Supplement to "Large dimensional Spearman's rank correlation matrices: The central limit theorem and its applications"

Hantao Chen and Cheng Wang

School of Mathematical Sciences, MOE-LSC, Shanghai Jiao Tong University

Supplementary material includes the detailed proofs of the main theorems and additional lemmas.

S1 Proof

S1.1 Proof sketch

The rigorous proof of Theorem 3.1 and Theorem 3.3 will be presented in this section. As can be seen, the Gram-type of Spearman's rank correlation matrix $\mathbf{g}_n$ is formulated as the sum of independent outer product matrices, which reveals similar structure with sample covariance matrix. Therefore, our methodology is originated from the proof in Bai and Silverstein (2004) that establish the CLT of LSS of large dimensional sample covariance matrices.

We denote $s_n^{(0)}(z)$ and $\underline{s}_n^{(0)}(z)$ as the Stieltjes transforms of $F_{n/p}$ and $\underline{F}_{n/p}$ respectively. By Cauchy's integral formula, we have

$$\int f(x)dG_n(x) = -\frac{1}{2\pi i}\int f(z)\cdot n\left(s_n(z) - s_n^{(0)}(z)\right)dz, \tag{S1.1}$$

where the contour of this integral is closed and enclose the extreme eigenvalues of $\mathbf{g}_n$. It is noted that if the limit superior and limit inferior of extreme eigenvalues of $\mathbf{g}_n$ are contained in the support of F_{y_0} with probability 1, then for any function f analytic on (3.8) and closed contour enclosing (3.8), the formula (S1.1) holds for all sufficiently large n with probability 1. However the concerntration of extreme eigenvalues are not trivial at all, and a more stronger control is presented in the following lemma.

Lemma S1.1. Under the same assumptions in Theorem 3.1, for any $\eta_l < (1-\sqrt{y})^2$, $\eta_r > (1+\sqrt{y})^2$ and any $m > 0$,

$$P\left(\lambda_1(\boldsymbol{\rho}_n) > \eta_r\right) = o(n^{-m}), \quad P\left(\lambda_{\min\{n,p\}}(\boldsymbol{\rho}_n) \leq \eta_l\right) = o(n^{-m}). \tag{S1.2}$$

Remark S1.1. The boundness of the largest eigenvalue is a direct corollary of Proposition 2.3 in Bao (2019). Due to the strong local law, the rigidity on the left edge can be derived with the same steps as right edge. Therefore, the boundness of the smallest eigenvalue can also be concluded.

As has been discussed in the above, the focus of the problem is shifted to establishing the asymptotic distribution of

$$M_n(z) \stackrel{\text{def}}{=} n\left(s_n(z) - s_n^{(0)}(z)\right) = p\left(\underline{s}_n(z) - \underline{s}_n^{(0)}(z)\right).$$

Since the CLT of LSS is obtained through a process of integration, we define a contour $\mathcal{C}$ enclosing interval (3.8) as follows. Let η_l and η_r be any two numbers such that (3.8) $\subset (\eta_l, \eta_r)$, and choose $v_0 > 0$. The contour is described as a rectangle,

$$\mathcal{C} = \{x \pm iv_0 : x \in [\eta_l, \eta_r]\} \cup \{x + iv : x \in \{\eta_l, \eta_r\}, v \in [-v_0, v_0]\}.$$

For further analysis, we consider $\widehat{M}_n(z)$ instead, a truncated version of $M_n(z)$, which is defined as

$$\widehat{M}_n(z) = \begin{cases} M_n(z), & z \in \mathcal{C}_n, \\ M_n(x + in^{-1}\varepsilon_n), & x \in \{\eta_l, \eta_r\} \text{ and } v \in [0, n^{-1}\varepsilon_n], \\ M_n(x - in^{-1}\varepsilon_n), & x \in \{\eta_l, \eta_r\} \text{ and } v \in [-n^{-1}\varepsilon_n, 0], \end{cases}$$

where $\mathcal{C}_n = \{x \pm iv_0 : x \in [\eta_l, \eta_r]\} \cup \{x \pm iv : x \in \{\eta_l, \eta_r\}, v \in [n^{-1}\varepsilon_n, v_0]\}$ and $\{\varepsilon_n\}$ is a sequence decreasing to zero satisfying $\varepsilon_n \geq n^{-\alpha}$ for some $\alpha \in (0, 1)$. It follows that $\widehat{M}_n(z)$ pauses at $x + in^{-1}\varepsilon_n$ when z tends to the real line, which makes the imaginary gap a natural bound to control the spectral norm or Euclidean distance of Stieltjes transforms. Besides, this truncation step have no influence on the limiting behavior of (S1.1) since for all sufficiently large n,

$$\left| \int_{\mathcal{C}} f(z) \left(M_n(z) - \widehat{M}_n(z) \right) dz \right| \leq C\varepsilon_n \left(\left| (1 + \sqrt{y_0})^2 \vee \lambda_1(\mathbf{g}_n) - \eta_r \right| \right. \\ \left. + \left| I(y_n > 1)(1 - \sqrt{y_0})^2 \wedge \lambda_p(\mathbf{g}_n) - \eta_l \right| \right), \tag{S1.3}$$

which converges to zero. So now we have prepared all ingredients and the proof of Theorem 3.1 can be completed by the following lemma establishing the convergence of $\widehat{M}_n(z)$ on $\mathcal{C}$.

Lemma S1.2. Under the same assumptions in Theorem 3.1, $\{\widehat{M}_n(\cdot)\}$, as a stochastic process on $\mathcal{C}$, converges weakly to a Gaussian process $M(\cdot)$ with mean function

$$\mathbb{E}M(z) = \mu(z),$$

and covariance function

$$\mathrm{cov}(M(z_1), M(z_2)) = \sigma(z_1, z_2),$$

where $\mu(z)$ and $\sigma(z_1, z_2)$ are defined in Theorem 3.1.

Theorem 3.2 is a corollary of Theorem 3.1 with the application of Cauchy's integral formula. More specifically, if $f(y\cdot)$ is analytic on an open interval containing (3.8), $f(y_n\cdot)$ converges to $f(y\cdot)$ uniformly and by the method of Stieltjes transform,

$$\begin{aligned} T(f) =& -\frac{1}{2\pi i}\int_{\mathcal{C}} f(y_n z)M_n(z)dz \\ =& -\frac{1}{2\pi i}\int_{\mathcal{C}} f(y_n z)\widehat{M}_n(z)dz + o_P(1) \\ =& -\frac{1}{2\pi i}\int_{\mathcal{C}} f(yz)\widehat{M}_n(z)dz + o_P(1), \end{aligned}$$

where the contour $\mathcal{C}$ encloses a neighborhoog of (3.8). The second equality holds by the same procedure of (S1.3), and the last equality holds by

$$\begin{aligned} \mathbb{E}\left|\int_{\mathcal{C}} \left(f(y_n z) - f(yz)\right)\widehat{M}_n(z)dz\right| \le& |\mathcal{C}| \cdot \sup_{z\in\mathcal{C}} |f(y_n z) - f(yz)| \cdot \sup_{z\in\mathcal{C}} \mathbb{E}\left|\widehat{M}_n(z)\right| \\ =& o(1). \end{aligned}$$

Therefore, by the convergence of $\widehat{M}_n(z)$ stated in Lemma S1.2, we obtain Theorem 3.2.

The proof of Theorem 3.3 basically follows the same approach as Lemma S1.2. With the help of Theorem 3.2, we only need to figure out the difference of $F^{\widetilde{\boldsymbol{\rho}}_n}$ and $F^{\boldsymbol{\rho}_n}$. By Cauchy's integral formula,

$$p\left(\int f(x)dF^{\widetilde{\boldsymbol{\rho}}_n}(x) - \int f(x)dF^{\boldsymbol{\rho}_n}(x)\right) = -\frac{1}{2\pi i}\int f(z)\cdot p\left(m_{F^{\widetilde{\boldsymbol{\rho}}_n}}(z) - m_{F^{\boldsymbol{\rho}_n}}(z)\right)dz,$$

and we are supposed to find the limit of

$$L_n(z) \stackrel{\text{def}}{=} p\left(m_{F^{\widetilde{\boldsymbol{\rho}}_n}}(z) - m_{F^{\boldsymbol{\rho}_n}}(z)\right).$$

As has been discussed before, we consider $\widehat{L}_n(z)$, a truncated version of $L_n(z)$, defined on $\mathcal{C}$ by the same way of $\widehat{M}_n(z)$. By the rigidity of the edge of Kendall's correlation matrix, we are able to control the extreme eigenvalues of $\widetilde{\boldsymbol{\rho}}_n$ as in Lemma S1.1, which implies $\int f(z)L_n(z)dz - \int f(z)\widehat{L}_n(z)dz \to 0$ almost surely. The following Lemma states the convergence of $\widehat{L}_n(z)$, which conclude Theorem 3.3.

Lemma S1.3. Under the same assumptions in Theorem 3.3, $\{\widehat{L}_n(\cdot)\}$, as a stochastic process on $\mathcal{C}$, converges weakly to a non-random function

$$L(z) = \widetilde{\mu}(z/y)/y,$$

where $\widetilde{\mu}(z)$ is defined in Theorem 3.3.

S1.2 Proof of Lemma S1.2

We decompose $\widehat{M}_n(z)$ into two parts as

$$\begin{aligned}\widehat{M}_n(z) =& n\left(s_n(z) - \mathbb{E}s_n(z)\right) + n\left(\mathbb{E}s_n(z) - s_n^{(0)}(z)\right)\\ \stackrel{\text{def}}{=}& M_n^{(1)}(z) + M_n^{(2)}(z),\end{aligned}$$

where $M_n^{(1)}(z)$ is the random part and $M_n^{(2)}(z)$ is the non-random part. For simplicity, denote

$$\mathbf{D}(z) = \mathbf{g}_n - z\mathbf{I}_n, \quad \mathbf{D}_j(z) = \mathbf{g}_n - z\mathbf{I}_n - \frac{1}{p}\mathbf{s}_j\mathbf{s}_j^\top,$$

$$\beta_j(z) = \frac{1}{1+\frac{1}{p}\mathbf{s}_j^\top \mathbf{D}_j^{-1}(z)\mathbf{s}_j}, \quad \overline{\beta}_j(z) = \frac{1}{1+\frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_j^{-1}(z)}, \quad b_n(z) = \frac{1}{1+\frac{1}{p}\mathbb{E}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_1^{-1}(z)},$$
$$\varepsilon_j(z) = \frac{1}{p}\mathbf{s}_j^\top \mathbf{D}_j^{-1}(z)\mathbf{s}_j - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_j^{-1}(z), \quad \delta_j(z) = \frac{1}{p}\mathbf{s}_j^\top \mathbf{D}_j^{-2}(z)\mathbf{s}_j - \frac{1}{p}\mathbb{E}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_j^{-2}(z).$$

Note that $\beta_j(z)$, $\overline{\beta}_j(z)$ and $b_n(z)$ are all bounded by $\frac{|z|}{\Im(z)}$, where $\Im(\cdot)$ is the imaginary part. And by some matrix identity, we have

$$\mathbf{D}_j^{-1}(z) = \mathbf{D}^{-1}(z) + \frac{1}{p}\beta_j(z)^\top \mathbf{D}_j^{-1}(z)\mathbf{s}_j\mathbf{s}_j^\top \mathbf{D}_j^{-1}(z), \tag{S1.4}$$
$$\underline{s}_n(z) = -\frac{1}{z}\cdot\frac{1}{p}\sum_{i=1}^{p}\beta_i(z). \tag{S1.5}$$

Step 1. Finite-dimensional weak convergence of $M_n^{(1)}(z)$.

Let $\mathbb{E}_k(\cdot)$ be the conditional expectation with respect to the σ-field generated by $\mathbf{s}_1, \cdots, \mathbf{s}_k$. By martingale difference decomposition,

$$\begin{aligned}
M_n^{(1)}(z) =& \sum_{j=1}^{p}\left(\mathbb{E}_j \mathrm{tr}\mathbf{D}^{-1}(z) - \mathbb{E}_{j-1}\mathrm{tr}\mathbf{D}^{-1}(z)\right) \\
=& -\frac{1}{p}\sum_{j=1}^{p}\left(\mathbb{E}_j - \mathbb{E}_{j-1}\right)\beta_j(z)\mathbf{s}_j^\top \mathbf{D}_j^{-2}(z)\mathbf{s}_j \\
=& -\frac{1}{p}\sum_{j=1}^{p}\left(\mathbb{E}_j - \mathbb{E}_{j-1}\right)\left(\overline{\beta}_j(z) - \overline{\beta}_j^2(z)\varepsilon_j(z) + \overline{\beta}_j^2(z)\beta_j(z)\varepsilon_j^2(z)\right)\mathbf{s}_j^\top \mathbf{D}_j^{-2}(z)\mathbf{s}_j \\
=& -\sum_{j=1}^{p}\mathbb{E}_j\left(\overline{\beta}_j(z)\delta_j(z) + \overline{\beta}_j^2(z)\varepsilon_j(z)\frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_j^{-2}(z)\right) + o_P(1),
\end{aligned}$$

where the second-to-last equality holds by the identity $\beta_j(z) = \overline{\beta}_j(z) - \overline{\beta}_j^2(z)\varepsilon_j(z) + \overline{\beta}_j^2(z)\beta_j(z)\varepsilon_j^2(z)$, and the last equality holds by Lemma S2.2. The dominant term denoted by

$$Y_j(z) = \mathbb{E}_j\left(\overline{\beta}_j(z)\delta_j(z) + \overline{\beta}_j^2(z)\varepsilon_j(z)\frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_j^{-2}(z)\right), \quad j = 1, \cdots, p,$$

is still a martingale difference sequence. For some fixed $r > 0$, since

$$\sum_{i=1}^{r} \alpha_i M_n^{(1)}(z_i) = \sum_{j=1}^{p} \sum_{i=1}^{r} \alpha_i Y_j(z_i) + o_P(1),$$

by martingale CLT in Lemma S2.3, it suffices to verify

$$\sum_{j=1}^{p} \mathbb{E} \left| \sum_{i=1}^{m} \alpha_i Y_j(z_i) \right|^4 \to 0, \tag{S1.6}$$

and find the limit of convergence in probability of

$$\sum_{j=1}^{n} \mathbb{E}_{j-1} Y_j(z_1) Y_j(z_2). \tag{S1.7}$$

By Lemma S2.2,

$$\mathbb{E} \left| Y_j(z) \right|^4 \leq C \left(\frac{|z|^4}{\Im^4(z)} \mathbb{E} |\delta_j(z)|^4 + \frac{|z|^8}{\Im^{16}(z)} \mathbb{E} |\varepsilon_j(z)|^4 \right) = o(n^{-1}),$$

which implies (S1.6). As for (S1.7), observe that

$$\overline{\beta}_j(z) \delta_j(z) + \overline{\beta}_j^2(z) \varepsilon_j(z) \frac{1}{p} \mathrm{tr} \mathbf{\Sigma} \mathbf{D}_j^{-2}(z) = \frac{d}{dz} \overline{\beta}_j(z) \varepsilon_j(z),$$

so we have

$$\frac{\partial^2}{\partial z_1 \partial z_2} \mathbb{E}_{j-1} \left[\mathbb{E}_j \left(\overline{\beta}_j(z_1) \varepsilon_j(z_1) \right) \mathbb{E}_j \left(\overline{\beta}_j(z_2) \varepsilon_j(z_2) \right) \right] = \mathbb{E}_{j-1} Y_j(z_1) Y_j(z_2).$$

With similar arguments on Page 571 of Bai and Silverstein (2004), it suffices to determine the limit of

$$\sum_{j=1}^{p} \mathbb{E}_{j-1} \left[\mathbb{E}_j \left(\overline{\beta}_j(z_1) \varepsilon_j(z_1) \right) \mathbb{E}_j \left(\overline{\beta}_j(z_2) \varepsilon_j(z_2) \right) \right]. \tag{S1.8}$$

Since by (S1.4) and (S1.5), we have

$$\mathbb{E} \left| \overline{\beta}_j(z) - b_n(z) \right|^2 \leq \frac{C}{p^2} \mathbb{E} \left| \mathrm{tr} \mathbf{\Sigma} \mathbf{D}_j^{-1}(z) - \mathbb{E} \mathrm{tr} \mathbf{\Sigma} \mathbf{D}_1^{-1}(z) \right|^2 = O(p^{-1}),$$

and

$$
\begin{aligned}
|b_n(z) + z\underline{s}(z)| &\leq |b_n(z) - \mathbb{E}\beta_1(z)| + |\mathbb{E}\beta_1(z) + z\underline{s}(z)| \\
&= o(1).
\end{aligned}
$$

Therefore, we only need find the limit of

$$
z_1 z_2 \underline{s}(z_1)\underline{s}(z_2) \sum_{j=1}^{p} \mathbb{E}_{j-1}\left[\mathbb{E}_j \varepsilon_j(z_1)\mathbb{E}_j \varepsilon_j(z_2)\right]. \tag{S1.9}
$$

By Lemma S2.1 and (S1.4),

$$
z_1 z_2 \underline{s}(z_1)\underline{s}(z_2) \sum_{j=1}^{p} \mathbb{E}_{j-1}\mathbb{E}_j \varepsilon_j(z_1)\mathbb{E}_j \varepsilon_j(z_2) = 2I_1 - \frac{6}{5}I_2 - \frac{4}{5}I_3 + O_P(p^{-1}),
$$

where

$$
\begin{aligned}
I_1 =& z_1 z_2 \underline{s}(z_1)\underline{s}(z_2)\frac{1}{p^2}\sum_{j=1}^{p} \operatorname{tr}\left(\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right), \\
I_2 =& z_1 z_2 \underline{s}(z_1)\underline{s}(z_2)\frac{1}{p^2}\sum_{j=1}^{p} \operatorname{tr}\left(\mathbb{E}_j \mathbf{D}^{-1}(z_1)\circ \mathbb{E}_j \mathbf{D}^{-1}(z_2)\right), \\
I_3 =& z_1 z_2 \underline{s}(z_1)\underline{s}(z_2)\frac{1}{np^2}\sum_{j=1}^{p} \operatorname{tr}(\mathbb{E}_j \mathbf{D}^{-1}(z_1))\operatorname{tr}(\mathbb{E}_j \mathbf{D}^{-1}(z_2)).
\end{aligned}
$$

For I_1, since that

$$
\begin{aligned}
&\left|\operatorname{tr}\left(\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right) - \operatorname{tr}\left(\mathbf{\Sigma}\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right)\right| \\
=&\left|-\frac{1}{n-1}\operatorname{tr}\left(\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right) + \frac{1}{n-1}\mathbf{1}_n^\top \mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\mathbf{1}_n\right| \\
=&O(1),
\end{aligned}
$$

and similarly

$$
\left|\operatorname{tr}\left(\mathbf{\Sigma}\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right) - \operatorname{tr}\left(\mathbf{\Sigma}\mathbb{E}_j \mathbf{D}^{-1}(z_1)\mathbf{\Sigma}\mathbb{E}_j \mathbf{D}^{-1}(z_2)\right)\right| = O(1),
$$

the effect of the multiplying $\mathbf{\Sigma}$ is negligible and

$$I_1 = z_1 z_2 \underline{s}(z_1)\underline{s}(z_2)\frac{1}{p^2}\sum_{j=1}^{p} \mathrm{tr}\left(\mathbf{\Sigma}\mathbb{E}_j\mathbf{D}^{-1}(z_1)\mathbf{\Sigma}\mathbb{E}_j\mathbf{D}^{-1}(z_2)\right) + O(n^{-1}).$$

With similar arguments on Pages 572-578, by Lemma S2.2,

$$I_1 = \log\frac{\underline{s}(z_1)-\underline{s}(z_2)}{\underline{s}(z_1)\underline{s}(z_2)(z_1-z_2)} + o_P(1). \tag{S1.10}$$

For I_2, following similar steps on Pages 1247-1249 of Pan and Zhou (2008) with applications of Lemma S2.2, we have

$$\begin{aligned} I_2 =& z_1 z_2 \underline{s}(z_1)\underline{s}(z_2)\frac{1}{p^2}\sum_{j=1}^{p} \mathrm{tr}\left(\mathbb{E}\mathbf{D}^{-1}(z_1)\circ\mathbb{E}\mathbf{D}^{-1}(z_2)\right) + o_P(1) \qquad \text{(S1.11)}\\ =& \underline{s}(z_1)\underline{s}(z_2)\frac{1}{p^2}\sum_{j=1}^{p} \mathrm{tr}\left((\underline{s}(z_1)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\circ(\underline{s}(z_2)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\right) + o_P(1). \end{aligned}$$

Denote $\mathbf{e}_k \in \mathbb{R}^n$ as the unit vector with k-th element being one and the rest being zero. Since that

$$\begin{aligned} &\left|\mathbf{e}_k^\top\left(\underline{s}(z)\mathbf{\Sigma}+\mathbf{I}_n\right)^{-1}\mathbf{e}_k - (\underline{s}(z)+1)^{-1}\right| \\ =&\left|\underline{s}(z)(\underline{s}(z)+1)^{-1}\mathbf{e}_k^\top\left(\left(\underline{s}(z)\mathbf{\Sigma}+\mathbf{I}_n\right)^{-1}\left(\mathbf{I}_n-\mathbf{\Sigma}\right)\right)\mathbf{e}_k\right| \\ \le&\frac{C}{n-1}\left|\mathbf{e}_k^\top\left(\underline{s}(z)\mathbf{\Sigma}+\mathbf{I}_n\right)^{-1}\mathbf{e}_k\right| + \frac{C}{n-1}\left|\mathbf{e}_k^\top\left(\underline{s}(z)\mathbf{\Sigma}+\mathbf{I}_n\right)^{-1}\mathbf{1}_n\mathbf{1}_n^\top\mathbf{e}_k\right| \\ =&O(n^{-\frac{1}{2}}), \end{aligned}$$

$\mathbf{\Sigma}$ can be approximated by $\mathbf{I}_n$ and

$$\begin{aligned} &\mathrm{tr}\left((\underline{s}(z_1)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\circ(\underline{s}(z_2)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\right) \\ =&\sum_{k=1}^{n}\mathbf{e}_k^\top(\underline{s}(z_1)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\mathbf{e}_k\mathbf{e}_k^\top(\underline{s}(z_2)\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\mathbf{e}_k \end{aligned}$$

$$=n\left(\underline{s}(z_1)+1\right)^{-1}\left(\underline{s}(z_2)+1\right)^{-1}+O(1).$$

Thus, we obtain

$$I_2=\frac{y_0\underline{s}(z_1)\underline{s}(z_2)}{\left(\underline{s}(z_1)+1\right)\left(\underline{s}(z_2)+1\right)}+o_P(1). \tag{S1.12}$$

I_3 can be simplified by the following approximation steps,

$$\mathrm{tr}\mathbf{D}^{-1}(z)=\mathrm{tr}\left(-z\underline{s}(z)\mathbf{\Sigma}-z\mathbf{I}\right)^{-1}+o_P(1)=n\left(z\underline{s}+z\right)^{-1}+o_P(1),$$

which implies

$$I_3=\frac{y_0\underline{s}(z_1)\underline{s}(z_2)}{\left(\underline{s}(z_1)+1\right)\left(\underline{s}(z_2)+1\right)}+o_P(1). \tag{S1.13}$$

Collecting (S1.10), (S1.12) and (S1.13), we have

$$(\text{S1.9})\to 2\log\frac{\underline{s}(z_1)-\underline{s}(z_2)}{\underline{s}(z_1)\underline{s}(z_2)(z_1-z_2)}-\frac{2y_0\underline{s}(z_1)\underline{s}(z_2)}{\left(\underline{s}(z_1)+1\right)\left(\underline{s}(z_2)+1\right)},\quad \text{in probability,}$$

which conclude by taking derivatives

$$(\text{S1.7})\to\sigma(z_1,z_2),\quad \text{in probability.} \tag{S1.14}$$

Step 2. Tightness of $M_n^{(1)}(z)$.

Combined with finite-dimensional weak convergence of $M_n^{(1)}(z)$ and tightness on $z\in\mathcal{C}_n$, we are able to prove the weak convergence of stochastic process $M_n^{(1)}(\cdot)$. To prove its tightness, by Theorem 12.3 of Billingsley (2013), it suffices to verify

$$\sup_{n;z_1,z_2\in\mathcal{C}_n}\frac{\mathbb{E}|M_n^{(1)}(z_1)-M_n^{(1)}(z_2)|^2}{|z_1-z_2|^2}<\infty.$$

By Lemma S1.1,

$$\mathbb{E}\|\mathbf{D}^{-1}(z)\|^k \leq C_1 + v^{-k} P\left(\|\mathbf{G}\| \geq \eta_r \text{ or } \lambda_{\min}(\mathbf{G}) \leq \eta_l\right) \leq C, \tag{S1.15}$$

for sufficiently large l. We emphasize that the moments bound here are uniform in n and $z \in \mathcal{C}_n$, that is, the constant C is independent of n and $z \in \mathcal{C}_n$. By the same way in (S1.15), one can prove that the moments of $\|\mathbf{D}_j^{-1}(z)\|$ is also bounded uniformly in n and $z \in \mathcal{C}_n$. Therefore, we extend Lemma S2.2 slightly as

$$\left|\mathbb{E}a(v)\prod_{l=1}^{q}\left(\mathbf{s}_1^\top \mathbf{B}_l(v)\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}_l(v)\right)\right| \leq Cn^{-\frac{q}{2}+\delta}, \tag{S1.16}$$

where $\mathbf{B}_l(v)$ is independent of $\mathbf{s}_1$ and $a(v)$ is some product of factors of the form $\beta_1(z)$ or $\mathbf{s}_1^\top \mathbf{B}_l(v)\mathbf{s}_1$. Following similar procedures on Pages 581-583 of Bai and Silverstein (2004) and applying (S1.16), we have

$$\mathbb{E}\left|\frac{M_n^{(1)}(z_1) - M_n^{(1)}(z_2)}{z_1 - z_2}\right|^2 = \mathbb{E}\left|\mathrm{tr}\mathbf{D}^{-1}(z_1)\mathbf{D}^{-1}(z_2) - \mathbb{E}\mathrm{tr}\mathbf{D}^{-1}(z_1)\mathbf{D}^{-1}(z_2)\right|^2 \leq C,$$

uniformly in $z_1, z_2 \in \mathcal{C}_n$.

Step 3. Uniform convergence of $M_n^{(2)}(z)$.

Before proceeding, we collect some necessary results as follows, whose proofs are omitted since one can verify them in the same approaches on Pages 584-586 of Bai and Silverstein (2004):

$$\sup_{z \in \mathcal{C}_n} |\mathbb{E}\underline{s}_n(z) - \underline{s}(z)| \to 0, \tag{S1.17}$$

$$\sup_{n; z \in \mathcal{C}_n} \left\|\left(\frac{1}{y_n}\mathbb{E}\underline{s}_n(z)\mathbf{I}_n + \mathbf{I}_n\right)^{-1}\right\| < \infty, \tag{S1.18}$$

$$\sup_{z\in\mathcal{C}_n}\left|\frac{\mathbb{E}\underline{s}_n^2(z)}{\left(1+\frac{1}{y_n}\mathbb{E}\underline{s}_n(z)\right)^2}\right|<\xi<1, \tag{S1.19}$$

$$\mathbb{E}\left|\mathrm{tr}\mathbf{D}^{-1}(z)\mathbf{M}-\mathbb{E}\mathrm{tr}\mathbf{D}^{-1}(z)\mathbf{M}\right|^2\leq C\|\mathbf{M}\|^2, \tag{S1.20}$$

where $\mathbf{M}$ is a non-random $n\times n$ matrix.

Next, we decompose $M_n^{(2)}(z)$ further into two parts as

$$M_n^{(2)}(z)=p\left(\mathbb{E}\underline{s}_n(z)-\underline{s}_n^{(1)}(z)\right)+p\left(\underline{s}_n^{(1)}(z)-\underline{s}_n^{(0)}(z)\right),$$

where $\underline{s}_n^{(1)}(z)\in\mathbb{C}^+$ is the unique solution to the following equation

$$z=-\frac{1}{\underline{s}_n^{(1)}(z)}+\frac{1}{y_n}\int\frac{t}{1+t\underline{s}_n^{(1)}(z)}dF^{\mathbf{\Sigma}}(t). \tag{S1.21}$$

It is noted that (S1.21) is a particular case of generalized MP equation formulated as

$$z=-\frac{1}{\underline{s}}+y_0\int\frac{t}{1+t\underline{s}}dH(t).$$

By the fact that $\mathcal{C}$ lies outside (3.8), one can verify that

$$\sup_{z\in\mathcal{C}}\left|\underline{s}_n^{(1)}(z)-\underline{s}(z)\right|\to 0,\quad \sup_{z\in\mathcal{C}}\left|\underline{s}_n^{(0)}(z)-\underline{s}(z)\right|\to 0.$$

Throughout the rest proof, all bounds and convergence statements hold uniformly in $z\in\mathcal{C}_n$, so we omit the argument z for simplicity of writing.

On the one hand, since $\mathbf{\Sigma}$ has one eigenvalue of 0 and $n-1$ of $n/(n-1)$, (S1.21) can be expressed as

$$z=-\frac{1}{\underline{s}_n^{(1)}}+\frac{\frac{1}{y_n}}{1+\frac{n}{n-1}\underline{s}_n^{(1)}}. \tag{S1.22}$$

Considering

$$\frac{\underline{s}_n^{(1)}-\underline{s}_n^{(0)}}{\underline{s}_n^{(0)}\underline{s}_n^{(1)}}=\frac{1}{\underline{s}_n^{(0)}}-\frac{1}{\underline{s}_n^{(1)}}=\frac{\frac{1}{y_n}}{1+\underline{s}_n^{(0)}}-\frac{\frac{1}{y_n}}{1+\frac{n}{n-1}\underline{s}_n^{(1)}},$$

we have

$$\begin{aligned}\underline{s}_n^{(1)}-\underline{s}_n^{(0)}=&\frac{\frac{1}{y_n}\underline{s}_n^{(0)}\underline{s}_n^{(1)}\left[\frac{n}{n-1}\underline{s}_n^{(1)}-\underline{s}_n^{(0)}\right]}{\left(1+\underline{s}_n^{(0)}\right)\left(1+\frac{n}{n-1}\underline{s}_n^{(1)}\right)}\\=&\frac{\frac{1}{y_n}\underline{s}_n^{(0)}\underline{s}_n^{(1)}\left[\underline{s}_n^{(1)}-\underline{s}_n^{(0)}\right]}{\left(1+\underline{s}_n^{(0)}\right)\left(1+\frac{n}{n-1}\underline{s}_n^{(1)}\right)}+\frac{1}{n-1}\cdot\frac{\frac{1}{y_n}\underline{s}_n^{(0)}\left(\underline{s}_n^{(1)}\right)^2}{\left(1+\underline{s}_n^{(0)}\right)\left(1+\frac{n}{n-1}\underline{s}_n^{(1)}\right)},\end{aligned}$$

which implies

$$\begin{aligned}p\left(\underline{s}_n^{(1)}-\underline{s}_n^{(0)}\right)=&\frac{\frac{p}{n-1}\frac{1}{y_n}\underline{s}_n^{(0)}\left(\underline{s}_n^{(1)}\right)^2}{\left(1+\underline{s}_n^{(0)}\right)\left(1+\frac{n}{n-1}\underline{s}_n^{(1)}\right)}\left(1-\frac{\frac{1}{y_n}\underline{s}_n^{(0)}\underline{s}_n^{(1)}}{\left(1+\underline{s}_n^{(0)}\right)\left(1+\frac{n}{n-1}\underline{s}_n^{(1)}\right)}\right)^{-1}\\\to&\frac{\underline{s}^3}{(1+\underline{s})^2-y_0\underline{s}^2}.\end{aligned}\tag{S1.23}$$

On the other hand, we consider

$$p\left(\mathbb{E}\underline{s}_n-\underline{s}_n^{(1)}\right)=-\left(1-\frac{1}{y_n}\int\frac{t^2\underline{s}_n^{(1)}\mathbb{E}\underline{s}_n}{\left(1+t\underline{s}_n^{(1)}\right)(1+t\mathbb{E}\underline{s}_n)}dF^{\boldsymbol{\Sigma}}(t)\right)^{-1}p\underline{s}_n^{(1)}\mathbb{E}\underline{s}_nR_n,\tag{S1.24}$$

where

$$R_n=\frac{1}{\mathbb{E}\underline{s}_n}+z-\frac{1}{y_n}\int\frac{t}{1+t\mathbb{E}\underline{s}_n}dF^{\boldsymbol{\Sigma}}(t).$$

Therefore, it suffices to analyze the limit of $n\underline{s}_n^{(1)}\mathbb{E}\underline{s}_nR_n$. Denote

$$\mathbf{K}(z)=\mathbb{E}\underline{s}_n(z)\boldsymbol{\Sigma}+\mathbf{I}_n.$$

We have

$$p\mathbb{E}\underline{s}_nR_n=p\mathbb{E}\underline{s}_n\left(\frac{1}{\mathbb{E}\underline{s}_n}+z-\frac{1}{y_n}\int\frac{t}{1+t\mathbb{E}\underline{s}_n}dF^{\boldsymbol{\Sigma}}(t)\right)$$

$$=p\mathbb{E}\mathbf{s}_1^\top\mathbf{D}^{-1}\mathbf{K}^{-1}\mathbf{s}_1+z\mathbb{E}\underline{s}_n\mathbb{E}\mathrm{tr}\mathbf{D}^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}$$

Applying (S1.4) and (S1.5), $\mathbf{s}_1^\top\mathbf{D}^{-1}=\beta_1\mathbf{s}_1^\top\mathbf{D}_1^{-1}$ and $z\mathbb{E}\underline{s}_n=-\mathbb{E}\beta_1$, which implies

$$\begin{aligned}p\mathbb{E}\underline{s}_nR_n=&p\mathbb{E}\beta_1\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\mathbf{D}^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}\\=&p\mathbb{E}\beta_1\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\mathbf{D}_1^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}+\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\left(\mathbf{D}_1^{-1}-\mathbf{D}^{-1}\right)\boldsymbol{\Sigma}\mathbf{K}^{-1}\end{aligned}$$

Considering respectively the following two terms,

$$p\mathbb{E}\beta_1\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\mathbf{D}_1^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}, \tag{S1.25}$$

$$\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\left(\mathbf{D}_1^{-1}-\mathbf{D}^{-1}\right)\boldsymbol{\Sigma}\mathbf{K}^{-1}. \tag{S1.26}$$

By identity $\beta_1=b_n-b_n^2\gamma_1+\beta_1b_n^2\gamma_1^2$, (S1.25) can be split into three parts. For the first part,

$$pb_n\mathbb{E}\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-b_n\mathbb{E}\mathrm{tr}\mathbf{D}_1^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}=0.$$

For the second part, by (S1.16) and Lemma S2.1,

$$\begin{aligned}&-pb_n^2\mathbb{E}\gamma_1\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1+b_n^2\mathbb{E}\gamma_1\mathbb{E}\mathrm{tr}\mathbf{D}_1^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}\\=&-pb_n^2\mathbb{E}\left(\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{s}_1-\frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_1^{-1}\right)\left(\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-\frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{D}_1^{-1}\mathbf{K}^{-1}\right)+o(1)\\=&-z^2\underline{s}^2\mathbb{E}\left(\frac{2}{p}\mathrm{tr}\mathbf{D}_1^{-2}\mathbf{K}^{-1}-\frac{6}{5p}\mathrm{tr}\left(\mathbf{D}_1^{-1}\circ\mathbf{D}_1^{-1}\mathbf{K}^{-1}\right)-\frac{4}{5np}\mathrm{tr}\mathbf{D}_1^{-1}\mathrm{tr}\mathbf{D}_1^{-1}\mathbf{K}^{-1}\right)+o(1).\end{aligned} \tag{S1.27}$$

For the third part, by (S1.16),

$$\left|pb_n^2\mathbb{E}\beta_1\gamma_1^2\mathbf{s}_1^\top\mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1-b_n^2\mathbb{E}\beta_1\gamma_1^2\mathbb{E}\mathrm{tr}\mathbf{D}_1^{-1}\boldsymbol{\Sigma}\mathbf{K}^{-1}\right|$$

$$
\begin{aligned}
=&pb_n^2 \left|\operatorname{cov}\left(\beta_1\gamma_1^2, \mathbf{s}_1^\top \mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1\right)\right| \\
\leq&pb_n^2\sqrt{\mathbb{E}\left|\beta_1\gamma_1^2\right|^2}\sqrt{\operatorname{var}\left(\mathbf{s}_1^\top \mathbf{D}_1^{-1}\mathbf{K}^{-1}\mathbf{s}_1\right)} \\
\to&0.
\end{aligned}
$$

(S1.26) can be expressed by (S1.4) as

$$
\mathbb{E}\beta_1\mathbb{E}\mathrm{tr}\left(\mathbf{D}_1^{-1}-\mathbf{D}^{-1}\right)\mathbf{\Sigma}\mathbf{K}^{-1} = \frac{z^2\underline{s}^2}{p}\mathbb{E}\mathrm{tr}\mathbf{\Sigma}\mathbf{D}_1^{-1}\mathbf{\Sigma}\mathbf{K}^{-1}\mathbf{D}_1^{-1}+o(1), \tag{S1.28}
$$

Collecting (S1.27) and (S1.28), by (S1.4) and (S1.16), we obtain

$$
p\mathbb{E}\underline{s}_nR_n = -J_1+\frac{6}{5}J_2+\frac{4}{5}J_3+o(1),
$$

where

$$
\begin{aligned}
J_1 =&\frac{z^2\underline{s}^2}{p}\left(2\mathbb{E}\mathrm{tr}\mathbf{D}^{-2}\mathbf{K}^{-1}-\mathbb{E}\mathrm{tr}\mathbf{\Sigma}\mathbf{D}^{-1}\mathbf{\Sigma}\mathbf{K}^{-1}\mathbf{D}^{-1}\right), \\
J_2 =&\frac{z^2\underline{s}^2}{p}\mathbb{E}\mathrm{tr}\left(\mathbf{D}^{-1}\circ\mathbf{D}^{-1}\mathbf{K}^{-1}\right), \\
J_3 =&\frac{z^2\underline{s}^2}{np}\mathbb{E}\mathrm{tr}\mathbf{D}^{-1}\mathrm{tr}\mathbf{D}^{-1}\mathbf{K}^{-1}.
\end{aligned}
$$

As has been discussed in **Step 1**, multiplying $\mathbf{\Sigma}$ has no influence on J_1, which implies

$$
J_1 = \frac{z^2\underline{s}^2}{p}\mathbb{E}\mathrm{tr}\mathbf{\Sigma}\mathbf{D}^{-1}\mathbf{\Sigma}\mathbf{K}^{-1}\mathbf{D}^{-1}+o(1).
$$

Following the same procedures on Pages 589-592 of Bai and Silverstein (2004) and applying (S1.16), we have

$$
J_1 = \frac{y_0\underline{s}^2}{\left((1+\underline{s})^2-y_0\underline{s}^2\right)(1+\underline{s})}+o(1). \tag{S1.29}
$$

With similar arguments in (S1.11)-(S1.13) and applying (S1.16), we have

$$\begin{aligned} J_2 =& \frac{z^2\underline{s}^2}{p}\mathrm{tr}\left(\mathbb{E}\mathbf{D}^{-1}\circ\mathbb{E}\mathbf{D}^{-1}\mathbf{K}^{-1}\right)+o(1) \\ =& \frac{z^2\underline{s}^2}{p}\mathrm{tr}\left((-z\underline{s}\mathbf{\Sigma}-z\mathbf{I}_n)^{-1}\circ(-z\underline{s}\mathbf{\Sigma}-z\mathbf{I}_n)^{-1}(\underline{s}\mathbf{\Sigma}+\mathbf{I}_n)^{-1}\right)+o(1) \\ =& \frac{y_0\underline{s}^2}{(1+\underline{s})^3}+o(1). \end{aligned} \tag{S1.30}$$

and

$$\begin{aligned} J_3 =& \frac{z^2\underline{s}^2}{np}\mathbb{E}\mathrm{tr}\mathbb{E}\mathbf{D}^{-1}\mathrm{tr}\mathbb{E}\mathbf{D}^{-1}\mathbf{K}^{-1}+o(1) \\ =& \frac{z^2\underline{s}^2}{np}\mathrm{tr}\left(-z\underline{s}\mathbf{\Sigma}-z\mathbf{I}_n\right)^{-1}\mathrm{tr}\left(-z\underline{s}\mathbf{\Sigma}-z\mathbf{I}_n\right)^{-1}\left(\underline{s}\mathbf{\Sigma}+\mathbf{I}_n\right)^{-1}+o(1) \\ =& \frac{y_0\underline{s}^2}{(1+\underline{s})^3}+o(1). \end{aligned} \tag{S1.31}$$

Collecting (S1.29)-(S1.31), we obtain

$$p\mathbb{E}\underline{s}_nR_n = -\frac{y_0\underline{s}^2}{\left((1+\underline{s})^2-y_0\underline{s}^2\right)(1+\underline{s})}+\frac{2y_0\underline{s}^2}{(1+\underline{s})^3}+o(1). \tag{S1.32}$$

Together with (S1.23), (S1.24) and (S1.32),

$$p\left(\mathbb{E}\underline{s}_n(z)-\underline{s}_n^{(0)}(z)\right)\to\mu(z).$$

S1.3 Proof of Lemma S1.3

The main process is similar with the proof of Lemma S1.2. First we prove the convergence of $\widehat{L}_n(z)$ in probability for each $z\in\mathcal{C}$, then we show its tightness on $\mathcal{C}$, which leads to the convergence of stochastic process.

Step 1. Convergence of $\widehat{L}_n(z)$.

Since that

$$\boldsymbol{\rho}_n - \widetilde{\boldsymbol{\rho}}_n = \frac{3}{n+1}\left(\mathbf{K}_n - \widetilde{\boldsymbol{\rho}}_n\right),$$

we have

$$\begin{aligned}\widehat{L}_n(z) =& \mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\left(\boldsymbol{\rho}_n - \widetilde{\boldsymbol{\rho}}_n\right)\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1} \\ =& \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1} \\ &- \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1}.\end{aligned}$$

We further expand $\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1}$,

$$\begin{aligned}&\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1} - \mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1} \\ =& \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1} \\ &- \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1}.\end{aligned}$$

By the rigidity of the edge of Kendall's rank correlation matrix, $\|\mathbf{K}_n\|$ is uniformly bounded almost surely, and then subsequently $\widetilde{\boldsymbol{\rho}}_n = (n+1)\boldsymbol{\rho}_n/(n-2) - 3\mathbf{K}_n/(n-2)$ also has uniformly bounded spectral norm. So we have

$$\begin{aligned}&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1} \\ =& \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1} + o_P(1),\end{aligned}$$

and similarly,

$$\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\boldsymbol{\rho}_n - z\mathbf{I}_p\right)^{-1}$$

$$=\frac{3}{n+1}\operatorname{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1} + o_P(1),$$

which implies

$$\begin{aligned}\widehat{L}_n(z) =& \frac{3}{n+1}\operatorname{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\\ &- \frac{3}{n+1}\operatorname{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1} + o_P(1).\end{aligned}$$

Denote by $\mathbf{A}_i$ the conditional expectation of $\mathbf{A}_{ij}$ given $\mathbf{X}_i$, $\mathbf{A}_i = \mathbb{E}[\mathbf{A}_{ij}|\mathbf{X}_i]$. By the Hoeffding's decomposition of $\mathbf{A}_{ij}$ illustrated in Bandeira et al. (2017), Wu and Wang (2022) and Li et al. (2023),

$$\mathbf{A}_{ij} = \mathbf{A}_i + \mathbf{A}_j + \boldsymbol{\varepsilon}_{ij},$$

where $\boldsymbol{\varepsilon}_{ij}$ is uncorrelated with $\mathbf{A}_i$ and $\mathbf{A}_j$. So we naturally approximate $\widetilde{\boldsymbol{\rho}}_n$ and $\mathbf{K}_n$ by

$$\mathbf{U}_n = \frac{3}{n}\sum_{i=1}^{n}\mathbf{A}_i\mathbf{A}_i^\top$$

and

$$\mathbf{V}_n = \frac{2}{n}\sum_{i=1}^{n}\mathbf{A}_i\mathbf{A}_i^\top + \frac{1}{3}\mathbf{I}_p$$

respectively. The error of this approximation can be well controlled as follows.

Lemma S1.4. Suppose $\mathbf{X}_1, \cdots, \mathbf{X}_n$ i.i.d. from a poplulation $\mathbf{X} \in \mathbb{R}^p$, whose entries are independent and absolutely continuous respect to the Lebesgue measure. Then we have

$$\mathbb{E}\left\|\widetilde{\boldsymbol{\rho}}_n - \mathbf{U}_n\right\|_F^2 = o(p), \tag{S1.33}$$

and

$$\mathbb{E}\left\|\mathbf{K}_n - \mathbf{V}_n\right\|_F^2 = o(p). \tag{S1.34}$$

To replace $\widetilde{\boldsymbol{\rho}}_n$ and $\mathbf{K}_n$ with $\mathbf{U}_n$ and $\mathbf{V}_n$, we consider

$$\begin{aligned}&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1} - \frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\\ =&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\left(\mathbf{U}_n - \widetilde{\boldsymbol{\rho}}_n\right)\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}.\end{aligned}$$

By Cauchy's inequality and Lemma S1.4,

$$\begin{aligned}&\mathbb{E}\left|\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\left(\mathbf{U}_n - \widetilde{\boldsymbol{\rho}}_n\right)\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\right|\\ \leq&\mathbb{E}\left(p\cdot\|\mathbf{K}_n\|\left\|\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\right\|^2\left\|\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\right\|\mathrm{tr}\left(\mathbf{U}_n - \widetilde{\boldsymbol{\rho}}_n\right)^2\right)^{\frac{1}{2}}\\ \lesssim&p^{\frac{1}{2}}\left(\mathbb{E}\mathrm{tr}\left(\mathbf{U}_n - \widetilde{\boldsymbol{\rho}}_n\right)^2\right)^{\frac{1}{2}}\\ =&o(p),\end{aligned}$$

which conclude by Markov's inequality that

$$\begin{aligned}&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\\ =&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1} + o_P(1).\end{aligned}$$

Following similar steps above, we obtain

$$\begin{aligned}&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\mathbf{K}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\\ =&\frac{3}{n+1}\mathrm{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{V}_n\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1} + o_P(1).\end{aligned}$$

And similarly,

$$\begin{aligned}&\frac{3}{n+1}\mathrm{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\widetilde{\boldsymbol{\rho}}_n\left(\widetilde{\boldsymbol{\rho}}_n - z\mathbf{I}_p\right)^{-1}\\ =&\frac{3}{n+1}\mathrm{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{U}_n\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1} + o_P(1).\end{aligned}$$

Observing that

$$\begin{aligned}\operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{V}_n\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1} =& \operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-2}\mathbf{V}_n \\ =& \frac{d}{dz}\operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{V}_n,\end{aligned}$$

so it suffices to find the limit of

$$\frac{3}{n+1}\operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{V}_n \tag{S1.35}$$

and

$$\frac{3}{n+1}\operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\mathbf{U}_n. \tag{S1.36}$$

For simplicity of writing, we denote

$$\mathbf{t}_i = \sqrt{\frac{3}{n}}\mathbf{A}_i, \quad \mathbf{H}(z) = \mathbf{U}_n - z\mathbf{I}_p, \quad \mathbf{H}_j(z) = \mathbf{U}_n - z\mathbf{I}_p - \mathbf{t}_j\mathbf{t}_j^\top.$$

As for the first term,

$$(\text{S1.35}) = \frac{2}{n+1}\sum_{j=1}^{n}\mathbf{t}_j^\top\mathbf{H}^{-1}(z)\mathbf{t}_j + \frac{1}{n+1}\operatorname{tr}\mathbf{H}^{-1}(z).$$

By the leave-one-out method,

$$\mathbf{H}^{-1}(z) = \mathbf{H}_j^{-1}(z) - \frac{\mathbf{H}_j^{-1}(z)\mathbf{t}_j\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)}{1+\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)\mathbf{t}_j},$$

and then

$$\mathbf{t}_j^\top\mathbf{H}^{-1}(z)\mathbf{t}_j = \frac{\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)\mathbf{t}_j}{1+\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)\mathbf{t}_j}.$$

Since that

$$\mathbb{E}\left|\frac{\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)\mathbf{t}_j - \frac{1}{n}\operatorname{tr}\mathbf{H}^{-1}(z)}{1+\mathbf{t}_j^\top\mathbf{H}_j^{-1}(z)\mathbf{t}_j}\right|$$

$$\lesssim \mathbb{E}\left|\mathbf{t}_j^\top \mathbf{H}_j^{-1}(z)\mathbf{t}_j - \frac{1}{n}\mathrm{tr}\mathbf{H}_j^{-1}(z)\right| + \frac{1}{n}\mathbb{E}\left|\mathrm{tr}\mathbf{H}_j^{-1}(z) - \mathrm{tr}\mathbf{H}^{-1}(z)\right|$$
$$= O(n^{-\frac{1}{2}}),$$

and

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{1+\mathbf{t}_j^\top \mathbf{H}_j^{-1}(z)\mathbf{t}_j} - \frac{1}{1+\frac{1}{n}\mathrm{tr}\mathbf{H}^{-1}}\right| \\
=&\mathbb{E}\left|\frac{\mathbf{t}_j^\top \mathbf{H}_j^{-1}(z)\mathbf{t}_j - \frac{1}{n}\mathrm{tr}\mathbf{H}^{-1}(z)}{\left(1+\mathbf{t}_j^\top \mathbf{H}_j^{-1}(z)\mathbf{t}_j\right)\left(1+\frac{1}{n}\mathrm{tr}\mathbf{H}^{-1}(z)\right)}\right| \\
\lesssim&\mathbb{E}\left|\mathbf{t}_j^\top \mathbf{H}_j^{-1}(z)\mathbf{t}_j - \frac{1}{n}\mathrm{tr}\mathbf{H}_j^{-1}(z)\right| + \frac{1}{n}\mathbb{E}\left|\mathrm{tr}\mathbf{H}_j^{-1}(z) - \mathrm{tr}\mathbf{H}^{-1}(z)\right| \\
=&O(n^{-\frac{1}{2}}),
\end{aligned}$$

we have

$$\text{(S1.35)} = \frac{2n}{n+1}\cdot\frac{\frac{1}{n}\mathrm{tr}\mathbf{H}^{-1}(z)}{1+\frac{1}{n}\mathrm{tr}\mathbf{H}^{-1}(z)} + \frac{1}{n+1}\mathrm{tr}\mathbf{H}^{-1}(z) + o_P(1).$$

By Theorem 2 of Wu and Wang (2022),

$$\frac{1}{p}\mathrm{tr}\mathbf{H}^{-1}(z) \to m(z) = \frac{1-y-z+\sqrt{(1+y-z)^2-4y}}{2yz}, \quad \text{almost surely.}$$

Therefore,

$$\text{(S1.35)} \to \frac{2ym(z)}{1+ym(z)} + ym(z), \quad \text{in probability.}$$

And similarly,

$$\text{(S1.36)} = \frac{3}{n+1}\sum_{j=1}^{n}\mathbf{t}_j^\top \mathbf{H}^{-1}(z)\mathbf{t}_j \to \frac{3ym(z)}{1+ym(z)}, \quad \text{in probability.}$$

To sum up,

$$\widehat{L}_n(z) = \frac{3}{n+1}\mathrm{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-2}\left(\mathbf{V}_n - \mathbf{U}_n\right) + o_P(1)$$

$$
\begin{aligned}
=&\frac{3}{n+1}\frac{d}{dz}\operatorname{tr}\left(\mathbf{U}_n - z\mathbf{I}_p\right)^{-1}\left(\mathbf{V}_n - \mathbf{U}_n\right) + o_P(1)\\
\to& ym'(z) - \frac{ym'(z)}{\left(1+ym(z)\right)^2}, \quad \text{in probability.}
\end{aligned}
$$

Since

$$
\underline{s}(z) = \frac{-1+y-z+\sqrt{(1+y-z)^2-4y}}{2z},
$$

we have

$$
\begin{aligned}
y_0\underline{s}(y_0z) =& \frac{-1+y_0-y_0z+\sqrt{(1+y_0-y_0z)^2-4y_0}}{2y_0z}\\
=& \frac{-y+1-z+\sqrt{(y+1-z)^2-4y}}{2z} = m(z).
\end{aligned}
$$

Therefore,

$$
ym'(z) - \frac{ym'(z)}{\left(1+ym(z)\right)^2} = y_0\underline{s}'(y_0z) - \frac{y_0\underline{s}'(y_0z)}{\left(1+\underline{s}(y_0z)\right)^2} = y_0\widetilde{\mu}(y_0z).
$$

Step 2. Tightness of $\widehat{L}_n(z)$.

Similar with the step 2 of the proof of Lemma S1.2, it suffices to verify

$$
\sup_{n;z_1,z_2\in\mathcal{C}_n} \frac{\mathbb{E}|\widehat{L}_n(z_1)-\widehat{L}_n(z_2)|^2}{|z_1-z_2|^2} < \infty.
$$

Since

$$
\begin{aligned}
&\frac{\mathbb{E}|\widehat{L}_n(z_1)-\widehat{L}_n(z_2)|^2}{|z_1-z_2|^2}\\
=&\mathbb{E}\left|\operatorname{tr}\left(\widetilde{\boldsymbol{\rho}}_n - z_1\mathbf{I}_p\right)^{-1}\left(\widetilde{\boldsymbol{\rho}}_n - z_2\mathbf{I}_p\right)^{-1} - \operatorname{tr}\left(\boldsymbol{\rho}_n - z_1\mathbf{I}_p\right)^{-1}\left(\boldsymbol{\rho}_n - z_2\mathbf{I}_p\right)^{-1}\right|^2\\
=&\mathbb{E}\left|\sum_{i=1}^p \frac{1}{(\widetilde{\lambda}_i - z_1)(\widetilde{\lambda}_i - z_2)} - \sum_{i=1}^p \frac{1}{(\lambda_i - z_1)(\lambda_i - z_2)}\right|^2
\end{aligned}
$$

$$=\mathbb{E}\left|\sum_{i=1}^{p}\frac{(\lambda_i+\widetilde{\lambda}_i-z_1-z_2)(\lambda_i-\widetilde{\lambda}_i)}{(\widetilde{\lambda}_i-z_1)(\widetilde{\lambda}_i-z_2)(\lambda_i-z_1)(\lambda_i-z_2)}\right|^2,$$

where $\lambda_1\geq\cdots\geq\lambda_p$ and $\widetilde{\lambda}_1\geq\cdots\geq\widetilde{\lambda}_p$ are the eigenvalues of $\boldsymbol{\rho}_n$ and $\widetilde{\boldsymbol{\rho}}_n$ respectively. By Weyl's inequality,

$$\left|\lambda_i-\widetilde{\lambda}_i\right|\leq\|\boldsymbol{\rho}_n-\widetilde{\boldsymbol{\rho}}_n\|=\frac{3}{n+1}\left\|\mathbf{K}_n+\widetilde{\boldsymbol{\rho}}_n\right\|.$$

By the rigidity of edge of $\mathbf{K}_n$ and $\boldsymbol{\rho}_n$ and the truncation of $\widehat{L}_n(z)$,

$$\mathbb{E}\left|\sum_{i=1}^{p}\frac{(\lambda_i+\widetilde{\lambda}_i-z_1-z_2)(\lambda_i-\widetilde{\lambda}_i)}{(\widetilde{\lambda}_i-z_1)(\widetilde{\lambda}_i-z_2)(\lambda_i-z_1)(\lambda_i-z_2)}\right|^2\leq C_1p\sum_{i=1}^{p}\mathbb{E}\left|\lambda_i-\widetilde{\lambda}_i\right|^2\leq C_2,$$

uniformly in $z_1,z_2\in\mathcal{C}_n$.

S2 Auxiliary lemmas

Lemma S2.1. For non-random $n\times n$ symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, we have

$$\begin{aligned}\mathbb{E}\mathbf{s}^\top\mathbf{A}\mathbf{s}=&\mathrm{tr}\boldsymbol{\Sigma}\mathbf{A};\\ \mathrm{cov}(\mathbf{s}^\top\mathbf{A}\mathbf{s},\mathbf{s}^\top\mathbf{B}\mathbf{s})=&2\mathrm{tr}(\mathbf{A}\mathbf{B})-\frac{6}{5}\mathrm{tr}(\mathbf{A}\circ\mathbf{B})-\frac{4}{5n}\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})+O(1)\|\mathbf{A}\|\|\mathbf{B}\|.\end{aligned}$$

Proof. Noting

$$\mathbb{E}\mathbf{s}=\mathbf{0}_n,\ \mathrm{cov}(\mathbf{s})=\boldsymbol{\Sigma}=\frac{n}{n-1}\left(\mathbf{I}_n-\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top\right),$$

we have $\mathbb{E}\mathbf{s}^\top\mathbf{A}\mathbf{s}=\mathrm{tr}\boldsymbol{\Sigma}\mathbf{A}$. Furthermore, we calculate the covariance of the interaction term $\mathrm{cov}(s_is_j,s_ks_l)$.

By Faulhaber's formula,

$$\mathrm{var}(s_1^2)=\mathbb{E}(s_1^2-1)^2=\frac{144}{n(n-1)^2(n+1)^2}\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)^4-1=\frac{4}{5}-\frac{12}{5(n^2-1)}.$$

Noting $s_1 + \ldots + s_n = 0$, we have

$$0 = \mathrm{cov}\left(s_1^2, s_1(s_1 + \ldots + s_n)\right) = \mathrm{var}(s_1^2) + (n-1)\mathrm{cov}(s_1^2, s_1 s_2),$$

which yields

$$\mathrm{cov}(s_1^2, s_1 s_2) = -\frac{1}{n-1}\mathrm{var}(s_1^2).$$

Similarly, since $s_1^2 + \ldots + s_n^2 = n$,

$$0 = \mathrm{cov}\left(s_1^2, s_1^2 + \ldots + s_n^2)\right) = \mathrm{var}(s_1^2) + (n-1)\mathrm{cov}(s_1^2, s_2^2),$$

which yields

$$\mathrm{cov}(s_1^2, s_2^2) = -\frac{1}{n-1}\mathrm{var}(s_1^2).$$

By exploiting this trick, we can get

$$\begin{aligned}
\mathrm{cov}(s_1^2, s_2 s_3) =& \frac{2}{(n-1)(n-2)}\mathrm{var}(s_1^2), \\
\mathrm{var}(s_1 s_2) =& \frac{n(n-2)}{(n-1)^2} - \frac{1}{n-1}\mathrm{var}(s_1^2), \\
\mathrm{cov}(s_1 s_2, s_1 s_3) =& -\frac{n}{(n-1)^2} + \frac{2}{(n-1)(n-2)}\mathrm{var}(s_1^2), \\
\mathrm{cov}(s_1 s_2, s_3 s_4) =& \frac{2n}{(n-1)^2(n-3)} - \frac{6}{(n-1)(n-2)(n-3)}\mathrm{var}(s_1^2).
\end{aligned}$$

Now, we are ready to derive the covariance

$$\mathrm{cov}(\mathbf{s}^\top \mathbf{A}\mathbf{s}, \mathbf{s}^\top \mathbf{B}\mathbf{s}) = \mathrm{cov}\left(\sum_{i=1}^n a_{ii} s_i^2 + 2\sum_{i<j} a_{ij} s_i s_j, \sum_{k=1}^n b_{kk} s_k^2 + 2\sum_{k<l} b_{kl} s_k s_l\right)$$

$$= \sum_{i,k} a_{ii} b_{kk} \mathrm{cov}(s_i^2, s_k^2) + 2\sum_{i,k<l} a_{ii} b_{kl} \mathrm{cov}(s_i^2, s_k s_l)$$

$$
\begin{aligned}
&+ 2\sum_{i<j,k} a_{ij}b_{kk}\mathrm{cov}(s_is_j, s_k^2) + 4\sum_{i<j,k<l} a_{ij}b_{kl}\mathrm{cov}(s_is_j, s_ks_l)\\
=&\mathrm{var}(s_1^2)\sum_{i=1}^n a_{ii}b_{ii} + \mathrm{cov}(s_1^2, s_2^2)\sum_{i,k}^{*} a_{ii}b_{kk}\\
&+ 2\mathrm{cov}(s_1^2, s_1s_2)\sum_{k,l}^{*} a_{kk}b_{kl} + \mathrm{cov}(s_1^2, s_2s_3)\sum_{i,k,l}^{*} a_{ii}b_{kl}\\
&+ 2\mathrm{cov}(s_1^2, s_1s_2)\sum_{i,j}^{*} a_{ij}b_{ii} + \mathrm{cov}(s_1^2, s_2s_3)\sum_{i,j,k}^{*} a_{ij}b_{kk}\\
&+ 2\mathrm{var}(s_1s_2)\sum_{i,j}^{*} a_{ij}b_{ij} + 4\mathrm{cov}(s_1s_2, s_1s_3)\sum_{i,j,k}^{*} a_{ij}b_{jk} + \mathrm{cov}(s_1s_2, s_3s_4)\sum_{i,j,k,l}^{*} a_{ij}b_{kl}\\
=& \big(\mathrm{var}(s_1^2) - \mathrm{cov}(s_1^2, s_2^2) - 4\mathrm{cov}(s_1^2, s_2s_3) + 4\mathrm{cov}(s_1^2, s_2s_3) - 2\mathrm{var}(s_1s_2) + 8\mathrm{cov}(s_1s_2, s_1s_3)\\
&-6\mathrm{cov}(s_1s_2, s_3s_4))\,\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) + \big(\mathrm{cov}(s_1^2, s_2^2) - 2(s_1^2, s_2s_3) + \mathrm{cov}(s_1s_2, s_3s_4)\big)\,\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\\
&+ (2\mathrm{var}(s_1s_2) - 4\mathrm{cov}(s_1s_2, s_1s_3) + 2\mathrm{cov}(s_1s_2, s_3s_4))\,\mathrm{tr}(\mathbf{AB})\\
&+ \big(2\mathrm{cov}(s_1^2, s_1s_2) - 2\mathrm{cov}(s_1^2, s_2s_3) - 4\mathrm{cov}(s_1s_2, s_1s_3) + 4\mathrm{cov}(s_1s_2, s_3s_4)\big)\,\mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A})\\
&+ \big(2\mathrm{cov}(s_1^2, s_1s_2) - 2\mathrm{cov}(s_1^2, s_2s_3) - 4\mathrm{cov}(s_1s_2, s_1s_3) + 4\mathrm{cov}(s_1s_2, s_3s_4)\big)\,\mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B})\\
&+ (4\mathrm{cov}(s_1s_2, s_1s_3) - 4\mathrm{cov}(s_1s_2, s_3s_4))\,\mathbf{1}_n^\top\mathbf{AB}\mathbf{1}_n + \mathrm{cov}(s_1s_2, s_3s_4)\mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n\\
=&2\mathrm{tr}(\mathbf{AB}) - \frac{6}{5}\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) - \frac{4}{5n}\mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) + O(1)\|\mathbf{A}\|\|\mathbf{B}\|,
\end{aligned}
$$

where we use the facts

$$
\begin{aligned}
\sum_{i,k}^{*} a_{ii}b_{kk} &= \sum_{i,j} a_{ii}b_{jj} - \sum_{i=1}^n a_{ii}b_{ii} = \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) - \mathrm{tr}(\mathbf{A}\circ\mathbf{B});\\
\sum_{k,l}^{*} a_{kk}b_{kl} &= \sum_{i,j} a_{ii}b_{ij} - \sum_{i=1}^n a_{ii}b_{ii} = \mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A}) - \mathrm{tr}(\mathbf{A}\circ\mathbf{B});\\
\sum_{i,k,l}^{*} a_{ii}b_{kl} &= \sum_{i=1}^n a_{ii}\left(\sum_{k,l} b_{kl} + 2b_{ii} - 2\sum_{k=1}^n b_{il} - \sum_{k=1}^n b_{kk}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{tr}(\mathbf{A})\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n + 2\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) - 2\mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A}) - \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B});\\
\sum_{i,j}^{*} a_{ij}b_{ii} &= \mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) - \mathrm{tr}(\mathbf{A}\circ\mathbf{B});\\
\sum_{i,j,k}^{*} a_{ij}b_{kk} &= \mathrm{tr}(\mathbf{B})\mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n + 2\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) - 2\mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) - \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B});\\
\sum_{i,j}^{*} a_{ij}b_{ij} &= \sum_{i,j} a_{ij}b_{ij} - \sum_{i=1}^{n} a_{ii}b_{ii} = \mathrm{tr}(\mathbf{A}\mathbf{B}) - \mathrm{tr}(\mathbf{A}\circ\mathbf{B});\\
\sum_{i,j,k}^{*} a_{ij}b_{jk} &= \sum_{i,k}^{*}\left(\sum_{j=1}^{n} a_{ij}b_{jk} - a_{ii}b_{ik} - a_{ik}b_{kk}\right)\\
&= \mathbf{1}_n^\top\mathbf{A}\mathbf{B}\mathbf{1}_n - \mathrm{tr}(\mathbf{A}\mathbf{B}) - \mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A}) - \mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) + 2\mathrm{tr}(\mathbf{A}\circ\mathbf{B})\\
\sum_{i,j,k,l}^{*} a_{ij}b_{kl} &= \sum_{i,j,k}^{*} a_{ij}\left(\sum_{l=1}^{n} b_{kl} - b_{ki} - b_{kj} - b_{kk}\right)\\
&= \sum_{i,j}^{*} a_{ij}\left(\sum_{k,l} b_{kl} - \sum_{l=1}^{n} b_{il} - \sum_{l=1}^{n} b_{jl}\right) - \sum_{i,j,k}^{*} a_{ij}\left(b_{ki} + b_{kj} + b_{kk}\right)\\
&= \mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n - 2\mathbf{1}_n^\top\mathbf{A}\mathbf{B}\mathbf{1}_n - \mathrm{tr}(\mathbf{A})\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n + 2\mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A})\\
&\quad - 2\left(\mathbf{1}_n^\top\mathbf{A}\mathbf{B}\mathbf{1}_n - \mathrm{tr}(\mathbf{A}\mathbf{B}) - \mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A}) - \mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) + 2\mathrm{tr}(\mathbf{A}\circ\mathbf{B})\right)\\
&\quad - \left(\mathrm{tr}(\mathbf{B})\mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n + 2\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) - 2\mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) - \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})\right)\\
&= \mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n - 4\mathbf{1}_n^\top\mathbf{A}\mathbf{B}\mathbf{1}_n + 4\mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) + 4\mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A})\\
&\quad - \mathrm{tr}(\mathbf{A})\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n - \mathrm{tr}(\mathbf{B})\mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n - 6\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) + 2\mathrm{tr}(\mathbf{A}\mathbf{B}) + \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B})
\end{aligned}
$$

and the bounds

$$
\mathrm{tr}(\mathbf{A}\circ\mathbf{B}) = O(n)\|\mathbf{A}\|\|\mathbf{B}\|, \mathrm{tr}(\mathbf{A})\mathrm{tr}(\mathbf{B}) = O(n^2)\|\mathbf{A}\|\|\mathbf{B}\|, \mathrm{tr}(\mathbf{A}\mathbf{B}) = O(n)\|\mathbf{A}\|\|\mathbf{B}\|,
$$

$$
\mathbf{1}_n^\top\mathbf{B}\mathrm{diag}(\mathbf{A}) = O(n)\|\mathbf{A}\|\|\mathbf{B}\|, \mathbf{1}_n^\top\mathbf{A}\mathrm{diag}(\mathbf{B}) = O(n)\|\mathbf{A}\|\|\mathbf{B}\|,
$$

$$\mathbf{1}_n^\top\mathbf{A}\mathbf{1}_n\mathbf{1}_n^\top\mathbf{B}\mathbf{1}_n = O(n^2)\|\mathbf{A}\|\|\mathbf{B}\|, \mathbf{1}_n^\top\mathbf{A}\mathbf{B}\mathbf{1}_n = O(n)\|\mathbf{A}\|\|\mathbf{B}\|.$$

The proof is completed. □

Lemma S2.2. For non-random $n \times n$ matrices $\mathbf{A}_k$, $k = 1, \cdots, r$ and $\mathbf{B}_l$, $l = 1, \cdots, q$, we have

$$\left|\mathbb{E}\prod_{k=1}^{r}\mathbf{s}_1^\top\mathbf{A}_k\mathbf{s}_1\prod_{l=1}^{q}\left(\mathbf{s}_1^\top\mathbf{B}_l\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}_l\right)\right| \le Cn^{-\frac{q}{2}+\delta}\prod_{k=1}^{r}\|\mathbf{A}_k\|\prod_{l=1}^{q}\|\mathbf{B}_l\|, \quad \text{(S2.37)}$$

for arbitrarily small $\delta > 0$.

Proof. For non-random $n \times n$ matrix $\mathbf{B}$, by Proposition 2.1 of Bao (2019),

$$\mathbb{E}\left|\mathbf{s}_1^\top\mathbf{B}\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}\right|^q \le Cn^{-\frac{q}{2}+\delta}\|\mathbf{B}\|^q, \quad \text{(S2.38)}$$

for arbitrarily small $\delta > 0$. When $r = 0$, $q > 1$, (S2.37) is a consequence of (S2.38) and Holder's inequality. If $r > 0$, by induction on r we have

$$\begin{aligned}
&\left|\mathbb{E}\prod_{k=1}^{r}\mathbf{s}_1^\top\mathbf{A}_k\mathbf{s}_1\prod_{l=1}^{q}\left(\mathbf{s}_1^\top\mathbf{B}_l\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}_l\right)\right| \\
\le &\left|\mathbb{E}\prod_{k=1}^{r-1}\mathbf{s}_1^\top\mathbf{A}_k\mathbf{s}_1(\mathbf{s}_1^\top\mathbf{A}_r\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{A}_r)\prod_{l=1}^{q}\left(\mathbf{s}_1^\top\mathbf{B}_l\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}_l\right)\right| \\
&+ \frac{n}{p}\|\mathbf{A}_r\|\left|\mathbb{E}\prod_{k=1}^{r-1}\mathbf{s}_1^\top\mathbf{A}_k\mathbf{s}_1\prod_{l=1}^{q}\left(\mathbf{s}_1^\top\mathbf{B}_l\mathbf{s}_1 - \frac{1}{p}\mathrm{tr}\boldsymbol{\Sigma}\mathbf{B}_l\right)\right| \\
\le &Cn^{-\frac{q}{2}+\delta}\prod_{k=1}^{r}\|\mathbf{A}_k\|\prod_{l=1}^{q}\|\mathbf{B}_l\|,
\end{aligned}$$

which conclude the result. □

Lemma S2.3 (Theorem 35.12 of Billingsley 2017)**.** Suppose for each n, $Y_{n1}, \cdots, Y_{nr_n}$ is a real martingale difference sequence with respect to σ-field $\{\mathcal{F}_{n,j}\}$ having finite second moments. If

$$\sum_{j=1}^{r_n} \mathbb{E}\left(Y_{nj}^2 | \mathcal{F}_{n,j-1}\right) \to \sigma^2, \quad \text{in probability,}$$

and for each $\varepsilon > 0$,

$$\sum_{j=1}^{r_n} \mathbb{E}\left(Y_{n,j}^2 I(|Y_{n,j}| \geq \varepsilon)\right) \to 0,$$

then

$$\sum_{j=1}^{r_n} Y_{n,j} \to N(o, \sigma^2), \quad \text{in distribution.}$$

Proof of Theorem 4.1. Letting $f(x) = x^k$, the centering term is

$$\int x^k dF_{y_n}(x) = \sum_{j=0}^{k-1} \frac{y_n^j}{j+1} \binom{k}{j} \binom{k-1}{j}.$$

The asymptotic mean and asymptotic variance can be derived with the same approach in Theorem 1.4 of Pan and Zhou (2008), In details, for any $k \geq 1$, we have

$$\begin{aligned}
& -\frac{1}{4\pi^2} \iint z_1^k z_2^k \frac{2\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_1) - \underline{s}(z_2))^2} dz_1 dz_2 \\
=& 2y_0^{2k} \sum_{j_1=0}^{k-1} \sum_{j_2=0}^{k} \binom{k}{j_1} \binom{k}{j_2} (\frac{1-y_0}{y_0})^{j_1+j_2} \sum_{l=1}^{k-j_1} l \binom{2k-1-j_1-l}{k-1} \binom{2k-1-j_2+l}{k-1}, \\
& -\frac{1}{4\pi^2} \iint z_1^k z_2^k \frac{2y_0\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2} dz_1 dz_2 \\
=& 2y_0^{2k+1} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} \binom{k}{j_1} \binom{k}{j_2} (\frac{1-y_0}{y_0})^{j_1+j_2} \binom{2k-j_1}{k-1} \binom{2k-j_2}{k-1}, \\
& -\frac{1}{2\pi i} \int \frac{z^k y_0 \underline{s}^3(z) \left(1+\underline{s}(z)\right)}{\left((1+\underline{s}(z))^2 - y_0\underline{s}^2(z)\right)^2} dz \\
=& \frac{1}{4}\left((1-\sqrt{y_0})^{2k} + (1+\sqrt{y_0})^{2k}\right) - \frac{1}{2}\sum_{j=0}^{k} \binom{k}{j} y_0^j,
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\int \frac{z^k y_0 \underline{s}^3(z)}{\left((1+\underline{s}(z))^2-y_0\underline{s}^2(z)\right)(1+\underline{s}(z))}dz\\
=&y_0^{k+1}\sum_{j=0}^{k}\binom{k}{j}\binom{2k-j}{k-1}(\frac{1-y_0}{y_0})^j-y_0^{k+1}\sum_{j=0}^{k}\binom{k}{j}\binom{2k+1-j}{k-1}(\frac{1-y_0}{y_0})^j,\\
&-\frac{1}{2\pi i}\int \frac{z^k \underline{s}^3(z)}{\left((1+\underline{s}(z))^2-y_0\underline{s}^2(z)\right)}dz\\
=&-y_0^{k}\sum_{j=0}^{k}\binom{k}{j}\binom{2k-j-1}{k-1}(\frac{1-y_0}{y_0})^j+y_0^{k}\sum_{j=0}^{k}\binom{k}{j}\binom{2k-j}{k-1}(\frac{1-y_0}{y_0})^j,\\
&-\frac{1}{2\pi i}\int \frac{z^k \underline{s}^3(z)\left(2+\underline{s}(z)\right)}{(1+\underline{s}(z))-y_0\underline{s}^2(z)}dz\\
=&y_0^{k}\sum_{j=0}^{k}\binom{k}{j}\binom{2k-j}{k-1}(\frac{1-y_0}{y_0})^j-y_0^{k}\sum_{j=0}^{k-1}\binom{k}{j}\binom{2k-j-2}{k-1}(\frac{1-y_0}{y_0})^j.
\end{aligned}
$$

Then, the corresponding asymptotic mean and asymptotic variance can be obtained directly.

Next we consider the case of $f(x)=\log(x)$. By (3.6) of Bai et al. (2009), the centering term is

$$
\int \log(x)dF_{y_n}(x)=\frac{y_n-1}{y_n}\log(1-y_n)-1.
$$

For the asymptotic mean and variance, we note that

$$
\underline{s}'(z)=\frac{\underline{s}^2(z)\left(1+\underline{s}(z)\right)^2}{\left(1+\underline{s}(z)\right)^2-y_0\underline{s}^2(z)}.
$$

Since we consider the case of $y<1$, then $y_0=1/y>1$. when $x>(1+\sqrt{y_0})^2$, we have $0>\underline{s}(x)>-1$, and when $x<(1-\sqrt{y_0})^2$, we have $\underline{s}(x)>1/(y_0-1)$. Then we calculate for $k\geq 2$,

$$
\int \frac{\log(z)\underline{s}'(z)}{(1+\underline{s})^k}dz=\int \frac{\log(z(\underline{s}))}{(1+\underline{s})^k}d\underline{s}
$$

$$
\begin{aligned}
=&\frac{1}{k-1}\int \frac{(\underline{s}+1)^2-y_0\underline{s}^2}{(\underline{s}+1)^k}\cdot\frac{1}{\underline{s}\left((y_0-1)\underline{s}-1\right)}d\underline{s}\\
=&\frac{2\pi i}{k-1}\left[1-(\frac{y_0-1}{y_0})^{k-1}\right],
\end{aligned}
$$

and for $k=1$,

$$
\begin{aligned}
\int \frac{\log(z)\underline{s}'(z)}{1+\underline{s}}dz =&\int \frac{\log(z(\underline{s}))}{1+\underline{s}}d\underline{s}\\
=&\int \frac{((\underline{s}+1)^2-y_0\underline{s}^2)\log(1+\underline{s})}{\underline{s}+1}\cdot\frac{1}{\underline{s}\left((y_0-1)\underline{s}-1\right)}d\underline{s}\\
=&-2\pi i\log(1-\frac{1}{y_0}).
\end{aligned}
$$

As for $\mu_{\log}$ and $\widetilde{\mu}_{\log}$, with similar rouine in Section 5 of Bai and Silverstein (2004),

$$
\begin{aligned}
-\frac{1}{2\pi i}\int \log(yz)\mu_1(z)dz =&\frac{1}{2\pi}\int_{(1-\sqrt{y_0})^2}^{(1+\sqrt{y_0})^2}\frac{1}{x}\arg\left(1-\frac{y_0\underline{s}^2(x)}{\left(1+\underline{s}(x)\right)^2}\right)dx\\
=&\frac{1}{2\pi}\int_{(1-\sqrt{y_0})^2}^{(1+\sqrt{y_0})^2}\frac{1}{x}\arctan\left(\frac{x-1-y_0}{\sqrt{4y_0-(x-1-y_0)^2}}\right)dx\\
=&\frac{1}{2}\log(1-\frac{1}{y_0}).
\end{aligned}
$$

For other terms,

$$
\begin{aligned}
-\frac{1}{2\pi i}\int \log(yz)\mu_2(z)dz =&\frac{y_0}{\pi i}\int \frac{\log(z)\underline{s}(z)\underline{s}'(z)}{\left(1+\underline{s}(z)\right)^3}dz\\
=&\frac{y_0}{\pi i}\int \frac{\log(z(\underline{s}))}{(1+\underline{s})^2}d\underline{s}-\frac{y_0}{\pi i}\int \frac{\log(z(\underline{s}))}{(1+\underline{s})^3}d\underline{s}\\
=&\frac{1}{y_0},\\
-\frac{1}{2\pi i}\int \log(yz)\mu_3(z)dz =&-\frac{1}{2\pi i}\int \frac{\log(z)\underline{s}(z)\underline{s}'(z)}{\left(1+\underline{s}(z)\right)^2}dz\\
=&-\frac{1}{2\pi i}\int \frac{\log(z(\underline{s}))}{1+\underline{s}}d\underline{s}+\frac{1}{2\pi i}\int \frac{\log(z(\underline{s}))}{(1+\underline{s})^2}d\underline{s}
\end{aligned}
$$

$$
\begin{aligned}
&= \log(1-\frac{1}{y_0}) + \frac{1}{y_0},\\
-\frac{1}{2\pi i}\int \log(yz)\widetilde{\mu}(z)dz &= -\frac{1}{2\pi i}\int \frac{\log(z)\underline{s}(z)\underline{s}'(z)}{1+\underline{s}(z)}dz - \frac{1}{2\pi i}\int \frac{\log(z)\underline{s}(z)\underline{s}'(z)}{\left(1+\underline{s}(z)\right)^2}dz\\
&= -\frac{1}{2\pi i}\int \log(z(\underline{s}))d\underline{s} + \frac{1}{2\pi i}\int \frac{\log(z(\underline{s}))}{(1+\underline{s})^2}d\underline{s}\\
&= \frac{1}{y_0-1} + \frac{1}{y_0}.
\end{aligned}
$$

As for $\sigma_{\log}$ and $\widetilde{\sigma}_{\log}$,

$$
\begin{aligned}
&-\frac{1}{4\pi^2}\iint \log(yz_1)\log(yz_2)\sigma_1(z_1,z_2)dz_1dz_2\\
=&-\frac{1}{2\pi^2}\int \log(z_2)\underline{s}'(z_2)\int \frac{\log(z_1)\underline{s}'(z_1)}{\left(\underline{s}(z_1)-\underline{s}(z_2)\right)^2}dz_1dz_2\\
=&-\frac{1}{\pi i}\int \log(z(\underline{s}_2))\left(\frac{1}{\underline{s}_2} - \frac{1}{\underline{s}_2 - \frac{1}{y_0-1}}\right)d\underline{s}_2\\
=&-\frac{1}{\pi i}\int \log(\frac{\underline{s}_2 - \frac{1}{y_0-1}}{\underline{s}_2})\left(\frac{1}{\underline{s}_2} - \frac{1}{\underline{s}_2 - \frac{1}{y_0-1}}\right)d\underline{s}_2\\
&+\frac{1}{\pi i}\int \log(\underline{s}_2+1)\left(\frac{1}{\underline{s}_2} - \frac{1}{\underline{s}_2 - \frac{1}{y_0-1}}\right)d\underline{s}_2\\
=&-2\log(1-\frac{1}{y_0}),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{4\pi^2}\iint \log(yz_1)\log(yz_2)\sigma_2(z_1,z_2)dz_1dz_2\\
=&\frac{y_0}{2\pi^2}\int \frac{\log(yz_1)\underline{s}'(z_1)}{\left(1+\underline{s}(z_1)\right)^2}dz_1 \int \frac{\log(yz_2)\underline{s}'(z_2)}{\left(1+\underline{s}(z_2)\right)^2}dz_2\\
=&-\frac{2}{y_0}.
\end{aligned}
$$

Collecting all the above terms, we complete our calculations. □

Proof of Lemma S1.4. For population $\mathbf{X} = (X_1, \cdots, X_p)^\top$, we write F_i as the distribution function of X_i. Since X_i is absolutely continuous respect to the Lebesgue measure, $F_i(X_i)$ is uniformly distributed on $[0, 1]$ and $Y_i = \Phi^{-1}(F_i(X_i))$ is a standard Gaussian distribution. Rank statistics are invariant under this monotonic transformation, that is $r(X_i) = r(Y_i)$ for $i = 1, \cdots, n$. Therefore, (S1.33) and (S1.34) are obtained in Wu and Wang (2022) and Li et al. (2023). □

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