Bootstrapping portmanteau tests for functional white noise under unknown dependence

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Supplementary Materials

Supplemental materials contain the technical proofs of Theorems **??**-**??** in this paper and the additional simulation studies of the BRWB method.

A. Technical Proofs

Let $\{u_i(\tau), \tau \in [0,1]\}_{i=1}^{\infty}$ be a sequence of orthonormal basis of the Hilbert space $L^2[0,1]$. Then $\{u_j(\tau_1)u_l(\tau_2), \tau_1, \tau_2 \in [0,1]\}_{j,l=1}^{\infty}$ forms a sequence of basis in $L^2\big([0,1]^2\big)$. Define $X_{t,j} = \langle X_t, u_j\rangle$, and define the space $\mathbb{H}_1$ as

$$\mathbb{H}_1 \equiv \left\{ f : [0,1]^2 \mapsto \mathbb{R}^K \,\middle|\, \iint f(\tau_1,\tau_2)^{\mathrm{T}} f(\tau_1,\tau_2) d\tau_1 d\tau_2 < \infty \right\}.$$

This space is a separable Hilbert space equipped with the inner product

$$\langle f, g\rangle_{\mathbb{H}_1} = \iint f(\tau_1,\tau_2)^{\mathrm{T}} g(\tau_1,\tau_2) d\tau_1 d\tau_2, \quad \forall f, g \in \mathbb{H}_1.$$

Let $\|\cdot\|_{\mathbb{H}_1}$ denote the norm induced by this inner product.

Without loss of generality, we assume that $\mathrm{E}(X_t)=0$ and define the following notations:

$$\begin{aligned}
\boldsymbol{X}_{t,K}(\tau_1,\tau_2) &= \big(X_t(\tau_1)X_{t-1}(\tau_2)-\mathrm{E}[X_t(\tau_1)X_{t-1}(\tau_2)],\ldots,X_t(\tau_1)X_{t-K}(\tau_2)-\mathrm{E}[X_t(\tau_1)X_{t-K}(\tau_2)]\big)^{\mathrm{T}},\\
\hat{\boldsymbol{\gamma}}_K(\tau_1,\tau_2) &= \big(\hat{\gamma}_1(\tau_1,\tau_2)-\gamma_1(\tau_1,\tau_2),\ldots,\hat{\gamma}_K(\tau_1,\tau_2)-\gamma_K(\tau_1,\tau_2)\big)^{\mathrm{T}},\\
\boldsymbol{X}_{t,K,j,l} &= \big(X_{t,j}X_{t-1,l}-\mathrm{E}[X_{t,j}X_{t-1,l}],\ldots,X_{t,j}X_{t-K,l}-\mathrm{E}[X_{t,j}X_{t-K,l}]\big)^{\mathrm{T}},\\
\hat{\boldsymbol{\gamma}}_{K,j,l} &= \Big(\frac{1}{T}\sum_{t=2}^{T}\big(X_{t,j}X_{t-1,l}-\mathrm{E}[X_{t,j}X_{t-1,l}]\big),\ldots,\frac{1}{T}\sum_{t=K+1}^{T}\big(X_{t,j}X_{t-K,l}-\mathrm{E}[X_{t,j}X_{t-K,l}]\big)\Big)^{\mathrm{T}},
\end{aligned}$$

where $\gamma_h(\tau_1,\tau_2)$ and $\hat{\gamma}_h(\tau_1,\tau_2)$ are defined in (**??**) and (**??**) respectively. Obviously,

$$\begin{aligned}
&\sqrt{T}\hat{\boldsymbol{\gamma}}_K(\tau_1,\tau_2)=\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\boldsymbol{X}_{t,K}(\tau_1,\tau_2)+o_p(1),\\
&\sqrt{T}\hat{\boldsymbol{\gamma}}_{K,j,l}=\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\boldsymbol{X}_{t,K,j,l}+o_p(1).
\end{aligned}$$

Lemma A.1. *Under Assumptions* **??** *and* **??***, as* $T\to\infty$*, we have*

$$\sqrt{T}\hat{\boldsymbol{\gamma}}_K(\tau_1,\tau_2)\xrightarrow{d}\boldsymbol{\Gamma}_K(\tau_1,\tau_2),\quad \tau_1,\tau_2\in[0,1],$$

where $\boldsymbol{\Gamma}_K(\tau_1,\tau_2)=\big(\Gamma_1(\tau_1,\tau_2),\ldots,\Gamma_K(\tau_1,\tau_2)\big)^{\mathrm{T}}$ *is a* $K\times 1$ *vector-valued mean-zero Gaussian process in* $\mathbb{H}_1$*, whose covariance structure is given in (5).*

Proof of Lemma A.1. This lemma can be derived from the following two conditions:

(A.i). Finite-dimensional convergence of the sequence $\hat{\boldsymbol{\gamma}}_K(\tau_1,\tau_2)$**.** According to Theorem A.1 in Aue et al. (2009), to show that the process $\sqrt{T}\hat{\gamma}_{K,j,l}$ converges to a multivariate normal distribution in $\mathbb{R}^K$, it suffices to show that the sequence $\{\boldsymbol{X}_{t,K,j,l}\}$ is L^2-m-approximable in $\mathbb{R}^K$. This condition is satisfied if the sequence $\{\boldsymbol{X}_{t,K}\}$ is L^2-m-approximable in the Hilbert space $\mathbb{H}_1$.

(A.ii). The tightness of the sequence $\hat{\gamma}_K(\tau_1,\tau_2)$ in $\mathbb{H}_1$. Following the Lemma 6.1 in Zhang (2016), it is sufficient to demonstrate the tightness of the sequence $\sqrt{T}\hat{\boldsymbol{\gamma}}_{h,j,l}$ for each $1 \le h \le K$ and for each $1 \le j, l < \infty$.

We begin by proving that the sequence $\{\boldsymbol{X}_{t,K}\}$ is L^2-m-approximable in $\mathbb{H}_1$. This is equivalent to demonstrating the L^2-m-approximability of the following sequence:

$$\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2) := \left(X_t(\tau_1)X_{t-1}(\tau_2), \ldots, X_t(\tau_1)X_{t-K}(\tau_2)\right)^{\mathrm{T}}$$

in $\mathbb{H}_1$, since the difference of $\boldsymbol{X}_{t,K}$ and $\widetilde{\boldsymbol{X}}_{t,K}$ is only a constant vector.

It follows from the assumption that the sequence $\{\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2)\}$ is strictly stationary. Using the Cauchy-Schwarz inequality and the assumption of L^4-m-approximability, we have:

$$\begin{aligned}
\mathrm{E}\iint \Big[\widetilde{\boldsymbol{X}}^{\mathrm{T}}_{t,K}(\tau_1,\tau_2)\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2)\Big] d\tau_1 d\tau_2 &= \sum_{h=1}^{K} \mathrm{E}\iint \Big[X_t(\tau_1)X_{t-h}(\tau_2)\Big]^2 d\tau_1 d\tau_2 \\
&\le \sum_{h=1}^{K} \mathrm{E}\Big[\int X_t^2(\tau_1)d\tau_1\Big]\Big[\int X_{t-h}^2(\tau_2)d\tau_2\Big] \\
&\le \sum_{h=1}^{K} \big(\mathrm{E}\,\|X_t\|^4\big)^{1/2}\big(\mathrm{E}\,\|X_{t-h}\|^4\big)^{1/2} \\
&= K\,\mathrm{E}\,\|X_t\|^4 < \infty.
\end{aligned}$$

Therefore, $\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2)$ is almost surely an element of $\mathbb{H}_1$.

Furthermore, by Assumption **??**, we have:

$$\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2) = f(\varepsilon_t, \varepsilon_{t-1}, \ldots),$$

for some measurable function $f : S^{\infty} \mapsto \mathbb{H}_1$. Let $\widetilde{\mathbf{X}}_{t,K}^{(m)}$ be defined as in (**??**), such that when $m > K$,

$$\widetilde{\boldsymbol{X}}_{t,K}^{(m)}(\tau_1,\tau_2) = \big(X_t^{(m)}(\tau_1)X_{t-1}^{(m-1)}(\tau_2),\ldots,X_t^{(m)}(\tau_1)X_{t-K}^{(m-K)}(\tau_2)\big)^{\mathrm{T}}.$$

Applying the triangle inequality in $L^2\big([0,1]^2\big)$, we have that for $m > K$,

$$\begin{aligned}
&\big(\mathrm{E}\,\|\widetilde{\boldsymbol{X}}_{t,K} - \widetilde{\boldsymbol{X}}_{t,K}^{(m)}\|_{\mathbb{H}_1}^2\big)^{1/2}\\
=&\Big\{\mathrm{E}\iint\Big[\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2) - \widetilde{\boldsymbol{X}}_{t,K}^{(m)}(\tau_1,\tau_2)\Big]^{\mathrm{T}}\Big[\widetilde{\boldsymbol{X}}_{t,K}(\tau_1,\tau_2) - \widetilde{\boldsymbol{X}}_{t,K}^{(m)}(\tau_1,\tau_2)\Big]d\tau_1 d\tau_2\Big\}^{1/2}\\
=&\Big\{\sum_{h=1}^{K}\mathrm{E}\,\big\|X_t(\tau_1)X_{t-h}(\tau_2) - X_t^{(m)}(\tau_1)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\Big\}^{1/2}\\
=&\Big\{\sum_{h=1}^{K}\mathrm{E}\,\big\|X_t(\tau_1)\big(X_{t-h}(\tau_2) - X_{t-h}^{(m-h)}(\tau_2)\big) + \big(X_t(\tau_1) - X_t^{(m)}(\tau_1)\big)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\Big\}^{1/2}\\
\leq&\sum_{h=1}^{K}\Big\{2\,\mathrm{E}\Big[\big\|X_t(\tau_1)X_{t-h}(\tau_2) - X_t(\tau_1)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\\
&+\big\|X_t(\tau_1)X_{t-h}^{(m-h)}(\tau_2) - X_t^{(m)}(\tau_1)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\Big]\Big\}^{1/2}\\
\leq&\sqrt{2}\sum_{h=1}^{K}\Big\{\Big[\mathrm{E}\,\big\|X_t(\tau_1)X_{t-h}(\tau_2) - X_t(\tau_1)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\Big]^{1/2}\\
&+\Big[\mathrm{E}\,\big\|X_t(\tau_1)X_{t-h}^{(m-h)}(\tau_2) - X_t^{(m)}(\tau_1)X_{t-h}^{(m-h)}(\tau_2)\big\|_{[0,1]^2}^2\Big]^{1/2}\Big\}\\
\equiv&\sqrt{2}\sum_{h=1}^{K}(J_1 + J_2),
\end{aligned} \tag{1}$$

where $\|\cdot\|_{[0,1]^2}$ denotes the norm in $L^2\big([0,1]^2\big)$.

By the Fubini theorem and the Cauchy-Schwarz inequality, we obtain:

$$J_2 = \big[\mathrm{E}\,\|X_{t-h}^{(m-h)}\|^2\|X_t - X_t^{(m)}\|^2\big]^{1/2} \leq \big(\mathrm{E}\,\|X_{t-h}\|^4\big)^{1/4}\big(\mathrm{E}\,\|X_t - X_t^{(m)}\|^4\big)^{1/4}. \tag{2}$$

Similarly, we have:

$$J_1 \leq \big(\mathrm{E} \, \|X_t\|^4 \big)^{1/4} \big(\mathrm{E} \, \|X_{t-h} - X_{t-h}^{(m-h)}\|^4 \big)^{1/4}. \tag{3}$$

Combining (1)-(3), we have:

$$\begin{aligned} \big(\mathrm{E} \, \|\widetilde{\boldsymbol{X}}_{t,K} - \widetilde{\boldsymbol{X}}_{t,K}^{(m)}\|_{\mathbb{H}_1}^2 \big)^{1/2} \leq & \sqrt{2} \big(\mathrm{E} \, \|X_0\|^4 \big)^{1/4} \sum_{h=1}^{K} \Big\{ \big(\mathrm{E} \, \|X_0 - X_0^{(m-h)}\|^4 \big)^{1/4} \\ & + \big(\mathrm{E} \, \|X_0 - X_0^{(m)}\|^4 \big)^{1/4} \Big\}. \end{aligned}$$

By the approximability condition, we have $\mathrm{E} \, \|X_t\|^4 < \infty$. When K is fixed, it follows that:

$$\begin{aligned} & \sum_{m=K+1}^{\infty} \big(\mathrm{E} \, \|\widetilde{\boldsymbol{X}}_{t,K} - \widetilde{\boldsymbol{X}}_{t,K}^{(m)}\|_{\mathbb{H}_1}^2 \big)^{1/2} \\ \leq & \sqrt{2} \sum_{h=1}^{K} \big(\mathrm{E} \, \|X_0\|^4 \big)^{1/4} \sum_{m=K+1}^{\infty} \Big\{ \big(\mathrm{E} \, \|X_0 - X_0^{(m-h)}\|^4 \big)^{1/4} + \big(\mathrm{E} \, \|X_0 - X_0^{(m)}\|^4 \big)^{1/4} \Big\} \\ < & \infty, \end{aligned}$$

which establishes that the sequence $\{\widetilde{\boldsymbol{X}}_{t,K}\}$ is L^2-m-approximable in $\mathbb{H}_1$.

As a result, the process $\sqrt{T}\hat{\gamma}_{K,j,l}$ converges to a multivariate normal distribution in $\mathbb{R}^K$, thus establishing the finite-dimensional convergence in **(A.i)**.

Next, for **(A.ii)**, we need to show that for any $1 \leq j, l < \infty$, the sequence $\sqrt{T}\hat{\boldsymbol{\gamma}}_{K,j,l}$ is tight in $\mathbb{R}^K$. Following the approach outlined in Eq. (36) of Zhang (2016), it is sufficient to show that for $1 \leq h \leq K$,

$$\lim_{\max(N_1,N_2)\to\infty} \sup_T \frac{1}{T} \mathrm{E} \sum_{j \geq N_1, l \geq N_2} \big[\sum_{t=h+1}^{T} (X_{t,j} X_{t-h,l} - \mathrm{E}[X_{t,j} X_{t-h,l}]) \big]^2 = 0. \tag{4}$$

Using the covariance structure, we have:

$$
\begin{aligned}
&\sup_T \frac{1}{T} \mathrm{E} \sum_{j\geq N_1, l\geq N_2} \Big(\sum_{t=h+1}^{T} [X_{t,j}X_{t-h,l} - \mathrm{E}(X_{t,j}X_{t-h,l})] \Big)^2 \\
=& \sup_T \frac{1}{T} \sum_{j\geq N_1, l\geq N_2} \sum_{t=h+1}^{T} \sum_{t'=h+1}^{T} \mathrm{Cov}(X_{t,j}X_{t-h,l}, X_{t',j}X_{t'-h,l}) \\
=& \sup_T \frac{1}{T} \sum_{j\geq N_1, l\geq N_2} \sum_{t=h+1}^{T} \sum_{t'=h+1}^{T} \big\{ \mathrm{Cum}(X_{t,j}, X_{t-h,l}, X_{t',j}, X_{t'-h,l}) + \mathrm{E}(X_{t,j}X_{t',j}) \,\mathrm{E}(X_{t-h,l}X_{t'-h,l}) \\
&+ \mathrm{E}(X_{t,j}X_{t'-h,l}) \,\mathrm{E}(X_{t-h,l}X_{t',j}) \big\} \\
=& \sup_T \frac{1}{T} \sum_{j\geq N_1, l\geq N_2} \sum_{t=h+1}^{T} \sum_{t'=h+1}^{T} \big\{ \langle \mathcal{R}_{t-t'+h, t-t', h}(u_j \otimes u_l), u_j \otimes u_l \rangle + \langle \gamma_{t-t'} u_j, u_j \rangle \langle \gamma_{t-t'} u_l, u_l \rangle \\
&+ \langle \gamma_{t-t'+h} u_l, u_j \rangle \langle \gamma_{t-t'-h} u_j, u_l \rangle \big\} \\
\leq& \sum_{j\geq N_1, l\geq N_2} \sum_{s=-\infty}^{\infty} \big| \langle \mathcal{R}_{s+h,s,h}(u_j \otimes u_l), u_j \otimes u_l \rangle + \langle \gamma_s u_j, u_j \rangle \langle \gamma_s u_l, u_l \rangle + \langle \gamma_{s+h} u_l, u_j \rangle \langle \gamma_{s-h} u_j, u_l \rangle \big|.
\end{aligned}
$$

By the summability condition, we have:

$$
\sum_{s=-\infty}^{\infty} \sum_{j,l=1}^{\infty} \big| \langle \gamma_s u_j, u_j \rangle \langle \gamma_s u_l, u_l \rangle \big| = \sum_{s=-\infty}^{\infty} \|\gamma_s\|_{TR}^2 < \infty,
$$

and by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\sum_{s=-\infty}^{\infty} \sum_{j,l=1}^{\infty} \big| \langle \gamma_{s+h} u_l, u_j \rangle \langle \gamma_{s-h} u_j, u_l \rangle \big| &\leq \sum_{s=-\infty}^{\infty} \Big(\sum_{j,l=1}^{\infty} |\langle \gamma_{s+h} u_l, u_j \rangle|^2 \Big)^{1/2} \Big(\sum_{j,l=1}^{\infty} |\langle \gamma_{s+h} u_l, u_j \rangle|^2 \Big)^{1/2} \\
&\leq \sum_{s=-\infty}^{\infty} \|\gamma_s\|_{\mathcal{S}}^2 \leq \sum_{s=-\infty}^{\infty} \|\gamma_s\|_{TR}^2 < \infty,
\end{aligned}
$$

$$
\sum_{s=-\infty}^{\infty} \sum_{j,l=1}^{\infty} \big| \langle \mathcal{R}_{s+h,s,h}(u_j \otimes u_l), u_j \otimes u_l \rangle \big| \leq \sum_{s=-\infty}^{\infty} \|\mathcal{R}_{s+h,s,h}\|_{TR} < \infty,
$$

where $\mathcal{R}_{l,r,p}$ is the 4th-order cumulant operator defined in (**??**).

Therefore, (4) holds, demonstrating that the sequence $\hat{\boldsymbol{\gamma}}_K(\tau_1, \tau_2)$ is tight in $\mathbb{H}_1$. Combining the results from **(A.i)** and **(A.ii)**, it follows that $\sqrt{T}\hat{\boldsymbol{\gamma}}_K(\tau_1, \tau_2) \xrightarrow{d} \boldsymbol{\Gamma}_K(\tau_1, \tau_2)$, $\tau_1, \tau_2 \in [0, 1]$, with the cross-covariance operator between $\Gamma_i(\tau_1, \tau_2)$ and $\Gamma_j(\tau_1, \tau_2)$, for any $1 \le i, j \le K$, given by

$$(\Psi_{i,j}\phi)(\tau_1, \tau_2) := \sum_{s=-\infty}^{\infty} \iint \mathrm{Cov}\left(X_i(\tau_1)X_0(\tau_2), X_{j+s}(\tau_1')X_s(\tau_2')\right)\phi(\tau_1', \tau_2')d\tau_1' d\tau_2'$$
$$= \iint \left[\psi_K(\tau_1, \tau_2, \tau_1', \tau_2')\right]_{ij} \phi(\tau_1', \tau_2')d\tau_1' d\tau_2', \tag{5}$$

for any $\phi(\tau_1, \tau_2) \in L^2\left([0, 1]^2\right)$. The $K \times K$ matrix $\psi_K(\tau_1, \tau_2, \tau_1', \tau_2')$ is the covariance kernel function of $\boldsymbol{\Gamma}_K(\tau_1, \tau_2)$ and its (i, j)-th element is implicitly defined.

The proof of Lemma A.1 is complete. ■

Proof of Theorem ??. The covariance operator of $\boldsymbol{\Gamma}_K(\tau_1, \tau_2)$ given by Lemma A.1, denoted by $\Psi_K : \mathbb{H}_1 \mapsto \mathbb{H}_1$, is induced by the kernel ψ_K, which takes the following form:

$$\Psi_K(\tilde{\phi})(\tau_1, \tau_2) = \iint \psi_K(\tau_1, \tau_2, \tau_1', \tau_2')\tilde{\phi}(\tau_1', \tau_2')d\tau_1' d\tau_2', \tag{6}$$

where $\tilde{\phi}(\tau_1', \tau_2') \in \mathbb{H}_1$.

Following the proof of Kokoszka et al. (2017), the operator Ψ_K is Hilbert-Schmidt, and thus compact, symmetric, and positive definite. Consequently, by Mercer's theorem, there exists non-negative eigenvalues $\{\xi_{K,l},\ 1 \le l < \infty\}$ and a corresponding collection of orthonormal eigenfunctions $\{\varphi_{K,l}(\tau_1, \tau_2), 1 \le l < \infty, 0 \le \tau_1, \tau_2 \le 1\}$, such that

$$\Psi_K(\varphi_{K,l})(\tau_1, \tau_2) = \iint \psi_K(\tau_1, \tau_2, \tau_1', \tau_2')\varphi_{K,l}(\tau_1', \tau_2')d\tau_1' d\tau_2' = \xi_{K,l}\varphi_{K,l}(\tau_1, \tau_2).$$

By the Karhunen-Loéve expansion,

$$\mathbf{\Gamma}_K(\tau_1, \tau_2) = \sum_{l=1}^{\infty} \xi_{K,l}^{1/2} \mathcal{N}_l \varphi_{K,l}(\tau_1, \tau_2),$$

where $\{\mathcal{N}_l\}_{l=1}^{\infty}$ are IID standard normal random variables. From Lemma A.1 and the continuous mapping theorem, it follows that, when $T \to \infty$,

$$T \sum_{h=1}^{K} \|\hat{\gamma}_h\|^2 \xrightarrow{d} \sum_{l=1}^{\infty} \xi_{K,l} \mathcal{N}_l^2.$$

Finally, since $\hat{\rho}_h = \dfrac{\|\hat{\gamma}_h\|}{\int \hat{\gamma}_0(\tau,\tau)d\tau}$ and $\int \hat{\gamma}_0(\tau,\tau)d\tau \xrightarrow{p} \int \mathrm{E}[X_0^2(t)]dt$, it follows from Slutsky's lemma that

$$T \sum_{h=1}^{K} \hat{\rho}_h^2 \xrightarrow{d} \frac{\sum_{l=1}^{\infty} \xi_{K,l} \mathcal{N}_l^2}{\left[\int \mathrm{E}[X_0^2(t)]dt\right]^2}.$$

This completes the proof of Theorem **??**. ■

Proof of Theorem ??. Since $V_{T,K} \geq Q_{T,h}$ for all $h \in \{1, \ldots, K\}$ and the denominator of $Q_{T,h}$ is finite, it follows that it is enough to show that, under the conditions of the theorem, $T\|\hat{\gamma}_h\|^2 \xrightarrow{p} \infty$ as $T \to \infty$. Simple algebra yields that

$$\begin{aligned}
T\|\hat{\gamma}_h\|^2 &= \frac{1}{T} \iint \Big\{ \sum_{t=1+h}^{T} \left[X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\right] \Big\}^2 d\tau_1 d\tau_2 \\
&= \iint \Big\{ \frac{1}{\sqrt{T}} \sum_{t=1+h}^{T} \left[X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\right] \Big\}^2 d\tau_1 d\tau_2 \\
&\quad + 2 \iint \Big\{ \frac{T-h}{T} \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right) \sum_{t=1+h}^{T} \left[X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\right] \Big\} d\tau_1 d\tau_2 \\
&\quad + \frac{(T-h)^2}{T} \left\| \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right) \right\|^2. \qquad (7)
\end{aligned}$$

By the arguments used to prove the condition **(A.i)** in Lemma A.1, the stationary sequence $\{X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\}$ is mean zero and L^2-m-approximable. From this we obtain that

$$\iint \Big\{ \frac{1}{\sqrt{T}} \sum_{t=1+h}^{T} \big[X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\big]\Big\}^2 d\tau_1 d\tau_2 = O_p(1),$$

and

$$\iint \Big\{ \frac{T-h}{T} \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right) \sum_{t=1+h}^{T} \big[X_t(\tau_1)X_{t-h}(\tau_2) - \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\big]\Big\} d\tau_1 d\tau_2 = O_p(\sqrt{T}).$$

The result then follows from (7) since $\frac{(T-h)^2}{T}\big\| \mathrm{E}\left(X_h(\tau_1)X_0(\tau_2)\right)\big\|^2$ diverges to infinity at the rate T. ■

To establish Theorem **??**, it is sufficient to demonstrate the following lemma.

Lemma A.2. *Under Assumptions* **??**, **??** *and* **??**, *when* $T \to \infty$, *we have*

$$d\Big\{\mathcal{L}\Big[\big(\sqrt{T}\hat{\gamma}_1^*(\tau_1,\tau_2), \ldots, \sqrt{T}\hat{\gamma}_K^*(\tau_1,\tau_2)\big)^{\mathrm{T}} \big| \mathbb{X}_T\Big], \mathcal{L}\Big[\big(\Gamma_1(\tau_1,\tau_2), \ldots, \Gamma_K(\tau_1,\tau_2)\big)^{\mathrm{T}}\Big]\Big\} \to 0,$$

in probability, where $\mathbf{\Gamma}_K(\tau_1,\tau_2) = \big(\Gamma_1(\tau_1,\tau_2), \ldots, \Gamma_K(\tau_1,\tau_2)\big)^{\mathrm{T}}$ *is a* $K \times 1$ *vector-valued mean-zero Gaussian process in* $\mathbb{H}_1$, *whose covariance structure is given in (5).*

Proof of Lemma A.2. To begin, we define the following random weighted quantities:

$$\widehat{\boldsymbol{X}}^*_{t,K} = \Big(w_t\big[X_t(\tau_1)X_{t-1}(\tau_2) - \hat{\gamma}_1(\tau_1,\tau_2)\big], \ldots, w_t\big[X_t(\tau_1)X_{t-K}(\tau_2) - \hat{\gamma}_K(\tau_1,\tau_2)\big]\Big)^{\mathrm{T}},$$
$$\widehat{\boldsymbol{X}}^*_{t,K,j,l} = \Big(w_t\big[X_{t,j}X_{t-1,l} - \langle\hat{\gamma}_1 u_l, u_j\rangle\big], \ldots, w_t\big[X_{t,j}X_{t-K,l} - \langle\hat{\gamma}_K u_l, u_j\rangle\big]\Big)^{\mathrm{T}},$$
$$\boldsymbol{X}^*_{t,K,j,l} = \Big(w_t\big[X_{t,j}X_{t-1,l} - \langle\gamma_1 u_l, u_j\rangle\big], \ldots, w_t\big[X_{t,j}X_{t-K,l} - \langle\gamma_K u_l, u_j\rangle\big]\Big)^{\mathrm{T}},$$

and their corresponding h-th elements are denoted respectively by $\widehat{\boldsymbol{X}}^*_{t,(h)}$, $\widehat{\boldsymbol{X}}^*_{t,(h),j,l}$, $\boldsymbol{X}^*_{t,(h),j,l}$.

Let E^*, Var^* denote the conditional expectation and variance given the original sample. To prove Lemma A.2, similarly to Lemma A.1, we need to show that, conditional on the original sample $\{X_t\}_{t=1}^T$, the following two conditions hold:

(A.iii). Finite dimensional convergence of the sequence $\left(\hat{\gamma}_1^*(\tau_1,\tau_2),\ldots,\hat{\gamma}_K^*(\tau_1,\tau_2)\right)^{\mathrm{T}}$**.**

Since

$$\left(\sqrt{T}\hat{\gamma}_1^*(\tau_1,\tau_2),\ldots,\sqrt{T}\hat{\gamma}_K^*(\tau_1,\tau_2)\right)^{\mathrm{T}} = \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\widehat{\boldsymbol{X}}^*_{t,K} + o_p(1),$$

it suffices to establish the CLT for the sequence $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\widehat{\boldsymbol{X}}^*_{t,K}$.

(A.iv). The tightness of the sequence $\left(\hat{\gamma}_1^*(\tau_1,\tau_2),\ldots,\hat{\gamma}_K^*(\tau_1,\tau_2)\right)^{\mathrm{T}}$**.**

We begin by proving **(A.iii)**. To establish the finite-dimensional CLT of $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\widehat{\boldsymbol{X}}^*_{t,K}$, it suffices to show the CLT for the sequence $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\widehat{\boldsymbol{X}}^*_{t,K,j,l}$. By the same argument as in the proof of Theorem **??**, $\{\boldsymbol{X}^*_{t,K,j,l}\}_{t=2}^{\infty}$ is an L^2-m-approximable process. Thus, $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\boldsymbol{X}^*_{t,K,j,l}$ converges to a multivariate normal distribution.

Next, we will calculate the asymptotic covariance matrix of $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\boldsymbol{X}^*_{t,K,j,l}$.

For any

$$\boldsymbol{V}_K := \left(v_1(\tau_1,\tau_2), v_2(\tau_1,\tau_2),\ldots,v_K(\tau_1,\tau_2)\right)^{\mathrm{T}} \in \mathbb{H}_1,$$

define $\boldsymbol{V}_{K,j,l} = (v_{1,j,l}, v_{2,j,l},\ldots,v_{K,j,l})^{\mathrm{T}} \in \mathbb{R}^K$, where $v_{h,j,l} = \langle v_h(\tau_1,\tau_2), u_j \otimes u_l\rangle,\ 1 \le h \le K$. The calculation of the asymptotic covariance matrix of $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\boldsymbol{X}^*_{t,K,j,l}$ is thus transformed to determining the asymptotic variance of $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\langle\boldsymbol{X}^*_{t,K,j,l}, \boldsymbol{V}_{K,j,l}\rangle$.

Note that

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\langle \boldsymbol{X}^*_{t,K,j,l}, \boldsymbol{V}_{K,j,l}\rangle &= \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\sum_{h=1}^{K}\boldsymbol{X}^*_{t,(h),j,l}v_{h,j,l}\\ &= \frac{1}{\sqrt{T}}\sum_{t=2}^{K+1}\Big(\sum_{h=1}^{(t-1)\wedge K}\boldsymbol{X}^*_{t,(h),j,l}v_{h,j,l}\Big) + \frac{1}{\sqrt{T}}\sum_{t=K+2}^{T}\Big(\sum_{h=1}^{(t-1)\wedge K}\boldsymbol{X}^*_{t,(h),j,l}v_{h,j,l}\Big) + o_p(1)\\ &:= \frac{1}{\sqrt{T}}\sum_{t=2}^{K+1}M^*_{t,K} + \frac{1}{\sqrt{T}}\sum_{t=K+2}^{T}M^*_{t,K} + o_p(1),\end{aligned}$$

where $\mathrm{E}\big[\mathrm{Var}^*(\frac{1}{\sqrt{T}}\sum_{t=2}^{K+1}M^*_{t,K})\big] = o(1)$, since this term only involves a finite number of terms. Thus, it suffices to show the asymptotic normality of $\frac{1}{\sqrt{T}}\sum_{t=K+2}^{T}M^*_{t,K}$.

We can write

$$\begin{aligned}J^*_{T,K} := \frac{1}{\sqrt{T}}\sum_{t=K+2}^{T}M^*_{t,K} &= \frac{1}{\sqrt{T}}\sum_{t=K+2}^{T}\sum_{h=1}^{K}w_t\big[X_{t,j}X_{t-h,l} - \langle\gamma_h u_l, u_j\rangle\big]v_{h,j,l}\\ &= \sum_{s=1}^{L_T}\frac{1}{\sqrt{T}}\delta_s\sum_{t\in B_s\cap[K+2,T]}\sum_{h=1}^{K}\big[X_{t,j}X_{t-h,l} - \langle\gamma_h u_l, u_j\rangle\big]v_{h,j,l}.\end{aligned}$$

Then, since δ_s, $s = 1,\ldots,L_T$ has mean 0 and variance 1, we have

$$\mathrm{Var}^*(J^*_{T,K}) = \frac{1}{T}\sum_{s=1}^{L_T}\Big(\sum_{t\in B_s\cap[K+2,T]}\sum_{h=1}^{K}\big[X_{t,j}X_{t-h,l} - \langle\gamma_h u_l, u_j\rangle\big]v_{h,j,l}\Big)^2.$$

In the following, we shall show that, when $T\to\infty$,

$$\mathrm{Var}^*(J^*_{T,K}) \xrightarrow{p} \sum_{h,h'=1}^{K}v_{h,j,l}v_{h',j,l}\big\langle\Psi_{h,h'}(u_j\otimes u_l), u_j\otimes u_l\big\rangle.$$

To this end, we calculate $\mathrm{E}\big[\mathrm{Var}^*(J^*_{T,K})\big]$ and $\mathrm{Var}\big[\mathrm{Var}^*(J^*_{T,K})\big]$ accordingly.

We begin by calculating $\mathrm{E}\big[\mathrm{Var}^*(J^*_{T,K})\big]$ as follows:

$$\mathrm{E}\big[\mathrm{Var}^*(J^*_{T,K})\big] = \frac{1}{T}\sum_{s=1}^{L_T}\sum_{t,t'\in B_s\cap[K+2,T]}\sum_{h,h'=1}^{K}\mathrm{Cov}(X_{t,j}X_{t-h,l},X_{t',j}X_{t'-h',l})v_{h,j,l}v_{h',j,l}$$

$$=\frac{1}{T}\sum_{s=1}^{L_T}\sum_{h,h'=1}^{K}v_{h,j,l}v_{h',j,l}\sum_{t\in B_s\cap[K+2,T],t\geq h+1}\sum_{t'\in B_s\cap[K+2,T],t'\geq h'+1}\mathrm{Cov}(X_{t,j}X_{t-h,l},X_{t',j}X_{t'-h',l})$$

$$\to\sum_{h,h'=1}^{K}v_{h,j,l}v_{h',j,l}\sum_{s=-\infty}^{\infty}\mathrm{Cov}(X_{h,j}X_{0,l},X_{s+h',j}X_{s,l})$$

$$=\sum_{h,h'=1}^{K}v_{h,j,l}v_{h',j,l}\sum_{s=-\infty}^{\infty}\mathrm{Cov}\big[\langle X_h(\tau_1)X_0(\tau_2),u_j\otimes u_l(\tau_1,\tau_2)\rangle,\langle X_{s+h'}(\tau_1')X_s(\tau_2'),u_j\otimes u_l(\tau_1',\tau_2')\big]$$

$$=\Big\langle\int\sum_{s=-\infty}^{\infty}\mathrm{Cov}\big(X_h(\tau_1)X_0(\tau_2),X_{s+h'}(\tau_1')X_s(\tau_2')\big)\big(u_j\otimes u_l(\tau_1',\tau_2')\big)d\tau_1'd\tau_2',u_j\otimes u_l(\tau_1,\tau_2)\Big\rangle$$

$$=\sum_{h,h'=1}^{K}v_{h,j,l}v_{h',j,l}\big\langle\Psi_{h,h'}(u_j\otimes u_l),u_j\otimes u_l\big\rangle.$$

Then, we will show that $\mathrm{Var}\big[\mathrm{Var}^*(J^*_{T,K})\big]=o(1)$.

Note that

$$\begin{aligned}
&\mathrm{Var}\big[\mathrm{Var}^*(J^*_{T,K})\big]\\
=&\frac{1}{T^2}\sum_{s,s'=1}^{L_T}\sum_{h_1,h_2=1}^{K}\sum_{h_1',h_2'=1}^{K}v_{h_1,j,l}v_{h_2,j,l}v_{h_1',j,l}v_{h_2',j,l}\sum_{t_1,t_2\in B_s\cap[K+2,T]}\sum_{t_1',t_2'\in B_{s'}\cap[K+2,T]}\\
&\times\mathrm{Cov}(\boldsymbol{X}_{t_1,(h_1),j,l}\boldsymbol{X}_{t_2,(h_2),j,l},\boldsymbol{X}_{t_1',(h_1'),j,l}\boldsymbol{X}_{t_2',(h_2'),j,l}\\
=&\frac{1}{T^2}\sum_{s,s'=1}^{L_T}\sum_{h_1,h_2=1}^{K}\sum_{h_1',h_2'=1}^{K}v_{h_1,j,l}v_{h_2,j,l}v_{h_1',j,l}v_{h_2',j,l}\sum_{t_1,t_2\in B_s\cap[K+2,T]}\sum_{t_1',t_2'\in B_{s'}\cap[K+2,T]}\\
&\times\Big[\mathrm{Cum}(\boldsymbol{X}_{t_1,(h_1),j,l},\boldsymbol{X}_{t_2,(h_2),j,l},\boldsymbol{X}_{t_1',(h_1'),j,l},\boldsymbol{X}_{t_2',(h_2'),j,l})+\mathrm{Cov}(\boldsymbol{X}_{t_1,(h_1),j,l},\boldsymbol{X}_{t_1',(h_1'),j,l})\\
&\times\mathrm{Cov}(\boldsymbol{X}_{t_2,(h_2),j,l},\boldsymbol{X}_{t_2',(h_2'),j,l})+\mathrm{Cov}(\boldsymbol{X}_{t_1,(h_1),j,l},\boldsymbol{X}_{t_2',(h_2'),j,l})\,\mathrm{Cov}(\boldsymbol{X}_{t_2,(h_2),j,l},\boldsymbol{X}_{t_1',(h_1'),j,l})\Big]\\
=&\frac{1}{T^2}\sum_{s,s'=1}^{L_T}\sum_{h_1,h_2=1}^{K}\sum_{h_1',h_2'=1}^{K}v_{h_1,j,l}v_{h_2,j,l}v_{h_1',j,l}v_{h_2',j,l}\sum_{t_1,t_2\in B_s\cap[K+2,T]}\sum_{t_1',t_2'\in B_{s'}\cap[K+2,T]}(I_{1,l}+I_{2,l}+I_{3,l}).
\end{aligned}\tag{8}$$

We now proceed to address $I_{i,l}$, $i = 1, 2, 3$, respectively. By applying Lemma F.9 in Panaretos and Tavakoli (2013), Theorem II.2 in Rosenblatt (1985), and utilizing the triangle inequality, we obtain that

$$
\begin{aligned}
|I_{1,l}| &= \big| \operatorname{Cum}(X_{t_1,j}X_{t_1-h_1,l}X_{t_2,j}X_{t_2-h_2,l}, X_{t_1',j}X_{t_1'-h_1',l}X_{t_2',j}X_{t_2'-h_2',l}) \big| \\
&\le \big\| \operatorname{Cum}(X_{t_1}X_{t_1-h_1}X_{t_2}X_{t_2-h_2}, X_{t_1'}X_{t_1'-h_1'}X_{t_2'}X_{t_2'-h_2'}) \big\|_{[0,1]^8} \\
&\le \sum_{\lambda=\lambda_1\cup\cdots\cup\lambda_p} \big\| \operatorname{Cum}(X_{\varsigma_1} : \varsigma_1 \in \lambda_1) \cdots \operatorname{Cum}(X_{\varsigma_p} : \varsigma_p \in \lambda_p) \big\|_{[0,1]^8} \\
&\le \sum_{\lambda=\lambda_1\cup\cdots\cup\lambda_p} \big\| \operatorname{Cum}(X_{\varsigma_1} : \varsigma_1 \in \lambda_1) \big\|_{[0,1]^{|\lambda_1|}} \cdots \big\| \operatorname{Cum}(X_{\varsigma_p} : \varsigma_p \in \lambda_p) \big\|_{[0,1]^{|\lambda_p|}},
\end{aligned}
$$

where $|\lambda_j|$ denotes the cardinality of the set λ_j and the summation is over all indecomposable partitions of the following two-way table:

$$
\begin{array}{ll}
t_1 & t_1 - h_1 \\
t_2 & t_2 - h_2 \\
t_1' & t_1' - h_1' \\
t_2' & t_2' - h_2'
\end{array}
$$

Under Assumption **??**, it is straightforward to verify that its contribution to the sum (8) is of order $o(1)$. The arguments for $I_{2,l}$ and $I_{3,l}$ are analogous, with the only difference being the interchange of t_1' and t_2'.

Specifically, we can write $I_{2,l}$ as follows:

$$
\begin{aligned}
I_{2,l} &= \operatorname{Cov}(X_{t_1,j}X_{t_1-h_1,l}, X_{t_1',j}X_{t_1'-h_1',l}) \operatorname{Cov}(X_{t_2,j}X_{t_2-h_2,l}, X_{t_2',j}X_{t_2'-h_2',l}) \\
&= \Big\{ \langle \mathcal{R}_{t_1-t_1'+h_1', t_1-t_1'-h_1+h_1', h_1'}(u_j \otimes u_l), u_j \otimes u_l \rangle + \langle \gamma_{t_1-t_1'} u_j, u_j \rangle \langle \gamma_{t_1-t_1'-h_1+h_1'} u_l, u_l \rangle \\
&\quad + \langle \gamma_{t_1-t_1'-h_1} u_j, u_l \rangle \langle \gamma_{t_1-t_1'+h_1'} u_l, u_j \rangle \Big\} \times \\
&\quad \Big\{ \langle \mathcal{R}_{t_2-t_2'+h_2', t_2-t_2'-h_2+h_2', h_2'}(u_j \otimes u_l), u_j \otimes u_l \rangle + \langle \gamma_{t_2-t_2'} u_j, u_j \rangle \langle \gamma_{t_2-t_2'-h_2+h_2'} u_l, u_l \rangle \\
&\quad + \langle \gamma_{t_2-t_2'-h_2} u_j, u_l \rangle \langle \gamma_{t_2-t_2'+h_2'} u_l, u_j \rangle \Big\}.
\end{aligned}
$$

When $s = s'$ in the summation of (8), the contribution of $I_{2,l}$ to $\operatorname{Var}\big[\operatorname{Var}^*(J_{T,K}^*)\big]$ is $O(T^{-2}b_T) = o(1)$. In the case of $s \neq s'$, Assumption **??** implies that the magnitude of these terms is bounded by $CT^{-2}L_T^2 = o(1)$. Consequently, we have $\operatorname{Var}\big[\operatorname{Var}^*(J_{T,K}^*)\big] = o(1)$.

As a result, when $T \to \infty$,

$$
\operatorname{Var}^*\Big(\frac{1}{\sqrt{T}} \sum_{t=2}^{T} \langle \boldsymbol{X}_{t,K,j,l}^*, \boldsymbol{V}_{K,j,l} \rangle\Big) \xrightarrow{p} \sum_{h,h'=1}^{K} v_{h,j,l} v_{h',j,l} \big\langle \Psi_{h,h'}(u_j \otimes u_l), u_j \otimes u_l \big\rangle. \tag{9}
$$

Let $\widetilde{\boldsymbol{X}}_{t,K,j,l}^* = \widehat{\boldsymbol{X}}_{t,K,j,l}^* - \boldsymbol{X}_{t,K,j,l}^*$. When $T \to \infty$, we have

$$
\mathrm{E}\Big[\operatorname{Var}^*\Big(\frac{1}{\sqrt{T}} \sum_{t=2}^{T} \langle \widetilde{\boldsymbol{X}}_{t,K,j,l}^*, \boldsymbol{V}_{K,j,l} \rangle\Big)\Big] \xrightarrow{p} 0. \tag{10}
$$

Combining the conclusions in (9) and (10), it is immediately obtained that

$$
\operatorname{Var}^*\Big(\frac{1}{\sqrt{T}} \sum_{t=2}^{T} \langle \widehat{\boldsymbol{X}}_{t,K,j,l}^*, \boldsymbol{V}_{K,j,l} \rangle\Big) \xrightarrow{p} \sum_{h,h'=1}^{K} v_{h,j,l} v_{h',j,l} \big\langle \Psi_{h,h'}(u_j \otimes u_l), u_j \otimes u_l \big\rangle,
$$

which is equivalent to

$$\mathrm{Var}^*\Big(\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\langle\widehat{\boldsymbol{X}}^*_{t,K},V_K\rangle_{\mathbb{H}_1}\Big)\xrightarrow{p}\langle\Psi_K(\boldsymbol{V}_K),\boldsymbol{V}_K\rangle_{\mathbb{H}_1},\quad T\to\infty.$$

The proof of **(A.iii)** is completed.

Next, we proceed to show **(A.iv)**, i.e., conditional on the sample, $\big\{\sqrt{T}\hat{\gamma}^*_h(\tau_1,\tau_2), 1\le h\le K\big\}$ is tight in $\mathbb{H}_1$ in probability.

For $1\le h\le K$, we observe that

$$\begin{aligned}\sqrt{T}\hat{\gamma}^*_h(\tau_1,\tau_2)&=\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T}w_t\big[X_t(\tau_1)X_{t-h}(\tau_2)-\hat{\gamma}_h(\tau_1,\tau_2)\big]\\&=L_T^{-1/2}\sum_{s=1}^{L_T}b_T^{-1/2}\delta_s\sum_{t\in B_s\cap[h+1,T]}\big[X_t(\tau_1)X_{t-h}(\tau_2)-\hat{\gamma}_h(\tau_1,\tau_2)\big]\\&:=L_T^{-1/2}\sum_{s=1}^{L_T}M^*_{sT}(\tau_1,\tau_2),\end{aligned}$$

where $M^*_{sT}(\tau_1,\tau_2)$ is implicitly defined. It is important to note that $M^*_{sT}(\tau_1,\tau_2)$ and $M^*_{s'T}(\tau_1,\tau_2)$, $s\neq s'$, are independent given the sample. In light of the argument in the proof of Theorem 4 of Escanciano and Velasco (2006) and Shao (2011), it suffices to verify that $\mathrm{E}^*\left\|M^*_{sT}(\tau_1,\tau_2)\right\|^2<\infty$ almost surely. To this end, following the analogous proof in Zhang (2016), we have $\mathrm{E}\big\{\mathrm{E}^*\left\|M^*_{sT}(\tau_1,\tau_2)\right\|^2\big\}$ $<\infty$.

By combining the conditions **(A.iii)** and **(A.iv)**, the proof of Lemma A.2 is complete. ■

Proof of Theorem ??. Following an argument similar to that of Theorem **??**, and given the original sample, we directly establish the conditional distribution of $V^*_{K,T}$ as $T\to\infty$ in Theorem **??**. ■

B. Additional Numerical Studies

In the following, we carry out some additional simulation studies to further investigate the influences of the parameters b_T and K on the testing performance. All the empirical rejection rates are average over 1000 Monte Carlo replicates following the same computing procedures used in Section **??**. Bootstrap conducts $B = 500$ replicates.

Table 1: Empirical sizes in the percentage of BRWB test.

			$K=1$			$K=3$			$K=5$			$K=8$			$K=10$			$K=15$			$K=20$		
	b_T	W	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
(a) IID-BM																							
$T=200$	3	$N(0,1)$	10.7	5.4	1.1	10.9	5.1	0.8	10.7	4.9	1.4	7.6	3.3	0.8	7.2	3.5	0.8	6.8	3.0	0.4	8.1	3.0	0.4
		Bernoulli	10.1	5.4	1.0	10.8	4.3	1.4	10.7	5.5	0.9	11.3	5.9	1.3	9.5	5.0	1.0	8.7	3.7	0.3	10.3	5.4	1.1
	6	$N(0,1)$	10.9	5.5	1.5	9.9	4.3	0.7	9.8	3.6	0.6	7.3	3.2	0.4	7.5	3.5	0.5	6.3	2.5	0.3	6.3	2.5	0.2
		Bernoulli	11.4	6.0	1.5	10.1	5.3	1.2	9.1	4.3	1.1	8.9	4.2	0.9	9.2	4.7	0.5	8.2	4.1	0.6	7.5	3.3	0.8
$T=800$	4	$N(0,1)$	11.1	5.3	1.4	9.4	4.7	1.2	10.4	5.0	1.1	9.4	4.2	1.5	9.2	4.0	1.0	9.7	4.4	1.0	7.3	3.1	0.8
		Bernoulli	10.1	5.6	1.2	11.0	6.0	1.4	8.9	3.2	0.7	9.5	5.3	0.9	9.0	5.2	0.7	9.6	5.6	1.7	8.5	4.6	0.7
	8	$N(0,1)$	9.7	5.4	1.5	10.3	4.8	1.3	10.2	4.1	0.5	8.7	3.5	0.6	8.7	3.9	0.8	7.5	3.3	0.4	9.1	3.0	0.4
		Bernoulli	11.0	5.6	1.3	9.9	5.0	1.2	9.3	4.7	0.8	11.0	5.1	1.2	10.3	4.7	1.0	9.8	4.0	0.3	7.7	2.5	0.3
(b) FGARCH(1,1)																							
$T=200$	3	$N(0,1)$	11.2	6.1	1.7	7.3	3.5	0.4	6.7	3.7	0.4	7.0	3.7	0.4	6.7	3.2	0.4	6.8	3.0	0.2	5.7	2.6	0.3
		Bernoulli	9.3	5.0	1.0	7.5	3.4	0.5	8.4	3.6	0.6	7.5	3.0	0.4	7.7	2.9	0.4	6.9	3.0	0.6	7.1	3.2	0.6
	6	$N(0,1)$	11.2	5.4	0.9	7.3	3.5	0.6	7.6	3.9	0.4	6.8	3.4	0.7	7.3	3.5	0.3	6.4	3.0	0.3	6.0	2.6	0.2
		Bernoulli	11.7	6.1	1.2	9.9	4.7	1.0	7.3	3.4	0.5	7.4	3.3	0.5	7.9	4.7	0.7	6.1	2.9	0.2	6.1	3.4	0.3
$T=800$	4	$N(0,1)$	9.5	4.7	0.9	8.3	4.4	0.7	8.3	4.6	0.5	8.5	4.4	0.3	7.2	3.5	0.5	6.5	2.8	0.4	6.8	3.0	0.3
		Bernoulli	9.9	5.1	1.1	9.8	4.5	0.7	9.2	4.4	0.7	7.8	4.6	0.4	7.5	3.6	0.7	7.7	3.6	0.7	6.2	3.2	0.3
	8	$N(0,1)$	9.2	4.5	1.0	7.8	3.8	0.7	8.5	4.1	0.5	7.8	4.0	0.6	7.4	4.1	0.6	6.4	3.0	0.4	6.2	3.2	0.3
		Bernoulli	11.6	5.3	1.3	9.0	4.0	0.7	9.3	4.0	0.5	8.8	4.3	0.7	8.8	4.4	0.7	7.6	3.6	0.5	6.4	3.3	0.4
(c) Fbilinear(1,0.3)																							
$T=200$	1	$N(0,1)$	11.3	5.3	1.6	8.7	4.1	0.9	9.8	4.9	1.1	10.5	5.1	1.0	8.7	4.5	1.0	10.5	5.0	1.4	7.2	3.1	0.6
		Bernoulli	11.8	5.5	1.7	12.0	6.0	1.3	11.4	5.8	1.5	11.7	6.3	1.2	8.9	5.5	1.1	8.7	3.9	0.7	10.0	4.5	1.0
$T=800$	1	$N(0,1)$	11.5	6.0	1.7	11.3	6.2	1.4	12.3	6.0	1.2	10.8	6.0	1.4	10.5	5.4	1.6	10.7	6.3	1.4	10.8	5.0	1.0
		Bernoulli	11.3	6.0	1.6	12.0	5.4	1.6	11.9	5.9	1.0	11.5	6.6	1.3	11.1	5.7	1.9	10.9	5.7	1.5	10.5	4.3	1.2

Table 2: Empirical powers in the percentage of BRWB test. Random weights are generated from $N(0, 1)$.

			$K = 1$			$K = 3$			$K = 5$			$K = 8$			$K = 10$			$K = 15$			$K = 20$		
	S	b_T	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
$T = 200$	0.2	1	78.5	64.0	38.9	60.8	45.0	21.1	50.6	35.7	16.4	43.5	29.7	10.9	40.1	25.5	9.1	33.3	21.8	8.5	27.9	15.2	4.0
		3	81.9	66.3	37.9	59.7	44.6	15.7	48.5	30.4	9.9	37.7	21.3	5.1	34.1	19.2	4.4	26.3	13.5	2.0	21.4	11.7	2.1
		6	79.5	65.8	38.8	57.9	40.7	15.3	46.5	29.0	7.0	39.7	19.8	4.4	29.1	13.6	2.4	22.9	9.7	1.3	16.2	6.8	0.9
$T = 500$	0.2	1	100.0	99.3	92.8	97.4	94.1	77.7	93.3	86.0	65.5	86.9	77.6	55.5	84.3	73.5	48.0	76.3	63.0	37.0	70.0	55.0	27.7
		3	100.0	99.2	91.7	97.8	93.2	78.1	92.7	82.9	58.1	87.5	78.3	48.2	80.4	67.1	35.7	70.3	52.6	25.0	60.8	45.4	20.1
		6	100.0	99.2	93.1	92.7	93.6	71.4	92.6	84.7	55.5	85.4	72.0	40.4	80.1	64.3	30.2	70.5	51.6	20.6	57.7	37.9	12.0
$T = 200$	0.4	1	100.0	100.0	99.9	100.0	99.9	98.0	99.9	99.1	93.9	99.2	97.2	87.9	97.9	94.7	79.2	95.5	89.9	71.0	92.2	84.1	62.0
		3	100.0	100.0	98.8	99.9	99.7	94.9	99.5	98.0	87.1	98.0	92.9	70.4	96.7	89.3	57.2	91.1	79.4	44.8	86.4	69.0	28.2
		6	100.0	99.9	98.7	99.9	98.8	90.1	99.4	96.3	78.0	97.4	89.6	55.6	94.1	84.7	49.0	87.0	65.8	26.4	78.4	55.6	17.3
$T = 500$	0.4	1	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9
		3	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.4
		6	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.9	100.0	100.0	99.5	100.0	99.9	98.7

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