Functional Tensor Regression

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Supplementary Material

This supplementary material contains the proofs of the main theorems and technical lemmas, and additional numerical results for observations on a non-uniform grid.

S1 Proofs

Proof of Theorem 1

Notice that

$$\begin{aligned}\|\boldsymbol{\Theta}^{k+1}-\boldsymbol{\Theta}\|_{\mathrm{F}} &= \|\mathcal{H}_{\boldsymbol{r}}(\check{\boldsymbol{\Theta}}^{k+1})-\boldsymbol{\Theta}\|_{\mathrm{F}} \quad \text{by (2.16)}\\ &\le \|\mathcal{H}_{\boldsymbol{r}}(\check{\boldsymbol{\Theta}}^{k+1})-\check{\boldsymbol{\Theta}}^{k+1}\|_{\mathrm{F}}+\|\check{\boldsymbol{\Theta}}^{k+1}-\boldsymbol{\Theta}\|_{\mathrm{F}}\\ &\le \{(D+1)^{1/2}+1\}\|\check{\boldsymbol{\Theta}}^{k+1}-\boldsymbol{\Theta}\|_{\mathrm{F}} \quad \text{by (2.15)}\\ &\le \{(D+1)^{1/2}+1\}R_l^{-1/2}p_0^{1/2}n^{-1/2}\|\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\boldsymbol{\Theta})\| \quad \text{by (3.3)} \quad . \end{aligned} \tag{S1}$$

To separate out the noise term, introduce the minimizer with $\boldsymbol{y}-\boldsymbol{\varepsilon}$ in place of $\boldsymbol{y}$, i.e.,

$$\tilde{\boldsymbol{\Theta}}^{k+1} = \mathscr{R}_{\boldsymbol{\Theta}^k}(\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\mathscr{Z}\mathscr{R}_{\boldsymbol{\Theta}^k} + n\rho\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}^k})^{-1}\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*(\boldsymbol{y}-\boldsymbol{\varepsilon}).$$

Since $\boldsymbol{y}-\boldsymbol{\varepsilon} = \big(\int_{\mathbb{T}}\langle\boldsymbol{\mathcal{X}}_i(t),\boldsymbol{\mathcal{B}}(t)\rangle\,\mathrm{d}t\big)_{i=1,\cdots,n} = \mathscr{Z}\boldsymbol{\Theta}+\boldsymbol{\delta}$, it follows that

$$\begin{aligned}
&(2n)^{-1}\|\mathscr{Z}(\boldsymbol{\Theta}-\tilde{\boldsymbol{\Theta}}^{k+1})+\boldsymbol{\delta}\|^2 + \rho\langle\mathscr{A}\tilde{\boldsymbol{\Theta}}^{k+1},\tilde{\boldsymbol{\Theta}}^{k+1}\rangle\\
&= (2n)^{-1}\|\boldsymbol{y}-\boldsymbol{\varepsilon}-\mathscr{Z}\tilde{\boldsymbol{\Theta}}^{k+1}\|^2 + \rho\langle\mathscr{A}\tilde{\boldsymbol{\Theta}}^{k+1},\tilde{\boldsymbol{\Theta}}^{k+1}\rangle\\
&\le (2n)^{-1}\|\boldsymbol{y}-\boldsymbol{\varepsilon}-\mathscr{Z}\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|^2 + \rho\langle\mathscr{A}\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta},\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\rangle \quad \text{recalling (2.11)}\\
&= (2n)^{-1}\|\mathscr{Z}(\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta})+\boldsymbol{\delta}\|^2 + \rho\langle\mathscr{A}\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta},\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\rangle.
\end{aligned}$$

Using the inequalities that

$$\|\mathscr{Z}(\boldsymbol{\Theta}-\tilde{\boldsymbol{\Theta}}^{k+1})+\boldsymbol{\delta}\|^2 \ge 2^{-1}\|\mathscr{Z}(\boldsymbol{\Theta}-\tilde{\boldsymbol{\Theta}}^{k+1})\|^2 - \|\boldsymbol{\delta}\|^2,$$

$$\|\mathscr{Z}(\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta})+\boldsymbol{\delta}\|^2 \le 2\|\mathscr{Z}(\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta})\|^2 + 2\|\boldsymbol{\delta}\|^2,$$

(3.1) and (3.5), we obtain that

$$\|\mathscr{Z}(\boldsymbol{\Theta}-\tilde{\boldsymbol{\Theta}}^{k+1})\|^2 \le 4nR_u p_0^{-1}\|\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|_{\mathrm{F}}^2 + 6\|\boldsymbol{\delta}\|^2 + 8n\rho C_m. \quad \text{(S2)}$$

Besides, by writing

$$\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\tilde{\boldsymbol{\Theta}}^{k+1}) = \mathscr{Z}\mathscr{R}_{\boldsymbol{\Theta}^k}\big(\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\mathscr{Z}\mathscr{R}_{\boldsymbol{\Theta}^k} + n\rho\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}^k}\big)^{-1}\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\boldsymbol{\varepsilon},$$

it can be seen that

$$\begin{aligned}\|\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\tilde{\boldsymbol{\Theta}}^{k+1})\|^2 &\le \boldsymbol{\varepsilon}^\top \mathscr{Z}\mathscr{R}_{\boldsymbol{\Theta}^k}\big(\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\mathscr{Z}\mathscr{R}_{\boldsymbol{\Theta}^k}+n\rho\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}^k}\big)^{-1}\mathscr{R}^*_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\boldsymbol{\varepsilon}\\ &\le p_0(c_m n\rho)^{-1}\|\mathscr{P}_{\boldsymbol{\Theta}^k}\mathscr{Z}^*\boldsymbol{\varepsilon}\|_{\mathrm{F}}^2 \quad \text{by (2.8)}\\ &\le c_m^{-1}p_0 n^{-1}\rho^{-1}\big\|(\mathscr{Z}^*\boldsymbol{\varepsilon})_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}^2\end{aligned} \tag{S3}$$

where the last step invokes the fact that $\mathrm{T}_{\boldsymbol{\Theta}^k}\mathbb{M}_{\boldsymbol{r}} \subset \bigcup_{\boldsymbol{s}\le 2\boldsymbol{r}}\mathbb{M}_{\boldsymbol{s}}$ (Luo and Zhang, 2023, Lemma 2). The combination of (S2) and (S3) leads to

$$\begin{aligned}&\|\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\boldsymbol{\Theta})\|^2 \le 2\|\mathscr{Z}(\boldsymbol{\Theta}-\tilde{\boldsymbol{\Theta}}^{k+1})\|^2+2\|\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\tilde{\boldsymbol{\Theta}}^{k+1})\|^2\\ &\le 8nR_u p_0^{-1}\|\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|_{\mathrm{F}}^2+12\|\boldsymbol{\delta}\|^2+16n\rho C_m+2c_m^{-1}p_0n^{-1}\rho^{-1}\big\|(\mathscr{Z}^*\boldsymbol{\varepsilon})_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}^2,\end{aligned}$$

so

$$\begin{aligned}\|\mathscr{Z}(\check{\boldsymbol{\Theta}}^{k+1}-\boldsymbol{\Theta})\| \le{}& 8^{1/2}n^{1/2}R_u^{1/2}p_0^{-1/2}\|\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|_{\mathrm{F}}\\ &+\big\{12\|\boldsymbol{\delta}\|^2+16n\rho C_m+2c_m^{-1}p_0n^{-1}\rho^{-1}\big\|(\mathscr{Z}^*\boldsymbol{\varepsilon})_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}^2\big\}^{1/2}.\end{aligned}$$

Plugging this into (S1), we have

$$\|\boldsymbol{\Theta}^{k+1}-\boldsymbol{\Theta}\|_{\mathrm{F}} \le \{(D+1)^{1/2}+1\}(8^{1/2}R_u^{1/2}R_l^{-1/2}\|\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|_{\mathrm{F}}+R_l^{-1/2}\eta).$$

Due to Luo and Zhang (2023, Lemma 9),

$$\|\boldsymbol{\Theta}-\mathscr{P}_{\boldsymbol{\Theta}^k}\boldsymbol{\Theta}\|_{\mathrm{F}} \le (D+1)\lambda_{\min}^{-1}\|\boldsymbol{\Theta}^k-\boldsymbol{\Theta}\|_{\mathrm{F}}^2,$$

and thus the proof is complete. □

Proof of Corollary 1

The argument is similar to the proof of Luo and Zhang (2023, Corollary 1), so we omit it here. □

Proof of Theorem 2

By writing the Riemann sum as the integral of a piece-wise constant function and recalling (2.5), it can be easily seen that the approximation errors satisfy that $|\delta_i| = \mathcal{O}_{\mathrm{pr}}(p_0^{-\kappa} C_m^{1/2})$, and thus

$$\|\boldsymbol{\delta}\|^2/n = \mathcal{O}_{\mathrm{pr}}(p_0^{-2\kappa} C_m).$$

Following the proof of Luo and Zhang (2023, Lemma 12),

$$\begin{aligned}\big\|(\mathscr{Z}^*\boldsymbol{\varepsilon})_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}} &= \max_{\boldsymbol{U}_d \in \mathbb{O}_{p_d, 2r_d},\, d=0,1,\cdots,D} \max_{\mathcal{T} \in \mathbb{R}^{p_0 \times p_1 \times \cdots \times p_D} : \|\mathcal{T}\|_{\mathrm{F}} \le 1} \langle \mathcal{T} \times_{d=0}^{D} (U_d U_d^\top), \mathscr{Z}^*\boldsymbol{\varepsilon} \rangle \\ &= \max_{\boldsymbol{U}_d \in \mathbb{O}_{p_d, 2r_d},\, d=0,1,\cdots,D} \max_{\mathcal{T} \in \mathbb{R}^{p_0 \times p_1 \times \cdots \times p_D} : \|\mathcal{T}\|_{\mathrm{F}} \le 1} \boldsymbol{\varepsilon}^\top \mathscr{Z}\{\mathcal{T} \times_{d=0}^{D} (U_d U_d^\top)\}.\end{aligned}$$

Since Lemma 1 provides the bounds for the operator norm of $\mathscr{Z}$, the proof of Han et al. (2022, Theorem 4.2) implies that

$$\big\|(\mathscr{Z}^*\boldsymbol{\varepsilon})_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}^2 = \mathcal{O}_{\mathrm{pr}}\Big\{ np_0^{-1}\Big(\sum_{d=0}^{D} p_d r_d + \prod_{d=0}^{D} r_d \Big) \Big\}.$$

Plugging these into (3.7) leads to the desired result, and (3.8) follows directly. The proof is then complete. □

Proof of Theorem 3

After rescaling the regression model using (3.1) and (3.3), it follows from Luo and Zhang (2023, Lemma 12) and the proof of Luo and Zhang (2023, Theorem 2) that when $n = \prod_{d=0}^{D} p_d$ we can choose $\boldsymbol{\mathcal{E}}_0 = \boldsymbol{\mathcal{E}}_{\max(2\boldsymbol{r})}$ for some $\boldsymbol{\mathcal{E}} \in \mathbb{R}^{p_0 \times p_1 \times \cdots \times p_D}$ with i.i.d. $\mathcal{N}(0,1)$ entries. In what follows, c_1 and c_2 denote universal positive constants. By Sudakov's minoration inequality (Vershynin, 2018, Theorem 7.4.1),

$$
\begin{aligned}
E(\big\|\boldsymbol{\mathcal{E}}_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}) &= E\big(\max_{\boldsymbol{U}_d \in \mathbb{O}_{p_d,2r_d},\, d=0,1,\cdots,D} \|\boldsymbol{\mathcal{E}} \times_{d=0}^{D} \boldsymbol{U}_d^{\top}\|_{\mathrm{F}}\big) \\
&\geq c_1 \epsilon \{\log N(\prod \mathbb{O}_{p_d,2r_d}, \epsilon)\}^{1/2}, \quad \epsilon > 0
\end{aligned}
$$

where $N(\prod \mathbb{O}_{p_d,2r_d}, \epsilon)$ is the ϵ-covering number of $\prod_{d=0}^{D} \mathbb{O}_{p_d,2r_d}$. It can be seen that

$$
N(\prod \mathbb{O}_{p_d,2r_d}, \epsilon) \geq N(\prod \mathbb{O}_{p_d,r_d}, \epsilon) \geq (c_2/\epsilon)^{\sum_{d=0}^{D}(p_d - r_d)r_d},
$$

and thus $E(\big\|\boldsymbol{\mathcal{E}}_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}) \geq 2c_0 \sum_{d=0}^{D}(p_d - r_d)r_d$. Besides, with some fixed $\boldsymbol{U}_d \in \mathbb{O}_{p_d,2r_d}$,

$$
E(\big\|\boldsymbol{\mathcal{E}}_{\max(2\boldsymbol{r})}\big\|_{\mathrm{F}}^2) \geq E(\|\boldsymbol{\mathcal{E}} \times_{d=0}^{D} \boldsymbol{U}_d^{\top}\|_{\mathrm{F}}^2) = \prod_{d=0}^{D}(2r_d).
$$

Combining the two bounds completes the proof. □

Proof of Lemma 1

By the law of large numbers,

$$n^{-1}\|\mathscr{Z}\boldsymbol{\Upsilon}\|^2 = n^{-1}(\langle \boldsymbol{\mathcal{Z}}_1, \boldsymbol{\Upsilon}\rangle^2 + \cdots + \langle \boldsymbol{\mathcal{Z}}_n, \boldsymbol{\Upsilon}\rangle^2) \to E(\langle \boldsymbol{\mathcal{Z}}_1, \boldsymbol{\Upsilon}\rangle^2)$$

in probability as $n \to \infty$. Hence we only need to bound $E(\langle \boldsymbol{\mathcal{Z}}_1, \boldsymbol{\Upsilon}\rangle^2)$. For convenience, let $\boldsymbol{\Gamma}_\ell$ and $\boldsymbol{\mathcal{E}}$ be the $p_0 \times p_1 \times \cdots \times p_D$ tensors such that $[\boldsymbol{\Gamma}_\ell]_{k,j_1,\ldots,j_D} = [\boldsymbol{\Xi}_{k+(\ell-1)p_0}]_{j_1,\ldots,j_D}$ and $[\boldsymbol{\mathcal{E}}]_{j,j_1,\ldots,j_D} = [\boldsymbol{\mathcal{E}}_{1j}]_{j_1,\ldots,j_D}$, and let $\boldsymbol{\Delta}^{1/2}$ be the diagonal matrix constructed from $(\Delta t_j)^{1/2}$, $j = 1, \cdots, p_0$. We have

$$\begin{aligned}
E(\langle \boldsymbol{\mathcal{Z}}_1, \boldsymbol{\Upsilon}\rangle^2) &= E\biggl\{\biggl\langle \sum_{\ell=1}^{\infty} \boldsymbol{\Gamma}_\ell \times_0 (\boldsymbol{\Delta}^{1/2}\boldsymbol{\Phi}_\ell) + \boldsymbol{\mathcal{E}} \times_0 \boldsymbol{\Delta}, \boldsymbol{\Upsilon}\biggr\rangle^2\biggr\} \quad \text{by (2.1) and (2.2)} \\
&= \sum_{\ell=1}^{\infty} E\{\langle \boldsymbol{\Gamma}_\ell \times_0 (\boldsymbol{\Delta}^{1/2}\boldsymbol{\Phi}_\ell), \boldsymbol{\Upsilon}\rangle^2\} + E(\langle \boldsymbol{\mathcal{E}} \times_0 \boldsymbol{\Delta}, \boldsymbol{\Upsilon}\rangle^2) \quad \text{by uncorrelatedness} \\
&= \sum_{\ell=1}^{\infty} E\{\langle \boldsymbol{\Gamma}_\ell, \boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}_\ell^\top \boldsymbol{\Delta}^{1/2})\rangle^2\} + E(\langle \boldsymbol{\mathcal{E}}, \boldsymbol{\Upsilon} \times_0 \boldsymbol{\Delta}\rangle^2) \quad \text{by algebra} \\
&\le \sum_{\ell=1}^{\infty} \frac{A}{\{1+(\ell-1)p_0\}^a} \bigl\|\boldsymbol{\Upsilon} \times_0 \boldsymbol{\Phi}_\ell^\top \times_0 \boldsymbol{\Delta}^{1/2}\bigr\|_{\mathrm{F}}^2 + \sigma_X^2 \|\boldsymbol{\Upsilon} \times_0 \boldsymbol{\Delta}\|_{\mathrm{F}}^2 \quad \text{by Assumptions 1 and 3} \\
&\le \sum_{\ell=1}^{\infty} \frac{A}{\{1+(\ell-1)p_0\}^a} \frac{C_0}{p_0} \bigl\|\boldsymbol{\Upsilon} \times_0 \boldsymbol{\Phi}_\ell^\top\bigr\|_{\mathrm{F}}^2 + \sigma_X^2 \frac{C_0^2}{p_0^2} \|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2 \quad \text{by (2.5)} \\
&\le \sum_{\ell=1}^{\infty} \frac{A}{\{1+(\ell-1)p_0\}^a} \frac{C_0}{p_0} C_\varphi \|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2 + \sigma_X^2 \frac{C_0^2}{p_0^2} \|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2 \quad \text{by Assumption 2} \\
&\le \biggl\{\Bigl(1 + \frac{a}{a-1}\frac{1}{p_0^a}\Bigr) A C_0 C_\varphi + \frac{C_0^2 \sigma_X^2}{p_0}\biggr\} \frac{\|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2}{p_0},
\end{aligned}$$

where the last step uses the fact that $\sum_{\ell=2}^{\infty}\{1/p_0 + (\ell-1)\}^{-a} < 1 + \int_1^{\infty} u^{-a}\,\mathrm{d}u = a/(a-1)$. This completes the proof of (3.1). In a similar manner, writing $\boldsymbol{\Gamma}_0$ for the $r_0 \times p_1 \times \cdots \times p_D$ tensor such that $[\boldsymbol{\Gamma}_0]_{k,j_1,\ldots,j_D} =$

$[\boldsymbol{\Xi}_k]_{j_1,\ldots,j_D}$, it can be seen that

$$
\begin{aligned}
E(\langle \boldsymbol{\mathcal{Z}}_1, \boldsymbol{\Upsilon}\rangle^2) &\geq E(\langle \boldsymbol{\Gamma}_0, \boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}_0^\top \boldsymbol{\Delta}^{1/2})\rangle^2) \\
&\geq \frac{1}{Ar_0^a} \big\| \boldsymbol{\Upsilon} \times_0 \boldsymbol{\Phi}_0^\top \times_0 \boldsymbol{\Delta}^{1/2} \big\|_{\mathrm{F}}^2 \quad \text{by Assumption 1} \\
&\geq \frac{1}{Ar_0^a} \frac{1}{C_0 p_0} \big\| \boldsymbol{\Upsilon} \times_0 \boldsymbol{\Phi}_0^\top \big\|_{\mathrm{F}}^2 \quad \text{by (2.5)} \\
&\geq \frac{c^2}{AC_0 r_0^a} \frac{\|\boldsymbol{\Upsilon}\|_{\mathrm{F}}^2}{p_0} \quad \text{by (3.2)} \quad .
\end{aligned}
$$

This completes the proof of (3.3). □

Proof of Lemma 2

Recall that $\boldsymbol{U}_0\boldsymbol{U}_0^\top + \boldsymbol{U}_{0\perp}\boldsymbol{U}_{0\perp}^\top = \boldsymbol{I}_{r_0}$. Using the triangle inequality,

$$
\begin{aligned}
\|\boldsymbol{\Upsilon} \times_0 \boldsymbol{\Phi}^\top\|_{\mathrm{F}} &= \|\boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}^\top \boldsymbol{U}_0 \boldsymbol{U}_0^\top) + \boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}^\top \boldsymbol{U}_{0\perp} \boldsymbol{U}_{0\perp}^\top)\|_{\mathrm{F}} \\
&\geq \|\boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}^\top \boldsymbol{U}_0 \boldsymbol{U}_0^\top)\|_{\mathrm{F}} - \|\boldsymbol{\Upsilon} \times_0 (\boldsymbol{\Phi}^\top \boldsymbol{U}_{0\perp} \boldsymbol{U}_{0\perp}^\top)\|_{\mathrm{F}} \\
&\geq \sigma_{r_0}(\boldsymbol{\Phi}^\top \boldsymbol{U}_0) \|\boldsymbol{\Upsilon} \times_0 \boldsymbol{U}_0^\top\|_{\mathrm{F}} - \sigma_1(\boldsymbol{\Phi}^\top \boldsymbol{U}_{0\perp}) \|\boldsymbol{\Upsilon} \times_0 \boldsymbol{U}_{0\perp}^\top\|_{\mathrm{F}} \\
&= \sigma_{r_0}(\boldsymbol{\Phi}^\top \boldsymbol{U}_0) \|\boldsymbol{\Upsilon} \times_0 (\boldsymbol{U}_0 \boldsymbol{U}_0^\top)\|_{\mathrm{F}} - \sigma_1(\boldsymbol{\Phi}^\top \boldsymbol{U}_{0\perp}) \|\boldsymbol{\Upsilon} \times_0 (\boldsymbol{U}_{0\perp} \boldsymbol{U}_{0\perp}^\top)\|_{\mathrm{F}} \\
&= \sigma_{r_0}(\boldsymbol{\Phi}^\top \boldsymbol{U}_0) \alpha \|\boldsymbol{\Upsilon}\|_{\mathrm{F}} - \sigma_1(\boldsymbol{\Phi}^\top \boldsymbol{U}_{0\perp}) (1-\alpha^2)^{1/2} \|\boldsymbol{\Upsilon}\|_{\mathrm{F}}.
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 1

Recalling (2.12), we have

$$\begin{aligned}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}},(\boldsymbol{D}_d)_{d=0}^D\big) &= \mathscr{A}\left\{\boldsymbol{\mathcal{C}}\times_{d=0}^D \boldsymbol{U}_d + \sum_{d=0}^D \mathscr{T}_d(\boldsymbol{U}_{d\perp}\boldsymbol{D}_d\boldsymbol{W}_d^\top)\right\}\\ &= \boldsymbol{\mathcal{C}}\times_0(\boldsymbol{A}\boldsymbol{U}_0)\times_{d=1}^D \boldsymbol{U}_d + \sum_{d=0}^D \mathscr{T}_d(\boldsymbol{U}_{d\perp}\boldsymbol{D}_d\boldsymbol{W}_d^\top)\times_0\boldsymbol{A}.\end{aligned}$$

The core part of $\mathscr{R}^*_{\boldsymbol{\Theta}}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}},(\boldsymbol{D}_d)_{d=0}^D\big)$ is

$$\begin{aligned}&\left\{\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}},(\boldsymbol{D}_d)_{d=0}^D\big)\right\}\times_{d=0}^D\boldsymbol{U}_d^\top\\ &= \boldsymbol{\mathcal{C}}\times_0(\boldsymbol{U}_0^\top\boldsymbol{A}\boldsymbol{U}_0)\times_{d=1}^D(\boldsymbol{U}_d^\top\boldsymbol{U}_d) + \mathscr{T}_0(\boldsymbol{U}_{0\perp}\boldsymbol{D}_0\boldsymbol{W}_0^\top)\times_0(\boldsymbol{U}_0^\top\boldsymbol{A})\times_{d=1}^D\boldsymbol{U}_d^\top\\ &\qquad + \sum_{d=1}^D\mathscr{T}_d(\boldsymbol{U}_d^\top\boldsymbol{U}_{d\perp}\boldsymbol{D}_d\boldsymbol{W}_d^\top)\times_0(\boldsymbol{U}_0^\top\boldsymbol{A})\times_{e\neq0,d}\boldsymbol{U}_e^\top\\ &= \boldsymbol{\mathcal{C}}\times_0(\boldsymbol{U}_0^\top\boldsymbol{A}\boldsymbol{U}_0) + \mathscr{T}_0(\boldsymbol{D}_0\boldsymbol{V}_0^\top)\times_0(\boldsymbol{U}_0^\top\boldsymbol{A}\boldsymbol{U}_{0\perp})\times_{d=1}^D(\boldsymbol{U}_d^\top\boldsymbol{U}_d)\\ &= \boldsymbol{\mathcal{C}}\times_0(\boldsymbol{U}_0^\top\boldsymbol{A}\boldsymbol{U}_0) + \mathscr{T}_0(\boldsymbol{D}_0\boldsymbol{V}_0^\top)\times_0(\boldsymbol{U}_0^\top\boldsymbol{A}\boldsymbol{U}_{0\perp}).\end{aligned}$$

The 0-mode part of $\mathscr{R}^*_{\boldsymbol{\Theta}}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}},(\boldsymbol{D}_d)_{d=0}^D\big)$ is

$$\begin{aligned}&\boldsymbol{U}_{0\perp}^\top\mathscr{M}_0\big\{\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}},(\boldsymbol{D}_d)_{d=0}^D\big)\big\}\boldsymbol{W}_0\\ &= \boldsymbol{U}_{0\perp}^\top\Big[\boldsymbol{A}\boldsymbol{U}_0\mathscr{M}_0(\boldsymbol{\mathcal{C}})(\otimes_{d=D}^1\boldsymbol{U}_d^\top) + \boldsymbol{A}\boldsymbol{U}_{0\perp}\boldsymbol{D}_0\boldsymbol{W}_0^\top + \sum_{d=1}^D\boldsymbol{A}\mathscr{M}_0\{\mathscr{T}_d(\boldsymbol{U}_{d\perp}\boldsymbol{D}_d\boldsymbol{W}_d^\top)\}\Big]\boldsymbol{W}_0\\ &= \boldsymbol{U}_{0\perp}^\top\boldsymbol{A}\boldsymbol{U}_0\mathscr{M}_0(\boldsymbol{\mathcal{C}})\boldsymbol{V}_0 + \boldsymbol{U}_{0\perp}^\top\boldsymbol{A}\boldsymbol{U}_{0\perp}\boldsymbol{D}_0 + \boldsymbol{U}_{0\perp}^\top\sum_{d=1}^D\boldsymbol{A}\mathscr{M}_0\big\{\mathscr{T}_d(\boldsymbol{U}_d^\top\boldsymbol{U}_{d\perp}\boldsymbol{D}_d\boldsymbol{W}_d^\top)\times_{e\neq0,d}\boldsymbol{U}_e^\top\big\}\boldsymbol{V}_0\\ &= \boldsymbol{U}_{0\perp}^\top\boldsymbol{A}\boldsymbol{U}_0\mathscr{M}_0(\boldsymbol{\mathcal{C}})\boldsymbol{V}_0 + \boldsymbol{U}_{0\perp}^\top\boldsymbol{A}\boldsymbol{U}_{0\perp}\boldsymbol{D}_0.\end{aligned}$$

For $d \neq 0$, the d-mode part of $\mathscr{R}^*_{\boldsymbol{\Theta}}\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}}, (\boldsymbol{D}_d)_{d=0}^D\big)$ is

$$
\begin{aligned}
&\boldsymbol{U}_{d\perp}^\top \mathscr{M}_d\big\{\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}}, (\boldsymbol{D}_d)_{d=0}^D\big)\big\}\boldsymbol{W}_d \\
&= \mathscr{M}_d\big[\,\big\{\mathscr{A}\mathscr{R}_{\boldsymbol{\Theta}}\big(\boldsymbol{\mathcal{C}}, (\boldsymbol{D}_d)_{d=0}^D\big)\big\} \times_d \boldsymbol{U}_{d\perp}^\top \times_{e\neq d} \boldsymbol{U}_e^\top\big]\boldsymbol{V}_d \\
&= \mathscr{M}_d\Big\{\boldsymbol{\mathcal{C}} \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}\boldsymbol{U}_0) \times_d (\boldsymbol{U}_{d\perp}^\top \boldsymbol{U}_d) \times_{e\neq 0,d} (\boldsymbol{U}_e^\top \boldsymbol{U}_e) \\
&\qquad + \sum_{c=0}^D \mathscr{T}_c(\boldsymbol{U}_{c\perp}\boldsymbol{D}_c\boldsymbol{W}_c^\top) \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}) \times_d \boldsymbol{U}_{d\perp}^\top \times_{e\neq 0,d} \boldsymbol{U}_e^\top\Big\}\boldsymbol{V}_d \\
&= \sum_{c=0}^D \mathscr{M}_d\Big\{\big(\mathscr{T}_c(\boldsymbol{U}_{c\perp}\boldsymbol{D}_c\boldsymbol{V}_c^\top) \times_{e\neq c} \boldsymbol{U}_e\big) \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}) \times_d \boldsymbol{U}_{d\perp}^\top \times_{e\neq 0,d} \boldsymbol{U}_e^\top\Big\}\boldsymbol{V}_d \\
&= \mathscr{M}_d\big\{\mathscr{T}_d(\boldsymbol{D}_d\boldsymbol{V}_d^\top) \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}\boldsymbol{U}_0) \times_d (\boldsymbol{U}_{d\perp}^\top \boldsymbol{U}_{d\perp}) \times_{e\neq 0,d} (\boldsymbol{U}_e^\top \boldsymbol{U}_e)\big\}\boldsymbol{V}_d \\
&\quad + \sum_{c\neq d} \mathscr{M}_d\Big\{\big(\mathscr{T}_c(\boldsymbol{U}_{c\perp}\boldsymbol{D}_c\boldsymbol{V}_c^\top) \times_{e\neq c,d} \boldsymbol{U}_e\big) \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}) \times_d (\boldsymbol{U}_{d\perp}^\top \boldsymbol{U}_d) \times_{e\neq 0,d} \boldsymbol{U}_e^\top\Big\}\boldsymbol{V}_d \\
&= \mathscr{M}_d\big\{\mathscr{T}_d(\boldsymbol{D}_d\boldsymbol{V}_d^\top) \times_0 (\boldsymbol{U}_0^\top \boldsymbol{A}\boldsymbol{U}_0)\big\}\boldsymbol{V}_d.
\end{aligned}
$$

The proof is then complete. □

S2 Additional Numerical Results

Here we present an example of non-uniform grid points. The setup follows the first simulation in Section 4, except we now choose an unbalanced set of t_j's. Specifically, $5p_0/12$ measurements are equally spaced within $(0, 1/3)$, another $5p_0/12$ measurements are equally spaced within $(2/3, 1)$, and $p_0/6$ measurements are equally spaced within $(1/3, 2/3)$. Despite this imbalance, the estimation performs similarly to the uniform case, as shown below,

analogous to the results in Figure 1.

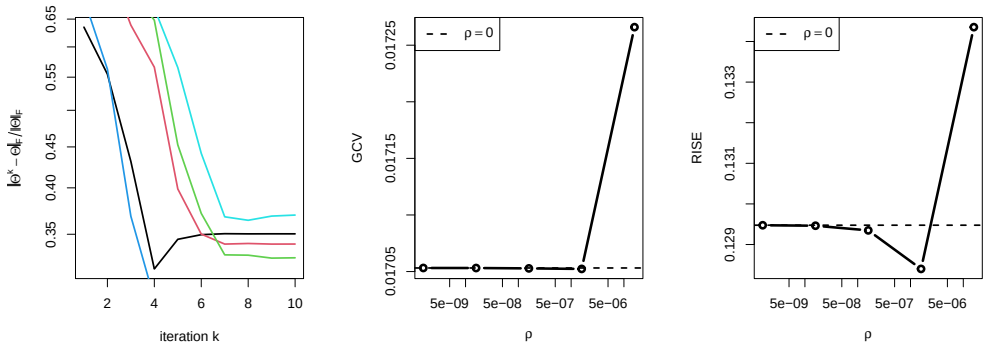

Figure S1: Left: Convergence performance of the functional Riemannian Gauss–Newton algorithm. Middle and Right: GCV and RISE versus the tuning parameter ρ. Displayed are averages based on 100 Monte Carlo replications of $(\boldsymbol{\mathcal{X}}_i, y_i)_{i=1,\ldots,500}$ with measurements on a non-uniform grid.

References

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