Supplementary Material for
"Quantile Residual Lifetime Regression for
Multivariate Failure Time Data"

Tonghui Yu[1], Liming Xiang[1] and Jong-Hyeon Jeong [2,3]

[1]School of Physical and Mathematical Sciences

Nanyang Technological University, Singapore

[2]Department of Biostatistics, Public Health

University of Pittsburgh, U.S.A.

[3]Biometric Research Program

Division of Cancer Treatment and Diagnosis

National Institutes of Health/National Cancer Institute, U.S.A.

1 Technical Proofs

Proof of Theorem 1. The strong consistency of $\widehat{\boldsymbol{\alpha}}_N$ can obtained from Lévy's theorem (Resnick, 2019), Lemma 2.2 of White (1980) and Theorem 5.9 of Van der Vaart (2000). In detail, it remains to verify 1) $||S_N(\boldsymbol{\alpha}) - \overline{S}_N(\boldsymbol{\alpha})|| \rightarrow_{a.s.} 0$ uniformly for all $\boldsymbol{\alpha} \in \mathcal{D}$; 2) $\inf_{\boldsymbol{\alpha}:||\boldsymbol{\alpha}-\boldsymbol{\alpha}_0||=\epsilon} ||\overline{S}_N(\boldsymbol{\alpha})||$ is strictly positive for any $\epsilon > 0$.

By Conditions 1-2, the proof of the former result is similar to the proof of consistency under an independent case, readers can refer to the appendix in Li et al. (2016). We now show the latter result. From the model (3) and the fact that $\Pr\left\{T_{ij} \geq t_0 + \exp(\mathbf{X}_{ij}^T \boldsymbol{\alpha}_0)|\mathbf{X}_{ij}\right\} - (1-\tau)\Pr\left\{T_{ij} \geq t_0|\mathbf{X}_{ij}\right\} = 0$, we can see that

$$\begin{aligned}
&\Pr\left\{T_{ij} \geq t_0 + \exp(\mathbf{X}_{ij}^T \boldsymbol{\alpha})|\mathbf{X}_{ij}\right\} - (1-\tau)\Pr\left\{T_{ij} \geq t_0|\mathbf{X}_{ij}\right\} \\
&= \Pr\left\{T_{ij} \geq t_0 + \exp(\mathbf{X}_{ij}^T \boldsymbol{\alpha})|\mathbf{X}_{ij}\right\} - \Pr\left\{T_{ij} \geq t_0 + \exp(\mathbf{X}_{ij}^T \boldsymbol{\alpha}_0)|\mathbf{X}_{ij}\right\} \\
&= \Pr(e_{ij}^{\tau} \geq \mathbf{X}_{ij}^T(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)|\mathbf{X}_{ij}) - \Pr(e_{ij}^{\tau} \geq 0|\mathbf{X}_{ij}) \\
&= F_e(0|\mathbf{X}_{ij}) - F_e(\mathbf{X}_{ij}^T(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0)|\mathbf{X}_{ij}) \\
&= f_e(0|\mathbf{X}_{ij})\mathbf{X}_{ij}^T(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0).
\end{aligned}$$

Thus, by Conditions 1 and 3, $\inf_{\boldsymbol{\alpha}:||\boldsymbol{\alpha}-\boldsymbol{\alpha}_0||=\epsilon} ||\overline{S}_N(\boldsymbol{\alpha})|| > 0$ for any $\epsilon > 0$. We now have completed the proof of the consistency of the proposed estimator under the multivariate setting. □

Proof of Lemma 1. $N^{1/2}S_N(\boldsymbol{\alpha}_0)$ can be expressed as

$$\begin{aligned}
N^{1/2}S_N(\boldsymbol{\alpha}_0) &= N^{-1/2}\sum_i \psi_i(\boldsymbol{\alpha}_0) \\
&+ N^{-1/2}\sum_i\sum_j \mathbf{X}_{ij}\Delta_{ij} I\left\{t_0 \leq Y_{ij} \leq t_0 + \exp(\mathbf{X}_{ij}^T\boldsymbol{\alpha}_0)\right\}\left[\frac{\widehat{G}(t_0)}{\widehat{G}(Y_{ij})} - \frac{G(t_0)}{G(Y_{ij})}\right],
\end{aligned} \tag{A.1}$$

where

$$\psi_i(\boldsymbol{\alpha}) = \sum_{j=1}^{m_i} \mathbf{X}_{ij} I(Y_{ij} \geq t_0)\left[\frac{\Delta_{ij} I\left\{Y_{ij} \leq t_0 + \exp(\mathbf{X}_{ij}^T\boldsymbol{\alpha})\right\}}{G(Y_{ij})/G(t_0)} - \tau\right].$$

The second term in (A.1) involves the estimator of G and can be represented by virtue of martingale processes. We see from (Fleming and Harrington, 2013) that

$$\widehat{G}_n(s) - G(s) = -G(s)\frac{1}{N}\sum_i \int_{-\infty}^{s} \frac{dM_i(u)}{y(u)} + o(1) \tag{A.2}$$

where $M_i(u) = \sum_j I(Y_{ij} \le u, \Delta_{ij} = 0) - \int_0^u I(Y_{ij} \ge t) d\wedge_G(t)$ and at-risk indicator $y(u) = \lim_{n\to\infty} N^{-1} \sum_{i,j} I(Y_{ij} \ge u)$, $\wedge_G(u)$ is the cumulative hazard function of the censoring variables C_{ij}. Through simple algebraic manipulations, (A.1) can be written as

$$N^{1/2} S_N(\boldsymbol{\alpha}_0) = N^{-1/2} \sum_i \boldsymbol{\omega}_i(\boldsymbol{\alpha}_0) + o(1), \tag{A.3}$$

where $\boldsymbol{\omega}_i(\boldsymbol{\alpha}) = \psi_i(\boldsymbol{\alpha}) + \eta_i(\boldsymbol{\alpha})$, and

$$\eta_i(\boldsymbol{\alpha}_0) = \sum_{k,j} \frac{\mathbf{X}_{kj}\delta_{kj} I\left\{t_0 \le Y_{kj} \le t_0 + \exp(\mathbf{X}_{kj}^T\boldsymbol{\alpha}_0)\right\}}{NG(Y_{kj})/G(t_0)} \int_{t_0}^{Y_{kj}} \frac{dM_i(u)}{y(u)}.$$

By Condition 1, the Lyapunov central limit theorem and martingale central limit theorem, the distribution of $N^{1/2}S_N(\boldsymbol{\alpha}_0)$ is asymptotically normal with mean zero and covariance matrix $\Sigma = N^{-1}\sum_{i=1}^n \boldsymbol{\omega}_i(\boldsymbol{\alpha}_0)\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0)^T$. □

Proof of Theorem 2. As seen from the proof of Lemma 1, $NS_N(\boldsymbol{\alpha})$ is asymptotically equivalent to sums of independent random vectors, $\boldsymbol{\omega}_i(\boldsymbol{\alpha})$, we then explore the limiting distribution of $\widehat{\boldsymbol{\alpha}}_N$ based on the asymptotic properties of $\boldsymbol{\omega}_i(\boldsymbol{\alpha}) = \psi_i(\boldsymbol{\alpha}) + \eta_i(\boldsymbol{\alpha})$. Since $\widehat{\boldsymbol{\alpha}}_N$ satisfies $S_N(\widehat{\boldsymbol{\alpha}}_N) = o(N^{-1/2})$, we see that

$$\begin{aligned}
\sum_{i=1}^n \psi_i(\widehat{\boldsymbol{\alpha}}_N) &= \sum_{i=1}^n\sum_{j=1}^{m_i} \mathbf{X}_{ij} I(Y_{ij} \ge t_0)\left[\frac{\Delta_{ij} I\left\{Y_{ij} \le t_0 + \exp(\mathbf{X}_{ij}^T\widehat{\boldsymbol{\alpha}}_N)\right\}}{G(Y_{ij})/G(t_0)} - \tau\right] \\
&= o(N^{1/2}) + \sum_i\sum_j \mathbf{X}_{ij}\Delta_{ij} I\left\{t_0 \le Y_{ij} \le t_0 + \exp(\mathbf{X}_{ij}^T\widehat{\boldsymbol{\alpha}}_N)\right\}\left[\frac{G(t_0)}{G(Y_{ij})} - \frac{\widehat{G}(t_0)}{\widehat{G}(Y_{ij})}\right] \\
&= o(N^{1/2}) - \sum_{i=1}^n \eta_i(\widehat{\boldsymbol{\alpha}}_N) + o(1),
\end{aligned}$$

where the second equality is derived from substituting $S_N(\widehat{\boldsymbol{\alpha}}_N)$ into, the third is from Equation (A.1). Note that $\boldsymbol{\omega}_i(\boldsymbol{\alpha})$'s are independent and $\sum\limits_{i=1} E\boldsymbol{\omega}_i(\boldsymbol{\alpha}) = N\overline{S}_N(\boldsymbol{\alpha})$. Thus, by Condition 4, we have $\sum\limits_{i=1}^n \boldsymbol{\omega}_i(\widehat{\boldsymbol{\alpha}}_N) = o(n^{1/2})$, which is same as (2.1) in He and Shao (1996). Using the similar techniques in Wang and Fygenson (2009), we also denote $u_i(\boldsymbol{\alpha}, d) = \sup\limits_{||\boldsymbol{\gamma}-\boldsymbol{\alpha}|| \le d} ||\boldsymbol{\omega}_i(\boldsymbol{\gamma}) - \boldsymbol{\omega}_i(\boldsymbol{\alpha})||$. According to He and

Shao (1996), it remains to show conditions B3-B4, B5' and B8 in their Theorem 2 are satisfied under our setting.

B3. Denote $\widetilde{e}_{ij}(\boldsymbol{\alpha}) = \log(Y_{ij} - t_0) - \mathbf{X}_{ij}^T\boldsymbol{\alpha}, e_{ij}(\boldsymbol{\alpha}) = \log(T_{ij} - t_0) - \mathbf{X}_{ij}^T\boldsymbol{\alpha}$. By the fact that $\sup_{t \le t_u} |\widehat{G}(t) - G(t)| = o(N^{-1/2+\epsilon})$ for any $\epsilon > 0$, we see that for $||\boldsymbol{\alpha} - \boldsymbol{\alpha}_0|| \le d_0$ and $d \le d_0$,

$$
\begin{aligned}
& u_i(\boldsymbol{\alpha}, d) \\
&= \sup_{||\boldsymbol{\gamma}-\boldsymbol{\alpha}|| \le d} \left|\left| \sum_j \mathbf{X}_{ij}\Delta_{ij}\delta_{ij} I(T_{ij} \ge t_0) \frac{\widehat{G}(t_0)}{\widehat{G}(Y_{ij})} \left[I\left\{e_{ij}(\alpha) \le \mathbf{X}_{ij}^T(\boldsymbol{\gamma} - \boldsymbol{\alpha})\right\} - I\left\{e_{ij}(\alpha) \le 0\right\} \right] \right|\right| \\
&\le c_1 \sum_j ||\mathbf{X}_{ij}||\Delta_{ij} I(T_{ij} \ge t_0) \frac{G(t_0)}{G(Y_{ij})} \sup_{||\boldsymbol{\gamma}-\boldsymbol{\alpha}|| \le d} I\left\{e_{ij}(\boldsymbol{\alpha}) \le \max(0, \mathbf{X}_{ij}^T(\boldsymbol{\gamma} - \boldsymbol{\alpha})\right\} \\
&\le c_1 \sum_j ||\mathbf{X}_{ij}||\Delta_{ij} I(T_{ij} \ge t_0) \frac{G(t_0)}{G(Y_{ij})} I\left\{e_{ij}(\boldsymbol{\alpha}) \le d||\mathbf{X}_{ij}||\right\} \\
&:= c_1 B_i,
\end{aligned}
$$

for some constant c_1. From Condition 1, there exists some constant c_2 such that

$$
\begin{aligned}
E(B_i^2|\mathbf{X}_{ij}, T_{ij} \ge t_0) &\le n_i \left(G(t_0)\right)^2 \sum_j ||\mathbf{X}_{ij}||^2 \Pr\left\{e_{ij}(\alpha) \le d||\mathbf{X}_{ij}||\right\} / G(Y_{ij}) \\
&\le c_2 n_i \sum_j ||\mathbf{X}_{ij}||^2 \Pr\left\{e_{ij}(\alpha) \le d||\mathbf{X}_{ij}||\right\} \\
&\le c_2 n_i \sum_j ||\mathbf{X}_{ij}||^2 \left[1 + f_e(0|\mathbf{X}_{ij}) d||\mathbf{X}_{ij}||\right] + O(d^2)
\end{aligned}
$$

Then, results in Condition B3 of He and Shao (1996) follows by fixing $a_i = c_3 d^{-1}\sqrt{n_i \sum_j ||\mathbf{X}_{ij}||^2 \left[1 + f_e(0|\mathbf{X}_{ij}) d||\mathbf{X}_{ij}||\right]}$ for some constant c_3.

B4. Under Conditions 3-4, $A_n = \sum_i a_i = O(n)$, Condition B4 of He and Shao (1996) can be obtained. Condition B5' follows by taking $d_n = n^{-1/4}\log n$.

B8. Taking the Taylor expansion of $\overline{S}_N(\boldsymbol{\alpha})$ at $\boldsymbol{\alpha}_0$ yields

$$
N^{1/2}(\overline{S}_N(\widehat{\boldsymbol{\alpha}}_N) - \overline{S}_N(\boldsymbol{\alpha}_0)) = N^{1/2}(\widehat{\boldsymbol{\alpha}}_N - \boldsymbol{\alpha}_0)\Lambda G(t_0) + o(1),
$$

where Λ is defined in Condition 3. Consequently, Condition B8 of He and Shao (1996) holds.

Therefore, all conditions in Theorem 2.2 of He and Shao (1996) hold, and we can apply their theorem, yielding the Bahadur representation in the form of

$$N^{1/2}(\widehat{\boldsymbol{\alpha}}_N - \boldsymbol{\alpha}_0) = -N^{-1/2}\Lambda^{-1}(G(t_0))^{-1}\sum_{i=1}\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0) + o_p(1) \tag{A.4}$$

Thus, by the Lyapunov central limit theorem, $N^{1/2}(\widehat{\boldsymbol{\alpha}}_N - \boldsymbol{\alpha})$ is asymptotically normal with covariance matrix as $\mathrm{Cov}\left\{N^{1/2}(\widehat{\boldsymbol{\alpha}}_N - \boldsymbol{\alpha})\right\} = \widetilde{\Lambda}^{-1}\Sigma\widetilde{\Lambda}^{-1}$, with $\widetilde{\Lambda} = G(t_0)\Lambda$.

□

Proof of Theorem 3. It can be straightforwardly verified that the difference between the perturbed G^* and the truth G can be asymptotically represented by sums of independent random processes in analogy to Eq(A.2). Then, given $E(\gamma_i) = 1$, following the arguments in proofs of Lemma 1 and Theorem 2, we can similarly get

$$N^{1/2}(\widehat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}_0) = -N^{-1/2}\Lambda^{-1}(G(t_0))^{-1}\sum_{i=1}\gamma_i\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0) + o_p(1). \tag{A.5}$$

When coupled with Eq (A.4), this implies that

$$N^{1/2}(\widehat{\boldsymbol{\alpha}}^* - \widehat{\boldsymbol{\alpha}}_N) = -N^{-1/2}\Lambda^{-1}(G(t_0))^{-1}\sum_{i=1}(\gamma_i - 1)\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0) + o_p(1). \tag{A.6}$$

Since $\mathrm{Var}(\gamma_i) = 1$, we have

$$\begin{aligned}
&\mathrm{Cov}\left[N^{-\frac{1}{2}}\sum_{i=1}(\gamma_i - 1)\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0)\middle|\{Y_{ij}, \Delta_{ij}, \mathbf{X}_{ij}\}_{i=1,\cdots,n;j=1,\cdots,m_i}\right] \\
&= N^{-1}\sum_{i=1}^{n}\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0)\boldsymbol{\omega}_i(\boldsymbol{\alpha}_0)^T = \Sigma.
\end{aligned}$$

It follows that given observed data, the conditional distribution of $\sqrt{N}(\widehat{\boldsymbol{\alpha}}^* - \widehat{\boldsymbol{\alpha}}_N)$ is asymptotically equivalent to the unconditional distribution of $\sqrt{N}(\widehat{\boldsymbol{\alpha}}_N - \boldsymbol{\alpha}_0)$, thereby justifying the perturbation-based covariance estimation procedure.

□

2 Additional Numerical Results

Scenario 5. We consider a Frank copula model for the error terms, keeping the scheme for data generation the same as in Scenario 2.

Scenario 6. Unlike the exchangeable dependence considered in Scenarios 1-3 and the new Scenario 5, we adopt an AR(1) dependence structure in this scenario. Particularly, the error terms satisfy $\exp(\epsilon_{ij}) \sim Exp(1)$ marginally and are jointly modeled by a Guassian copula with AR(1) correlation of $0.7^{|j-k|}$ between ϵ_{ij} and ϵ_{ik} for $j \neq k$. All the other setups remain the same as in Scenario 2.

Scenario 7. A cluster-level covariate $x_{ij} = x_i$ is considered, where x_i is independently generated from a Bernoulli(0.5). The error terms $(\epsilon_{i1}, \cdots, \epsilon_{im})$ follow a multivariate normal distribution with mean zero and covariance matrix $0.64\Sigma_\epsilon$, where the (j,k)-th entry of Σ_ϵ is $0.7^{|j-k|}$. The failure time outcomes T_{ij} are generated from the AFT model $\log T_{ij} = \beta_0 + \beta_1 x_i + \epsilon_{ij}$ with coefficients $\beta_0 = 0.5$ and $\beta_1 = 1$. In this setting, parameters in the QRL model (3) are obtained by

$$\alpha_0(\tau, t_0) = \begin{cases} \beta_0 + \Phi_{0.8}^{-1}(\tau), & t_0 = 0, \\ \log\Big\{ -t_0 + \exp\left[\beta_0 + \Phi_{0.8}^{-1}\{(1-\tau)\Phi_{0.8}(\log t_0 - \beta_0) + \tau\}\right] \Big\}, & t_0 \neq 0, \end{cases}$$

$$\alpha_1(\tau, t_0) = \begin{cases} \beta_1, & t_0 = 0, \\ \log\left\{ \frac{-t_0 + \exp\left[\beta_0 + \beta_1 + \Phi_{0.8}^{-1}\{(1-\tau)\Phi_{0.8}(\log t_0 - \beta_0 - \beta_1) + \tau\}\right]}{-t_0 + \exp\left[\beta_0 + \Phi_{0.8}^{-1}\{(1-\tau)\Phi_{0.8}(\log t_0 - \beta_0) + \tau\}\right]} \right\}, & t_0 \neq 0, \end{cases}$$

where $\Phi_{0.8}(\cdot)$ is the cumulative distribution function of a zero-mean normal variable with a standard deviation of 0.8.

Scenario 8. The multivariate failure times is generated from the model: $\log T_{ij} = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3ij} + \epsilon_{ij}$, where $x_{1i} \sim$ Bernoulli(0.5), $x_{2i} \sim$ Uniform$[0,1]$, $x_{3ij} \sim N(0,1)$, $\beta_0 = \beta_1 = 0.6$, $\beta_2 = 0.8$, $\beta_3 = 0.4$. The assumption for the error terms is the same as in Scenario 2. Under this setup, $\alpha_0(\tau, t_0) = \log[-\lambda^{-1}\log(1-\tau)] + \beta_0$, $\alpha_1(\tau, t_0) = \beta_1$, $\alpha_2(\tau, t_0) = \beta_2$ and $\alpha_3(\tau, t_0) = \beta_3$.

Estimation results under Scenarios 5-7 are summarized in Table S.3, which consistently highlight the outperformance of the proposed marginal method in addressing diverse dependence structures for multivariate failure times even based on an independent working model. These promising findings further exhibit a degree of robustness of the method across different types of copula. Simulation results for Scenario 8, summarized in Table S.4, also demonstrate that the overall performance of the proposed estimators is promising compared to the IFR estimator. Particularly, its superiority for the coefficients associated with cluster-level covariates x_{1i} and x_{2i} is greater than for the individual-level covariate x_{3ij}.

Table S.1: Estimation results based on 500 replicates for quantile level $\tau = 0.5$ under Scenario 1 with Kendall's tau=0.

			$\alpha_0(0.5, t_0)$			$\alpha_1(0.5, t_0)$			runtime
(n, m)			$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	(s)
(200,3)	bias		-0.013	-0.011	-0.015	0.026	0.024	0.028	
	MCSD		0.13	0.147	0.17	0.238	0.263	0.29	
	ASE	IFR	0.129	0.148	0.175	0.231	0.261	0.300	
		FR	0.132	0.15	0.176	0.236	0.267	0.305	5.523
		CFS	0.128	0.156	0.191	0.23	0.273	0.327	0.285
		RBS	0.126	0.143	0.165	0.224	0.251	0.284	3.665
	CP	IFR	0.946	0.946	0.971	0.952	0.946	0.958	
		FR	0.96	0.95	0.958	0.95	0.962	0.958	
		CFS	0.946	0.954	0.978	0.952	0.954	0.972	
		RBS	0.944	0.94	0.936	0.936	0.942	0.944	
(500,3)	bias		-0.001	-0.003	-0.01	0.006	0.01	0.015	
	MCSD		0.081	0.092	0.112	0.148	0.164	0.191	
	ASE	IFR	0.082	0.094	0.109	0.148	0.166	0.188	
		FR	0.082	0.092	0.107	0.147	0.163	0.186	10.104
		CFS	0.081	0.098	0.119	0.146	0.173	0.206	0.683
		RBS	0.08	0.089	0.103	0.142	0.157	0.178	6.714
	CP	IFR	0.952	0.958	0.959	0.948	0.938	0.943	
		FR	0.96	0.954	0.934	0.95	0.956	0.946	
		CFS	0.952	0.964	0.968	0.946	0.946	0.97	
		RBS	0.95	0.946	0.922	0.944	0.946	0.938	
(200,10)	bias		-0.004	-0.006	-0.004	0.009	0.014	0.011	
	MCSD		0.071	0.079	0.092	0.127	0.136	0.155	
	ASE	IFR	0.071	0.082	0.092	0.128	0.145	0.161	
		FR	0.071	0.081	0.091	0.128	0.143	0.159	11.762
		CFS	0.07	0.085	0.101	0.127	0.149	0.176	1.207
		RBS	0.069	0.079	0.089	0.124	0.139	0.153	6.511
	CP	IFR	0.952	0.952	0.962	0.964	0.968	0.964	
		FR	0.94	0.958	0.95	0.95	0.96	0.952	
		CFS	0.948	0.962	0.972	0.964	0.968	0.976	
		RBS	0.938	0.944	0.942	0.946	0.954	0.946	

Table S.2: Estimation results based on 500 replicates for quantile level $\tau = 0.5$ under Scenario 1 with Kendall's tau=0.8.

			$\alpha_0(0.5, t_0)$			$\alpha_1(0.5, t_0)$			runtime
(n, m)			$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	(s)
(200,3)	bias		-0.011	-0.007	-0.023	0.022	0.014	0.017	
	MCSD		0.145	0.154	0.177	0.229	0.253	0.28	
	ASE	IFR	0.127	0.147	0.171	0.23	0.259	0.295	
		FR	0.151	0.171	0.16	0.237	0.27	0.255	5.905
		CFS	0.15	0.178	0.176	0.232	0.279	0.275	0.288
		RBS	0.143	0.162	0.15	0.223	0.251	0.236	3.603
	CP	IFR	0.914	0.95	0.935	0.94	0.956	0.957	
		FR	0.958	0.976	0.921	0.954	0.964	0.931	
		CFS	0.956	0.978	0.947	0.942	0.974	0.945	
		RBS	0.954	0.96	0.919	0.934	0.946	0.911	
(500,3)	bias		0.003	0.006	0.007	0	-0.006	-0.005	
	MCSD		0.091	0.113	0.124	0.143	0.174	0.198	
	ASE	IFR	0.082	0.094	0.107	0.147	0.166	0.187	
		FR	0.096	0.108	0.113	0.149	0.17	0.181	10.822
		CFS	0.095	0.113	0.123	0.147	0.176	0.196	0.636
		RBS	0.093	0.104	0.108	0.143	0.163	0.173	5.685
	CP	IFR	0.926	0.892	0.908	0.954	0.926	0.945	
		FR	0.96	0.934	0.923	0.956	0.926	0.936	
		CFS	0.958	0.948	0.947	0.952	0.936	0.949	
		RBS	0.952	0.928	0.915	0.944	0.924	0.919	
(200,10)	bias		0.001	0.001	-0.005	0.003	0.004	-0.005	
	MCSD		0.112	0.124	0.131	0.122	0.144	0.178	
	ASE	IFR	0.069	0.081	0.092	0.125	0.142	0.16	
		FR	0.114	0.126	0.121	0.13	0.155	0.162	13.032
		CFS	0.113	0.131	0.134	0.127	0.159	0.176	0.765
		RBS	0.108	0.12	0.116	0.127	0.15	0.156	6.002
	CP	IFR	0.776	0.8	0.822	0.948	0.938	0.921	
		FR	0.954	0.944	0.932	0.956	0.948	0.935	
		CFS	0.954	0.958	0.955	0.952	0.956	0.955	
		RBS	0.944	0.93	0.926	0.952	0.944	0.917	

Table S.3: Estimation results based on 500 replicates for quantile level $\tau = 0.5$ under Scenarios 5-7 ($n = 200, m = 10$).

			$\alpha_0(0.5, t_0)$			$\alpha_1(0.5, t_0)$			runtime
Scenario			$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	(s)
5	bias		-0.002	-0.006	-0.006	0.001	0.011	0.012	
	MCSD		0.16	0.159	0.163	0.286	0.281	0.284	
	ASE	IFR	0.071	0.081	0.093	0.129	0.144	0.163	32.295
		FR	0.16	0.157	0.162	0.282	0.277	0.284	27.902
		CFS	0.161	0.165	0.18	0.282	0.292	0.314	1.234
		RBS	0.155	0.151	0.156	0.27	0.265	0.273	11.775
	CP	IFR	0.6	0.684	0.748	0.598	0.682	0.75	
		FR	0.956	0.954	0.95	0.954	0.952	0.946	
		CFS	0.956	0.96	0.964	0.954	0.966	0.97	
		RBS	0.954	0.94	0.936	0.952	0.942	0.94	
6	bias		0.002	0	-0.007	-0.006	-0.007	0.001	
	MCSD		0.113	0.106	0.121	0.203	0.194	0.213	
	ASE	IFR	0.069	0.081	0.099	0.122	0.141	0.167	31.735
		FR	0.116	0.113	0.125	0.204	0.199	0.215	36.391
		CFS	0.116	0.12	0.139	0.203	0.21	0.238	1.888
		RBS	0.113	0.111	0.122	0.197	0.194	0.209	19.061
	CP	IFR	0.754	0.872	0.898	0.742	0.86	0.866	
		FR	0.97	0.966	0.968	0.96	0.95	0.96	
		CFS	0.968	0.97	0.98	0.956	0.966	0.978	
		RBS	0.958	0.96	0.964	0.948	0.946	0.942	
7	bias		-0.006	-0.012	-0.009	0.008	0.013	0.011	
	MCSD		0.059	0.086	0.099	0.082	0.11	0.132	
	SE	IFR	0.033	0.057	0.083	0.049	0.073	0.101	29.449
		FR	0.057	0.08	0.1	0.082	0.108	0.129	28.179
		CFS	0.056	0.085	0.112	0.082	0.114	0.143	1.384
		RBS	0.056	0.078	0.097	0.08	0.106	0.125	14.203
	CP	IFR	0.744	0.798	0.896	0.764	0.816	0.874	
		FR	0.936	0.942	0.948	0.952	0.958	0.946	
		CFS	0.936	0.956	0.966	0.952	0.966	0.976	
		RBS	0.932	0.924	0.942	0.95	0.95	0.94	

Table S.4: Estimation results based on 500 replicates for quantile level $\tau = 0.5$ under Scenario 8.

			$\alpha_0(0.5, t_0)$			$\alpha_1(0.5, t_0)$			$\alpha_2(0.5, t_0)$			$\alpha_3(0.5, t_0)$		
(n, m)			$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$	$t_0 = 0$	$t_0 = 1$	$t_0 = 2$
(200,3)	bias		-0.015	-0.006	-0.012	0.005	-0.003	0.008	0.016	0.014	0.007	0.004	0.005	0.001
	MCSD		0.195	0.206	0.243	0.17	0.179	0.204	0.296	0.307	0.351	0.066	0.084	0.099
	ASE	IFR	0.151	0.18	0.214	0.138	0.159	0.184	0.242	0.277	0.323	0.07	0.081	0.094
		FR	0.203	0.211	0.233	0.184	0.189	0.204	0.322	0.329	0.357	0.07	0.082	0.095
		CFS	0.196	0.212	0.245	0.179	0.192	0.217	0.309	0.333	0.379	0.067	0.082	0.1
		RBS	0.191	0.199	0.219	0.174	0.177	0.191	0.3	0.308	0.336	0.069	0.081	0.095
	CP	IFR	0.882	0.918	0.924	0.882	0.926	0.928	0.884	0.928	0.924	0.956	0.948	0.954
		FR	0.962	0.954	0.95	0.968	0.96	0.954	0.962	0.966	0.956	0.958	0.952	0.958
		CFS	0.952	0.956	0.96	0.964	0.96	0.97	0.954	0.966	0.97	0.944	0.948	0.966
		RBS	0.942	0.942	0.938	0.952	0.95	0.934	0.952	0.942	0.94	0.95	0.946	0.958
(500,3)	bias		-0.007	-0.006	-0.004	0.005	0	-0.006	0.002	0.009	0.016	0.003	0.005	0.002
	MCSD		0.127	0.13	0.144	0.119	0.123	0.126	0.195	0.193	0.21	0.044	0.052	0.062
	ASE	IFR	0.092	0.111	0.133	0.085	0.099	0.115	0.148	0.173	0.2	0.044	0.051	0.059
		FR	0.126	0.132	0.146	0.115	0.119	0.129	0.199	0.207	0.224	0.044	0.052	0.059
		CFS	0.124	0.135	0.157	0.113	0.122	0.139	0.195	0.212	0.241	0.043	0.053	0.065
		RBS	0.12	0.126	0.141	0.11	0.115	0.124	0.189	0.198	0.214	0.043	0.051	0.06
	CP	IFR	0.85	0.92	0.928	0.842	0.894	0.924	0.86	0.934	0.938	0.936	0.94	0.932
		FR	0.944	0.954	0.952	0.94	0.934	0.958	0.962	0.968	0.97	0.936	0.946	0.932
		CFS	0.942	0.958	0.966	0.932	0.95	0.972	0.962	0.972	0.978	0.934	0.954	0.946
		RBS	0.936	0.948	0.94	0.924	0.932	0.938	0.958	0.96	0.958	0.934	0.94	0.932
(200,10)	bias		-0.006	-0.01	-0.011	-0.001	-0.002	-0.003	-0.005	0.004	0.01	0.001	-0.001	0.001
	MCSD		0.175	0.152	0.15	0.161	0.143	0.142	0.28	0.252	0.244	0.037	0.043	0.051
	ASE	IFR	0.08	0.097	0.115	0.074	0.086	0.099	0.128	0.15	0.174	0.037	0.044	0.051
		FR	0.175	0.161	0.16	0.159	0.147	0.144	0.274	0.256	0.253	0.038	0.046	0.053
		CFS	0.172	0.166	0.174	0.157	0.153	0.158	0.269	0.265	0.276	0.037	0.047	0.058
		RBS	0.165	0.155	0.155	0.151	0.142	0.139	0.261	0.244	0.244	0.038	0.046	0.054
	CP	IFR	0.636	0.81	0.862	0.642	0.752	0.836	0.636	0.754	0.834	0.94	0.954	0.952
		FR	0.944	0.958	0.978	0.96	0.962	0.95	0.944	0.962	0.97	0.944	0.958	0.958
		CFS	0.938	0.97	0.984	0.956	0.972	0.974	0.94	0.968	0.986	0.94	0.958	0.976
		RBS	0.932	0.95	0.968	0.946	0.962	0.942	0.932	0.956	0.96	0.944	0.958	0.96

Figure S.1: Estimation results with different quantile levels and residual time points for the Framingham heart data.

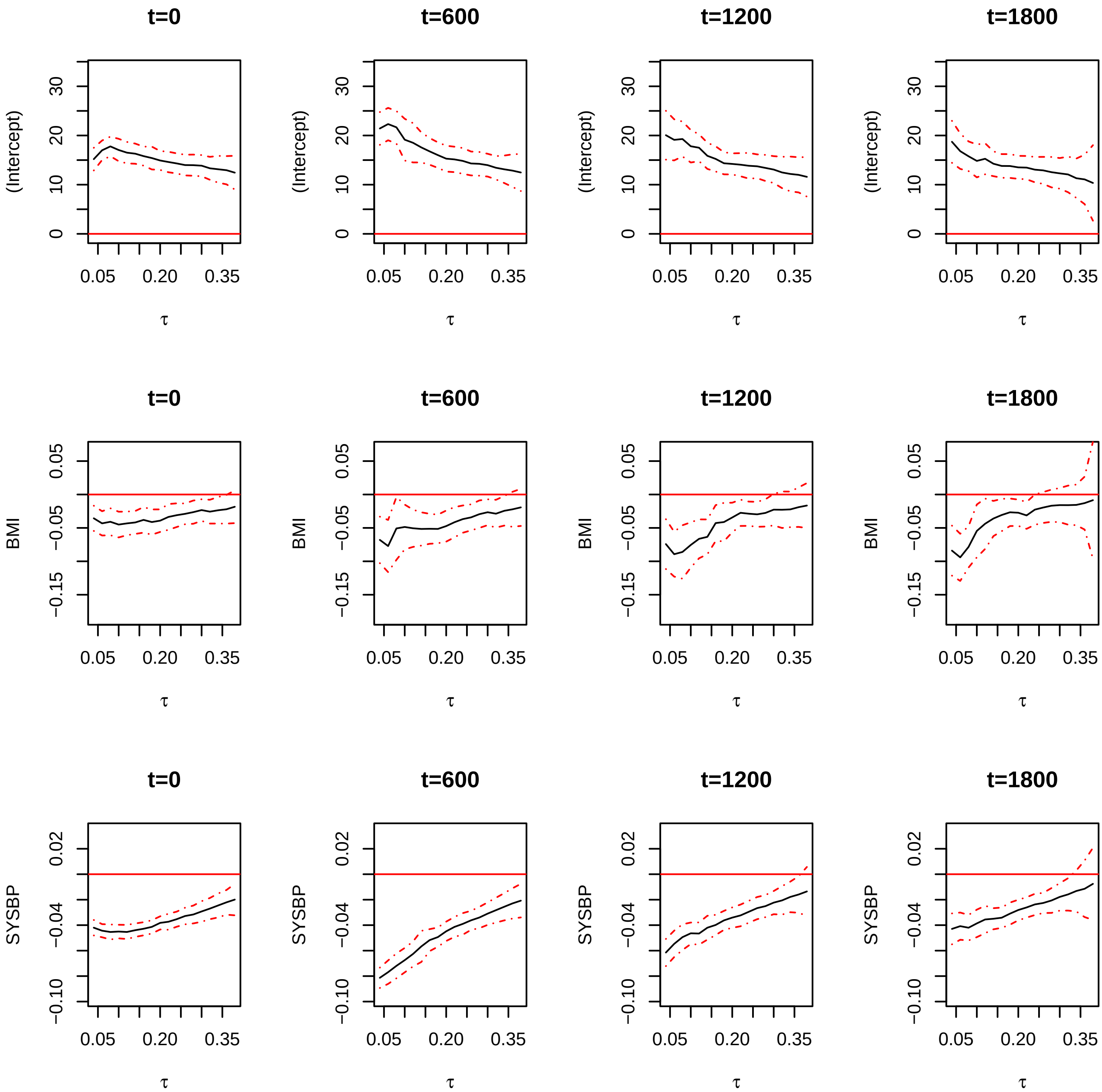
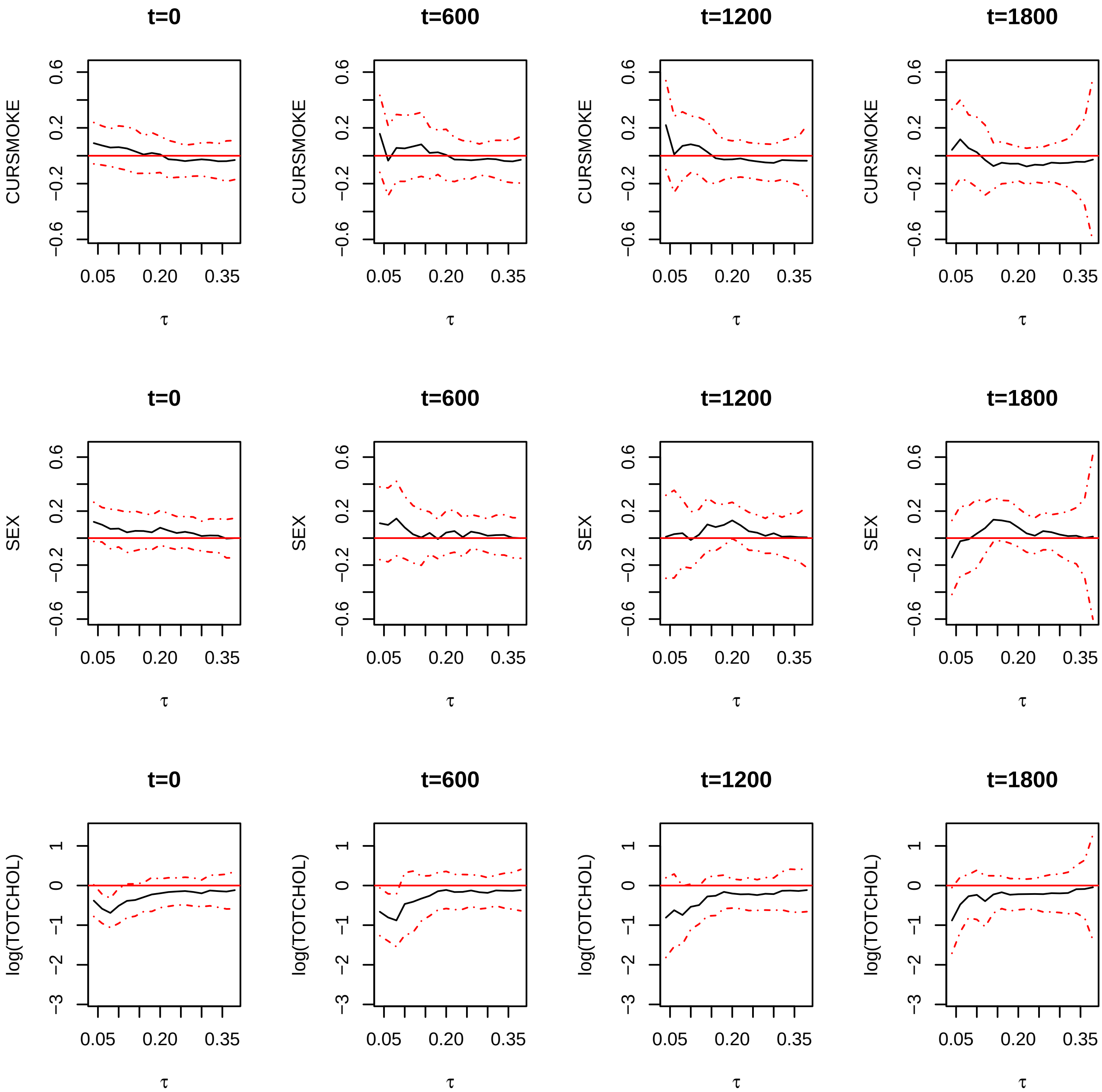


Figure S.2: Estimation results with different quantile levels and residual time points for the Framingham heart data.



References

Fleming, T. R. and Harrington, D. P. (2013). *Counting processes and survival analysis*. John Wiley & Sons.

He, X. and Shao, Q.-M. (1996). A general bahadur representation of m-estimators and its application to linear regression with nonstochastic designs. *The Annals of Statistics*, 24(6):2608–2630.

Li, R., Huang, X., and Cortes, J. (2016). Quantile residual life regression with longitudinal biomarker measurements for dynamic prediction. *Journal of the Royal Statistical Society Series C: Applied Statistics*, 65(5):755–773.

Resnick, S. I. (2019). *A probability path*. Springer Science & Business Media.

Van der Vaart, A. W. (2000). *Asymptotic statistics*. Cambridge university press.

Wang, H. J. and Fygenson, M. (2009). Inference for censored quantile regression models in longitudinal studies. *The Annals of Statistics*, 37(2):756–781.

White, H. (1980). Nonlinear regression on cross-section data. *Econometrica*, 48(3):721–746.