(g'g)^2 - 2g'g + n\}$. It follows that $\mathrm{E}(\xi_j^2|\Gamma_j) = \mathrm{E}(\xi_{j,g}^2)|_{g=\Gamma_j} = (\sum_{k=1}^{j-1}\sum_{l<k} d_{kl}^2)\{(\Gamma_j'\Gamma_j)^2 - 2\Gamma_j'\Gamma_j + n\}$, hence

$$\begin{aligned}
\mathrm{E}(\xi_j^2) &= \left(\sum_{k=1}^{j-1}\sum_{l<k} d_{kl}^2\right)\{\mathrm{E}(\Gamma_j'\Gamma_j)^2 - 2\mathrm{E}(\Gamma_j'\Gamma_j) + n\} \\
&= n(n-2+\mu_4)\left(\sum_{k=1}^{j-1}\sum_{l<k} d_{kl}^2\right).
\end{aligned}$$

Thus, we have $\mathrm{E}(S_{3,3,I}^2) = n(n-2+\mu_4)\sum_{l<k<j} d_{kl}^2 \le cn^2p^3$. It then follows that $S_{3,3,I} = np^{3/2}O_{\mathrm{P}}(1)$.

Next, we can write $S_{3,3,II} = \sum_{k=3}^p \eta_k$ with $\eta_k = (\sum_{j<k}\sum_{l<j} d_{kl}\Gamma_l' D_j)\Gamma_k$. It is easy to see that $\eta_k, \mathcal{F}_k, 3 \le k \le p$ is a sequence of martingale differences. Thus, we have $\mathrm{E}(S_{3,3,II}^2) = \sum_{k=3}^p \mathrm{E}(\eta_k^2)$. Furthermore, we have

$$\mathrm{E}(\eta_k^2) = \sum_{j_1,j_2=2}^{k-1} \mathrm{tr}(\mathrm{E}(\eta_{j_1k}\eta_{j_2k}')).$$

It can be shown that $\mathrm{E}(\eta_{j_1k}\eta_{j_2k}') = 0$ if $j_1 \ne j_2$, and

$$\mathrm{E}(\eta_{jk}\eta_{jk}') = \left(\sum_{l<j} d_{kl}^2\right)\mathrm{E}(D_j^2).$$

It follows that $\mathrm{E}(\eta_k^2) = n(n-2+\mu_4)\sum_{j=2}^{k-1}\sum_{l<j} d_{kl}^2$, hence

$$\mathrm{E}(S_{3,3,II}^2) = n(n-2+\mu_4)\sum_{l<j<k} d_{kl}^2 \le cn^2p^3.$$

Thus, similarly, we have $S_{3,3,II} = np^{3/2}O_{\mathrm{P}}(1)$.

Similarly, we have $S_{3,3,III} = \sum_{k=3}^{p}\zeta_k$ with

$$\zeta_k = \left\{\sum_{l<k} d_{kl}\Gamma_l'\left(\sum_{j<l} D_j\right)\right\}\Gamma_k;$$

$\mathrm{E}(S_{3,3,III}^2) = \sum_{k=3}^{p}\mathrm{E}(\zeta_k^2)$; and $\mathrm{E}(\zeta_k^2) = \sum_{l_1,l_2=2}^{k-1}\mathrm{tr}(\mathrm{E}(\zeta_{kl_2}\zeta_{kl_1}'))$, where

$$\zeta_{kl} = d_{kl}\Gamma_l'\sum_{j<l} D_j.$$

It can be shown that for $l_1 \ne l_2$, we have $\mathrm{E}(\zeta_{kl_2}\zeta_{kl_1}') = 0$; and

$$\begin{aligned}
\mathrm{tr}(\mathrm{E}(\zeta_{kl}\zeta_{kl}')) &= d_{kl}^2\mathrm{tr}\left(\mathrm{E}\left(\sum_{j<l} D_j\right)^2\right) \\
&= d_{kl}^2\sum_{j<l}\mathrm{E}\{\mathrm{tr}(D_j^2)\} \\
&= n(n-2+\mu_4)(l-1)d_{kl}^2.
\end{aligned}$$

It then follows that $\mathrm{E}(S_{3,3,III}^2) \le cn^2p^3$, hence $S_{3,3,III} = np^{3/2}O_{\mathrm{P}}(1)$.

Thus, in conclusion, we have

$$(np^2)^{-1}\delta_{3,3} = (np^2)^{-1}np^{3/2}O_{\mathrm{P}}(1) = \frac{O_{\mathrm{P}}(1)}{\sqrt{p}} = o_{\mathrm{P}}(1).$$

Combining the above results, we have $(np^2)^{-1}\delta_3 = o_{\mathrm{P}}(1)$. This, again combined with an earlier results, imply that $(np^2)^{-1}\mathrm{tr}(\Gamma'\Delta\Gamma) = o_{\mathrm{P}}(1)$.

Now consider $\mathrm{tr}(\Delta^2) = \mathrm{tr}((\Gamma D\Gamma')^2)$. It is easy to show $\Gamma D\Gamma' = \sum_{k=1}^{p} W_k$, where $W_k = U_k + U_k'$ with $U_k = \Gamma_k(\sum_{l<k} d_{kl}\Gamma_l')$. Note that W_k, $\mathcal{F}_k$, $1 \le k \le p$ is a sequence of matrix-valued martingale differences. Thus, we have

$$\begin{aligned}
&\mathrm{E}\{\mathrm{tr}(\Delta^2)\} = \mathrm{tr}\left(\mathrm{E}\left(\sum_{k=1}^{p} W_k\right)^2\right) = \mathrm{tr}\left(\sum_{k=1}^{p}\mathrm{E}(W_k^2)\right) \\
&= \sum_{k=1}^{p}\mathrm{tr}(\mathrm{E}(W_k^2)) = 2\sum_{k=1}^{p}[\mathrm{E}\{\mathrm{tr}(U_k^2)\} + \mathrm{E}\{\mathrm{tr}(U_k'U_k)\}].
\end{aligned}$$

It is can be shown $\mathrm{E}\{\mathrm{tr}(U_k^2)\} = \sum_{l<k} d_{kl}^2 \mathrm{E}(\Gamma_l'\Gamma_l) = n\sum_{l<k} d_{kl}^2$. On the other hand, we have $U_k'U_k = \Gamma_k'\Gamma_k(\sum_{l<k} d_{kl}\Gamma_l)(\sum_{l<k} d_{kl}\Gamma_l')$. Thus, we have

$$\begin{aligned}\mathrm{E}(U_k'U_k) &= \mathrm{E}(\Gamma_k'\Gamma_k)\mathrm{E}\left\{\left(\sum_{l<k} d_{kl}\Gamma_l\right)\left(\sum_{l<k} d_{kl}\Gamma_l'\right)\right\} \\ &= n\sum_{l<k} d_{kl}^2 \mathrm{E}(\Gamma_l\Gamma_l') = n\left(\sum_{l<k} d_{kl}^2\right) I_n.\end{aligned}$$

It follows that $\mathrm{E}\{\mathrm{tr}(U_k'U_k)\} = \mathrm{tr}(\mathrm{E}(U_k'U_k)) = n^2\sum_{l<k} d_{kl}^2$; hence

$$\mathrm{E}\{\mathrm{tr}(\Delta^2)\} = n(n+1)\sum_{k\neq l} d_{kl}^2 = n(n+1)\mathrm{tr}(D^2) = n(n+1)\{\mathrm{tr}(R^2) - p\}.$$

Therefore, we have, by the assumptions,

$$\mathrm{E}\{(np^2)^{-1}\mathrm{tr}(\Delta^2)\} = \frac{n+1}{p}\left\{\frac{\mathrm{tr}(R^2)}{p} - 1\right\} \to \gamma(\rho - 1). \tag{39}$$

Next, we have $\mathrm{tr}(\Delta^2) = \sum_{k_1\neq l_1, k_2\neq l_2} d_{k_1 l_1} d_{k_2 l_2} \Gamma_{k_1}'\Gamma_{l_2}\Gamma_{k_2}'\Gamma_{l_1}$; thus, we have

$$\begin{aligned}\mathrm{var}(\mathrm{tr}(\Delta^2)) \;&=\; \sum_{k_1\neq l_1, k_2\neq l_2}\;\sum_{k_3\neq l_3, k_4\neq l_4} d_{k_1 l_1} d_{k_2 l_2} d_{k_3 l_3} d_{k_4 l_4} \\ &\quad \times\mathrm{cov}(\Gamma_{k_1}'\Gamma_{l_2}\Gamma_{k_2}'\Gamma_{l_1}, \Gamma_{k_3}'\Gamma_{l_4}\Gamma_{k_4}'\Gamma_{l_3}).\end{aligned} \tag{40}$$

The covariance in (40) has the expression

$$\begin{aligned}&\mathrm{cov}(\Gamma_{k_1}'\Gamma_{l_2}\Gamma_{k_2}'\Gamma_{l_1}, \Gamma_{k_3}'\Gamma_{l_4}\Gamma_{k_4}'\Gamma_{l_3}) = \mathrm{E}((\Gamma_{k_1}'\Gamma_{l_2}\Gamma_{k_2}'\Gamma_{l_1}\Gamma_{k_3}'\Gamma_{l_4}\Gamma_{k_4}'\Gamma_{l_3}) \\ &-\mathrm{E}(\Gamma_{k_1}'\Gamma_{l_2}\Gamma_{k_2}'\Gamma_{l_1})\mathrm{E}(\Gamma_{k_3}'\Gamma_{l_4}\Gamma_{k_4}'\Gamma_{l_3}).\end{aligned} \tag{41}$$

Consider the indexes $k_s, l_s, s = 1, 2, 3, 4$. There are five cases:

I. An index appears exactly once.

II. Not I, but an index appears exactly twice.

III. Not I or II, but an index appears exactly three times.

IV. Not I, II, III, but an index appears exactly four times.

V. All indexes are equal.

Note that these are all possible cases because, if an index appears more than 4 times, but not all indexes are equal, it must appear either (exactly) 5, or 6, or 7 times; in each case, one of the cases I, II, or III will happen.

Case I. It is easy to see that, on the right side of (41), the first expectation is zero, and at least one of the product expectations is zero; thus, the covariance is 0.

Case II. There are two sub-cases.

(A) If the identical indexes appear in the same pair of indexes, j_1, j_2, corresponding to some $\Gamma'_{j_1}\Gamma_{j_2}$, the first expectation on the right side of (41) is equal to n times the expectation of the product with $\Gamma'_{j_1}\Gamma_{j_2}$ removed; one of the product expectations is equal to n times the expectation with $\Gamma'_{j_1}\Gamma_{j_2}$ removed, and the other product expection is unchanged. Thus, the covariance reduces to an expression like

$$n\{\mathrm{E}(\Gamma'_{a_1}\Gamma_{b_1}\Gamma'_{a_2}\Gamma_{b_2}\Gamma'_{a_3}\Gamma_{b_3}) - \mathrm{E}(\Gamma'_{a_1}\Gamma_{b_1})\mathrm{E}(\Gamma'_{a_2}\Gamma_{b_2}\Gamma'_{a_3}\Gamma_{b_3})\}. \tag{42}$$

Similarly, there are four cases:

i. An index appears exactly once.

ii. Not I, but an index appears exactly twice.

iii. Not I or II, but an index appears exactly three times.

iv. All indexes are equal.

In case i, (42) is zero. In case ii, if $a_1 = b_1$ are the index, again, (42) is zero. Otherwise, w.l.o.g., let $a_1 = a_2 = a$. Then, it can be shown that the expression inside the $\{\cdots\}$ in (42) is equal to $\mathrm{E}(\Gamma'_{b_1}\Gamma_{b_2}\Gamma'_{a_3}\Gamma_{b_3})$. Now there are three cases, still denoted by i, ii, and iv as above. In case i, the last expectation is zero. In case ii, the expectation is either something like $\mathrm{E}(\Gamma'_{b_1}\Gamma_{b_1})\mathrm{E}(\Gamma'_{a_3}\Gamma_{a_3}) = n^2$ with $b_1 \neq a_3$, or something like $\mathrm{E}(\Gamma'_{b_1}\Gamma_{b_2})^2 = n$ with $b_1 \neq b_2$. In case iv, the expression inside $\{\cdots\}$ in (42) is equal to something like $\mathrm{E}(\Gamma'_j\Gamma_j)^3 - \mathrm{E}(\Gamma'_j\Gamma_j)\mathrm{E}(\Gamma'_j\Gamma_j)^2 \leq cn^3$. It can now be summarized that the total covariances associated with the non-zero scenarios under II-A-ii is bounded in absolute value by

$$cn(n^2p^2 + n^3p) = cn^3p(n+p). \tag{43}$$

In case iii, it can be seen that the remaining indexes much also be the same (but not the same as the other three); otherwise, either i or ii will happen. There are then two scenarios that the expression inside the $\{\cdots\}$ in (42) is either something like $\mathrm{E}(\Gamma_a'\Gamma_a\Gamma_a')\mathrm{E}(\Gamma_b\Gamma_b'\Gamma_b) = \mu_3^2 n$, or something like

$$\mathrm{E}(\Gamma_a'\Gamma_b)^3 - \mathrm{E}(\Gamma_a'\Gamma_b)\mathrm{E}(\Gamma_a'\Gamma_b)^2 = \mathrm{E}(\Gamma_a'\Gamma_b)^3 = \mu_3 n$$

for some $a \neq b$. It follows that the total covariances associated with the non-zero scenarios under II-A-iii is bounded in absolute value by cn^2p^2. Finally, the total covariances associated with the non-zero scenarios under II-A-iv is bounded in absolute value by cn^4p. Thus, in summary, the total covariances associated with the non-zero scenarios under II-A is bounded in absolute value by (43).

(B) If the identical indexes appear in different pair of indexes, say, $\Gamma_{j_1}'\Gamma_j$ and $\Gamma_j'\Gamma_{j_2}$ (note that $\Gamma_{j_1}'\Gamma_{j_2} = \Gamma_{j_2}'\Gamma_{j_1}$ for any j_1, j_2). It follows that the first expectation on the right side of (41) is equal to

$$\mathrm{E}(\Gamma_{j_1}'\Gamma_j\Gamma_j'\Gamma_{j_2}\cdots) = \mathrm{E}\{\Gamma_{j_1}'\mathrm{E}(\Gamma_j\Gamma_j'|\Gamma_k, k \neq j)\Gamma_{j_2}\cdots\} = \mathrm{E}(\Gamma_{j_1}'\Gamma_{j_2}\cdots),$$

because $\mathrm{E}(\Gamma_j\Gamma_j'|\Gamma_k, k \neq j) = \mathrm{E}(\Gamma_j\Gamma_j') = I_n$. Thus, the first expectation reduces to the same form as in in the expression inside $\{\cdots\}$ in (42). As for the product of expectations on the right side of (41), there are three cases. Let us call the original indexes $k_s, l_s, s = 1, 2$ Group 1 and $k_s, l_s, s = 3, 4$ Group 2. If j_1, j_2 are both in Group 1, or both in Group 2, the product of expectations are in the same form as in the expression inside $\{\cdots\}$ in (42). Then, by the established result, the total covariances associated with these cases is bounded in absolute value by $cn^2p(n+p)$, that is, (43) with one less n factor. If the j_1, j_2 are in different groups, the product of expectations is zero. In this case, again according to the established result, it can be shown that the total covariances associated with these cases is bounded in absolute value by (43). Thus, in summary, the total non-zero covariances associated with the non-zero scenarios under II-A is bounded in absolute value by (43).

Case III. It is easy to see the only possible scenario is that the remaining 5 indexes are the identical, but not the same as the 3. Thus, the covariance is either something like

$$\mathrm{E}(\Gamma_a'\Gamma_a\Gamma_a')\mathrm{E}\{\Gamma_b(\Gamma_b'\Gamma_b)^2\} = \mu_3 n\{2(n-1)\mu_3 + \mu_5\} \le cn^2,$$

or something like

$$\mathrm{E}\{(\Gamma_a'\Gamma_b)^3\Gamma_b'\Gamma_b\} = \mu_3 n\{(n-1)\mu_3 + \mu_5\} \le cn^2$$

for $a \neq b$. It follows tha the total non-zero covariances associated with case III is bounded in absolute value by cn^2p^2.

Case IV. It is easy to see the only possible scenario is that the remaining 4 indexes are also identical, but not the same as the other 4. If the two 4's are not in the same index group (see above), the covariance is zero. Now suppose the two 4's cross the groups. There are three possible scenarios in this case, where the covariance has the following forms, respectively, for $a \neq b$:

i. $\mathrm{E}\{(\Gamma_a'\Gamma_a)(\Gamma_a'\Gamma_b)^2(\Gamma_b'\Gamma_b)\} \le n^3 + (1+\mu_4)n^2 + \mu_4^2 n \le cn^3$;

ii $\mathrm{E}(\Gamma_a'\Gamma_b)^4 - \{\mathrm{E}(\Gamma_a'\Gamma_b)^2\}^2 = (\mu_4 - 1)n$;

iii. $\mathrm{E}(\Gamma_a'\Gamma_a)^2\mathrm{E}(\Gamma_b'\Gamma_b)^2 - \{\mathrm{E}(\Gamma_a'\Gamma_a)\mathrm{E}(\Gamma_b'\Gamma_b)\}^2 = (\mu_4 - 1)n^2(2n - 1 + \mu_4)$.

Thus, the total non-zero covariances associated with case IV is bounded in absolute value by cn^3p^2.

Case V. It is easy to see the total covariances associated with case IV is bounded in absolute value by cn^4p.

In conclusion, it follows by (41) and the above results, noting that $|d_{kl}| \le 1$, that $\mathrm{var}(\mathrm{tr}(\Delta^2)) \le cn^3p(n+p)$. Therefore, we have

$$\mathrm{var}((np^2)^{-1}\mathrm{tr}(\Delta^2)) \le \frac{cn^3p(n+p)}{n^2p^4} = \left(\frac{c}{p}\right)\left(\frac{n}{p}\right)\left(1 + \frac{n}{p}\right) \to 0. \tag{44}$$

Combining (39) and (44), we have

$$(np^2)^{-1}\mathrm{tr}(\Delta^2) \xrightarrow{\mathrm{P}} \gamma(\rho - 1). \tag{45}$$

Combining the results, and in view of (37), the proof is complete.

4 Proof of Theorem 5.1

We focus on the parts of Theorem 3.1 that need to be modified; the other parts remain unchanged as in the proof of Theorem 3.1.

Part (I): First, we extend (1) in the proof of Theorem 3.1. We have

$$\mathrm{tr}(S_z^2) = p^{-2} \sum_{j_1,j_2=1}^{p} \left(\sum_{i=1}^{n} z_{ij_1} z_{ij_2} \right)^2 .$$

Also, we have $\mathrm{E}(\sum_{i=1}^{n} z_{ij_1} z_{ij_2})^2 = n(n-1)r_{2,j_1j_2}^2 + nr_{4,j_1j_2}$. Thus, we have

$$\begin{aligned}
\frac{1}{n}\mathrm{E}\{\mathrm{tr}(S_z^2)\} &= \frac{1}{np^2} \sum_{j_1,j_2=1}^{p} \mathrm{E} \left(\sum_{i=1}^{n} z_{ij_1} z_{ij_2} \right)^2 \\
&= \frac{1}{np^2} \sum_{j_1,j_2=1}^{p} \{n(n-1)r_{2,j_1j_2}^2 + nr_{4,j_1j_2}\} \\
&= \frac{n-1}{p^2} \sum_{j_1,j_2=1}^{p} r_{2,j_1j_2}^2 + \frac{1}{p^2} \sum_{j_1,j_2=1}^{p} r_{4,j_1j_2}.
\end{aligned}$$

If $|j_1 - j_2| > C$, we have $r_{2,j_1,j_2} = \mathrm{E}(z_{1j_1})\mathrm{E}(z_{1j_2}) = 0$, and $r_{4,j_1j_2} = \mathrm{E}(z_{ij_1}^2)\mathrm{E}(z_{ij_2}^2) = 1$; and $r_{4,j_1j_2} \leq \{\mathrm{E}(z_{ij_1}^4)\}^{1/2}\{\mathrm{E}(z_{ij_2}^4)\}^{1/2} \leq c$ if $|j_1 - j_2| \leq C$. It follows that

$$\frac{1}{n}\mathrm{E}\{\mathrm{tr}(S_z^2)\} = \frac{n-1}{p^2} \sum_{(j_1,j_2)\in\mathcal{S}_{1,p}} r_{2,j_1j_2}^2 + \frac{d_{1,p}}{p^2} + O(p^{-1}) \longrightarrow d_1 + \gamma f_2.$$

Next, we can write $\mathrm{tr}(S_z^2) = p^{-2} \sum_{j_1,j_2=1}^{p} Z_{j_1}' Z_{j_2} Z_{j_2}' Z_{j_1}$. It is easy to see that the conditions of Lemma 3 of Jiang *et al.* (2023) are satisfied. By the latter lemma, we have

$$\mathrm{var}(\mathrm{tr}(S_z^2)) = \frac{1}{p^4}\mathrm{var}\left(\sum_{j_1,j_2=1}^{p} Z_{j_1}' Z_{j_2} Z_{j_2}' Z_{j_1} \right) \leq \frac{c}{p^4}(n \vee p)^4 = O(1),$$

hence $\mathrm{var}(n^{-1}\mathrm{tr}(S_z^2)) = n^{-2}\mathrm{tr}(\mathrm{tr}(S_z^2)) = O(n^{-2}) \to 0$.

Combining the above results, we have shown that

$$\frac{1}{n}\mathrm{tr}(S_z^2) \xrightarrow{\mathrm{P}} d_1 + \gamma f_2. \tag{46}$$

Next, we have $s_{z,i,i} = p^{-1} \sum_{j=1}^{p} z_{ij}^2$, $\mathrm{E}(s_{z,i,i}^2) = p^{-2}\mathrm{E}(\sum_{j=1}^{p} z_{ij}^2)^2$, and

$$\mathrm{E} \left(\sum_{j=1}^{p} z_{ij}^2 \right)^2 = \sum_{j_1,j_2=1}^{p} r_{4,j_1j_2} = d_{1,p} + r_{1,i},$$

by the earlier arguments, where $|r_{1,i}| \leq cp$. It follows that

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}(s_{z,i,i}^2) = \frac{d_{1,p}}{p^2} + \frac{1}{np^2}\sum_{i=1}^{n} r_{1,i} = \frac{d_{1,p}}{p^2} + O(p^{-1}) \longrightarrow d_1.$$

On the other hand, we have, by Jensen's inequality,

$$\mathrm{E}(s_{z,i,i}^4) = \mathrm{E}\left(\frac{1}{p}\sum_{j=1}^{p} z_{ij}^2\right)^4 \leq \frac{1}{p}\sum_{j=1}^{p} \mathrm{E}(z_{ij}^8) \leq c.$$

Thus, we have $\mathrm{var}(n^{-1}\sum_{i=1}^n s_{z,i,i,}^2) = n^{-2}\sum_{i=1}^n \mathrm{var}(s_{z,i,i,}^2) \leq n^{-2}\sum_{i=1}^n \mathrm{E}(s_{z,i,i,}^4) \leq cn^{-1}$. Combining the above results, we have shown that

$$\frac{1}{n}\sum_{i=1}^{n} s_{z,i,i,}^2 \xrightarrow{\mathrm{P}} d_1. \tag{47}$$

It follows that (note that $A = S_{z,\mathrm{o}}$ in the above)

$$\frac{1}{n}\mathrm{tr}(S_{z,\mathrm{o}}^2) = \frac{1}{n}\mathrm{tr}(S_z^2) - \frac{1}{n}\sum_{i=1}^{n} s_{z,i,i,}^2 \xrightarrow{\mathrm{P}} \gamma f_2. \tag{48}$$

Furthermore, by Corollary 1 of Jiang *et al.* (2023; supplementary material), we have $\lambda_{\max}(n^{-1}Z'Z) = O_{\mathrm{P}}(1)$ under the C-dependent assumption and *A3*. Note that the condition $n/p \to \tau \in (0,1]$ can be relaxed to $\tau \in (0,\infty)$ for the conclusion $\lambda_{\max}(n^{-1}Z'Z) = O_{\mathrm{P}}(1)$ to hold. With that, it can be shown that $\lambda_{\max}(S_z^2)$ and $\lambda_{\max}(S_{z,\mathrm{o}}^2)$ are $O_{\mathrm{P}}(1)$; and $\mathrm{tr}(S_z) = nO_{\mathrm{P}}(1)$. It then follows, by similar arguments as in the proof of Theorem 3.1, that $\mathrm{var}(\hat{\sigma}^2|Z) = \mathrm{var}(y'S_{z,\mathrm{o}}y|Z)/\{\mathrm{tr}(S_{z,\mathrm{o}}^2)\}^2 = O_{\mathrm{P}}(1)/n$. It then follows, again by the similar arguments, that $\hat{\sigma}^2 \xrightarrow{\mathrm{P}} \omega\sigma^2$ and $\hat{\tau}^2 \xrightarrow{\mathrm{P}} \tau^2$.

Part (II): Now $A = a_0A_0 + a_1A_1 = B + \Delta$, where

$$\begin{aligned} B &= \frac{a_0}{n}I_n + \frac{a_1 - a_0}{n}\cdot\frac{S_{z,\mathrm{o}}}{\gamma f_2}, \\ \Delta &= \frac{a_1 S_{z,\mathrm{o}}}{n}\left\{\frac{n}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} - \frac{1}{\gamma f_2}\right\} - \frac{a_0 S_{z,\mathrm{o}}}{n}\left\{\frac{\mathrm{tr}(S_z)}{\mathrm{tr}(S_{z,\mathrm{o}}^2)} - \frac{1}{\gamma f_2}\right\}. \end{aligned}$$

By the previous results, it can be shown that

$$n\|B\|_2^2 \xrightarrow{\mathrm{P}} a_0^2 + \frac{(a_1 - a_0)^2}{\gamma f_2}. \tag{49}$$

Next, we have $n^{-1}\mathrm{tr}(S_z) \xrightarrow{\mathrm{P}} 1$, $n^{-1}\mathrm{tr}(S_{z,\mathrm{o}}^2) \xrightarrow{\mathrm{P}} \gamma f_2$ by the previous results. Also, we have $\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}}) = \mathrm{tr}(S_z^3) - 2\mathrm{tr}(S_z^2 dS_z) + \mathrm{tr}(dS_zS_zdS_z)$. By the proof of Theorem 2 in Jiang *et al.* (2023), it can be shown that

$$\frac{1}{n}\mathrm{tr}(S_z^3) \xrightarrow{\mathrm{P}} f_6\gamma^2 + f_7\gamma + f_8.$$

Next, we have $\mathrm{tr}(S_z^2 dS_z) = p^{-2}\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}Z_{j_2}'dS_zZ_{j_1}$ and

$$Z_{j_2}'dS_zZ_{j_1} = \sum_{i=1}^{n} s_{z,i,i}z_{ij_2}z_{ij_1} = Z_{j_2}'Z_{j_1} + \sum_{i=1}^{n}(s_{z,i,i}-1)z_{ij_2}z_{ij_1}.$$

Thus, we have the following expression:

$$\begin{aligned}
\frac{1}{n}\mathrm{tr}(S_z^2 dS_z) &= \frac{1}{np^2}\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}Z_{j_2}'Z_{j_1} \\
&\quad + \frac{1}{np^2}\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}\sum_{i=1}^{n}(s_{z,i,i}-1)z_{ij_2}z_{ij_1} \\
&= \frac{1}{n}\mathrm{tr}(S_z^2) + \frac{1}{np^2}\sum_{i=1}^{n}(s_{z,i,i}-1)\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}z_{ij_2}z_{ij_1} \\
&= I_1 + I_2,
\end{aligned}$$

I_1, I_2 defined in an obviou ways. By (46), we have $I_1 \xrightarrow{\mathrm{P}} d_1 + \gamma f_2$. On the other hand, as argued in the proof of Theorem 3.1 by the Marcinkiewicz-Zygmund inequality (e.g., Jiang 2022, p. 161), it can be shown that, for any $\delta > 0$, we have

$$\mathrm{P}\left(\max_{1\le i\le n}|s_{z,i,i}-1| > \delta\right) \le \frac{c}{\delta^4 p^2} \longrightarrow 0.$$

It follows that $\max_{1\le i\le n}|s_{z,i,i,}-1| = o_{\mathrm{P}}(1)$. Thus, we have

$$\begin{aligned}
|I_2| &\le \left(\frac{1}{np^2}\max_{1\le i\le n}|s_{z,i,i}-1|\right)\sum_{i=1}^{n}\left|\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}z_{ij_2}z_{ij_1}\right| \\
&\le \frac{o_{\mathrm{P}}(1)}{np^2}\sum_{i=1}^{n}\left\{\left|\sum_{j_1,j_2=1}^{p}\mathrm{E}(Z_{j_1}'Z_{j_2}z_{ij_2}z_{ij_1})\right|\right. \\
&\quad \left. + \left|\sum_{j_1,j_2=1}^{p} Z_{j_1}'Z_{j_2}z_{ij_2}z_{ij_1} - \sum_{j_1,j_2=1}^{p}\mathrm{E}(Z_{j_1}'Z_{j_2}z_{ij_2}z_{ij_1})\right|\right\}.
\end{aligned}$$

It can be shown that $\mathrm{E}(Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1}) = (n-1)r^2_{2,j_1j_2}1_{(|j_1-j_2|\le C)} + r_{4,j_1j_2}$. Thus, it can be shown that $\sum_{i=1}^n |\sum_{j_1,j_2=1}^p \mathrm{E}(Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1})| \le cnp(n+p)$. On the other hand, by Lemma 3 of Jiang *et al.* (2023), we have

$$
\begin{aligned}
&\mathrm{E}\left\{\left|\sum_{j_1,j_2=1}^p Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1} - \sum_{j_1,j_2=1}^p \mathrm{E}(Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1})\right|^2\right\} \\
&= \mathrm{var}\left(\sum_{j_1,j_2=1}^p Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1}\right) \le c(n\vee p)^4,
\end{aligned}
$$

hence $\mathrm{E}\{|\sum_{j_1,j_2=1}^p Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1} - \sum_{j_1,j_2=1}^p \mathrm{E}(Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1})| \le c(n\vee p)^2$ by Jensen's inequality. It follows that

$$
\sum_{i=1}^n \mathrm{E}\left\{\left|\sum_{j_1,j_2=1}^p Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1} - \sum_{j_1,j_2=1}^p \mathrm{E}(Z'_{j_1}Z_{j_2}z_{ij_2}z_{ij_1})\right|\right\} \le cn(n\vee p)^2.
$$

Therefore, we have $|I_2| \le o_{\mathrm{P}}(1)[n/p+1+\{(n/p)\vee 1\}^2 O_{\mathrm{P}}(1)] = o_{\mathrm{P}}(1)$. Thus, combining the results, we have shown

$$
\frac{1}{n}\mathrm{tr}(S_z^2 dS_z) \xrightarrow{\mathrm{P}} d_1 + f_2\gamma.
$$

Finally, it is easy to show, by *A1*, that

$$
\frac{1}{n}\mathrm{tr}(dS_zS_zdS_z) \xrightarrow{\mathrm{P}} f_4.
$$

Combining the results, we have shown that

$$
\begin{aligned}
\frac{1}{n}\mathrm{tr}(S_{z,\mathrm{o}}S_zS_{z,\mathrm{o}}) &\xrightarrow{\mathrm{P}} f_6\gamma^2 + f_7\gamma + f_8 - 2(d_1+f_2\gamma) + f_4 \\
&= f_6\gamma^2 + (f_7-2f_2)\gamma + f_4 + f_8 - 2d_1. \qquad (50)
\end{aligned}
$$

[It can be verified that, in the case of independence entries of Z, the right side of (50) is equal to $\gamma(\gamma+1)$.] It then follows that

$$
\begin{aligned}
n\|B\tilde{Z}\|_2^2 &\xrightarrow{\mathrm{P}} a_0^2 + 2a_0(a_1-a_0) \\
&+\frac{f_6\gamma^2 + (f_7-2f_2)\gamma + f_4 + f_8 - 2d_1}{f_2^2\gamma^2}(a_1-a_0)^2. \qquad (51)
\end{aligned}
$$

Next, by the established results, it can be shown that

$$\frac{1}{n}\mathrm{tr}(S_z S_{z,\mathrm{o}} S_z) \xrightarrow{\mathrm{P}} f_6\gamma^2 + (f_7 - f_2)\gamma + f_8 - d_1.$$

Also, by the proof of Theorem 2 in Jiang *et al.* (2023), it can be shown that

$$\frac{1}{n}\mathrm{tr}(S^4) \xrightarrow{\mathrm{P}} f_1\gamma^3 + 6f_3\gamma^2 + (4f_5 + 2f_9 + f_{10})\gamma + f_{11}.$$

With this, combined with the earlier results, it can be shown that

$$\begin{aligned}\frac{1}{n}\mathrm{tr}(S_{z,\mathrm{o}} S_z S_{z,\mathrm{o}} S_z) \quad \xrightarrow{\mathrm{P}} \quad & f_1\gamma^3 + 2(3f_3 - f_6)\gamma^2 \\ & +(f_2 + 4f_5 - 2f_7 + 2f_9 + f_{10})\gamma \\ & +d_1 - 2f_8 + f_{11}. \end{aligned} \tag{52}$$

Combining the above results, it follows that

$$\begin{aligned} n\|\tilde{Z}'B\tilde{Z}\|_2^2 \quad \xrightarrow{\mathrm{P}} \quad & (f_2\gamma + d_1)a_0^2 \\ & +2\left(f_6\gamma + f_7 - f_2 + \frac{f_8 - d_1}{\gamma}\right) a_0(a_1 - a_0) \\ & +\left\{f_1\gamma + 2(3f_3 - f_6) + \frac{f_2 + 4f_5 - 2f_7 + 2f_9 + f_{10}}{\gamma}\right. \\ & \left. + \frac{d_1 - 2f_8 + f_{11}}{\gamma^2}\right\}(a_1 - a_0)^2. \end{aligned} \tag{53}$$

Furthermore, with some tedious derivations, it can be shown that

$$\begin{aligned} \frac{n}{p^2}\sum_{j=1}^{p}(Z_j B Z_j)^2 \quad \xrightarrow{\mathrm{P}} \quad & \{a_0^2 + 2f_2 a_0(a_1 - a_0) + f_{12}(a_1 - a_0)^2\}\gamma \\ & +(f_2 - f_{13})(a_1 - a_0)^2 \\ & +(d_1 - 2f_4 + f_{14})\frac{(a_1 - a_0)^2}{\gamma}. \end{aligned} \tag{54}$$

Note that, in the special case of independent SNPs, we have $d_1 = f_2 = f_4 = f_{12} = f_{13} = f_{14} = 1$; it follows that the right side of (54) reduces to γa_1^2, which is what we got in the proof of Theorem 3.1.

Combining the above results, it can be shown that

$$\sum_{k=1}^{n+p} \mathrm{E}(\zeta_{nk}^2|Z) \xrightarrow{\mathrm{P}} \lambda_0 a_0^2 + 2\lambda_1 a_0(a_1 - a_0) + \lambda_2(a_1 - a_0)^2 = (a_0\ a_1)\Sigma \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \quad (55)$$

where Σ is defined in Theorem 5.1.

Next, we can write $\sum_{k=1}^{n+p} \zeta_{nk}^2 - \sum_{k=1}^{n+p} \mathrm{E}(\zeta_{nk}^2|Z) =$

$$\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\} + \sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\}.$$

Furthermore, we have $\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) = n\{\mathrm{var}(\xi_k^2)w_{kk}^2 + 4\mathrm{E}(\xi_k^2)(\sum_{l<k} w_{kl}\xi_l)^2\}$. Thus, we have $\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z) =$

$$4n\mathrm{E}(\xi_k^2)\left[\left(\sum_{l<k} w_{kl}\xi_l\right)^2 - \mathrm{E}\left\{\left(\sum_{l<k} w_{kl}\xi_l\right)^2 \middle| Z\right\}\right].$$

It can then be shown that $\sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\} = \xi' A\xi - \mathrm{tr}(AD)$, where $A = 4nW_{\mathrm{L}}' D W_{\mathrm{L}}$ and $D = \mathrm{diag}(\mathrm{E}(\xi_k^2), 1 \le k \le n+p)$. By Jiang *et al.* (2023; Lemma 1 in the supplement), we have $\xi' A\xi - \mathrm{tr}(AD) \xrightarrow{\mathrm{P}} 0$ provided that $\mathrm{tr}(ADAD) \xrightarrow{\mathrm{P}} 0$. It can be shown that

$$\mathrm{tr}(ADAD) = 16n^2\mathrm{tr}(W_{\mathrm{L}}' D W_{\mathrm{L}} D W_{\mathrm{L}}' D W_{\mathrm{L}} D) \le 16\lambda_{\max}^4(D)\|W_{\mathrm{L}}' W_{\mathrm{L}}\|_2^2.$$

Furhermore, by Lemma 5.3 of Jiang (2022), we have

$$\|W_{\mathrm{L}}' W_{\mathrm{L}}\|_2^2 \le 2\|W_{\mathrm{L}}' + W_{\mathrm{L}}\|^2\|W_{\mathrm{L}}\|_2^2.$$

Note that $\|W_{\mathrm{L}}' + W_{\mathrm{L}}\| = \|W - dW\| \le 2\|W\|$ and, by an earlier result, $\|W\| = n^{-1}O_{\mathrm{P}}(1)$. Also, again by an earlier result, it can be shown that $\|W_{\mathrm{L}}\|_2^2 \le (1/2)\|W\|_2^2 = n^{-1}O_{\mathrm{P}}(1)$. It follows that $\mathrm{tr}(ADAD) \le n^{-1}O_{\mathrm{P}}(1) \xrightarrow{\mathrm{P}} 0$, hence $\sum_{k=1}^{n+p}\{\mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1}) - \mathrm{E}(\zeta_{nk}^2|Z)\} \xrightarrow{\mathrm{P}} 0$. Also, we have

$$\begin{aligned} &\mathrm{E}\left(\left[\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\}\right]^2 \middle| Z\right) = \sum_{k=1}^{n+p}\mathrm{E}[\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\}^2|Z] \\ &= \sum_{k=1}^{n+p}\mathrm{E}\{\mathrm{var}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})|Z\} \le \sum_{k=1}^{n+p}(\zeta_{nk}^4|Z) \le cn^2\sum_{k=1}^{n+p}\left\{w_{kk}^4 + \left(\sum_{l<k} w_{kl}^2\right)^2\right\} \\ &\le cn^2\|W\|^2\|W\|_2^2 = n^{-1}O_{\mathrm{P}}(1), \end{aligned}$$

according to the earlier results. It can then be shown, by the dominated convergence theorem that $\sum_{k=1}^{n+p}\{\zeta_{nk}^2 - \mathrm{E}(\zeta_{nk}^2|\mathcal{F}_{n,k-1})\} \xrightarrow{\mathrm{P}} 0$.

Combining the above results, we have $\sum_{k=1}^{n+p}\zeta_{nk}^2 \xrightarrow{\mathrm{P}} (a_0\ a_1)\Sigma(a_0\ a_1)'$.

Furthermore, with some careful evaluations, it can be shown that

$$\mathrm{E}\left(\max_{1\leq k\leq n+p}\zeta_{nk}^2\right) \leq \sum_{k=1}^{n+p}\mathrm{E}(\zeta_{nk}^2) \leq cn\mathrm{E}(\|W\|_2^2) = O(1).$$

Thus, all conditions of the martingale CLT are verified.

5 Additional simulation studies

5.1 Simulation design

We follow the general setting described in Section 1 to generate the simulated SNPs. Specifically, the SNPs are treated as independent Binomial random variable with two trials and the allele frequency being the probability of success. The genotype values for each SNP are thus coded as 0, 1, and 2. We first simulate the minor allele frequency of SNP j, denoted as f_j, such that $f_j \sim \mathrm{Uniform}(0.05, 0.5)$ for all $j \in \{1, 2, \cdots, p\}$. Given f_j, we simulate the genotype matrix $U \in \{0, 1, 2\}^{n\times p}$, where the rows correspond to the individuals and columns to the SNPs. According to the Hardy-Weinberg equilibrium, for the jth SNP, the probabilities of the genotype being 0, 1, and 2 are $(1 - f_j)^2$, $2f_j(1 - f_j)$, and f_j^2, respectively. We then standardize each column of U so that it has zero mean and unit variance. The standardized genotype matrix is denoted as Z. We let $X = 0$ for simplicity. Then, by Section 1 of MS, the true underlying model describing the relationship between phenotype vector y and the standardized genotype matrix $\tilde{Z}$ can be expressed as

$$y = \tilde{Z}\alpha + \epsilon, \quad \alpha = b \circ \zeta, \quad \epsilon \sim N(0, \tau^2 I_n), \tag{56}$$

where $\tilde{Z} = p^{-1/2}Z$, $b = (b_1, \cdots, b_p)'$, whose components are independent Bernoulli(ω), $\zeta = (\zeta_1, \ldots, \zeta_p)' \sim N(0, \sigma^2 I_p)$, and $b \circ \zeta = (b_j\zeta_j)_{1\leq j\leq p}$. Let $S \subset \{1, 2, \cdots, p\}$ such that

$b_j = 1$ for $j \in S$ and $|S| = s$. Then, the above model can be expressed as:

$$y = \tilde{Z}_S \zeta_S + \epsilon, \quad \zeta_S \sim N(0, \sigma^2 I_s), \quad \epsilon \sim N(0, \tau^2 I_n), \tag{57}$$

where Z_S is the sub-matrix of Z consisting columns Z_j of Z such that $j \in S$, and ζ_S the sub-vector of ζ consisting components ζ_j of ζ such that $j \in S$.

Under the true underlying model, the (true) heritability is

$$h^2_{\text{true}} = \frac{\omega\sigma^2}{\omega\sigma^2 + \tau^2}. \tag{58}$$

In practice, the true casual SNPs in S are unknown. Therefore, while we simulate phenotype value under the true model, but pretend that we do not know the true causal SNPs. Thus, following the usual practice, we simply utilize all the SNPs in Z to estimate σ^2 and τ^2 (e.g., Jiang *et al.* 2016). The estimated heritability is thus

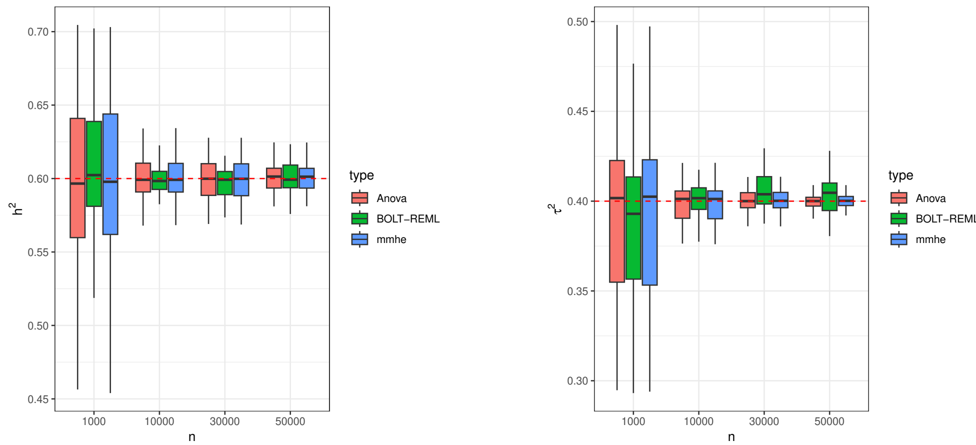
$$\hat{h}^2 = \frac{\hat{\sigma}^2}{\hat{\sigma}^2 + \hat{\tau}^2}. \tag{59}$$

5.2 The all-SNPs-causal scenario

An additional simulation study is carried out to investigate the performance of the three methods (see Section 6 of MS) when all SNPs being causal. Specifically, we let $\omega = 1$ and $\sigma^2 = 0.6$, with other parameters unchanged. Figure 1 presents the side-by-side boxplots. In terms of heritability estimation, all three methods have smaller bias and variance compared to results in the previous simulation study. When comparing the estimation of τ^2, the three methods also have smaller variation. Still, the performance of ANOVA and mmhe keeps improving as n increases, while that of BOLT-REML remains similar once n is greater than 10000.

5.3 Correlated SNPs

We also investigate the performance of the three methods when SNPs are correlated. The simulation settings are as follows. We generated the correlted SNPs matrix Z

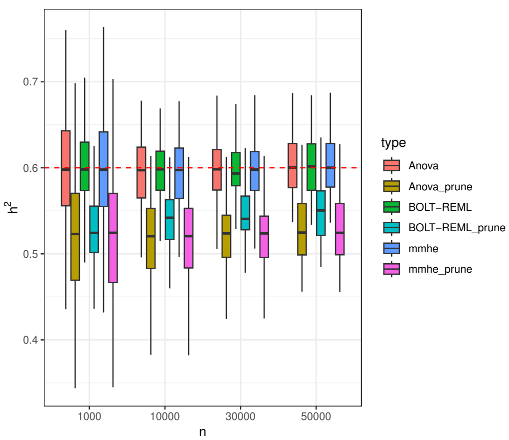


(a) (b)

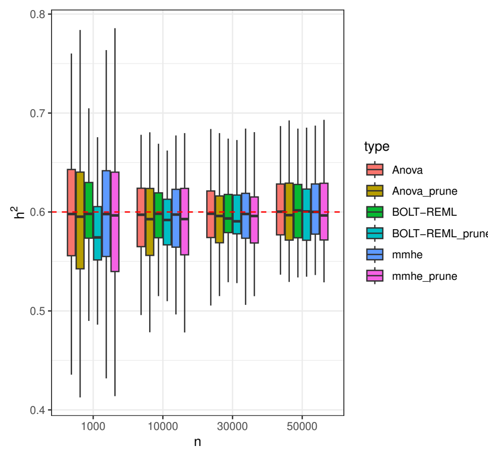
Figure 1: Comparison of ANOVA, BOLT-REML and mmhe: $p = 1000$; All SNPs causal. (a) Heritability; (b) Environmental Variance (τ^2).

under the first-order autoregressive [AR(1)] structure with parameter $\rho = 0.2$. The other parameters remain unchanged as in Section 6 of MS. To assess the estimation performance of the three methods under the pruned SNPs setting, we pruned the correlated SNPs by setting different correlation thresholds, namely, 0, 4×10^{-5}, and 1×10^{-4}, after marginally significant SNPs were identified with 0.005 significance level. More specifically, in terms of heritability estimation, all three methods have apparent consistent estimates when directly using the unpruned SNP matrix. However, the heritability is underestimated when only independent SNPs (i.e., threshold 0) are retained for all three methods, and a little overestimated when the prune correlation threshold is 1×10^{-4}. The under/overstimated issue is more severe for ANOVA and mmhe compared to BOLT-REML. When the correlation threshold is 4×10^{-5}, the three methods are seen to perform similarly. See Figure 2.

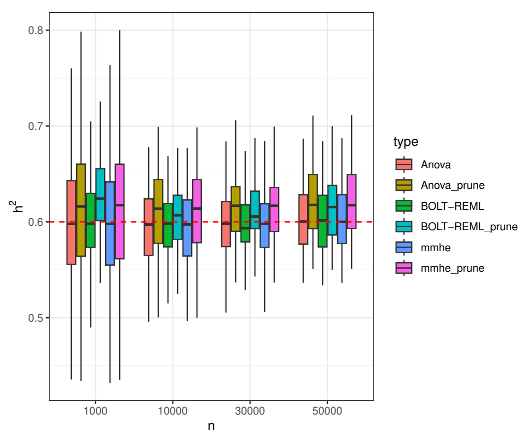
We also report the performance of τ^2 estimation under unpruned and pruned SNPs with 4×10^{-5} threshold, as shown in Figure 4. In this case, the BOLT-REML estimator (with or without prunning) seem to have relatively smaller variation for



(a)



(b)



(c)

Figure 2: Estimation of h^2 using ANOVA, BOLT-REML and mmhe Methods with unpruned/pruned SNPs Across Different Pruning Correlation Thresholds: $p = 1000$; SNPs Are Correlated under AR(1) Structure with $\rho = 0.2$. (a) $r^2 = 0$; (b) $r^2 = 4 \times 10^{-5}$; (c) $r^2 = 1 \times 10^{-4}$.

smaller n ($n = 1000$), but much larger variation for larger n ($n \geq 10000$).

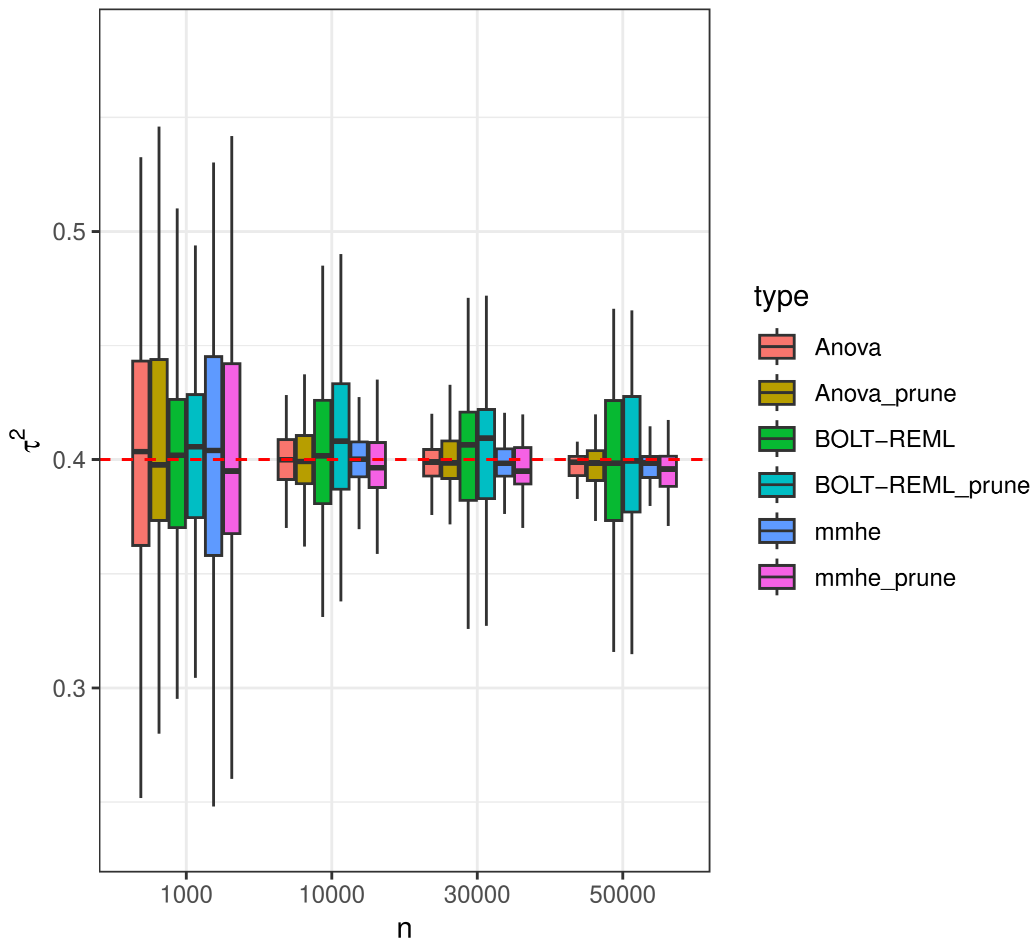


Figure 3

Figure 4: Estimation of τ^2 using ANOVA, BOLT-REML and mmhe Methods with Unpruned/Pruned SNPs: $p = 1000$; SNPs Are Correlated with AR(1) Structure with $\rho = 0.2$. SNPs Are Pruned with $r^2 = 4 \times 10^{-5}$ Correlation Threshold.

The performance of ANOVA confidence intervals based on the asymptotic theory (Theorem 5.1 in MS) under correlated SNPs is also investigated in a similar way as described in Section 6 of MS. Here, again, we generated SNPs under the AR(1) correlation structure with $\rho = 0.2$. For the pruned SNPs setting, we use the 4×10^{-5}

correlation threshold. We treat the pruned SNPs as independent and the asymptotic variance is computed based on Theorem 3.1. For the un-pruned SNPs setting, we use the empirical estimation of ω and keep $C = 5$, which are described in Theorem 5.1. Table 1 [the parts under Anova and Anova-prune ($r^2 = 4 \times 10^{-5}$)] reports the results based on 500 simulations runs. It is seen that ANOVA estimator with pruning (Anova-prune) has generally smaller ML; however, it is not as accurate in terms of CP compared to the ANOVA estimator without pruning (Anova) in most cases. Furthermore, for τ^2 estimation, there seems to be some over-coverage in CP by Anova, but severe under-coverage in CP by Anova-prune; for h^2 estimation, the CP of Anova is nearly correct, especially for larger n, but that of Anova-prune seems to be significantly lower than the nominal level (95%) for larger $n \geq 10000$.

As in MS (see Table 1 of MS), for comparison purpose, in Table 1 (last four columns) we also include confidence interval results based on the mmhe method. As can be seen, in terms of the CP, the performance of the mmhe method is not comparable to (any of) the ANOVA methods.

Table 1: Empirical Mean Length (ML) and Coverage Probability (CP) of 95% Confidence Intervals for τ^2 and h^2 Constructed via Un-pruned ANOVA (Anova), ANOVA with Prune Threshold 4×10^{-5} [Anova_prune ($r^2 = 4 \times 10^{-5}$)], Matching-pruned ANOVA (Anova_prune), and Un-pruned mmhe(mmhe). Methods: $p = 1000$, $\omega = 0.1$, $\tau^2 = 0.4$, $h^2 = 0.6$, AR(1) SNPs with $\rho = 0.2$.

n	Anova				Anova_prune (4×10^{-5})				Anova_prune				mmhe			
	τ^2		h^2		τ^2		h^2		τ^2		h^2		τ^2		h^2	
	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP	ML	CP
1000	.287	.992	.336	.976	.217	.945	.285	.940	.217	.924	.285	.950	.067	.353	.169	.784
10000	.113	.987	.185	.950	.043	.823	.168	.932	.043	.605	.171	.952	.007	.273	.017	.147
30000	.106	.989	.174	.960	.023	.654	.162	.931	.024	.617	.164	.945	.002	.082	.006	.075
50000	.102	.995	.180	.958	.018	.561	.160	.936	.018	.548	.163	.946	.001	.147	.003	.069

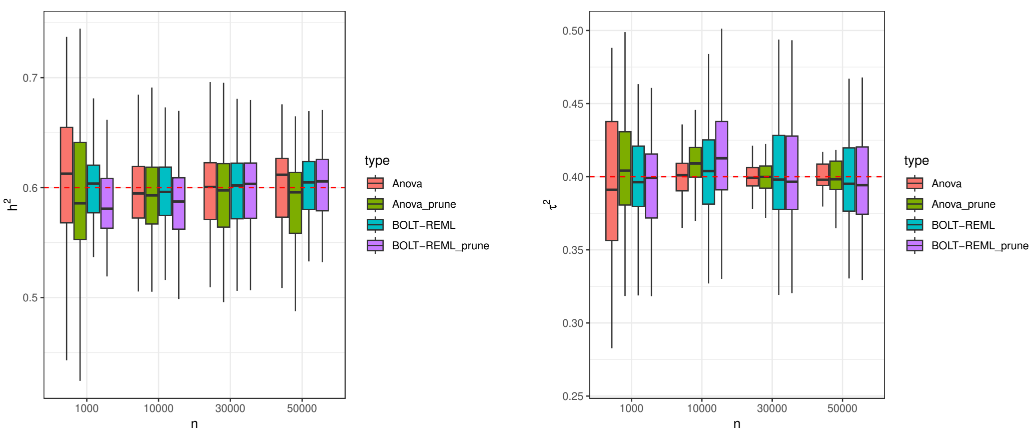
5.4 The matching-prune strategy

In this subsection, we carry out another simulation study, this time on the performance of the matching-prune strategy proposed in Section 7 of MS. To this end, we consider a grid of candidate correlation thresholds for the pruning (see Section 7 of MS), namely, $\mathcal{G} = \{0, 5 \times 10^{-5}, 10^{-4}, 5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}, 10^{-2}\}$. It should be noted that the grid is not very dense; nevertheless, the goal is to have some idea on how the matching-prune strategy performs compared to no pruning and a fixed threshold pruning strategy. From Section 5.3, in particular, Figure 2, it appeared that the threshold $r^2_{\text{opt}} = 4 \times 10^{-5}$ is optimal among the three thresholds considered (the subscript opt refers to "optimal"). However, this was based on the simulation results, in which one knows the truth (that is, the true underlying model that generates the data). In practice, however, one may not be able to determine the "optimal" threshold this way. Nevertheless, here, we compare the matching strategy with the optimal pruning, r_{opt}, and with the method without pruning.

We first consider the estimation performance. In Section 5.3, in particular, Figure 2 (b) and Figure 4, we already compared method without pruning (Anova) with the r_{opt} pruning method and several other methods. Thus, here, we can focus on comparing the matching strategy with Anova to see their relative performance. The boxplots for the matching-prune based method (Anova_prune), the method without pruning (Anova), and the corresponding BOLT-REML methods (with the matching-prune and without pruning) are presented in Figure 5. It is seen that Anova_prune does not perform as well as the other three methods for estimating h^2, although for $n = 30000$ the four methods seem to perform similarly. On the other hand, for estimating τ^2, the two ANOVA methods seem to perform much better than the REML methods once n is greater than 1000, although Anova_prune still appears to perform slightly worse than Anova.

Next, we compare the inferential performance of the matching-pruned ANOVA

(Anova_prune) with the un-pruned ANOVA (Anova) and ANOVA with the "optimal" pruning threshold $r^2_{\text{opt}} = 4 \times 10^{-5}$ [Anova_prune ($r^2 = 4 \times 10^{-5}$)] in terms of the confidence intervals. The results are presented in Table 1. It appears that the ANOVA method with the matching pruning and that with the "optimal" pruning perform similarly in both ML and CP; both are not as accurate as the un-pruned ANOVA in terms of CP, but having smaller ML compared to the un-pruned ANOVA. As noted in MS (see the discussion about Table 1 in MS), because p is fixed ($= 1000$), the ML for h^2 does not necessarily decrease with n.



(a) (b)

Figure 5: Estimation of h^2 and τ^2 using Un-pruned ANOVA (Anova), Matching-pruned ANOVA (Anova_prune), Un-pruned BOLT-REML (BOLT-REML), and Matching-pruned BOLT-REML (BOLT-REML_prune) Methods: $p = 1000$, $\omega = 0.1$, $\tau^2 = 0.4$, $h^2 = 0.6$, SNPs Are Correlated with AR(1) Structure with $\rho = 0.2$.

6 Additional results of Big-data application

We applied the variance estimator under the C-dependent assumption (Theorem 5.1) to real data from the UK Biobank, examining BMI and height in European samples. The analysis follows the same matching settings as the main big-data application section. We first conducted a genome-wide association study (GWAS) to derive marginal association p-values for each SNP, retaining those with $p < 0.01$ as marginally significant associations. The selected SNPs were then further filtered using a correlation threshold of $r^2 \leq 4 \times 10^{-5}$ and pruned within a 100 kb window.

We note that the ANOVA point estimator is unaffected by the choice of dependence assumption; only the variance estimator and its associated standard error (s.e.) change. For clarity, we refer to the ANOVA estimator paired with the s.e. derived under the independence assumption as *ANOVA-independent*, and the ANOVA estimator paired with the s.e. derived under the C-dependent assumption as *ANOVA-C-dependent*, for $C \in \{0, 5, 10, 15, 20, 25, 30\}$. The parameter C controls the number of neighboring SNPs incorporated into the dependence structure and represents the proposed extension of the variance estimation framework. For each trait, we report the heritability estimates together with both the independence-based and correlation-adjusted standard errors.

As shown in Figure 6, the estimated standard errors increase monotonically with C, consistent with the simulation results reported in simulations. While theoretically the s.e. under $C = 0$ and the s.e. under the independence assumption should coincide (since both rely on the same r values defined in Assumption A.2), this equivalence holds only for the population-level (theoretical) values of those quantities. In practice, these expectations are approximated by sample means, which introduces a discrepancy. In the simulation study, the z-scores were generated from a known distribution with a known dependence structure, so the sample means closely approximated the corresponding theoretical r values, yielding near-identical results under $C = 0$ and under independence. In the real-data analysis, however, the true distribution of the z-scores and the true value of C are unknown, which may explain why the s.e. under $C = 0$ does not coincide exactly with the s.e. under the independence assumption.

Additionally, the filtering and pruning procedure is designed to approximate independence among retained SNPs (targeting $r^2 \leq 4 \times 10^{-5}$), yet the degree to which this is achieved appears to differ across traits. Specifically, the SNPs appear closer to independence for BMI than for height, as evidenced by the marginally significant difference between the ANOVA and BOLT-REML estimates in the height case ($p < 0.01$), compared to the closer agreement observed for BMI. This may further explain why the s.e. under the independence assumption is nearly indistinguishable from the s.e. under $C = 0$ for BMI, whereas a more pronounced discrepancy is observed for height. Taken together, these results suggest that the appropriate choice of C is trait dependent, and that selecting C based on the observed agreement between the independence-based and C-dependent standard errors may serve as a practical diagnostic in applications.

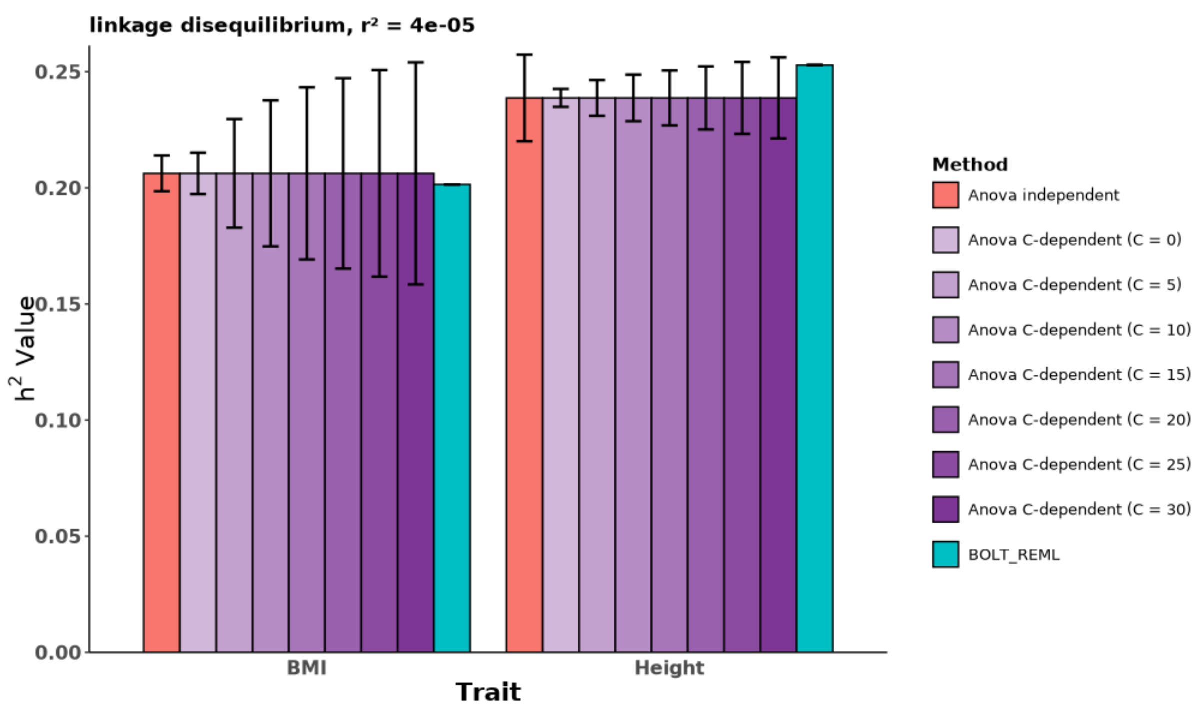


Figure 6: Heritability Estimates for BMI and Height Using ANOVA independent, ANOVA C-dependent, and BOLT-REML

7 Other extensions

This section supplements Section 5 of MS to discuss other directions.

First, as in Jiang *et al.* (2016), we can relax the normality assumption about Z to sub-Gaussian random variables with mean 0 and variance 1. A random variable, Y, is sub-Gaussian if there is a constant $d > 0$ such that for any real number λ we have $\mathrm{E}(e^{\lambda Y}) \leq e^{\lambda^2 d^2/2}$. The arguments of the proofs (see the supplement) change slightly, because now the entries of W and δ in (3.17) in MS are not automatically independent random variables. Nevertheless, it can be shown, as in Jiang *et al.* (2016; Remark 3.5), that all of the results established in the previous sections remain valid provided that (3.27) of MS is strengthened to

$$q = o(\sqrt{n}). \tag{60}$$

Also, as in Jiang *et al.* (2016), under (60) we can relax the assumption that X is non-random. Note that this assumption is equivalent to that X is independent with $\eta = (Z, b, \zeta, \epsilon)$. However, as argued in Jiang *et al.* (2016; Remark 3.5), under (60), the independence of X with η is not needed.

Furthermore, as argued in Jiang *et al.* (2016; 5th paragraph of sec. 1.3), the above results can be extended to the case where Z is standardized such that the sample mean and sample variance of each column of Z is 0 and 1, respectively.

Finally, it is possible to extend the asymptotic theory to the case where the columns of Z are correlated. Typically in GWAS, the entries of Z are standardized so that $\mathrm{E}(z_{ij}) = 0$ and $\mathrm{var}(z_{ij}) = 1$. However, there may be correlations among $z_{ij}, 1 \leq j \leq p$. This is reasonable, because the columns of Z, are associated with the SNPs, and there may be correlations among the nearby SNPs. On the other hand, the rows of Z correspond to different individuals; it is reasonable to assume that the individuals are independent, provided that there are no family associations among the individuals. Let z_i' denote the ith row of Z, $1 \leq i \leq n$. In view of the above discussion, we assume that $z_i, 1 \leq i \leq n$ are independent such that $\mathrm{E}(z_i) = 0$ and $\mathrm{Var}(z_i) = R$, where $R = (r_{jk})_{1\leq j,k\leq p}$ is a non-singular correlation matrix (which implies that $r_{jj} = 1, 1 \leq j \leq p$). More specifically, let W be a symmetric square root of R such that $R = W^2$. We assume that $z_i = W\gamma_i$, where $\gamma_i = (\gamma_{ij})_{1\leq j\leq p}$ such that the entries of $\Gamma = (\gamma_{ij})_{1\leq i\leq n, 1\leq j\leq p}$ are i.i.d. with $\mathrm{E}(\gamma_{ij}) = 0$, $\mathrm{E}(\gamma_{ij}^2) = 1$ and $\mathrm{E}(\gamma_{ij}^8) < \infty$. It follows that $Z = \Gamma W$, where the ith row of Γ is γ_i' and jth column of Γ is denoted by Γ_j, and W a non-random, positive definite matrix. It can be shown that the consistency result, that is, part (I) of Theorem 3.1, remains valid provided that

$$p^{-1}\mathrm{tr}(R^2) = O(1) \text{ and } p^{-1}\lambda_{\max}(R^2) = o(1). \tag{61}$$

The asymptotic normality result, that is, part (II) of Theorem 3.1, holds with a different asymptotic covariance matrix, provided that (61) holds but with the first part of it strengthened to that

$$p^{-1}\mathrm{tr}(R^2) \longrightarrow \rho \tag{62}$$

for some constant $\rho \in (0, \infty)$. The result follows from the proof of Theorem 3.1 with the help from extensions of some intermediate results such as the following, which itself may be of interest. The proof is given in the supplement.

Lemma A.1. Under the above setting of correlation structure for Z, we have $(np)^{-1}\mathrm{tr}(ZZ') \xrightarrow{\mathrm{P}} 1$. Furthermore, if (61) holds with the first part strengthened to (62), where ρ is either finite or infinity, then, we have

$$(np^2)^{-1}\mathrm{tr}((ZZ')^2) \xrightarrow{\mathrm{P}} 1 + \rho\gamma, \tag{63}$$

which is understood as ∞ if $\rho = \infty$.

The other main theoretical results can also be extended accordingly. We defer the details to a future publication.

Condition (61) holds for some of the popular correlation structures, but not for the others. For example, it holds for exponentially-decaying correlations, that is,

$|r_{jk}| \leq e^{-a|j-k|}$ for some $a > 0$, which is often associated with an autoregressive (AR) process; on the other hand, the condition does not hold for the equal-correlation structure, that is, $r_{jk} = \varrho$, $j \neq k$ for some $0 < \varrho < 1$. The latter may arise in the case of intraclass correlation (e.g., Diggle *et al.* 2002).

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