TAIL RISK EQUIVALENT LEVEL TRANSITION AND ITS APPLICATION FOR ESTIMATING EXTREME L_p-QUANTILES

Qingzhao Zhong and Yanxi Hou

School of Data Science, Fudan University, Shanghai, China.

Supplementary Material

The theoretical statements of CTRELT and dual CTRELT for case of $p-\frac{1}{\gamma}<1-q$ are included in Section S1. We provide some auxiliary results in Section S2 to support our assertions in the main paper. All technical details of main results are collected in Section S3. Section S4 contains necessary additional simulation results and analyses.

S1 CTRELT and dual TRELT in another case

In this section, we start a discussion of CTRELT and dual CTRELT for the case of $p-\frac{1}{\gamma}<1-q$, which can be defined in a similar way to the case of $1-q<p-\frac{1}{\gamma}<1$. Hence, we only exhibit these definitions without further elaboration. On the one hand, the CTRELT $\Pi'_{p,q}(\varepsilon)$ is a multiplier

Corresponding author: Yanxi Hou, yxhou@fudan.edu.cn.

$c' \in \left[1, \frac{1-\tau_0'}{\varepsilon}\right]$ solving

$$\theta_q(1 - c'\varepsilon) = \theta_p(1 - \varepsilon), \quad \varepsilon \in (0, 1 - \tau_0'), \tag{S1.1}$$

where τ_0' can be defined by

$$\tau_0' = \begin{cases} \sup_{\tau \in (0,1)} \left\{ \tau \mid \theta_q(\tau) \le \theta_p(\tau) \right\}, & \text{if } \left\{ \tau \mid \theta_q(\tau) \le \theta_p(\tau) \right\} \ne \emptyset, \\ 0, & \text{if } \left\{ \tau \mid \theta_q(\tau) \le \theta_p(\tau) \right\} = \emptyset. \end{cases} \tag{S1.2}$$

Its formal definition is given by

$$\Pi'_{p,q}(\varepsilon) = \inf_{c' \in \left[1, \frac{1-\tau_0'}{\varepsilon}\right]} \left\{ c' \mid \theta_q(1 - c'\varepsilon) \le \theta_p(1 - \varepsilon) \right\}. \tag{S1.3}$$

On the other hand, the dual CTRELT $\pi'_{p,q}(\varepsilon)$ is a multiplier $d' \in [1, \infty)$ solving

$$\theta_q(1 - \varepsilon) = \theta_p\left(1 - \frac{\varepsilon}{d'}\right), \quad \varepsilon \in (0, 1 - \tau_0'), \tag{S1.4}$$

with the formal definition given by

$$\pi'_{p,q}(\varepsilon) = \inf_{d' \in [1,\infty)} \left\{ d' \,\middle|\, \theta_q(1 - \varepsilon) \le \theta_p\left(1 - \frac{\varepsilon}{d'}\right) \right\}. \tag{S1.5}$$

The analyses of the existence and uniqueness for $\Pi'_{p,q}(\varepsilon)$, $\pi'_{p,q}(\varepsilon)$ are quite similar to $\Pi_{p,q}(\varepsilon)$, $\pi_{p,q}(\varepsilon)$ and we just present them below without proof.

Assumption S1. *For all $p > q \ge 1$, there exists a threshold τ_0' such that* *(a)* $\theta_q(\tau_0') \leq \theta_p(1-\varepsilon)$ *for all* $\varepsilon \in (0, 1-\tau_0')$*;*

(b) both $\theta_p(\tau)$ *and* $\theta_q(\tau)$ *are not constants on* $[\tau_0', 1]$.

Proposition S1 (Existence and Uniqueness of CTRELT)**.** *Suppose* F *satisfies both Assumptions 1 and 2 with* p, q *satisfying* $1 \leq q < p$, $p - \frac{1}{\gamma} < 1 - q$. *Then, for all* $\varepsilon \in (0, 1-\tau_0')$*, there exists* $c' \in \left[1, \frac{1-\tau_0'}{\varepsilon}\right]$ *such that* (S1.1) *holds for* $c' = \Pi'_{p,q}(\varepsilon)$ *if and only if Assumption S1 (a) holds. Moreover, if Assumption S1 (b) also holds, then the* $c' \in \left[1, \frac{1-\tau_0'}{\varepsilon}\right]$ *in* (S1.1) *is unique.*

Proposition S2 (Existence and Uniqueness of dual CTRELT)**.** *Suppose* F *satisfies both Assumptions 1 and 2 with* p, q *satisfying* $1 \leq q < p$, $p - \frac{1}{\gamma} < 1 - q$*. Then, for all* $\varepsilon \in (0, 1-\tau_0')$*, there exists* $d' \in [1, \infty)$ *such that* (S1.4) *holds for* $d' = \pi'_{p,q}(\varepsilon)$*. Moreover, if Assumption S1 (b) holds, then the* $d' \in [1, \infty)$ *in* (S1.4) *is unique.*

S2 Auxiliary results

Lemma S1. *Let* X *be a random variable with distribution* F *satisfying Assumption 1 for some* $\gamma > 0$*, then for all* $-\infty < x < U(\infty)$*(*$U(\infty)$ *is the right endpoint of* F*),*

$$E\left(|X|^{\iota}\mathbb{1}_{\{X>x\}}\right) < \infty,$$

if $0 < \iota < \frac{1}{\gamma}$ *and* $E\left(|X|^{\iota}\mathbb{1}_{\{X>x\}}\right) = \infty$ *if* $\iota > \frac{1}{\gamma}$.

Proof of Lemma S1. At first, we notice that

$$\begin{cases} E\left(|X|^\iota \mathbb{1}_{\{X>x\}}\right) = E\left(X^\iota \mathbb{1}_{\{X>x\}}\right), & \text{if } x \geq 0, \\ E\left(|X|^\iota \mathbb{1}_{\{X>x\}}\right) = E\left(X^\iota \mathbb{1}_{\{X\geq 0\}}\right) + E\left(|X|^\iota \mathbb{1}_{\{x<X<0\}}\right), & \text{if } x < 0. \end{cases}$$

Hence it's sufficient to consider $E\left(X^\iota \mathbb{1}_{\{X>x\}}\right)$ for $x \geq 0$ as the term $E\left(|X|^\iota \mathbb{1}_{\{x<X<0\}}\right)$ can be bounded by $|x|^\iota$ if $x < 0$. Since $X\mathbb{1}_{\{X>x\}} \geq 0$, we have

$$\begin{aligned} E\left(X^\iota \mathbb{1}_{\{X>x\}}\right) &= E\left((X\mathbb{1}_{\{X>x\}})^\iota\right) \\ &= x^\iota \mathbb{P}(X > x) + \iota \int_x^\infty s^{\iota-1}(1 - F(s))\, ds, \end{aligned}$$

by Fubini's theorem. Assumption 1 implies $s^{\iota-1}(1-F(s))$ is regular varying with index $\iota - 1 - \frac{1}{\gamma}$ and $\iota - 1 - \frac{1}{\gamma} < -1$ when $0 < \iota < \frac{1}{\gamma}$, we gets

$$\int_x^\infty s^{\iota-1}(1 - F(s))\, ds = \int_x^t s^{\iota-1}(1 - F(s))\, ds + \int_t^\infty s^{\iota-1}(1 - F(s))\, ds < \infty$$

for t sufficiently large by Proposition B.1.9 (4) in De Haan and Ferreira (2006). Otherwise, $\int_t^\infty s^{\iota-1}(1 - F(s))\, ds$ doesn't exist. □

Proposition S3. *Let X be a random variable with continuous distribution function F, $x_* = \inf_{x\in\mathbb{R}}\{\, x \mid F(x) > 0\}$ and $x^* = \inf_{x\in\mathbb{R}}\{\, x \mid F(x) \geq 1\}$ denote the left and right endpoints of X respectively. For $p > 1$, we have the following statements.*

1. *(Existence and Uniqueness) For all $\tau \in (0, 1)$, (1.2) has a unique solution $\theta_p(\tau) \in (x_*, x^*)$;*

2. *(Monotonicity in level) The mapping* $\tau \in (0,1) \mapsto \theta_p(\tau) \in \mathbb{R}$ *is strictly increasing, and*

$$\lim_{\tau \uparrow 1} \theta_p(\tau) = x^*, \ \ \lim_{\tau \downarrow 0} \theta_p(\tau) = x_*;$$

3. *(Continuity) The mapping* $\tau \in (0,1) \mapsto \theta_p(\tau) \in \mathbb{R}$ *is continuous.*

Proof of Proposition S3. The first statement for existence and uniqueness, and the second statement for monotonicity are straightforward consequences of Proposition 3.1 in Mao, Stupfler and Yang (2023) and Proposition 1 in Chen (1996) respectively. It remains to show the continuity. We define a monotonically increasing sequence $\{\tau_n\}$ such that $\tau_n \uparrow \tau$. We denote $\theta^- := \lim_{n\to\infty} \theta_p(\tau_n)$. Obviously, we have $\theta_p(\tau_n) \le \theta^- \le \theta_p(\tau)$. As shown in Mao, Stupfler and Yang (2023), the mappings $\theta \mapsto E\left((X-\theta)_+^{p-1}\right)$ and $\theta \mapsto E\left((X-\theta)_-^{p-1}\right)$ are strictly decreasing and increasing on (x_*, x^*). Then, we have,

$$\begin{aligned}\tau_n E\left((X-\theta_p(\tau_n))_+^{p-1}\right) \ge \tau_n E\left((X-\theta_p(\tau))_+^{p-1}\right) =& \tau_n \frac{1-\tau}{\tau} E\left((X-\theta_p(\tau))_-^{p-1}\right) \\ \ge& \tau_n \frac{1-\tau}{\tau} E\left((X-\theta_p(\tau_n))_-^{p-1}\right).\end{aligned}$$

We take $n \to \infty$ both sides of the above equation, and we get

$$\begin{aligned}\tau E\left((X-\theta^-)_+^{p-1}\right) = \tau E\left((X-\theta_p(\tau))_+^{p-1}\right) =& (1-\tau) E\left((X-\theta_p(\tau))_-^{p-1}\right) \\ =& (1-\tau) E\left((X-\theta^-)_-^{p-1}\right),\end{aligned}$$

which implies $\theta^- = \theta_p(\tau)$, that is $\theta_p(\tau)$ is left-continuous. On the other hand, we can also take $\tau_n \downarrow \tau$ and denote $\theta^+ := \lim_{n\to\infty} \theta_p(\tau_n)$. Apparently $\theta_p(\tau_n) \geq \theta^+ \geq \theta_p(\tau)$, and we also have,

$$\begin{aligned}(1-\tau_n)E\left((X-\theta_p(\tau_n))_-^{p-1}\right) &\geq (1-\tau_n)E\left((X-\theta_p(\tau))_-^{p-1}\right)\\ &= (1-\tau_n)\frac{\tau}{1-\tau}E\left((X-\theta_p(\tau))_+^{p-1}\right)\\ &\geq (1-\tau_n)\frac{\tau}{1-\tau}E\left((X-\theta_p(\tau_n))_+^{p-1}\right).\end{aligned}$$

We can obtain $\theta^+ = \theta_p(\tau)$ by taking the limit on both sides, which shows the right-continuity of $\theta_p(\tau)$.

□

Proposition S4. *Under the conditions of Theorem 2, for all $q > 1$, we have,*

$$\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} - 1\right) \xrightarrow{d} \mathcal{N}\left(\frac{\lambda(q-1)}{\gamma^{\rho-1}\rho}\frac{B(q-1,-(\rho-1)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}} - \frac{\lambda}{\rho}, \frac{\gamma}{4}\frac{[B(q,(2\gamma)^{-1}-q+1)]^2}{B(q,\gamma^{-1}-q+1)}\right). \tag{S2.6}$$

Proof of Proposition S4. In the proofs of Theorem 1 and Lemma 8 in Daouia, Girard and Stupfler (2019), it is the term $I_{2,n}$ that determines its asymptotic distribution, which is given by

$$
\begin{aligned}
I_n &:= \frac{1}{\sqrt{n\varepsilon_n}} \sum_{i=1}^{n} \frac{1}{[\theta_q(1-\varepsilon_n)]^{q-1}} \varphi_{1-\varepsilon_n}(X_i - \theta_q(1-\varepsilon_n); q) \\
&= \frac{1}{\sqrt{n\varepsilon_n}} \sum_{i=1}^{n} \frac{1}{[\theta_q(1-\varepsilon_n)]^{q-1}} \left\{ \varphi_{1-\varepsilon_n}(X_i - \theta_q(1-\varepsilon_n); q) \mathbb{1}_{\{X_i \le \theta_q(1-\varepsilon_n)\}} \right. \\
&\left. - E\left(\varphi_{1-\varepsilon_n}(X - \theta_q(1-\varepsilon_n); q) \mathbb{1}_{\{X_i \le \theta_q(1-\varepsilon_n)\}} \right) \right\} \\
&+ \frac{1}{\sqrt{n\varepsilon_n}} \sum_{i=1}^{n} \frac{1}{[\theta_q(1-\varepsilon_n)]^{q-1}} \left\{ \varphi_{1-\varepsilon_n}(X_i - \theta_q(1-\varepsilon_n); q) \mathbb{1}_{\{X_i > \theta_q(1-\varepsilon_n)\}} \right. \\
&\left. - E\left(\varphi_{1-\varepsilon_n}(X - \theta_q(1-\varepsilon_n); q) \mathbb{1}_{\{X_i > \theta_q(1-\varepsilon_n)\}} \right) \right\} \\
&=: I_{1,n} + I_{2,n},
\end{aligned}
$$

where $\varphi_\tau(y;q) = |\tau - \mathbb{1}_{\{y \le 0\}}||y|^{q-1}\text{sign}(y)$. It has been shown $I_{1,n} \xrightarrow{\mathbb{P}} 0$ and we need only discuss $I_{2,n}$,

$$
\begin{aligned}
I_{2,n} &= \frac{1}{\sqrt{n\varepsilon_n}} \sum_{i=1}^{n} \left\{ (1-\varepsilon_n) \left| \frac{X_i}{\theta_q(1-\varepsilon_n)} - 1 \right|^{q-1} \mathbb{1}_{\{X_i > \theta_q(1-\varepsilon_n)\}} \right. \\
&\left. -(1-\varepsilon_n) E\left(\left| \frac{X}{\theta_q(1-\varepsilon_n)} - 1 \right|^{q-1} \mathbb{1}_{\{X > \theta_q(1-\varepsilon_n)\}} \right) \right\} \\
&= (1-\varepsilon_n)\sqrt{\frac{n}{\varepsilon_n}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - 1 \right)_+^{q-1} - E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1 \right)_+^{q-1} \right) \right\}.
\end{aligned}
$$

Then the result follows from (S3.20) and (S3.21),

$$
\begin{aligned}
\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} - 1 \right) &= \gamma(q-1) \int_1^\infty (s-1)^{q-2} W_n\left(s^{-1/\gamma} \cdot \frac{\gamma}{B(q, \gamma^{-1} - q + 1)} \right) ds \\
&+ \frac{\lambda(q-1)}{\gamma^{\rho-1}\rho} \frac{B(q-1, -(\rho-1)/\gamma - q + 1)}{[B(q, \gamma^{-1} - q + 1)]^{1-\rho}} - \frac{\lambda}{\rho} + o_{\mathbb{P}}(1).
\end{aligned}
\tag{S2.7}
$$

□

Proposition S5. *Suppose F is strictly increasing and satisfies both Assumption 4 for some $\gamma > 0$ and Assumption 2 with an order $p \in (1, 1+1/\gamma)$. Then, for all $q \in [1, p)$, we have that,*

$$\frac{\theta_p(\tau)}{\theta_q(\tau)} = \mathcal{L}(\gamma, p, q)\left(1 - \gamma R(p, q, \gamma, \tau) + A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)\left\{\frac{1}{\rho}\left[\left[\frac{B(p, \gamma^{-1} - p + 1)}{B(q, \gamma^{-1} - q + 1)}\right]^{\rho} - 1\right] + o(1)\right\}\right), \tag{S2.8}$$

as $\tau \uparrow 1$, where

$$R(p, q, \gamma, \tau) = -\frac{\gamma}{B(p, \gamma^{-1} - p + 1)} A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right) K(p, q, \gamma, \rho)(1 + o(1))$$

$$-(p-1) r(\mathcal{L}(\gamma, p, q)\theta_q(\tau), p, \gamma, X),$$

$$K(p, q, \gamma, \rho) = \begin{cases} \left[\frac{B(p, \gamma^{-1} - p + 1)}{B(q, \gamma^{-1} - q + 1)}\right]^{\rho} \frac{1}{\gamma^2 \rho}[(1-\rho)B(p, (1-\rho)/\gamma - p + 1) \\ - B(p, \gamma^{-1} - p + 1)], & \text{if } \rho < 0, \\ \frac{p-1}{\gamma^2}\int_1^{+\infty} (x-1)^{p-2} x^{-\frac{1}{\gamma}} \log x \, dx, & \text{if } \rho = 0, \end{cases}$$

$$r(\theta_q(\tau), p, \gamma, X) = \begin{cases} \frac{E\left(X \mathbb{1}_{\{0 < X < \theta_q(\tau)\}}\right)}{\theta_q(\tau)}(1 + o(1)), & \text{if } \gamma \leq 1, \\ \overline{F}(\theta_q(\tau)) B(p - 1, 1 - \gamma^{-1})(1 + o(1)), & \text{if } \gamma > 1. \end{cases}$$

In particular, if $q = 1$, then (S2.8) *becomes, as $\tau \uparrow 1$,*

$$\frac{\theta_p(\tau)}{\theta_1(\tau)} = \mathcal{L}(\gamma, p, 1)\left(1 - \gamma R(p, 1, \gamma, \tau) + A\left(\frac{1}{1-\tau}\right)\left\{\frac{1}{\rho}\left[\left[\frac{B(p, \gamma^{-1} - p + 1)}{\gamma}\right]^{\rho} - 1\right] + o(1)\right\}\right). \tag{S2.9}$$

Proof of Proposition S5. From the definition of $\theta_p(\tau)$, it follows that,

$$E\left(\left(\frac{X}{\theta_p(\tau)}-1\right)^{p-1}\mathbb{1}_{\{X>\theta_p(\tau)\}}\right)=(1-\tau)E\left(\left|\frac{X}{\theta_p(\tau)}-1\right|^{p-1}\right). \quad \text{(S2.10)}$$

Step 1: we work on the left-hand side of (S2.10)**.** We first put a useful result here

$$\frac{\overline{F}(\theta_p(\tau))}{\overline{F}(\theta_q(\tau))}=\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}(1+o(1)), \quad \text{(S2.11)}$$

which can be derived from the proof of Proposition 1 in Daouia, Girard and Stupfler (2019). Then, by regular variation of A (for example, Theorem 2.3.3 in De Haan and Ferreira (2006)), we have

$$A\left(\frac{1}{\overline{F}(\theta_p(\tau))}\right)=\left[\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}\right]^{\rho}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)(1+o(1)). \quad \text{(S2.12)}$$

If we apply second-order regular variation condition, the left-hand side of (S2.10) becomes

$$\begin{aligned}
&E\left(\left(\frac{X}{\theta_p(\tau)}-1\right)^{p-1}\mathbb{1}_{\{X>\theta_p(\tau)\}}\right)\\
=&\overline{F}(\theta_p(\tau))\left[\int_1^{+\infty}(p-1)(x-1)^{p-2}x^{-\frac{1}{\gamma}}\,dx+\int_1^{+\infty}\left[\frac{\overline{F}(\theta_p(\tau)x)}{\overline{F}(\theta_p(\tau))}-x^{-\frac{1}{\gamma}}\right](p-1)(x-1)^{p-2}\,dx\right]\\
=&\overline{F}(\theta_p(\tau))\left[\frac{B(p,\gamma^{-1}-p+1)}{\gamma}+A\left(\frac{1}{\overline{F}(\theta_p(\tau))}\right)\int_1^{+\infty}(p-1)(x-1)^{p-2}x^{-\frac{1}{\gamma}}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx(1+o(1))\right]\\
=&\overline{F}(\theta_p(\tau))\left[\frac{B(p,\gamma^{-1}-p+1)}{\gamma}+A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)(1+o(1))\right],
\end{aligned} \quad \text{(S2.13)}$$

where the last step follows that, by (S2.12),

$$
\begin{aligned}
&A\left(\frac{1}{\overline{F}(\theta_p(\tau))}\right)\int_1^{+\infty}(p-1)(x-1)^{p-2}x^{-\frac{1}{\gamma}}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\,dx\\
=&\begin{cases}\left[\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}\right]^{\rho}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)\frac{1}{\gamma^2\rho}[(1-\rho)B(p,(1-\rho)/\gamma-p+1)\\
-B(p,\gamma^{-1}-p+1)](1+o(1)), & \text{if } \rho<0,\\
A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)\frac{p-1}{\gamma^2}\int_1^{+\infty}(x-1)^{p-2}x^{-\frac{1}{\gamma}}\log x\,dx(1+o(1)), & \text{if } \rho=0,\end{cases}\\
=&A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)(1+o(1)).
\end{aligned}
$$

Step 2: we work on the right-hand side of (S2.10)**.** It can be decomposed as,

$$
\begin{aligned}
E\left(\left|\frac{X}{\theta_p(\tau)}-1\right|^{p-1}\right)&=E\left(\left(\frac{X}{\theta_p(\tau)}-1\right)^{p-1}\mathbb{1}_{\{X>\theta_p(\tau)\}}\right)\\
&+E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{0\le X\le\theta_p(\tau)\}}\right)\\
&+E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{X\le 0\}}\right)\\
&=:I+II+III.
\end{aligned}
$$

For I, same arguments in proof of Proposition 1 in Daouia, Girard and Stupfler (2019) yield $I\to 0$ as $\tau\uparrow 1$.

For III, we note that

$$
\begin{aligned}
&E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{X\leq 0\}}\right)\\
=&E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{X\leq -\theta_p(\tau)\}}\right)+E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{-\theta_p(\tau)\leq X\leq 0\}}\right)\\
=&o(1)+\mathbb{P}(X\leq 0)(1+o(1)),
\end{aligned}
$$

by dominated convergence theorem.

For II, we also note

$$
\begin{aligned}
E\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}\mathbb{1}_{\{0\leq X\leq \theta_p(\tau)\}}\right)=&E\left(\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}-1\right)\mathbb{1}_{\{0\leq X\leq \theta_p(\tau)\}}\right)\\
&+1-\overline{F}(\theta_p(\tau))-\mathbb{P}(X\leq 0).
\end{aligned}
$$

Let $H(x)=-(p-1)^{-1}(1-x)^{p-1}\mathbb{1}_{\{0\leq x\leq 1\}}$, we further apply Lemma 1 $(iii),(iv)$ and (v) given in the supplementary material of Daouia, Girard and Stupfler (2019), to obtain

$$
\begin{aligned}
&E\left(\left(\left(1-\frac{X}{\theta_p(\tau)}\right)^{p-1}-1\right)\mathbb{1}_{\{0\le X\le\theta_p(\tau)\}}\right)\\
=&-(p-1)E\left(\left(H\left(\frac{X}{\theta_p(\tau)}\right)-H(0)\right)\mathbb{1}_{\{X>0\}}\right)+\mathbb{P}(X>\theta_p(\tau))\\
=&-(p-1)\begin{cases}\frac{E\left(X\mathbb{1}_{\{X>0\}}\right)}{\theta_p(\tau)}(1+o(1)), & \text{if } \gamma<1 \text{ or } \gamma=1 \text{ and } E\left(X_+\right)<\infty\\ \frac{E\left(X\mathbb{1}_{\{0<X<\theta_p(\tau)\}}\right)}{\theta_p(\tau)}(1+o(1)), & \text{if } \gamma=1 \text{ and } E\left(X_+\right)=\infty\\ \overline{F}(\theta_p(\tau))B(p-1,1-\gamma^{-1})(1+o(1)), & \text{if } \gamma>1.\end{cases}\\
&+\overline{F}(\theta_p(\tau)).
\end{aligned}
$$

We can denote $\theta_p(\tau)$ in terms of $\theta_q(\tau)$ via (2.11), and let

$$
\begin{aligned}
r(\theta_p(\tau),p,\gamma,X)=&\begin{cases}\frac{E\left(X\mathbb{1}_{\{X>0\}}\right)}{\theta_p(\tau)}(1+o(1)), & \text{if } \gamma<1 \text{ or } \gamma=1 \text{ and } E\left(X_+\right)<\infty\\ \frac{E\left(X\mathbb{1}_{\{0<X<\theta_p(\tau)\}}\right)}{\theta_p(\tau)}(1+o(1)), & \text{if } \gamma=1 \text{ and } E\left(X_+\right)=\infty\\ \overline{F}(\theta_p(\tau))B(p-1,1-\gamma^{-1})(1+o(1)), & \text{if } \gamma>1\end{cases}\\
=&\begin{cases}\frac{E\left(X\mathbb{1}_{\{0<X<\theta_p(\tau)\}}\right)}{\theta_p(\tau)}(1+o(1)), & \text{if } \gamma\le 1\\ \overline{F}(\theta_p(\tau))B(p-1,1-\gamma^{-1})(1+o(1)), & \text{if } \gamma>1\end{cases}\\
=&\begin{cases}\mathcal{L}(\gamma,p,q)^{-1}\frac{E\left(X\mathbb{1}_{\{0<X<\mathcal{L}(\gamma,p,q)\theta_q(\tau)\}}\right)}{\theta_q(\tau)}(1+o(1)), & \text{if } \gamma\le 1\\ \mathbb{P}(X>\mathcal{L}(\gamma,p,q)\theta_q(\tau))B(p-1,1-\gamma^{-1})(1+o(1)), & \text{if } \gamma>1\end{cases}\\
=&\,r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X).
\end{aligned}
$$

From above arguments, we can claim that,

$$E\left(\left|\frac{X}{\theta_p(\tau)}-1\right|^{p-1}\right)=1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X). \tag{S2.14}$$

Step 3: we derive the final conclusion. Combining (S2.10), (S2.13) and (S2.14), (S2.10) becomes,

$$\begin{aligned}&\overline{F}(\theta_p(\tau))\left[\frac{B(p,\gamma^{-1}-p+1)}{\gamma}+A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)(1+o(1))\right]\\=&(1-\tau)(1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X)).\end{aligned}$$

This indicates that,

$$\begin{aligned}&\frac{\overline{F}(\theta_p(\tau))}{\overline{F}(\theta_q(\tau))}\frac{\gamma(1-\tau)}{B(q,\gamma^{-1}-q+1)}\left[\frac{B(p,\gamma^{-1}-p+1)}{\gamma}+A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)\right](1+o(1))\\=&(1-\tau)(1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X)),\end{aligned}$$

by substituting $\overline{F}(\theta_q(\tau))=\frac{\gamma(1-\tau)}{B(q,\gamma^{-1}-q+1)}(1+o(1))$ (use (S2.11) again with $q=1$).

Therefore, it follows that

$$\begin{aligned}
&\frac{\overline{F}(\theta_p(\tau))}{\overline{F}(\theta_q(\tau))}\\
=&\left[\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}+\frac{\gamma}{B(q,\gamma^{-1}-q+1)}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)\right]^{-1}\\
&\times(1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X))(1+o(1))\\
=&\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\left[1+\frac{\gamma}{B(p,\gamma^{-1}-p+1)}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)\right]^{-1}\\
&\times(1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X))(1+o(1))\\
=&\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\left[1-\frac{\gamma}{B(p,\gamma^{-1}-p+1)}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)\right]\\
&\times(1-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X))(1+o(1))\\
=&\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\left(1-\frac{\gamma}{B(p,\gamma^{-1}-p+1)}A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)K(p,q,\gamma,\rho)(1+o(1))\right.\\
&\left.-(p-1)r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X)\right)\\
=&\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}(1+R(p,q,\gamma,\tau)),
\end{aligned}$$

where the third equality follows from a straightforward Taylor's expansion of $x\mapsto(1+x)^{-1}$ in a neighborhood of 0. Moreover, it is readily to check that the remainder term $R(p,q,\gamma,\tau)\to 0$ as $\tau\to 1$ since $A\left(\frac{1}{\overline{F}(\theta_q(\tau))}\right)$ and $r(\mathcal{L}(\gamma,p,q)\theta_q(\tau),p,\gamma,X)$ indeed tend to 0 as $\tau\to 1$. Applying Taylor's expansion again, we have

$$\frac{\overline{F}(\theta_q(\tau))}{\overline{F}(\theta_p(\tau))}=\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}(1-R(p,q,\gamma,\tau)(1+o(1))),$$

and the final result follows from the Lemma 2 in the supplementary material of Daouia, Girard and Stupfler (2019) immediately. □

S3 Proofs of main results

S3.1 Some intermediate lemmas

Lemma S2. *Under the conditions of Theorem 2, for all $\alpha \geq 0$ and $q > 1$, then we have,*

$$\begin{aligned}
&\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{\alpha} - E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{\alpha}\right)\right)\\
=&\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{\alpha} - E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{\alpha}\right)\right) + o_{\mathbb{P}}(1).
\end{aligned}$$

Proof of Lemma S2. First, by Chebyshev's inequality and Proposition S4, it follows, for a $e > 0$,

$$\mathbb{P}\left(\left|\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} - 1\right| \geq e\right) = \mathbb{P}\left(\left|\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} - 1\right)\right| \geq \sqrt{n\varepsilon_n}e\right) \leq \frac{V(\lambda,\gamma,q)}{e^2 n\varepsilon_n} \to 0,$$

where $V(\lambda,\gamma,q)$ is the second moment of the limit distribution in Proposition S4. This implies $\eta_n := \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \xrightarrow{\mathbb{P}} 1$ and hence we can find a $\delta > 0$ such that $\eta_n \in [1-\delta, 1+\delta]$ in probability. We consider the changing classes $\mathcal{F}_n = \left\{\left(\frac{x}{\theta_q(1-\varepsilon_n)} - t\right)_+^{\alpha} : t \in T =: [1-\delta, 1+\delta]\right\}$, which is a sequence of classes of measurable functions $x \mapsto \left(\frac{x}{\theta_q(1-\varepsilon_n)} - t\right)_+^{\alpha}$ indexed by the parameter t, belonging to a common index set T. We note that the absolute metric $|\cdot|$ always makes T into a totally bounded space such that, for a positive sequence δ_n,

$$
\begin{aligned}
&\sup_{|t-1|<\delta_n} E\left(\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-t\right)_+^{\alpha}-\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{\alpha}\right)^2\right)\\
\leq &\begin{cases}\sup_{|t-1|<\delta_n} E\left(2\alpha^2\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{2(\alpha-1)}(t-1)^2+O((t-1)^4)\right), & \text{if } \alpha>0\\ \sup_{|t-1|<\delta_n}\left\{\mathbb{P}\left(t<\frac{X}{\theta_q(1-\varepsilon_n)}<1, t<1\right)+\mathbb{P}\left(1<\frac{X}{\theta_q(1-\varepsilon_n)}<t, t>1\right)\right\}, & \text{if } \alpha=0\end{cases}\\
\leq &\begin{cases}2\alpha^2\delta_n^2 E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{2(\alpha-1)}\right)+O(\delta_n^4), & \text{if } \alpha>0\\ \mathbb{P}\left(1-\delta_n<\frac{X}{\theta_q(1-\varepsilon_n)}<1\right)+\mathbb{P}\left(1<\frac{X}{\theta_q(1-\varepsilon_n)}<1+\delta_n\right), & \text{if } \alpha=0\end{cases}\\
\to &0, \text{ as } \delta_n\downarrow 0.
\end{aligned}
$$

Here, the first step for $\alpha>0$ follows from the Taylor's expansion on $\theta\mapsto\left(\frac{X}{\theta_q(1-\varepsilon_n)}-\theta\right)_+^{\alpha}$ and the inequality $(a+b)^2\leq 2a^2+2b^2$, and last step for $\alpha=0$ follows from dominated convergence theorem.

Moreover, for $\alpha>0$, there exists constants M_1, M_2 such that, for all

$t, t' \in T$,

$$\begin{aligned}
&\left|\left(\frac{x}{\theta_q(1-\varepsilon_n)} - t\right)_+^\alpha - \left(\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right)_+^\alpha\right| \\
=&\left|(-\alpha)\left(\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right)_+^{\alpha-1}(t-t') + O((t-t')^2)\right| \\
\leq&\alpha\left|\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right|^{\alpha-1}|t-t'| + O(2\delta\cdot|t-t'|) \\
\leq&\begin{cases} M_1\left|\frac{x}{\theta_q(1-\varepsilon_n)} - (1-\delta)\right|^{\alpha-1}|t-t'|, & \text{if } \alpha \geq 1, \\ M_2\left|\frac{x}{\theta_q(1-\varepsilon_n)} - (1+\delta)\right|^{\alpha-1}|t-t'|, & \text{if } \alpha < 1, \end{cases}
\end{aligned}$$

and

$$E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - (1-\delta)\right|^{2(\alpha-1)}\right) < \infty \quad \text{and} \quad E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - (1+\delta)\right|^{2(\alpha-1)}\right) < \infty.$$

Then by Example 19.7 in Van der Vaart (2000), there exists a constant K, depending on T and its dimension, such that the bracketing numbers satisfy,

$$N_{[\,]}(\epsilon, \mathcal{F}_n, L_2(\mathbb{P})) \leq K\left(\frac{\delta}{\epsilon}\right),$$

for every $0 < \epsilon < 2\delta$. Then the bracketing integral satisfies

$$J_{[\,]}(1, \mathcal{F}_n, L_2(\mathbb{P})) = \int_0^1 \sqrt{\log N_{[\,]}(\epsilon, \mathcal{F}_n, L_2(\mathbb{P}))}\, d\epsilon \leq K' \int_0^1 \sqrt{-\log\epsilon}\, d\epsilon < \infty. \tag{S3.15}$$

where K' is a general constant.

The changing class will reduce to $\mathcal{F}_n = \left\{\mathbb{1}_{\left\{\frac{x}{\theta_q(1-\varepsilon_n)} > t\right\}} : t \in T =: [1-\delta, 1+\delta]\right\}$ when $\alpha = 0$. Example 19.11 in Van der Vaart (2000) implies that the bracketing numbers satisfy,

$$N_{[\,]}(\epsilon, \mathcal{F}_n, L_2(\mathbb{P})) \leq K' \left(\frac{1}{\epsilon}\right),$$

and the bracketing integral is given by

$$J_{[\,]}(1, \mathcal{F}_n, L_2(\mathbb{P})) = \int_0^1 \sqrt{\log N_{[\,]}(\epsilon, \mathcal{F}_n, L_2(\mathbb{P}))}\, d\epsilon \leq K' \int_0^1 \sqrt{-\log \epsilon}\, d\epsilon < \infty. \tag{S3.16}$$

Both (S3.15) and (S3.16) imply $\mathcal{F}_n$ is a Donsker class for any given n. Then this result follows from Lemma 19.24 in Van der Vaart (2000).

□

Lemma S3. *Under the conditions of Theorem 2, for all* $\alpha > 0$ *and* $q > 1$*, then we have,*

$$\begin{aligned}
&\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right|^{\alpha} - E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right|^{\alpha}\right)\right) \\
=&\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_q(1-\varepsilon_n)} - 1\right|^{\alpha} - E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{\alpha}\right)\right) + o_{\mathbb{P}}(1).
\end{aligned}$$

Proof of Lemma S3. We consider $\mathcal{G}_n = \left\{\left|\frac{x}{\theta_q(1-\varepsilon_n)} - t\right|^{\alpha} : t \in T =: [1-\delta, 1+\delta]\right\}$. We note that, for a positive sequence δ_n,

$$\begin{aligned}
&\sup_{|t-1|<\delta_n} E\left(\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - t\right|^{\alpha} - \left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{\alpha}\right)^2\right) \\
\leq &\sup_{|t-1|<\delta_n} E\left(4\alpha^2\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{2(\alpha-1)}(t-1)^2 + O((t-1)^4)\right) \\
\leq &4\alpha^2\delta_n^2 E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{2(\alpha-1)}\right) + O(\delta_n^4) \to 0, \text{ as } \delta_n \downarrow 0.
\end{aligned}$$

Moreover, there exists M_3, M_4 such that, for all $t, t' \in T$,

$$
\begin{aligned}
&\left|\left|\frac{x}{\theta_q(1-\varepsilon_n)} - t\right|^{\alpha} - \left|\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right|^{\alpha}\right| \\
=&\left|\alpha\left|\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right|^{\alpha-1}\left(\mathbb{1}_{\left\{t' \geq \frac{x}{\theta_q(1-\varepsilon_n)}\right\}} - \mathbb{1}_{\left\{t' < \frac{x}{\theta_q(1-\varepsilon_n)}\right\}}\right)(t-t') + O((t-t')^2)\right| \\
\leq& 2\alpha\left|\frac{x}{\theta_q(1-\varepsilon_n)} - t'\right|^{\alpha}|t-t'| + O(|t-t'|^2) \\
\leq& \begin{cases} M_3\left|\frac{x}{\theta_q(1-\varepsilon_n)} - (1-\delta)\right|^{\alpha-1}|t-t'|, & \text{if } \alpha \geq 1, \\ M_4\left|\frac{x}{\theta_q(1-\varepsilon_n)} - (1+\delta)\right|^{\alpha-1}|t-t'|, & \text{if } \alpha < 1, \end{cases}
\end{aligned}
$$

and

$$
E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - (1-\delta)\right|^{2(\alpha-1)}\right) < \infty \quad \text{and} \quad E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - (1+\delta)\right|^{2(\alpha-1)}\right) < \infty.
$$

Then using Example 19.7 in Van der Vaart (2000) again, the bracketing integral is given by

$$
J_{[\,]}(1, \mathcal{G}_n, L_2(\mathbb{P})) = \int_0^1 \sqrt{\log N_{[\,]}(\epsilon, \mathcal{G}_n, L_2(\mathbb{P}))}\, d\epsilon \leq K' \int_0^1 \sqrt{-\log \epsilon}\, d\epsilon < \infty.
$$

Thus, $\mathcal{G}_n$ is a Donsker class for any given n. Then final result follows from Lemma 19.24 in Van der Vaart (2000).

□

Lemma S4. *Under the conditions of Theorem 3, for all* $\alpha \geq 0$ *and* $q > 1$, *then we have,*

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon_n')}-\frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right)_+^{\alpha}-E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n')}-\frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right)_+^{\alpha}\right)\right)$$
$$=\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon_n')}-1\right)_+^{\alpha}-E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n')}-1\right)_+^{\alpha}\right)\right)+o_{\mathbb{P}}(1).$$

Proof of Lemma S4. This proof is similar to that of Lemma S2 and is therefore omitted here. □

Lemma S5. *Under the conditions of Theorem 3, for all $\alpha > 0$ and $q > 1$, then we have,*

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_q(1-\varepsilon_n')}-\frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right|^{\alpha}-E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')}-\frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right|^{\alpha}\right)\right)$$
$$=\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_q(1-\varepsilon_n')}-1\right|^{\alpha}-E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')}-1\right|^{\alpha}\right)\right)+o_{\mathbb{P}}(1).$$

Proof of Lemma S5. This proof is similar to that of Lemma S3 and is therefore omitted here. □

S3.2 Proofs

Proof of Proposition 1. Note that (2.11) is a straightforward consequence of Corollary 1 in Daouia, Girard and Stupfler (2019). We hence only need to show the comparison between $\mathcal{L}(\gamma, p, q)$ and 1 for the cases of $1-q < p-\frac{1}{\gamma} < 1$ and $p-\frac{1}{\gamma} < 1-q$. First of all, we check the function $x \mapsto \frac{\Gamma(x+\varepsilon)}{\Gamma(x)}$ is increasing for $x \geq 1$ given a $\varepsilon > 0$, where $\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}\,dt$ is Gamma function. We notice the fact by Cauchy-Schwarz inequality,

$$
\begin{aligned}
[\Gamma'(s)]^2 &= \left(\int_0^\infty t^{s-1} e^{-t} \log t \, dt \right)^2 \\
&= \left(\int_0^\infty t^{\frac{s-1}{2}} e^{-\frac{t}{2}} \log t \cdot t^{\frac{s-1}{2}} e^{-\frac{t}{2}} \, dt \right)^2 \\
&\leq \int_0^\infty t^{s-1} e^{-t} (\log t)^2 \, dt \int_0^\infty t^{s-1} e^{-t} \, dt \\
&= \Gamma''(s)\Gamma(s).
\end{aligned}
$$

Hence, we have

$$
\left(\frac{\Gamma'(s)}{\Gamma(s)} \right)' = \frac{\Gamma''(s)\Gamma(s) - [\Gamma'(s)]^2}{[\Gamma(s)]^2} \geq 0,
$$

which implies

$$
\frac{\Gamma'(x+\varepsilon)}{\Gamma(x+\varepsilon)} \geq \frac{\Gamma'(x)}{\Gamma(x)}, \quad x \geq 1,
$$

that is, $\Gamma'(x+\varepsilon)\Gamma(x) \geq \Gamma'(x)\Gamma(x+\varepsilon)$. Therefore, the monotonicity of $x \mapsto \frac{\Gamma(x+\varepsilon)}{\Gamma(x)}$ is given by

$$
\left(\frac{\Gamma(x+\varepsilon)}{\Gamma(x)} \right)' = \frac{\Gamma'(x+\varepsilon)\Gamma(x) - \Gamma'(x)\Gamma(x+\varepsilon)}{[\Gamma(x)]^2} \geq 0.
$$

Next, we let $p - q = m + \varepsilon$ where $m \in \mathbb{N}$ and $\varepsilon \in [0, 1)$ and then we have

$$\begin{aligned}
&\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}\\
=&\frac{\Gamma(p)\Gamma(\gamma^{-1}-p+1)}{\Gamma(q)\Gamma(\gamma^{-1}-q+1)}=\frac{\Gamma(q+m+\varepsilon)}{\Gamma(q)}\times\frac{\Gamma(\gamma^{-1}+1-q-m-\varepsilon)}{\Gamma(\gamma^{-1}-q+1)}\\
=&\frac{(q+m+\varepsilon-1)(q+m+\varepsilon-2)\cdots(q+\varepsilon)\Gamma(q+\varepsilon)}{\Gamma(q)}\\
\times&\frac{\Gamma(\gamma^{-1}+1-q-m-\varepsilon)}{(\gamma^{-1}-q+1-1)(\gamma^{-1}-q+1-2)\cdots(\gamma^{-1}-q+1-m)\Gamma(\gamma^{-1}-q+1-m)}\\
=&\frac{(q+m+\varepsilon-1)\cdots(q+\varepsilon)}{(\gamma^{-1}-q+1-1)\cdots(\gamma^{-1}-q+1-m)}\times\frac{\Gamma(q+\varepsilon)}{\Gamma(q)}\times\frac{\Gamma(\gamma^{-1}+1-p)}{\Gamma(\gamma^{-1}+1-p+\varepsilon)}.
\end{aligned}$$

On the one hand, if $1-q<p-1/\gamma$, then $1/\gamma-q+1<p=q+m+\varepsilon$ and $1/\gamma+1-p<q$, so by monotonicity, we have

$$\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}>1\times\frac{\Gamma(q+\varepsilon)}{\Gamma(q)}\times\frac{\Gamma(\gamma^{-1}+1-p)}{\Gamma(\gamma^{-1}+1-p+\varepsilon)}>1.$$

On the other hand, if $1-q>p-1/\gamma$, then $1/\gamma-q+1>q+m+\varepsilon$ and $1/\gamma+1-p>q$, so by monotonicity again, we have

$$\frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}<1\times\frac{\Gamma(q+\varepsilon)}{\Gamma(q)}\times\frac{\Gamma(\gamma^{-1}+1-p)}{\Gamma(\gamma^{-1}+1-p+\varepsilon)}<1.$$

□

Proof of Proposition 2. For the existence, we use the notation $\inf\{\emptyset\}=\infty$. On the one hand, we have

$$\theta_p(\tau_0)=\inf_{c\in\left[1,\frac{1-\tau_0}{\varepsilon}\right]}\theta_p(1-c\varepsilon),$$

by monotonicity of $\theta_p(1-c\varepsilon)$ in c. If $\theta_p(\tau_0) > \theta_q(1-\varepsilon)$, then

$$\inf_{c\in\left[1,\frac{1-\tau_0}{\varepsilon}\right]} \theta_p(1-c\varepsilon) > \theta_q(1-\varepsilon),$$

which implies the set $\{\ c \mid \theta_p(1-c\varepsilon) \le \theta_q(1-\varepsilon)\}$ is empty and hence $c = \Pi_{p,q}(\varepsilon) = \infty \notin \left[1, \frac{1-\tau_0}{\varepsilon}\right]$, leading to a contradiction. On the other hand, if $\theta_p(\tau_0) \le \theta_q(1-\varepsilon)$, then $\left\{ c \in \left[1, \frac{1-\tau_0}{\varepsilon}\right] \middle| \theta_p(1-c\varepsilon) \le \theta_q(1-\varepsilon) \right\}$ is not empty and hence $\Pi_{p,q}(\varepsilon) < \infty$. Using intermediate value theorem, there exists $c \in \left[1, \frac{1-\tau_0}{\varepsilon}\right]$ such that (2.13) holds since

$$\theta_p(1-\varepsilon) > \theta_q(1-\varepsilon) \ge \theta_p(\tau_0). \tag{S3.17}$$

Moreover, it attains infimum when $\theta_p(1-c\varepsilon) = \theta_q(1-\varepsilon)$ holds implying that $c = \Pi_{p,q}(\varepsilon)$. If not, we can find a $c_0 < c$ such that $\theta_p(1-c_0\varepsilon) < \theta_q(1-\varepsilon)$, then we have,

$$\theta_q(1-\varepsilon) = \theta_p(1-c\varepsilon) \le \theta_p(1-c_0\varepsilon),$$

which leads to a contradiction. Even if there are multiple c such that (2.13) holds, we take the smallest one.

The uniqueness follows from (S3.17) and the strict monotonicity of $\theta_p(1-\varepsilon)$ on $(0, 1-\tau_0)$ when $\theta_p(\tau)$ or $\theta_q(\tau)$ are not constants on $[\tau_0, 1]$.

□

Proof of Proposition 3. For the existence, recall that $\theta_q\left(1-\frac{\varepsilon}{d}\right)$ is increasing and continuous in d. By intermediate value theorem, there exists $d \in [1, \infty)$

such that (2.15) holds since

$$\theta_q(1-\varepsilon) < \theta_p(1-\varepsilon) < \theta_q(1) := \lim_{d\to\infty} \theta_q\left(1-\frac{\varepsilon}{d}\right). \tag{S3.18}$$

It remains to show $d = \pi_{p,q}(\varepsilon)$. Note that d always attains its infimum when $\theta_p(1-\varepsilon) = \theta_q\left(1-\frac{\varepsilon}{d}\right)$ holds. If not, we can find $d_0 < d$ such that $\theta_p(1-\varepsilon) < \theta_q\left(1-\frac{\varepsilon}{d_0}\right)$, then we have $\theta_p(1-\varepsilon) = \theta_q\left(1-\frac{\varepsilon}{d}\right) \geq \theta_q\left(1-\frac{\varepsilon}{d_0}\right)$, which leads to a contradiction. Even if there are multiple d such that (2.15) holds, we take the smallest one.

The uniqueness follows from (S3.18) and the strict monotonicity of $\theta_p(1-\varepsilon)$ on $(0, 1-\tau_0)$ when $\theta_p(\tau)$ or $\theta_q(\tau)$ are not constant on $[\tau_0, 1]$. □

Proof of Proposition 4. By the relationship (2.13), we have,

$$1 = \frac{\theta_p(1-c\varepsilon)}{\theta_q(1-\varepsilon)} = \frac{\frac{\theta_p(1-c\varepsilon)}{\theta_1(1-c\varepsilon)}}{\frac{\theta_q(1-\varepsilon)}{\theta_1(1-\varepsilon)}} \times \frac{\theta_1(1-c\varepsilon)}{\theta_1(1-\varepsilon)} = \frac{\frac{\theta_p(1-c\varepsilon)}{\theta_1(1-c\varepsilon)}}{\frac{\theta_q(1-\varepsilon)}{\theta_1(1-\varepsilon)}} \times \frac{U(1/(c\varepsilon))}{U(1/\varepsilon)}.$$

We take limit $\varepsilon \to 0$ on the right-hand side of the above equation, and the conclusion follows from (2.11) and regular variation on $U(\cdot)$. The argument for $\pi_{p,q}(\varepsilon)$ is completely similar and is therefore omitted here.

□

Proof of Proposition 5. The assertion (2.20) follows from

$$\theta_q\left(1-\frac{c\varepsilon}{d}\right) = \theta_p(1-c\varepsilon) = \theta_q(1-\varepsilon),$$

where $c = \Pi_{p,q}(\varepsilon)$ and $d = \pi_{p,q}(\Pi_{p,q}(\varepsilon)\varepsilon)$.

Then assertion (2.19) follows from

$$\theta_p(1-\varepsilon) = \theta_q\left(1-\frac{\varepsilon}{d}\right) = \theta_p\left(1-\frac{c\varepsilon}{d}\right),$$

where $d = \pi_{p,q}(\varepsilon)$ and $c = \Pi_{p,q}(\varepsilon/\pi_{p,q}(\varepsilon))$. □

Proof of Theorem 1. We consider the function $t \mapsto \ell(t,p,q) = \frac{B(p,t^{-1}-p+1)}{B(q,t^{-1}-q+1)}$, and its derivative is given by,

$$\frac{\partial}{\partial t}\ell(t,p,q) = \frac{\Gamma(p)}{\Gamma(q)}\left(\frac{\Gamma(t^{-1}-q+1)\left(\frac{\partial}{\partial t}\Gamma(t^{-1}-p+1)\right)}{[\Gamma(t^{-1}-q+1)]^2}\right.$$
$$\left. - \frac{\Gamma(t^{-1}-p+1)\left(\frac{\partial}{\partial t}\Gamma(t^{-1}-q+1)\right)}{[\Gamma(t^{-1}-q+1)]^2}\right)$$

where $\frac{\partial}{\partial t}\Gamma(t^{-1}-p+1) = -\frac{1}{t^2}\int_0^\infty s^{t^{-1}-p}e^{-s}\log s\, ds$. Through Taylor's expansion and the fact $\hat{\gamma}_H - \gamma = o_{\mathbb{P}}(1)$, we have

$$\widehat{\Pi}_{p,q} = \ell(\hat{\gamma}_H,p,q) = \ell(\gamma,p,q) + \frac{\partial}{\partial\gamma}\ell(\gamma,p,q)(\hat{\gamma}_H-\gamma) + o((\hat{\gamma}_H-\gamma)),$$

which implies that, by the asymptotic normality for $\hat{\gamma}_H$ (for example, we refer to Theorem 3.2.5 of De Haan and Ferreira (2006)) and Delta-method on map: $t \mapsto \ell(t,p,q)$, it follows from

$$\sqrt{k}\left(\widehat{\Pi}_{p,q} - \ell(\gamma,p,q)\right) \xrightarrow{d} \mathcal{N}\left(\frac{\partial}{\partial\gamma}\ell(\gamma,p,q)\frac{\lambda}{1-\rho}, \left(\frac{\partial}{\partial\gamma}\ell(\gamma,p,q)\right)^2\gamma^2\right).$$

□

Proof of Theorem 2. We divide this proof into two scenarios: $\{q>1\}$ and $\{q=1\}$.

Case I: $q > 1$. We consider the following transformation of $\widetilde{\Pi}_{p,q}(\varepsilon_n)$ and $\Pi_{p,q}(\varepsilon_n)$,

$$\widetilde{\Pi}_{p,q}(\varepsilon_n) = \frac{\frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{p-1}}{\frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{q-1}} \frac{\frac{1}{n}\sum_{i=1}^n \left|\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right|^{q-1}}{\frac{1}{n}\sum_{i=1}^n \left|\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right|^{p-1}} =: \frac{\hat{A}_{n,p-1}\cdot \hat{B}_{n,q-1}}{\hat{A}_{n,q-1}\cdot \hat{B}_{n,p-1}}$$

and

$$\Pi_{p,q}(\varepsilon_n) = \frac{E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{p-1}\right)}{E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{q-1}\right)} \times \frac{E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{q-1}\right)}{E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{p-1}\right)} =: \frac{A_{n,p-1}\cdot B_{n,q-1}}{A_{n,q-1}\cdot B_{n,p-1}},$$

where

$$\begin{cases} \hat{A}_{n,a} = \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{a}, \\ \hat{B}_{n,b} = \frac{1}{n}\sum_{i=1}^n \left|\frac{X_i}{\theta_q(1-\varepsilon_n)} - \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right|^{b}, \end{cases} \quad \text{and} \quad \begin{cases} A_{n,a} = E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{a}\right), \\ B_{n,b} = E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right|^{b}\right). \end{cases}$$

We always consider $\theta_q(1-\varepsilon_n)$ on the right tail, hence we don't need to worry about the negative values of $\theta_q(1-\varepsilon_n)$. Hence, the main task is to find asymptotic results for $\hat{A}_{n,p-1}, \hat{A}_{n,q-1}, \hat{B}_{n,p-1}, \hat{B}_{n,q-1}$ separately.

Step 1: we work on $\hat{A}_{n,p-1}$ and $\hat{A}_{n,q-1}$. By Fubini's theorem, regular variation on $\overline{F}$ and (2.11), we have that,

$$\begin{aligned} A_{n,p-1} &= E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+^{p-1}\right) = (p-1)\int_0^\infty t^{p-2}\mathbb{P}\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)} - 1\right)_+ > t\right) dt \\ &= (p-1)\int_0^\infty t^{p-2}\mathbb{P}\left(X > (1+t)\theta_q(1-\varepsilon_n)\right) dt \\ &= \varepsilon_n(p-1)\int_0^\infty t^{p-2}\frac{1-F((1+t)\theta_q(1-\varepsilon_n))}{1-F(\theta_1(1-\varepsilon_n))} dt = \frac{B(p,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}\varepsilon_n(1+o(1)), \end{aligned}$$

and

$$
\begin{aligned}
A_{n,p-2} &= E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{p-2}\right) = \varepsilon_n(p-2)\int_0^\infty t^{p-3}\frac{1-F((1+t)\theta_q(1-\varepsilon_n))}{1-F(\theta_1(1-\varepsilon_n))}\,dt \\
&= \varepsilon_n\frac{\gamma(p-2)}{B(q,\gamma^{-1}-q+1)}\int_0^\infty t^{p-3}(t+1)^{-1/\gamma}\,dt(1+o(1)) \\
&=: \frac{I(p,\gamma)}{B(q,\gamma^{-1}-q+1)}\varepsilon_n(1+o(1)),
\end{aligned}
$$

where $I(p,\gamma)=\gamma(p-2)\int_0^\infty t^{p-3}(t+1)^{-1/\gamma}\,dt$. Now we need to discuss the convergence of $I(p,\gamma)$. Obviously, $I(p,\gamma)$ will becomes $B(p-1,\gamma^{-1}-p+2)$ when $p\geq 2$. For $1<p<2$, it's sufficient to consider the integral $\int_0^1(u^{-\gamma}-1)^{p-3}u^{-\gamma}\,du$, which is integrable since the order satisfies $-\gamma(p-3)-\gamma>-1$.

Then, by Lemma S2 and Delta-method on map: $\theta\mapsto E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-\theta\right)_+^{p-1}\right)$, we have the following decomposition,

$$
\begin{aligned}
&\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}-1\right) \\
=&\frac{\sqrt{n\varepsilon_n}}{A_{n,p-1}}\left(\frac{1}{n}\sum_{i=1}^n\left(\frac{X_i}{\theta_q(1-\varepsilon_n)}-\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{p-1}-E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{p-1}\right)\right) \\
=&\frac{\sqrt{n\varepsilon_n}}{A_{n,p-1}}\left(\frac{1}{n}\sum_{i=1}^n\left(\frac{X_i}{\theta_q(1-\varepsilon_n)}-\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{p-1}-E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\right)_+^{p-1}\right)\right) \\
&+\frac{\sqrt{n\varepsilon_n}}{A_{n,p-1}}\left(E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-\theta\right)_+^{p-1}\right)\Bigg|_{\theta=\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}}-E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-\theta\right)_+^{p-1}\right)\Bigg|_{\theta=1}\right) \\
=:&\frac{(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}-1\right)(1+o(1))+\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\cdot T_{n,p}+o_{\mathbb{P}}(1),
\end{aligned}
\tag{S3.19}
$$

from which the term $T_{n,p}$ can also be reformulated as

$$
\begin{aligned}
T_{n,p} &= \sqrt{\frac{n}{\varepsilon_n}}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon_n)}-1\right)_+^{p-1} - E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{p-1}\right)\right) \\
&= (p-1)\sqrt{\frac{n}{\varepsilon_n}}\int_0^\infty t^{p-2}[1-F_n((t+1)\theta_q(1-\varepsilon_n)) - (1-F((t+1)\theta_q(1-\varepsilon_n)))]\,dt.
\end{aligned}
\tag{S3.20}
$$

Recall the asymptotic result in (2.11) again: $z_n := \frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)} \to \mathcal{L}(\gamma, q, 1)$, we have

$$
\begin{aligned}
&\sqrt{\frac{n}{\varepsilon_n}}\{1-F_n((t+1)\theta_q(1-\varepsilon_n)) - (1-F((t+1)\theta_q(1-\varepsilon_n)))\} \\
=&\sqrt{\frac{n}{\varepsilon_n}}\left\{1-F_n\left((t+1)z_nU(\varepsilon_n^{-1})\right) - \left(1-F\left((t+1)z_nU(\varepsilon_n^{-1})\right)\right)\right\} \\
=&\sqrt{n\varepsilon_n}\left\{\frac{1}{\varepsilon_n}\left[1-F_n\left((t+1)z_nU(\varepsilon_n^{-1})\right)\right] - [(t+1)z_n]^{-1/\gamma}(1+o(1))\right\} \\
=&W_n\left([(t+1)z_n]^{-1/\gamma}\right) + \sqrt{n\varepsilon_n}A_0(\varepsilon_n^{-1})[(t+1)z_n]^{-1/\gamma}\frac{[(t+1)z_n]^{\rho/\gamma}-1}{\gamma\rho} \\
&+o_{\mathbb{P}}\left([(t+1)z_n]^{\frac{\epsilon-1/2}{\gamma}}\right),
\end{aligned}
$$

where the last step follows from Theorem 5.1.4 of De Haan and Ferreira (2006) with intermediate sequence $n\varepsilon_n$. Here, $A_0(\cdot)$ is another auxiliary function with definition in Theorem 2.3.9 of De Haan and Ferreira (2006), serving a similar role as $A(\cdot)$ and also satisfying $\sqrt{n\varepsilon_n}A_0(1/\varepsilon_n)$ bounded. Moreover, W_n are denoted as a sequence of Brownian motions. We substitute this result into (S3.20), and obtain

$$\begin{aligned} T_{n,p} = &(p-1)\int_1^\infty (s-1)^{p-2} W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds \\ &+\frac{\lambda(p-1)}{\gamma\rho}\left[\frac{B(q,\gamma^{-1}-q+1)}{\gamma}\right]^{\rho-1} B(p-1,-(\rho-1)/\gamma-p+1) \\ &-\frac{\lambda(p-1)}{\rho}\frac{B(p-1,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}+o_{\mathbb{P}}(1). \end{aligned} \tag{S3.21}$$

We combine $T_{n,p}$ with (S2.7) (denote the right-hand side of (S2.7) as ξ_n) to get

$$\begin{aligned} &\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}-1\right) \\ =&(1-p)\frac{I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}\cdot\xi_n \\ &+(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty (s-1)^{p-2} W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds \\ &+\frac{\lambda(p-1)}{\gamma^\rho\rho}\frac{[B(q,\gamma^{-1}-q+1)]^\rho B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}-\frac{\lambda}{\rho\gamma}+o_{\mathbb{P}}(1). \end{aligned} \tag{S3.22}$$

The same arguments for $\hat{A}_{n,q-1}$ yield that,

$$\begin{aligned} \sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}-1\right) = &(1-q)\frac{I(q,\gamma)}{B(q,\gamma^{-1}-q+1)}\cdot\xi_n \\ &+(q-1)\int_1^\infty (s-1)^{q-2} W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds \\ &+\frac{\lambda(q-1)}{\gamma^\rho\rho}\frac{B(q-1,-(\rho-1)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}}-\frac{\lambda}{\rho\gamma}+o_{\mathbb{P}}(1). \end{aligned} \tag{S3.23}$$

Step 2: we work on $\hat{B}_{n,p-1}$ and $\hat{B}_{n,q-1}$. We first state some limit relationships,

$$E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-2}\mathbb{1}_{\{X\geq\theta_q(1-\varepsilon_n)\}}\right)=E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n)}-1\right)_+^{p-2}\right)\to 0 \text{ as } n\to\infty.$$

Besides of this, since

$$\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-2}\mathbb{1}_{\{X\leq\theta_q(1-\varepsilon_n)\}}<\infty \quad \text{and} \quad \left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-2}\mathbb{1}_{\{X\leq\theta_q(1-\varepsilon_n)\}}\to 1,$$

then we have that,

$$E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-2}\mathbb{1}_{\{X\leq\theta_q(1-\varepsilon_n)\}}\right)\to 1, \text{ as } n\to\infty.$$

Indeed, it follows that,

$$B_{n,p-1}=E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-1}\right)\to 1, \text{ as } n\to\infty.$$

Then, similar to (S3.19), by Lemma S3 and Delta-method on map: $\theta\mapsto E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-\theta\right|^{p-1}\right)$, we have that,

$$\begin{aligned}
&\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}-1\right)\\
=&\frac{p-1}{B_{n,p-1}}E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-2}\left(\mathbb{1}_{\{X\leq\theta_q(1-\varepsilon_n)\}}-\mathbb{1}_{\{X\geq\theta_q(1-\varepsilon_n)\}}\right)\right)\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}-1\right)\\
+&\frac{\sqrt{n\varepsilon_n}}{B_{n,p-1}}\left(\frac{1}{n}\sum_{i=1}^n\left|\frac{X_i}{\theta_q(1-\varepsilon_n)}-1\right|^{p-1}-E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n)}-1\right|^{p-1}\right)\right)+o_{\mathbb{P}}(1)\\
=:&(p-1)\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}-1\right)(1+o(1))+S_{n,p}+o_{\mathbb{P}}(1).
\end{aligned}\tag{S3.24}$$

The term $S_{n,p}$ also can be reformulated as follows,

$$
\begin{aligned}
S_{n,p} &= (p-1)\sqrt{n\varepsilon_n}\int_0^\infty t^{p-2}[1-F_n((t+1)\theta_q(1-\varepsilon_n))+F_n((1-t)\theta_q(1-\varepsilon_n)) \\
&- \{1-F((t+1)\theta_q(1-\varepsilon_n))+F((1-t)\theta_q(1-\varepsilon_n))\}]\,dt \\
&= (p-1)\varepsilon_n\int_0^\infty t^{p-2}\sqrt{\frac{n}{\varepsilon_n}}[1-F_n((t+1)\theta_q(1-\varepsilon_n))-(1-F((t+1)\theta_q(1-\varepsilon_n)))]\,dt \\
&+ (p-1)\sqrt{\varepsilon_n}\int_0^\infty t^{p-2}\sqrt{n}[F_n((1-t)\theta_q(1-\varepsilon_n))-F((1-t)\theta_q(1-\varepsilon_n)))]\,dt = o_{\mathbb{P}}(1),
\end{aligned}
\tag{S3.25}
$$

where the last step follows from the arguments (S3.20), (S3.21) and central limit theorem for empirical process.

Then (S3.24) becomes

$$
\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}-1\right) = (p-1)\cdot\xi_n + o_{\mathbb{P}}(1), \tag{S3.26}
$$

as well as,

$$
\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}-1\right) = (q-1)\cdot\xi_n + o_{\mathbb{P}}(1). \tag{S3.27}
$$

Step 3: we work on $\widetilde{\Pi}_{p,q}(\varepsilon_n)$***.*** Our object can be given by an expression as follows,

$$
\begin{aligned}
\sqrt{n\varepsilon_n}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)}{\Pi_{p,q}(\varepsilon_n)}-1\right) &= \sqrt{n\varepsilon_n}\left(\frac{\frac{\hat{A}_{n,p-1}}{\hat{A}_{n,q-1}}\times\frac{\hat{B}_{n,q-1}}{\hat{B}_{n,p-1}}}{\frac{A_{n,p-1}}{A_{n,q-1}}\times\frac{B_{n,q-1}}{B_{n,p-1}}}-1\right) \\
&= \sqrt{n\varepsilon_n}\left(\frac{\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}\times\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}-\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}\times\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}}{\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}\times\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}}\right).
\end{aligned}
$$

Moreover, we have the following decompositions,

$$\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}\times\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}-1\right)=\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}-1\right)\cdot\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}+\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}-1\right), \tag{S3.28}$$

and

$$\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}\times\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}-1\right)=\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}-1\right)\cdot\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}+\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}-1\right). \tag{S3.29}$$

Combining (S3.22), (S3.23), (S3.26), (S3.27), (S3.28), (S3.29) and the fact $\frac{\hat{B}_{n,q-1}}{B_{n,q-1}}\xrightarrow{\mathbb{P}}1$ in (S3.27), $\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}\xrightarrow{\mathbb{P}}1$ in (S3.26) and $\frac{\hat{A}_{n,q-1}}{A_{n,q-1}}\xrightarrow{\mathbb{P}}1$ in (S3.23), we have that,

$$\begin{aligned}
&\sqrt{n\varepsilon_n}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)}{\Pi_{p,q}(\varepsilon_n)}-1\right)\\
=&\left\{\frac{(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}-\frac{(1-q)I(q,\gamma)}{B(q,\gamma^{-1}-q+1)}+q-p\right\}\cdot\xi_n\\
&+(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty(s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds\\
&-(q-1)\int_1^\infty(s-1)^{q-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds\\
&+\frac{\lambda}{\gamma^\rho\rho}\left\{\frac{(p-1)[B(q,\gamma^{-1}-q+1)]^\rho B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}\right.\\
&\left.-\frac{(q-1)B(q-1,-(\rho-1)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}}\right\}+o_{\mathbb{P}}(1).
\end{aligned} \tag{S3.30}$$

We get the final result by replacing ξ_n by the right-hand side of (S2.7). If $p>q\geq 2$, the coefficient of term ξ_n will vanish in which case $I(p,\gamma)=$

$B(p-1,\gamma^{-1}-p+2)$ and $I(q,\gamma)=B(q-1,\gamma^{-1}-q+2)$. So far, the proof for the case of $q>1$ is finished.

Case II: $q=1$. In this case, $\hat{A}_{n,0}=\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>\hat{\theta}_1(1-\varepsilon_n)\}}$, $\hat{B}_{n,0}=1$ and the estimator $\hat{\theta}_1(1-\varepsilon_n)$ will be taken as $X_{n-[n\varepsilon_n],n}$ (tail empirical quantile process) with asymptotic property following from Theorem 2.4.8 in De Haan and Ferreira (2006),

$$\left|\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-1\right)-\gamma W_n(1)\right|\xrightarrow{\mathbb{P}}0. \tag{S3.31}$$

Step 1: we work on $\hat{A}_{n,p-1}$. After some similar calculations to (S3.20), (S3.21), (S3.22), we have,

$$\begin{aligned}&\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}-1\right)\\=&\frac{\gamma(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}W_n(1)+\frac{\gamma(p-1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty(s-1)^{p-2}W_n(s^{-1/\gamma})\,ds\\+&\frac{\lambda(p-1)}{\rho}\frac{B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}-\frac{\lambda}{\gamma\rho}+o_{\mathbb{P}}(1).\end{aligned}\tag{S3.32}$$

Step 2: we work on $\hat{A}_{n,0}$. Using Proposition 5.1.4 of De Haan and

Ferreira (2006) again, we have,

$$\begin{aligned}\sqrt{n\varepsilon_n}\left(\frac{\hat{A}_{n,0}}{A_{n,0}}-1\right) &= \sqrt{\frac{n}{\varepsilon_n}}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>\theta_1(1-\varepsilon_n)\}}-\mathbb{P}(X>\theta_1(1-\varepsilon_n))\right)+o_{\mathbb{P}}(1)\\ &= \sqrt{n\varepsilon_n}\left\{\frac{1}{\varepsilon_n}(1-F_n(\theta_1(1-\varepsilon_n)))-1\right\}+o_{\mathbb{P}}(1)\\ &= W_n(1)+o_{\mathbb{P}}(1).\end{aligned}\tag{S3.33}$$

Step 3: we work on $\hat{B}_{n,p-1}$. Similar to (S3.24), we have that,

$$\begin{aligned}\sqrt{n\varepsilon_n}\left(\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}-1\right) =&\sqrt{n\varepsilon_n}\left(\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_1(1-\varepsilon_n)}-1\right|^{p-1}-E\left(\left|\frac{X}{\theta_1(1-\varepsilon_n)}-1\right|^{p-1}\right)\right)\\ &+(p-1)\sqrt{n\varepsilon_n}\left(\frac{\hat{\theta}_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-1\right)+o_{\mathbb{P}}(1)\\ =&(p-1)\gamma W_n(1)+o_{\mathbb{P}}(1).\end{aligned}\tag{S3.34}$$

Step 4: we work on $\widetilde{\Pi}_{p,1}(\varepsilon_n)$. Combining (S3.28), (S3.29), (S3.32), (S3.33) and (S3.34), it follows,

$$\begin{aligned}&\sqrt{n\varepsilon_n}\left(\frac{\widetilde{\Pi}_{p,1}(\varepsilon_n)}{\Pi_{p,1}(\varepsilon_n)}-1\right)\left(=\sqrt{n\varepsilon_n}\left(\frac{\frac{\hat{A}_{n,p-1}}{A_{n,p-1}}-\frac{\hat{A}_{n,0}}{A_{n,0}}\times\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}}{\frac{\hat{A}_{n,0}}{A_{n,0}}\times\frac{\hat{B}_{n,p-1}}{B_{n,p-1}}}\right)\right)\\ =&\left[\frac{\gamma(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}-(p-1)\gamma-1\right]W_n(1)+\frac{\gamma(p-1)}{B(p,\gamma^{-1}-p+1)}\int_1^{\infty}(s-1)^{p-2}W_n(s^{-1/\gamma})ds\\ &+\frac{\lambda(p-1)}{\rho}\frac{B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}-\frac{\lambda}{\gamma\rho}+o_{\mathbb{P}}(1).\end{aligned}$$

Note that the coefficient of term $W_n(1)$ will become -2 when $p\geq 2$.

In a summary, after a series of complex calculations, we can assert that,

$$\sqrt{n\varepsilon_n}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)}{\Pi_{p,q}(\varepsilon_n)}-1\right)\xrightarrow{d}\begin{cases}\mathcal{N}(\mathcal{E}_1(p,q,\gamma),\mathcal{V}_1(p,q,\gamma)), & \text{if } q>1,\\ \mathcal{N}(\mathcal{E}_2(p,q,\gamma),\mathcal{V}_2(p,q,\gamma)), & \text{if } q=1,\end{cases}$$

where the asymptotic means $\mathcal{E}_1(p,q,\gamma)$, $\mathcal{E}_2(p,q,\gamma)$, and variances $\mathcal{V}_1(p,q,\gamma)$, $\mathcal{V}_2(p,q,\gamma)$ are given by

$$\begin{aligned}\mathcal{E}_1(p,q,\gamma)=&\left[\frac{(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}-\frac{(1-q)I(q,\gamma)}{B(q,\gamma^{-1}-q+1)}+q-p\right]\\ &\times\left[\frac{\lambda(q-1)}{\gamma^{\rho-1}\rho}\frac{B(q-1,-(\rho-1)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}}-\frac{\lambda}{\rho}\right]\\ &+\frac{\lambda}{\gamma^{\rho}\rho}\left\{\frac{(p-1)[B(q,\gamma^{-1}-q+1)]^{\rho}B(p-1,(1-\rho)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}\right.\\ &\left.-\frac{(q-1)B(q-1,(1-\rho)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}}\right\},\\ \mathcal{V}_1(p,q,\gamma)=&\left\{\frac{\sqrt{\gamma}}{2}\left[\frac{(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}-\frac{(1-q)I(q,\gamma)}{B(q,\gamma^{-1}-q+1)}+q-p\right]\frac{B(q,(2\gamma)^{-1}-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1/2}}\right.\\ &\left.+\frac{[B(q,\gamma^{-1}-q+1)]^{1/2}B(p,(2\gamma)^{-1}-p+1)}{2\sqrt{\gamma}B(p,\gamma^{-1}-p+1)}-\frac{B(q,(2\gamma)^{-1}-q+1)}{2\sqrt{\gamma}[B(q,\gamma^{-1}-q+1)]^{1/2}}\right\}^2,\\ \mathcal{E}_2(p,q,\gamma)=&\frac{\lambda(p-1)}{\rho}\frac{B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}-\frac{\lambda}{\gamma\rho},\\ \mathcal{V}_2(p,q,\gamma)=&\left[\frac{\gamma(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)}+\frac{B(p,(2\gamma)^{-1}-p+1)}{2B(p,\gamma^{-1}-p+1)}-(p-1)\gamma-1\right]^2.\end{aligned}\tag{S3.35}$$

Moreover, when $2\le q<p$, the asymptotic means and variances will

reduce to, by noting $I(p,\gamma) = B(p-1,\gamma^{-1}-p+2)$,

$$\begin{aligned}
\mathcal{E}_1(p,q,\gamma) &= \frac{\lambda}{\gamma^\rho \rho}\left\{\frac{(p-1)[B(q,\gamma^{-1}-q+1)]^\rho B(p-1,(1-\rho)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)}\right. \\
&\quad \left. -\frac{(q-1)B(q-1,(1-\rho)/\gamma-q+1)}{[B(q,\gamma^{-1}-q+1)]^{1-\rho}}\right\}, \\
\mathcal{V}_1(p,q,\gamma) &= \left\{\frac{[B(q,\gamma^{-1}-q+1)]^{1/2}B(p,(2\gamma)^{-1}-p+1)}{2\sqrt{\gamma}B(p,\gamma^{-1}-p+1)}\right. \\
&\quad \left. -\frac{B(q,(2\gamma)^{-1}-q+1)}{2\sqrt{\gamma}[B(q,\gamma^{-1}-q+1)]^{1/2}}\right\}^2, \\
\mathcal{E}_2(p,q,\gamma) &= \frac{\lambda(p-1)}{\rho}\frac{B(p-1,-(\rho-1)/\gamma-p+1)}{B(p,\gamma^{-1}-p+1)} - \frac{\lambda}{\gamma\rho}, \\
\mathcal{V}_2(p,q,\gamma) &= \left[\frac{B(p,(2\gamma)^{-1}-p+1)}{2B(p,\gamma^{-1}-p+1)} - 2\right]^2.
\end{aligned} \tag{S3.36}$$

This proof is complete.

□

Proof of Theorem 3. We divide this proof into four scenarios: $\{a>0, q>1\}$, $\{a>0, q=1\}$, $\{a=0, q>1\}$ and $\{a=0, q=1\}$.

Case I: $\boldsymbol{a > 0, q > 1}$**.** The equivalent formulations for $\widetilde{\Pi}_{p,q}(\varepsilon_n')$ and $\Pi_{p,q}(\varepsilon_n')$ are given by,

$$\widetilde{\Pi}_{p,q}(\varepsilon_n') = \frac{\hat{C}_{n,p-1}\cdot \hat{D}_{n,q-1}}{\hat{C}_{n,q-1}\cdot \hat{D}_{n,p-1}}, \quad \text{and} \quad \Pi_{p,q}(\varepsilon_n') = \frac{C_{n,p-1}\cdot D_{n,q-1}}{C_{n,q-1}\cdot D_{n,p-1}},$$

where

$$\begin{cases} \hat{C}_{n,c} = \frac{1}{n}\sum_{i=1}^n \left(\frac{X_i}{\theta_q(1-\varepsilon_n')} - \frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right)^c, \\ \hat{D}_{n,d} = \frac{1}{n}\sum_{i=1}^n \left|\frac{X_i}{\theta_q(1-\varepsilon_n')} - \frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}\right|^d, \end{cases} \quad \text{and} \quad \begin{cases} C_{n,c} = E\left(\left(\frac{X}{\theta_q(1-\varepsilon_n')} - 1\right)_+^c\right), \\ D_{n,d} = E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')} - 1\right|^d\right). \end{cases}$$

Step 1: we work on $\hat{C}_{n,p-1}$ and $\hat{C}_{n,q-1}$. By Delta-method on map: $\theta \mapsto E\left(\left(\frac{X}{\theta_q(1-\varepsilon'_n)} - \theta\right)_+^{p-1}\right)$ and Lemma S4, we have the following decomposition,

$$\sqrt{n\varepsilon'_n}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}} - 1\right) = \frac{(1-p)I(p,\gamma)}{B(p,\gamma^{-1}-p+1)} \frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon'_n]} \left(\frac{\hat{\theta}_q(1-\varepsilon'_n)}{\theta_q(1-\varepsilon'_n)} - 1\right) \frac{\log[\varepsilon_n/\varepsilon'_n]}{\sqrt{\varepsilon_n/\varepsilon'_n}}(1+o(1))$$
$$+ \frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)} \cdot \sqrt{\frac{\varepsilon_n}{\varepsilon'_n}} \cdot T'_{n,p} + o_{\mathbb{P}}(1). \tag{S3.37}$$

Similar to $T_{n,p}$, $T'_{n,p}$ can be reformulated as,

$$T'_{n,p} = \sqrt{\frac{n}{\varepsilon_n}}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\theta_q(1-\varepsilon'_n)} - 1\right)_+^{p-1} - E\left(\left(\frac{X}{\theta_q(1-\varepsilon'_n)} - 1\right)_+^{p-1}\right)\right)$$
$$= (p-1)\sqrt{\frac{n}{\varepsilon_n}}\int_0^\infty t^{p-2}[1 - F_n((t+1)\theta_q(1-\varepsilon'_n)) - (1 - F((t+1)\theta_q(1-\varepsilon'_n)))]\, dt. \tag{S3.38}$$

Using (2.11) ($z'_n := \frac{\theta_q(1-\varepsilon'_n)}{\theta_1(1-\varepsilon_n)} \sim \mathcal{L}(\gamma,q,1)(\varepsilon_n/\varepsilon'_n)^\gamma$) and Proposition 5.1.4 of

De Haan and Ferreira (2006) with $k = n\varepsilon_n$ again, we have

$$
\begin{aligned}
&\sqrt{\frac{n}{\varepsilon_n}}\{1-F_n((t+1)\theta_q(1-\varepsilon'_n))-(1-F((t+1)\theta_q(1-\varepsilon'_n)))\}\\
=&\sqrt{\frac{n}{\varepsilon_n}}\left\{1-F_n\left((t+1)z'_nU(\varepsilon_n^{-1})\right)-\left(1-F\left((t+1)z'_nU(\varepsilon_n^{-1})\right)\right)\right\}\\
=&\sqrt{n\varepsilon_n}\left\{\frac{1}{\varepsilon_n}\left[1-F_n\left((t+1)z'_nU(\varepsilon_n^{-1})\right)\right]-\frac{1}{\varepsilon_n}\left[1-F\left((t+1)z'_nU(\varepsilon_n^{-1})\right)\right]\right\}\\
=&\sqrt{n\varepsilon_n}\left\{\frac{1}{\varepsilon_n}\left[1-F_n\left((t+1)z'_nU(\varepsilon_n^{-1}\right)\right]-[(t+1)z'_n]^{-1/\gamma}(1+o(1))\right\}\\
=&W_n\left([(t+1)z'_n]^{-1/\gamma}\right)+\sqrt{n\varepsilon_n}A_0(\varepsilon_n^{-1})[(t+1)z'_n]^{-1/\gamma}\frac{[(t+1)z'_n]^{\rho/\gamma}-1}{\gamma\rho}\\
&+o_{\mathbb{P}}\left([(t+1)z'_n]^{\frac{\epsilon-1/2}{\gamma}}\right).
\end{aligned}
$$

We substitute this result into (S3.38),

$$
\begin{aligned}
T'_{n,p}=&(p-1)\int_1^\infty(s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\cdot\frac{\varepsilon'_n}{\varepsilon_n}\right)ds\\
&+\frac{\lambda(p-1)}{\gamma\rho}\left[\frac{B(q,\gamma^{-1}-q+1)}{\gamma}\right]^{\rho-1}\left(\frac{\varepsilon'_n}{\varepsilon_n}\right)^{1-\rho}B(p-1,-(\rho-1)/\gamma-p+1)\\
&-\frac{\lambda(p-1)}{\rho}\frac{B(p-1,\gamma^{-1}-p+1)}{B(q,\gamma^{-1}-q+1)}\left(\frac{\varepsilon'_n}{\varepsilon_n}\right)+o_{\mathbb{P}}\left(\left(\frac{\varepsilon_n}{\varepsilon'_n}\right)^{\epsilon-1/2}\right).
\end{aligned}
\tag{S3.39}
$$

We combine this result with Theorem 1 of Daouia, Girard and Stupfler

(2019) to obtain that,

$$\begin{aligned}&\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)\\=&(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty(s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds\\&+o_{\mathbb{P}}(1).\end{aligned}\tag{S3.40}$$

The same arguments for $\hat{C}_{n,q-1}$ yield that,

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}-1\right)=(q-1)\int_1^\infty(s-1)^{q-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+o_{\mathbb{P}}(1).\tag{S3.41}$$

Step 2: we work on $\hat{\boldsymbol{D}}_{\boldsymbol{n,p-1}}$ and $\hat{\boldsymbol{D}}_{\boldsymbol{n,q-1}}$. It follows from Lemma S5 that,

$$\begin{aligned}&\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)\\=&\frac{(p-1)}{D_{n,p-1}}E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')}-1\right|^{p-2}(\mathbb{1}_{\{X\leq\theta_q(1-\varepsilon_n')\}}-\mathbb{1}_{\{X\geq\theta_q(1-\varepsilon_n')\}})\right)\\&\times\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left(\frac{\hat{\theta}_q(1-\varepsilon_n')}{\theta_q(1-\varepsilon_n')}-1\right)\frac{\log[\varepsilon_n/\varepsilon_n']}{\sqrt{\varepsilon_n/\varepsilon_n'}}(1+o(1))\\&+\frac{\sqrt{n\varepsilon_n'}}{D_{n,p-1}}\left(\frac{1}{n}\sum_{i=1}^n\left|\frac{X_i}{\theta_q(1-\varepsilon_n')}-1\right|^{p-1}-E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')}-1\right|^{p-1}\right)\right)+o_{\mathbb{P}}(1)\\=&o_{\mathbb{P}}(1),\end{aligned}\tag{S3.42}$$

where the last step follows from,

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left|\frac{X_i}{\theta_q(1-\varepsilon_n')}-1\right|^{p-1}-E\left(\left|\frac{X}{\theta_q(1-\varepsilon_n')}-1\right|^{p-1}\right)\\
=&(p-1)\int_0^\infty t^{p-2}[1-F_n((t+1)\theta_q(1-\varepsilon_n'))+F_n((1-t)\theta_q(1-\varepsilon_n'))\\
-&\{1-F((t+1)\theta_q(1-\varepsilon_n'))+F((1-t)\theta_q(1-\varepsilon_n'))\}]\,dt\\
=&(p-1)\sqrt{\frac{\varepsilon_n}{n}}\int_0^\infty t^{p-2}\sqrt{\frac{n}{\varepsilon_n}}[1-F_n((t+1)\theta_q(1-\varepsilon_n'))-(1-F((t+1)\theta_q(1-\varepsilon_n')))]\,dt\\
+&(p-1)\frac{1}{\sqrt{n}}\int_0^\infty t^{p-2}\sqrt{n}[F_n((1-t)\theta_q(1-\varepsilon_n'))-F((1-t)\theta_q(1-\varepsilon_n')))]\,dt\\
=&o_{\mathbb{P}}(1).
\end{aligned}
\tag{S3.43}
$$

The same arguments for $\hat{D}_{n,q-1}$ yield that,

$$
\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)=o_{\mathbb{P}}(1). \tag{S3.44}
$$

Step 3: we work on $\widetilde{\Pi}_{p,q}(\varepsilon_n')$***.*** Our object is given by,

$$
\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')}{\Pi_{p,q}(\varepsilon_n')}=\frac{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}\times\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)+\sqrt{n\varepsilon_n'}}{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}\times\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}.
$$

On the one hand, we have that,

$$
\begin{aligned}
&\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}\times\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)\\
=&\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)\cdot\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}+\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)\\
=&(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty(s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+o_{\mathbb{P}}(1).
\end{aligned}
\tag{S3.45}
$$

On the other hand, we also have that,

$$\begin{aligned}
&\sqrt{n\varepsilon'_n}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}\times\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)\\
=&\sqrt{n\varepsilon'_n}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}-1\right)\cdot\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}+\sqrt{n\varepsilon'_n}\left(\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)\\
=&(q-1)\int_1^\infty (s-1)^{q-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+o_{\mathbb{P}}(1).
\end{aligned} \tag{S3.46}$$

Therefore, the final result follows from,

$$\begin{aligned}
\frac{\widetilde{\Pi}_{p,q}(\varepsilon'_n)}{\Pi_{p,q}(\varepsilon'_n)}&=\frac{(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty (s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+\sqrt{a}}{(q-1)\int_1^\infty (s-1)^{q-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+\sqrt{a}}+o_{\mathbb{P}}(1)\\
&=\Delta+o_{\mathbb{P}}(1).
\end{aligned}$$

Case II: $\boldsymbol{a>0, q=1}$***.*** In this case, $\hat{C}_{n,0}=\frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\{X_i>\hat{\theta}_1(1-\varepsilon'_n)\}}$ and $\hat{D}_{n,0}=1$. The estimator $\hat{\theta}_1(1-\varepsilon'_n)$ will be taken as

$$\hat{\theta}_1(1-\varepsilon'_n)=\left(\frac{\varepsilon_n}{\varepsilon'_n}\right)^{\hat{\gamma}_H}\cdot\hat{\theta}_1(1-\varepsilon_n)=\left(\frac{\varepsilon_n}{\varepsilon'_n}\right)^{\hat{\gamma}_H}\cdot X_{n-[n\varepsilon_n],n},$$

and its asymptotic normality is also given in Theorem 1 of Daouia, Girard and Stupfler (2019).

Step 1: we work on $\boldsymbol{\hat{C}_{n,p-1}}$***.*** It follows that,

$$\sqrt{n\varepsilon'_n}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)=\frac{\gamma(p-1)}{B(p,\gamma^{-1}-p+1)}\int_1^\infty (s-1)^{p-2}W_n\left(s^{-1/\gamma}\right)ds+o_{\mathbb{P}}(1). \tag{S3.47}$$

Step 2: we work on $\hat{C}_{n,0}$. We have that,

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,0}}{C_{n,0}}-1\right)=\sqrt{\frac{\varepsilon_n}{\varepsilon_n'}}\left[\sqrt{n\varepsilon_n}\left\{\frac{1}{\varepsilon_n}\left(1-F_n\left(\left(\frac{\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_1(1-\varepsilon_n)\right)\right)-\frac{\varepsilon_n'}{\varepsilon_n}\right\}\right]+o_{\mathbb{P}}(1)$$
$$=W_n(1)+o_{\mathbb{P}}(1). \tag{S3.48}$$

Step 3: we work on $\hat{D}_{n,p-1}$. We have that,

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)=o_{\mathbb{P}}(1). \tag{S3.49}$$

Step 4: we work on $\widetilde{\Pi}_{p,1}(\varepsilon_n')$. Combining (S3.47), (S3.48) and (S3.49) yields that,

$$\begin{aligned}\frac{\widetilde{\Pi}_{p,1}(\varepsilon_n')}{\Pi_{p,1}(\varepsilon_n')}&=\frac{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,0}}{C_{n,0}}\times\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}\\&=\frac{\frac{\gamma(p-1)}{B(p,\gamma^{-1}-p+1)}\int_1^{\infty}(s-1)^{p-2}W_n(s^{-1/\gamma})\,ds+\sqrt{a}}{W_n(1)+\sqrt{a}}+o_{\mathbb{P}}(1)\\&=\Delta+o_{\mathbb{P}}(1).\end{aligned}$$

Case III: $a=0,q>1$. Similar to the above arguments, we have that,

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)=(p-1)\frac{B(q,\gamma^{-1}-q+1)}{B(p,\gamma^{-1}-p+1)}\int_1^{\infty}(s-1)^{p-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds$$
$$+o_{\mathbb{P}}(1), \tag{S3.50}$$

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}-1\right)=(q-1)\int_1^{\infty}(s-1)^{q-2}W_n\left(s^{-1/\gamma}\cdot\frac{\gamma}{B(q,\gamma^{-1}-q+1)}\right)ds+o_{\mathbb{P}}(1), \tag{S3.51}$$

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)=o_{\mathbb{P}}(1), \tag{S3.52}$$

and

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)=o_{\mathbb{P}}(1). \tag{S3.53}$$

Hence, it follows from,

$$\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')}{\Pi_{p,q}(\varepsilon_n')}=\frac{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}\times\frac{\hat{D}_{n,q-1}}{D_{n,q-1}}-1\right)+\sqrt{n\varepsilon_n'}}{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,q-1}}{C_{n,q-1}}\times\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}=\frac{B(q,\gamma^{-1}-q+1)B(p,(2\gamma)^{-1}-p+1)}{B(p,\gamma^{-1}-p+1)B(q,(2\gamma)^{-1}-q+1)}+o_{\mathbb{P}}(1).$$

Case IV: $\boldsymbol{a=0,q=1}$***.*** Similarly, we have that,

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)=(p-1)\frac{\gamma}{B(p,\gamma^{-1}-p+1)}\int_1^\infty(s-1)^{p-2}W_n\left(s^{-1/\gamma}\right)\,ds+o_{\mathbb{P}}(1), \tag{S3.54}$$

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,0}}{C_{n,0}}-1\right)=W_n(1)+o_{\mathbb{P}}(1), \tag{S3.55}$$

and

$$\sqrt{n\varepsilon_n'}\left(\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)=o_{\mathbb{P}}(1). \tag{S3.56}$$

Then, it follows from,

$$\frac{\widetilde{\Pi}_{p,1}(\varepsilon_n')}{\Pi_{p,1}(\varepsilon_n')}=\frac{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,p-1}}{C_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}{\sqrt{n\varepsilon_n'}\left(\frac{\hat{C}_{n,0}}{C_{n,0}}\times\frac{\hat{D}_{n,p-1}}{D_{n,p-1}}-1\right)+\sqrt{n\varepsilon_n'}}=\frac{B(p,(2\gamma)^{-1}-p+1)}{2B(p,\gamma^{-1}-p+1)}+o_{\mathbb{P}}(1).$$

So far, the proof is complete.

□

Proof of Theorem 4. **Firstly**, we have the following decompositions,

$$\frac{\widetilde{\theta}_p^{\text{int}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')}=\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\hat{\gamma}_H-\gamma}\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}, \tag{S3.57}$$

and take the logarithm on both sides,

$$\log \frac{\tilde{\theta}_p^{\mathrm{int}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')} = (\hat{\gamma}_H - \gamma) \log \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'} + \log \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} + \log \frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')} \left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'} \right)^{\gamma} . \tag{S3.58}$$

Secondly, by the asymptotic normality of Hill estimator (for example, Theorem 3.2.5 in De Haan and Ferreira (2006), we denote its limit distribution as ζ), it follows that

$$\begin{aligned} \frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']} (\hat{\gamma}_H - \gamma) \log \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'} &= \frac{\sqrt{n\varepsilon_n}(\hat{\gamma}_H - \gamma)}{\log[\varepsilon_n/\varepsilon_n']} \left(\log \widetilde{\Pi}_{p,q}(\varepsilon_n) + \log[\varepsilon_n/\varepsilon_n'] \right) \\ &= \sqrt{n\varepsilon_n}(\hat{\gamma}_H - \gamma) \left(O_{\mathbb{P}}\left(\frac{1}{\log[\varepsilon_n/\varepsilon_n']} \right) + 1 \right) \\ &= \zeta + o_{\mathbb{P}}(1). \end{aligned} \tag{S3.59}$$

Thirdly, using (S2.6), we have that,

$$\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']} \log \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} = O_{\mathbb{P}}\left(\frac{1}{\log[\varepsilon_n/\varepsilon_n']} \right) = o_{\mathbb{P}}(1). \tag{S3.60}$$

Finally, using (S2.9), Theorem 2, (2.11) and Assumption 5, it follows that,

$$
\begin{aligned}
&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\log\frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\\
=&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-\log\frac{\theta_p(1-\varepsilon_n')}{\theta_1(1-\varepsilon_n)}+\log\frac{\theta_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n')}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\right]\\
=&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-\log\frac{\theta_p(1-\varepsilon_n')}{\theta_1(1-\varepsilon_n)}+\gamma\log\widetilde{\Pi}_{p,q}(\varepsilon_n)(1+o(1))\right]\\
=&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\mathcal{L}(\gamma,q,1)}{\mathcal{L}(\gamma,p,1)}+\gamma\log\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)}{\Pi_{p,q}(\varepsilon_n)}(1+o(1))+\gamma\log\Pi_{p,q}(\varepsilon_n)(1+o(1))\right]\\
&+O\left(\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left\{R(p,1,\gamma,1-\varepsilon_n')-R(q,1,\gamma,1-\varepsilon_n)+A(1/\varepsilon_n)-A(1/\varepsilon_n')\right\}\right)\\
=&\sqrt{n\varepsilon_n}\log\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)}{\Pi_{p,q}(\varepsilon_n)}\frac{\gamma}{\log[\varepsilon_n/\varepsilon_n']}(1+o(1))+O\left(\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left\{R(p,1,\gamma,1-\varepsilon_n)+A(1/\varepsilon_n)\right\}\right)\\
=&o_{\mathbb{P}}(1).
\end{aligned}
\tag{S3.61}
$$

It is worth noting that $R(q,1,\gamma,1-\varepsilon_n)=O(R(p,1,\gamma,1-\varepsilon_n))$, $R(p,1,\gamma,1-\varepsilon_n')=O(R(p,1,\gamma,1-\varepsilon_n))$ and $A(1/\varepsilon_n')=O(A(1/\varepsilon_n))$. Combining (S3.58), (S3.59), (S3.60) and (S3.61) yields the result.

□

Proof of Theorem 5. We also note that,

$$
\frac{\tilde{\theta}_p^{\text{ext}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')}=\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'}\right)^{\hat{\gamma}_H-\gamma}\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)}\frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}\left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'}\right)^{\gamma},
\tag{S3.62}
$$

and

$$\log \frac{\tilde{\theta}_p^{\text{ext}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')} = (\hat{\gamma}_H - \gamma) \log \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'} + \log \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} + \log \frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')} \left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'} \right)^{\gamma}. \tag{S3.63}$$

For the first term in (S3.63), it follows that,

$$(\hat{\gamma}_H - \gamma) \log \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'} = \sqrt{n\varepsilon_n}(\hat{\gamma}_H - \gamma) \left(\frac{\log \widetilde{\Pi}_{p,q}(\varepsilon_n')}{\sqrt{n\varepsilon_n}} + \frac{\log[\varepsilon_n/\varepsilon_n']}{\sqrt{n\varepsilon_n}} \right) = o_{\mathbb{P}}(1). \tag{S3.64}$$

For the second term in (S3.63), by (S2.6), we have that,

$$\log \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} = \sqrt{n\varepsilon_n} \log \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \frac{1}{\sqrt{n\varepsilon_n}} = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n\varepsilon_n}} \right) = o_{\mathbb{P}}(1). \tag{S3.65}$$

For the last term in (S3.63), using (S2.9) and Theorem 3 again, we have that,

$$
\begin{aligned}
&\log \frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon'_n)} \left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon'_n)\varepsilon_n}{\varepsilon'_n}\right)^{\gamma} \\
=&\log \frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)} \frac{\theta_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon'_n)} \frac{\theta_1(1-\varepsilon'_n)}{\theta_p(1-\varepsilon'_n)} \left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon'_n)\varepsilon_n}{\varepsilon'_n}\right)^{\gamma} \\
=&\log \frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)} - \log \frac{\theta_p(1-\varepsilon'_n)}{\theta_1(1-\varepsilon_n)} + \log \frac{\theta_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon'_n)} \left(\frac{\widetilde{\Pi}_{p,q}(\varepsilon'_n)\varepsilon_n}{\varepsilon'_n}\right)^{\gamma} \\
=&\log \frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)} - \log \frac{\theta_p(1-\varepsilon'_n)}{\theta_1(1-\varepsilon_n)} + \gamma \log \widetilde{\Pi}_{p,q}(\varepsilon'_n)(1+o(1)) \\
=&\log \frac{\mathcal{L}(\gamma,q,1)}{\mathcal{L}(\gamma,p,1)} + \gamma \log \frac{\widetilde{\Pi}_{p,q}(\varepsilon'_n)}{\Pi_{p,q}(\varepsilon'_n)}(1+o(1)) + \gamma \log \Pi_{p,q}(\varepsilon'_n)(1+o(1)) \\
&+O\left(\sqrt{n\varepsilon_n}\left\{R(p,1,\gamma,1-\varepsilon_n)+A(1/\varepsilon_n)\right\}\frac{1}{\sqrt{n\varepsilon_n}}\right) \\
=&\begin{cases} \gamma\log\Delta, & \text{if } a>0 \\ \gamma\log\frac{B(q,\gamma^{-1}-q+1)B(p,(2\gamma)^{-1}-p+1)}{B(p,\gamma^{-1}-p+1)B(q,(2\gamma)^{-1}-q+1)}, & \text{if } a=0. \end{cases} + o_{\mathbb{P}}(1).
\end{aligned}
\tag{S3.66}
$$

Then the final result follows from (S3.63), (S3.64), (S3.65) and (S3.66). □

Proof of Theorem 6. This proof is quite similar to that of Theorem 4. We also have the following decompositions,

$$
\frac{\tilde{\theta}_p^{\lim}(1-\varepsilon'_n)}{\theta_p(1-\varepsilon'_n)} = \left(\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon'_n}\right)^{\hat{\gamma}_H-\gamma} \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon'_n)} \left(\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon'_n}\right)^{\gamma}, \tag{S3.67}
$$

and

$$
\log \frac{\tilde{\theta}_p^{\lim}(1-\varepsilon'_n)}{\theta_p(1-\varepsilon'_n)} = (\hat{\gamma}_H-\gamma)\log\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon'_n} + \log\frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} + \log\frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon'_n)}\left(\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon'_n}\right)^{\gamma}. \tag{S3.68}
$$

First, by Theorem 1, it follows that

$$\begin{aligned}\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}(\hat{\gamma}_H-\gamma)\log\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon_n'} &= \frac{\sqrt{n\varepsilon_n}(\hat{\gamma}_H-\gamma)}{\log[\varepsilon_n/\varepsilon_n']}\left(\log\widehat{\Pi}_{p,q}+\log[\varepsilon_n/\varepsilon_n']\right)\\ &= \sqrt{n\varepsilon_n}(\hat{\gamma}_H-\gamma)\left(O_{\mathbb{P}}\left(\frac{1}{\log[\varepsilon_n/\varepsilon_n']}\right)+1\right)\\ &= \zeta+o_{\mathbb{P}}(1).\end{aligned} \tag{S3.69}$$

Second, using (S2.9), Theorem 1 (take $k=n\varepsilon_n$), (2.11) and Assumption 5, we have that,

$$\begin{aligned}&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\log\frac{\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}\left(\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\\ =&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-\log\frac{\theta_p(1-\varepsilon_n')}{\theta_1(1-\varepsilon_n')}+\log\frac{\theta_1(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n')}\left(\frac{\widehat{\Pi}_{p,q}\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\right]\\ =&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\theta_q(1-\varepsilon_n)}{\theta_1(1-\varepsilon_n)}-\log\frac{\theta_p(1-\varepsilon_n')}{\theta_1(1-\varepsilon_n')}+\gamma\log\widehat{\Pi}_{p,q}(1+o(1))\right]\\ =&\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left[\log\frac{\mathcal{L}(\gamma,q,1)}{\mathcal{L}(\gamma,p,1)}+\gamma\log\frac{\widehat{\Pi}_{p,q}}{\ell(\gamma,p,q)}(1+o(1))+\gamma\log\ell(\gamma,p,q)(1+o(1))\right]\\ &+O\left(\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left\{R(p,1,\gamma,1-\varepsilon_n')-R(q,1,\gamma,1-\varepsilon_n)+A(1/\varepsilon_n)-A(1/\varepsilon_n')\right\}\right)\\ =&\sqrt{n\varepsilon_n}\log\frac{\widehat{\Pi}_{p,q}}{\ell(\gamma,p,q)}\frac{\gamma}{\log[\varepsilon_n/\varepsilon_n']}(1+o(1))+O\left(\frac{\sqrt{n\varepsilon_n}}{\log[\varepsilon_n/\varepsilon_n']}\left\{R(p,1,\gamma,1-\varepsilon_n)+A(1/\varepsilon_n)\right\}\right)\\ =&O\left(\frac{1}{\log[\varepsilon_n/\varepsilon_n']}\right)=o(1).\end{aligned} \tag{S3.70}$$

Hence, the conclusion follows from (S3.68), (S3.69), (S3.60) and (S3.70).

□

S4 Additional simulation results

S4.1 On Assumption 5

There are two sets of conditions (Assumptions 4 and 5) that are quite troublesome to verify. Table 1 has summarized the values of auxiliary function $A(\cdot)$ and parameter ρ in second-order regular variation for the three distributions used in our simulation. Next, we will further illustrate the feasibility and reasonableness of Assumption 5.

We continue considering Pareto, Fréchet and Student-t distributions with parameter γ, whose CDF or PDF are given at the beginning of Section 5. Following the expression of $R(p, 1, \gamma, 1-\varepsilon_n)$, it is sufficient to verify that $\sqrt{n\varepsilon_n}A(1/\varepsilon_n) = O(1)$ and $\sqrt{n\varepsilon_n}r(\mathcal{L}(\gamma, p, 1)\theta_1(1-\varepsilon_n), p, \gamma, X) = O(1)$. For the former, it follows that, by checking the values of $A(\cdot)$,

$$\sqrt{n\varepsilon_n}A(1/\varepsilon_n) = \begin{cases} 0, & \text{Pareto} \\ \frac{\gamma}{2}\sqrt{n\varepsilon_n}\varepsilon_n, & \text{Fréchet} \\ C_0\sqrt{n\varepsilon_n}\varepsilon_n^{2\gamma}, & \text{Student-}t, \end{cases}$$

with a constant C_0, which can be bounded by choosing a suitable ε_n and meets the condition $\sqrt{n\varepsilon_n}A(1/\varepsilon_n) \to \lambda$ as $n \to \infty$ in Theorems 1 - 6. For the latter, as $n \to \infty$, it follows that,

$$\sqrt{n\varepsilon_n} r(\mathcal{L}(\gamma, p, 1)\theta_1(1-\varepsilon_n), p, \gamma, X) \sim \begin{cases} C_1\sqrt{n\varepsilon_n}\varepsilon_n^{\min\{1,\gamma\}}, & \text{Pareto} \\ C_2\sqrt{n\varepsilon_n} \max\left\{\varepsilon_n, \left(\log\left(\frac{1}{1-\varepsilon_n}\right)\right)^{\gamma}\right\}, & \text{Fréchet} \\ C_3\sqrt{n\varepsilon_n}\varepsilon_n^{\min\{1,\gamma\}}, & \text{Student-}t, \end{cases}$$

by noting that

$$\theta_1(1-\varepsilon_n) \sim \begin{cases} \varepsilon_n^{-\gamma}, & \text{Pareto,} \\ (\log(1/(1-\varepsilon_n)))^{-\gamma}, & \text{Fréchet,} \\ \varepsilon_n^{-\gamma}, & \text{Student-}t. \end{cases}$$

where C_1, C_2, C_3 are all constants depending on γ, p. The quantities involved above can all be bounded by choosing a suitable ε_n. In a summary, Pareto, Fréchet, and Student-t distributions can all satisfy Assumption 5.

S4.2 Simulation results for TRELT

In this section, we provide a brief analysis of experimental results for TRELT. We implement the following methods for comparison,

- LimTRELT-I: the estimator $\widehat{\Pi}_{p,q}$ (3.28) for $\Pi_{p,q}(\varepsilon_n)$;
- LimTRELT-II: the estimator $\widehat{\Pi}_{p,q}$ (3.28) for $\Pi_{p,q}(\varepsilon'_n)$;
- IntTRELT: the estimator $\widetilde{\Pi}_{p,q}(\varepsilon_n)$ (3.32) for $\Pi_{p,q}(\varepsilon_n)$;

- ExtTRELT: the estimator $\widetilde{\Pi}_{p,q}(\varepsilon_n')$ (3.33) for $\Pi_{p,q}(\varepsilon_n')$.

The values of MSREs are all collected in Tables 1 and 2. What can be observed is that, for the intermediate level, LimTRELT-I usually presents higher MSREs than IntTRELT when γ is smaller. This may be attributed to the fact that LimTRELT-I is more suitable for smaller levels, whereas the intermediate level ε_n is not sufficiently small, especially for a smaller γ. For the extreme level, it can be observed that LimTRELT-II usually reports the lowest MSERs while ExtTRELT reports the highest MSREs. This could be due to two main reasons: firstly, the extreme level is small enough to render LimTRELT-II more appropriate and effective; secondly, as observed, the BM method indeed displays some biases, which may contribute to the unsatisfactory performance of ExtTRELT, since $\widetilde{\Pi}_{p,q}(\varepsilon_n')$ is exactly established by substituting BM into (3.33). Overall, the four methods demonstrate lower MSREs with heavier-tailed populations and larger sample sizes.

S4.3 Additional analyses for ExtraM-I, ExtraM-II, ExtraM-III

In our simulation, it has been shown that one well-performed extrapolation for $\theta_p(1-\varepsilon_n')$ is the ExtraM-II. It appears to be counterintuitive, as it is an extrapolation based on ExtTRELT, whose performance is not really

Table 1: The MSREs of LimTRELT-I, LimTRELT-II, IntTRELT, and ExtTRELT for Pareto, Fréchet and Student-t distributions with $\gamma = 1/3$. The bold numbers are the smallest values in each row.

	Methods		LimTRELT-I	LimTRELT-II	IntTRELT	ExtTRELT
	$\varepsilon_n(k)$	(p,q)		$n = 2000$		
Pareto	0.0290(58)	(2.4,1.8)	0.08621	**0.04659**	0.05564	0.14720
		(2.4,2.0)	0.04212	**0.02356**	0.02800	0.06712
Fréchet	0.0385(77)	(2.4,1.8)	0.06887	**0.03260**	0.07509	0.19160
		(2.4,2.0)	0.03247	**0.01620**	0.02952	0.07440
Student-t	0.0265(53)	(2.4,1.8)	0.14574	0.13605	**0.05006**	0.12116
		(2.4,2.0)	0.07289	0.06461	**0.02530**	0.05273
				$n = 5000$		
Pareto	0.0110(55)	(2.4,1.8)	0.05923	**0.04620**	0.05719	0.08763
		(2.4,2.0)	0.02959	**0.02345**	0.02841	0.04048
Fréchet	0.0160(80)	(2.4,1.8)	0.05337	**0.03591**	0.07362	0.12345
		(2.4,2.0)	0.02590	**0.01788**	0.02886	0.04643
Student-t	0.0120(60)	(2.4,1.8)	0.06626	0.06415	**0.05096**	0.08526
		(2.4,2.0)	0.03317	0.03131	**0.02589**	0.03842

Table 2: The MSREs of LimTRELT-I, LimTRELT-II, IntTRELT, and ExtTRELT for Pareto, Fréchet and Student-t distributions with $\gamma = 0.45$. The bold numbers are the smallest values in each row.

Methods			LimTRELT-I	LimTRELT-II	IntTRELT	ExtTRELT
	$\varepsilon_n(k)$	(p,q)	$n = 2000$			
Pareto	0.0290(58)	(2.0,1.5)	0.04157	**0.02456**	0.03483	0.12541
		(2.0,1.8)	0.00787	**0.00516**	0.00832	0.02145
Fréchet	0.0385(77)	(2.0,1.5)	0.03193	**0.01658**	0.05719	0.20204
		(2.0,1.8)	0.00559	**0.00342**	0.00855	0.02331
Student-t	0.0285(58)	(2.0,1.5)	0.04222	0.03816	**0.03614**	0.12617
		(2.0,1.8)	0.00896	**0.00757**	0.00862	0.01888
			$n = 5000$			
Pareto	0.0108(54)	(2.0,1.5)	0.02949	**0.02531**	0.03804	0.06494
		(2.0,1.8)	0.00607	**0.00537**	0.00861	0.01252
Fréchet	0.0160(80)	(2.0,1.5)	0.02460	**0.01813**	0.04939	0.10478
		(2.0,1.8)	0.00469	**0.00369**	0.00819	0.01375
Student-t	0.0110(55)	(2.0,1.5)	0.03075	**0.03003**	0.04876	0.09202
		(2.0,1.8)	0.00628	**0.00604**	0.01036	0.01431

satisfactory. Now, we try to provide a reasonable explanation for this observation from a numerical perspective. To do this, we report the values of following quantities in Tables 3 and 4,

$$\{\Pi_{p,q}(\varepsilon_n),\ \Pi_{p,q}(\varepsilon_n'),\ \theta_p(1-\varepsilon_n'),\ \text{Ratio}_1,\ \text{Ratio}_2,\ \text{Ratio}_3\}, \tag{S4.71}$$

where

$$\text{Ratio}_1 = \frac{\left(\frac{\Pi_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}, \tag{S4.72}$$

$$\text{Ratio}_2 = \frac{\left(\frac{\Pi_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}, \tag{S4.73}$$

$$\text{Ratio}_3 = \frac{\left(\frac{\ell(\gamma,p,q)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}. \tag{S4.74}$$

We consider the decompositions for ExtraM-I, ExtraM-II, ExtraM-III:

$$\begin{cases} \frac{\tilde{\theta}_p^{\text{int}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')} = \left(\frac{\varepsilon_n}{\varepsilon_n'}\right)^{\hat{\gamma}_H-\gamma} \times \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \times \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n)^{\hat{\gamma}_H}}{\Pi_{p,q}(\varepsilon_n)^{\gamma}} \times \frac{\left(\frac{\Pi_{p,q}(\varepsilon_n)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}, \\ \frac{\tilde{\theta}_p^{\text{ext}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')} = \left(\frac{\varepsilon_n}{\varepsilon_n'}\right)^{\hat{\gamma}_H-\gamma} \times \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \times \frac{\widetilde{\Pi}_{p,q}(\varepsilon_n')^{\hat{\gamma}_H}}{\Pi_{p,q}(\varepsilon_n')^{\gamma}} \times \frac{\left(\frac{\Pi_{p,q}(\varepsilon_n')\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}, \\ \frac{\tilde{\theta}_p^{\text{lim}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')} = \left(\frac{\varepsilon_n}{\varepsilon_n'}\right)^{\hat{\gamma}_H-\gamma} \times \frac{\hat{\theta}_q(1-\varepsilon_n)}{\theta_q(1-\varepsilon_n)} \times \frac{\widehat{\Pi}_{p,q}^{\hat{\gamma}_H}}{\ell(\gamma,p,q)^{\gamma}} \times \frac{\left(\frac{\ell(\gamma,p,q)\varepsilon_n}{\varepsilon_n'}\right)^{\gamma}\theta_q(1-\varepsilon_n)}{\theta_p(1-\varepsilon_n')}. \end{cases}$$

Note that only the last two terms are different in the above decompositions. It is readily to see that $\widehat{\Pi}_{p,q}$ and $\widetilde{\Pi}_{p,q}(\varepsilon_n')$ are more downward biased below 1 than $\widetilde{\Pi}_{p,q}(\varepsilon_n)$, and Ratio$_1$, Ratio$_2$, Ratio$_3$ are almost larger than 1 from Tables 3 - 4, especially for Pareto and Fréchet distributions. Thus, the products of the last two terms probably result in that $\frac{\tilde{\theta}_p^{\text{int}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')}$ is sometimes upward bias while $\frac{\tilde{\theta}_p^{\text{ext}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')}$ and $\frac{\tilde{\theta}_p^{\text{lim}}(1-\varepsilon_n')}{\theta_p(1-\varepsilon_n')}$ mostly stabilize around 1.

Table 3: The true values of quantities given in (S4.71) for Pareto, Fréchet and Student-t with $n = 2000$, $\varepsilon_n = k/n$ and $\varepsilon'_n = 0.005$.

	γ (ε_n)	(p, q)	$\Pi_{p,q}(\varepsilon_n)$	$\Pi_{p,q}(\varepsilon'_n)$	$\theta_p(1-\varepsilon'_n)$	Ratio_1	Ratio_2	Ratio_3
Pareto	1/3	(2.4,1.8)	1.49171	1.30619	5.53376	1.13952	1.09017	1.02371
	(0.0290)	(2.4,2.0)	1.36311	1.25541	5.53376	1.13168	1.10105	1.05675
	0.45	(2.0,1.5)	1.36022	1.23646	10.72927	1.10970	1.06308	1.01997
	(0.0290)	(2.0,1.8)	1.18339	1.14471	10.72927	1.09922	1.08291	1.06607
Fréchet	1/3	(2.4,1.8)	1.45972	1.27669	5.45012	1.13905	1.08930	1.03071
	(0.0385)	(2.4,2.0)	1.18339	1.14471	10.72927	1.09922	1.08291	1.06607
	0.45	(2.0,1.5)	1.34688	1.21409	10.58741	1.11111	1.06041	1.02581
	(0.0385)	(2.0,1.8)	1.17675	1.13418	10.58741	1.10232	1.08419	1.07178
Student-t	1/3	(2.4,1.8)	1.06031	1.07328	4.81372	0.92663	0.93040	0.93278
	(0.0265)	(2.4,2.0)	1.08074	1.09849	4.81372	0.92520	0.93024	0.93344
	0.45	(2.0,1.5)	1.10219	1.11855	7.68907	0.95357	0.95991	0.96347
	(0.0285)	(2.0,1.8)	1.08480	1.09828	7.68907	0.95443	0.95975	0.96259

Table 4: The true values of quantities given in (S4.71) for Pareto, Fréchet and Student-t with $n = 5000$, $\varepsilon_n = k/n$ and $\varepsilon'_n = 0.005$.

	γ (ε_n)	(p, q)	$\Pi_{p,q}(\varepsilon_n)$	$\Pi_{p,q}(\varepsilon'_n)$	$\theta_p(1-\varepsilon'_n)$	Ratio_1	Ratio_2	Ratio_3
Pareto	1/3	(2.4,1.8)	1.37728	1.30619	5.53376	1.05992	1.04136	0.97787
	(0.0110)	(2.4,2.0)	1.29794	1.25541	5.53376	1.05613	1.04447	1.00245
	0.45	(2.0,1.5)	1.28038	1.23646	10.72927	1.04282	1.02657	0.98494
	(0.0108)	(2.0,1.8)	1.15913	1.14471	10.72927	1.03778	1.03195	1.01590
Fréchet	1/3	(2.4,1.8)	1.36968	1.27669	5.45012	1.07389	1.04901	0.99259
	(0.0160)	(2.4,2.0)	1.29205	1.23646	5.45012	1.07033	1.05476	1.01747
	0.45	(2.0,1.5)	1.27918	1.21409	10.58741	1.05664	1.03210	0.99842
	(0.0160)	(2.0,1.8)	1.15633	1.13418	10.58741	1.05160	1.04249	1.03055
Student-t	1/3	(2.4,1.8)	1.06788	1.07328	4.81372	0.97114	0.97277	0.97527
	(0.0120)	(2.4,2.0)	1.09110	1.09849	4.81372	0.97022	0.97241	0.97576
	0.45	(2.0,1.5)	1.11385	1.11855	7.68907	0.98713	0.98900	0.99267
	(0.0110)	(2.0,1.8)	1.09437	1.09828	7.68907	0.98751	0.98909	0.99203

S4.4 On the choice of q

In this subsection, we explore the impact of the choice of q on the estimation of $\theta_p(1-\varepsilon'_n)$ from a simulation-based viewpoint. To describe the influence on the performance of our proposed methods, we plot the curves of the MSREs (Mean Squared Relative Error) against a series of discrete q. We follow the parameters with $p = 2.4$ for $\gamma = 1/3$ and $p = 2$ for $\gamma = 0.45$. By setting step as 0.1, the range of q is $\{1.7, 1.8.1.9, 2.0, 2.1, 2.2, 2.3, 2.4\}$ for $p = 2.4$ and $\gamma = 1/3$ while $\{1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}$ for $p = 2$ and $\gamma = 0.45$. Note that the three extrapolative estimations $\tilde{\theta}_p^{\text{int}}(1-\varepsilon'_n)$, $\tilde{\theta}_p^{\text{ext}}(1-\varepsilon'_n)$ and $\tilde{\theta}_p^{\text{lim}}(1-\varepsilon'_n)$ will reduce to the standard one $\tilde{\theta}_p^{\text{sta}}(1-\varepsilon'_n)$ when $p = q$.

Figure 1 shows the curves of MSREs against q for Pareto, Fréchet and Student-t distributions. It can be observed that under different distributions or different parameters of the same distribution, the three estimations exhibit varying performance. To be more specific, the MSREs of ExtraM-I decrease for Pareto distribution but increase for Fréchet and Student-t distributions; the MSREs of ExtraM-II show a trend of first decreasing and then increasing for all three distributions; the MSREs of ExtraM-III exhibit a increasing trend for Pareto and Fréchet distributions, while for Student-t distribution, they show a decreasing trend with a small sample but first decreasing and then increasing trend with a large sample. Overall, for Pareto, Fréchet distributions, ExtraM-III performs better, while for Student-t distribution, ExtraM-I and ExtraM-II yield better results.

In summary, given a specific distribution along with p and γ, the experimental results suggest that, q should be chosen as small as possible to minimize the MSRE. However, this does not always work for all three extrapolative estimations. For Pareto, Fréchet distributions, if ExtraM-III is used, q should be chosen as small as possible, whereas for the other two methods, a larger q would yield better results. For Student-t distribution, a smaller q is more suitable for ExtraM-I, while ExtraM-II and ExtraM-III perform better with a larger q.

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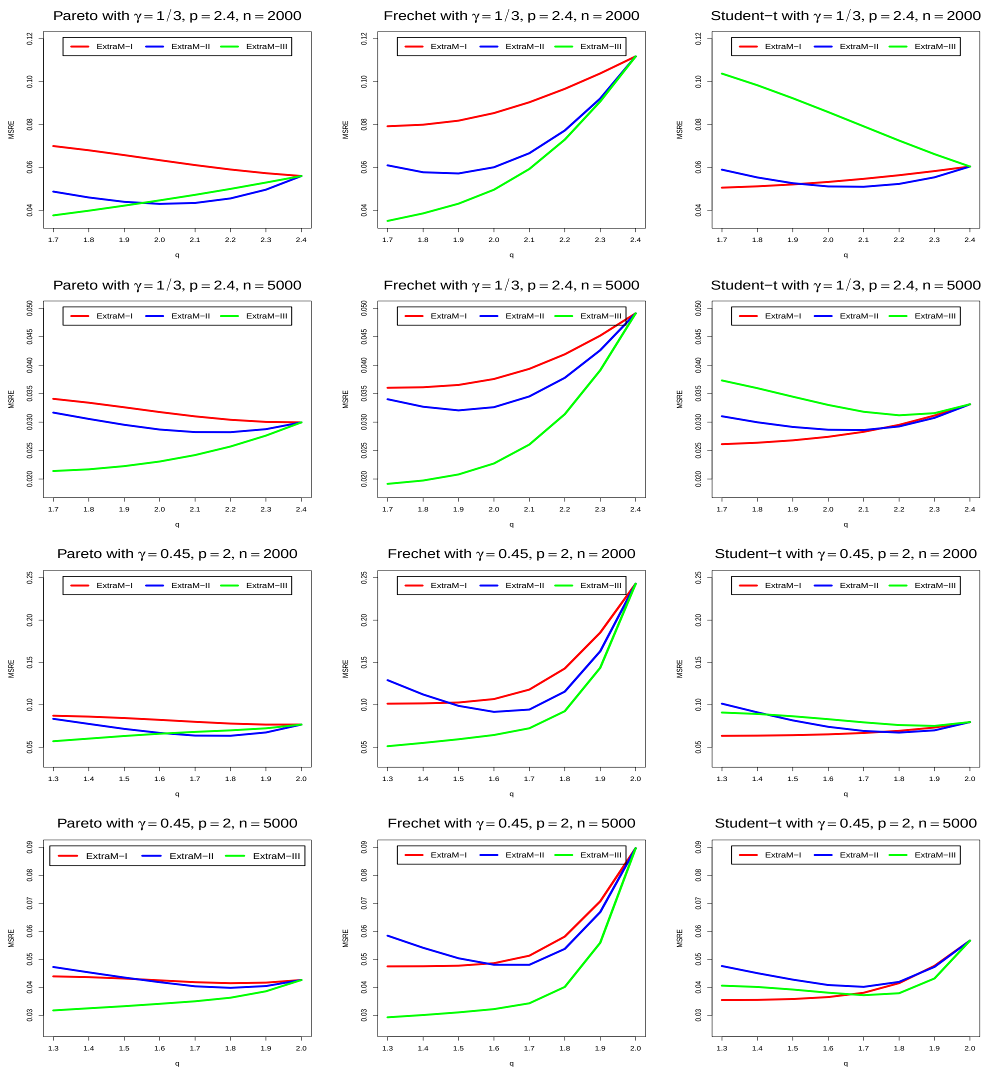

Figure 1: The curves of MSRE against q for ExtraM-I, ExtraM-II, and ExtraM-III under Pareto (left column), Fréchet (middle column) and Student-t (right column) distributions. The plots in the top two lines are drawn for $\gamma = 1/3$ and $p = 2.4$ with $n = 2000, 5000$ while the plots in the bottom two lines are drawn for $\gamma = 0.45$ and $p = 2$ with $n = 2000, 5000$, respectively.

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