Minimax Rates of Convergence for Nonparametric Regression Under Adversarial Attacks

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Supplementary Material

The Supplementary Material provides the detailed proofs for the lemmas, theorems, and two examples discussed in the main text.

S1 Proof of Lemma 1

Recall the definition of the adversarial sup-norm loss in (2.2). By the change of variable $x' = x + \delta$, we have $x' \in (x + \Delta_n) \cap \mathcal{X}$ since $\delta \in \Delta_n$. Therefore, the adversarial loss can be expressed as

$$L_{\Delta_n}(f, \hat{f}) = \sup_{x \in \mathcal{X}} \sup_{x' \in (x+\Delta_n) \cap \mathcal{X}} \left| f(x) - \hat{f}(x') \right|. \tag{S1.1}$$

To prove the equivalence (3.1), it remains to show

$$\sup_{x \in \mathcal{X}} \sup_{x' \in (x+\Delta_n) \cap \mathcal{X}} \left| f(x) - \hat{f}(x') \right| = \sup_{x' \in \mathcal{X}} \sup_{x \in (x'-\Delta_n) \cap \mathcal{X}} \left| f(x) - \hat{f}(x') \right|. \tag{S1.2}$$

Assume that

$$\sup_{x\in\mathcal{X}}\sup_{x'\in(x+\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|>\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|.$$

Then there must exist $x_1\in\mathcal{X}$ and $x_1'\in(x_1+\Delta_n)\cap\mathcal{X}$ such that $|f(x_1)-\hat{f}(x_1')|>\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|$. On the other hand, based on the definition of (x_1,x_1'), we have $x_1'\in\mathcal{X}$ and $x_1=x_1'-\delta_1$ for some $\delta_1\in\Delta_n$, implying that $x_1\in(x_1'-\Delta_n)\cap\mathcal{X}$. This leads to

$$|f(x_1)-\hat{f}(x_1')|\le\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|,$$

which is a contradiction. Likewise, we can prove that $\sup_{x\in\mathcal{X}}\sup_{x'\in(x+\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|<\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}|f(x)-\hat{f}(x')|$ is also impossible. Therefore, (S1.2) is proved.

S2 Proof of Theorem 1

Based on the results in Lemma 1, we have

$$L_{\Delta_n}(f,f')=\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}|f(x)-f'(x')|\,. \tag{S2.1}$$

For any given $x' \in \mathcal{X}$, note that

$$\begin{aligned}
&\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} |f(x) - f'(x')| \\
&= \max\left\{\left|\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) - f'(x')\right|, \left|\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) - f'(x')\right|\right\} \\
&= \left\{\left|\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) + \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2} - f'(x')\right|\right. \\
&\quad \left. + \frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) - \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}\right\},
\end{aligned} \tag{S2.2}$$

where the first equality follows from the fact that $|f(x) - f'(x')|$, as a piecewise linear function of $f(x)$, achieves the supremum when $f(x)$ attains either its supremum $\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)$ or its infimum $\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)$, and the second equality is established by analyzing the relative values of $\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)$, $\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)$, and $f'(x')$.

Combining (S2.1) with (S2.2), we obtain

$$\begin{aligned}
L_{\Delta_n}(f, f') = \sup_{x'\in\mathcal{X}} &\left\{\left|\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) + \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2} - f'(x')\right|\right. \\
&\left. + \frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) - \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}\right\}.
\end{aligned} \tag{S2.3}$$

Since f' appears only in the absolute value term in (S2.3), the infimum $\inf_{f'} L_{\Delta_n}(f, f')$ is therefore obtained when

$$f'(x') = f^*(x') = \frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) + \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}$$

for any $x' \in \mathcal{X}$. And the ideal adversarial risk is given by

$$\frac{1}{2}\sup_{x'\in\mathcal{X}}\left\{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x) - \inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)\right\},$$

which completes the proof of this theorem.

S3 Proof of Theorem 2

From (S2.3), we see

$$R_{\Delta_n}(f,\hat{f}_{\mathrm{PI}}) = \mathbb{E}\sup_{x'\in\mathcal{X}}\left\{\left|\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)+\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}-\hat{f}_{\mathrm{PI}}(x')\right|\right.$$
$$\left.+\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)-\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}\right\}. \tag{S3.1}$$

Based on the definition (3.4) of $\hat{f}_{\mathrm{PI}}(x')$, the first term in the square bracket of (S3.1) can be upper bounded by

$$\begin{aligned}
&\left|\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)+\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}-\hat{f}_{\mathrm{PI}}(x')\right| \\
&= \left|\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)+\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)}{2}\right. \\
&\quad \left.-\frac{\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} \tilde{f}(x)+\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} \tilde{f}(x)}{2}\right| \\
&\le \frac{1}{2}\left|\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)-\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}} \tilde{f}(x)\right| \\
&\quad +\frac{1}{2}\left|\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} f(x)-\inf_{x\in(x'-\Delta_n)\cap\mathcal{X}} \tilde{f}(x)\right| \\
&\le \sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}\left|f(x)-\tilde{f}(x)\right|.
\end{aligned} \tag{S3.2}$$

Combining (S3.2) with (S3.1), we have

$$\begin{aligned}
R_{\Delta_n}(f,\hat{f}_{\mathrm{PI}}) &\le \mathbb{E}\sup_{x'\in\mathcal{X}}\sup_{x\in(x'-\Delta_n)\cap\mathcal{X}}\left|f(x)-\tilde{f}(x)\right| + L^*_{\Delta_n}(f) \\
&\le R(f,\tilde{f}) + L^*_{\Delta_n}(f),
\end{aligned}$$

where the first inequality follows from (S3.2)–(S3.1) and the definition of $L^*_{\Delta_n}(f)$ in (3.2), and the second inequality follows from

$$\sup_{x' \in \mathcal{X}} \sup_{x \in (x'-\Delta_n) \cap \mathcal{X}} \left| f(x) - \tilde{f}(x) \right| \leq \sup_{x' \in \mathcal{X}} \sup_{x \in \mathcal{X}} \left| f(x) - \tilde{f}(x) \right| = \sup_{x \in \mathcal{X}} \left| f(x) - \tilde{f}(x) \right|.$$

Thus, we complete the proof of (3.5).

The second part of this theorem is proved by taking upper bound on both sides of (3.5) with respect to $f \in \mathcal{F}$ and then using the condition (3.6). Specifically, we have

$$\begin{aligned} \sup_{f \in \mathcal{F}} R_{\Delta_n}(f, \hat{f}_{\mathrm{PI}}) &\lesssim \sup_{f \in \mathcal{F}} R(f, \tilde{f}) + \sup_{f \in \mathcal{F}} L^*_{\Delta_n}(f) \\ &\asymp \inf_{\hat{f}} \sup_{f \in \mathcal{F}} R(f, \hat{f}) + \sup_{f \in \mathcal{F}} L^*_{\Delta_n}(f), \end{aligned}$$

which leads to (3.7).

S4 Proof of Theorem 3

Based on the relation (S2.3), we have for any $\hat{f}$,

$$\begin{aligned} R_{\Delta_n}(f, \hat{f}) &\geq \sup_{x' \in \mathcal{X}} \left\{ \frac{\sup_{x \in (x'-\Delta_n) \cap \mathcal{X}} f(x) - \inf_{x \in (x'-\Delta_n) \cap \mathcal{X}} f(x)}{2} \right\} \\ &= L^*_{\Delta_n}(f), \end{aligned} \tag{S4.1}$$

where the equality follows from (3.2). In addition, the adversarial risk is always lower bounded by the standard risk, i.e.,

$$R_{\Delta_n}(f, \hat{f}) = \mathbb{E} \sup_{x \in \mathcal{X}} \sup_{\delta \in \Delta_n} \left| f(x) - \hat{f}(x+\delta) \right| \geq \mathbb{E} \sup_{x \in \mathcal{X}} \left| f(x) - \hat{f}(x) \right| = R(f, \hat{f}). \tag{S4.2}$$

Combining (S4.1) and (S4.2) yields the lower bound (3.8). The minimax lower bound (3.9) follows directly from (3.8).

S5 Proof of Example 1

To simplify the notation, for any d-dimensional multi-index $l = (l_1, l_2, \ldots, l_d) \in \mathbb{N}_0^d$, we define $|l| = l_1 + l_2 + \cdots + l_d$, and $l! = l_1! l_2! \ldots l_d!$. Derivatives and powers of order l are denoted by $D^l = \frac{\partial^{|l|}}{\partial x_1^{l_1} \partial x_2^{l_2} \ldots \partial x_d^{l_d}}$ and $x^l = x_1^{l_1} x_2^{l_2} \ldots x_d^{l_d}$, respectively.

For any function f in $\mathcal{F}_1(\beta, L)$, let

$$g_k(x;t) = \sum_{|l| \le k} \frac{D^l f(t)}{l!} (x-t)^l \tag{S5.1}$$

denote its Taylor polynomial of degree $k = \lfloor \beta \rfloor$ at point t. Using results from the approximation theory (see, e.g., DeVore and Lorentz, 1993), we know that

$$|f(x) - g_k(x;t)| \le L \sum_{|l|=k} \frac{1}{l!} |x-t|^l \cdot \|x-t\|^\alpha , \tag{S5.2}$$

where $\alpha = \beta - k$. For completeness, we provide a simplified proof for (S5.2) based on the similar technique in Lemma 11.1 of Györfi et al. (2002). When $k = 0$, we have $\beta = \alpha$, then (S5.2) follows from the assumption that f is

(β, L)-smooth. In the case $k \geq 1$, we have

$$
\begin{aligned}
&|f(x) - g_k(x;t)| \\
&= \left| f(x) - \sum_{|l|\leq k-1} \frac{D^l f(t)}{l!}(x-t)^l - \sum_{|l|=k} \frac{D^l f(t)}{l!}(x-t)^l \right| \\
&= \left| \sum_{|l|=k} \frac{k}{l!}(x-t)^l \int_0^1 (1-z)^{k-1} D^l f[t+z(x-t)]dz \right. \\
&\quad \left. - \sum_{|l|=k} \frac{k}{l!}(x-t)^l \int_0^1 (1-z)^{k-1} D^l f(t) dz \right| \\
&= \left| \sum_{|l|=k} \frac{k}{l!}(x-t)^l \int_0^1 (1-z)^{k-1} \left\{ D^l f[t+z(x-t)] - D^l f(t) \right\} dz \right| \\
&\leq L \sum_{|l|=k} \frac{1}{l!} |x-t|^l \cdot \|x-t\|^\alpha ,
\end{aligned}
$$

where the second equality follows from the integral form of the Taylor series remainder, and the last inequality follows from the definition of $\mathcal{F}_1(\beta, L)$.

We first construct an upper bound on $L^*_{\Delta_n}(f)$ for $f \in \mathcal{F}_1(\beta, L)$. Recall the definitions

$$
2L^*_{\Delta_n}(f) = \sup_{x' \in \mathcal{X}} \left\{ \sup_{x \in (x'-\Delta_n)\cap\mathcal{X}} f(x) - \inf_{x \in (x'-\Delta_n)\cap\mathcal{X}} f(x) \right\}.
$$

and $r_n \triangleq \max_{\delta,\delta' \in \Delta_n} \|\delta - \delta'\|$. In addition, define $\bar{x} = (x + x')/2$. Then we

have

$$
\begin{aligned}
2L^*_{\Delta_n}(f) &\le \sup_{\|x-x'\|\le 2r_n} |f(x)-f(x')| \\
&= \sup_{\|x-x'\|\le 2r_n} |f(x)-g_k(x;\bar{x})+g_k(x;\bar{x})-g_k(x';\bar{x})+g_k(x';\bar{x})-f(x')| \\
&\le \sup_{\|x-x'\|\le 2r_n} |f(x)-g_k(x;\bar{x})| + \sup_{\|x-x'\|\le 2r_n} |g_k(x;\bar{x})-g_k(x';\bar{x})| \\
&+ \sup_{\|x-x'\|\le 2r_n} |g_k(x';\bar{x})-f(x')| .
\end{aligned} \tag{S5.3}
$$

The first term at the right side of (S5.3) is upper bounded by

$$
\begin{aligned}
\sup_{\|x-x'\|\le 2r_n} |f(x)-g_k(x';\bar{x})| &\le \frac{L}{2^\beta} \sup_{\|x-x'\|\le 2r_n} \sum_{|l|=k} \frac{1}{l!} |x-x'|^l \cdot \|x-x'\|^\alpha \\
&\le \frac{Lr_n^\alpha}{2^k} \sup_{\|x-x'\|\le 2r_n} \sum_{|l|=k} \frac{1}{l!} |x-x'|^l \\
&= \frac{Lr_n^\alpha}{2^k k!} \sup_{\|x-x'\|\le 2r_n} \left(|x_1-x_1'|+\cdots+|x_d-x_d'|\right)^k \\
&\le \frac{Lr_n^\alpha d^{\frac{k}{2}}}{2^k k!} \left(|x_1-x_1'|^2+\cdots+|x_d-x_d'|^2\right)^{\frac{k}{2}} \\
&\le \frac{Ld^{\frac{k}{2}} r_n^\alpha (2r_n)^k}{2^k k!} \le C d^{\frac{k}{2}} r_n^\beta,
\end{aligned} \tag{S5.4}
$$

where the first inequality follows from (S5.2) and the definition of $\bar{x}$, the second inequality follows from $\|x-x'\| \le 2r_n$ and $\beta = k+\alpha$, and the third inequality follows from Jensen's inequality. The second term of (S5.3) is

upper bounded by

$$
\begin{aligned}
&\sup_{\|x-x'\|\le 2r_n} |g_k(x;\bar{x}) - g_k(x';\bar{x})| \\
&= \sup_{\|x-x'\|\le 2r_n} \left| \sum_{|l|\le k} \frac{D^l f(\bar{x})}{l!} \left[(x-\bar{x})^l - (x'-\bar{x})^l \right] \right| \\
&= \sup_{\|x-x'\|\le 2r_n} \left| \sum_{|l|\le k} \frac{D^l f(\bar{x})}{2^{|l|} l!} \left[1 + (-1)^{|l|+1} \right] (x-x')^l \right| \\
&\le C \sup_{\|x-x'\|\le 2r_n} \sum_{1\le s\le k} \sum_{|l|=s} \frac{1}{l!} |x-x'|^l \\
&\le C \sup_{\|x-x'\|\le 2r_n} \sum_{1\le s\le k} \left(|x_1 - x_1'|^2 + \cdots + |x_d - x_d'|^2 \right)^{\frac{s}{2}} \\
&\le C r_n,
\end{aligned}
\tag{S5.5}
$$

where the first equality follows from (S5.1), the second equality is due to the definition of $\bar{x}$, the first inequality follows from $D^l f(\bar{x})$ is bounded and $\frac{1}{2^{|l|}} \le 1$, and the second inequality follows the similar reasoning as in the third line of (S5.4). Based on the same technique in (S5.4), we see the third term of (S5.3) is upper bounded by

$$
\sup_{\|x-x'\|\le 2r_n} |g_k(x';\bar{x}) - f(x')| \le C d^{\frac{k}{2}} r_n^{\beta}. \tag{S5.6}
$$

Combining (S5.3) with (S5.4)–(S5.6), we have for all $f \in \mathcal{F}_1(\beta, L)$, $L^*_{\Delta_n}(f) \le C d^{\frac{k}{2}} r_n^{1\wedge\beta}$, i.e.,

$$
\sup_{f\in\mathcal{F}_1(\beta,L)} L^*_{\Delta_n}(f) \le C d^{\frac{k}{2}} r_n^{1\wedge\beta} \le C_{d,\beta} r_n^{1\wedge\beta},
$$

where $C_{d,\beta}$ is a constant depending on d and β, but independent of n.

To lower bound $\sup_{f\in\mathcal{F}_1(\beta,L)} L^*_{\Delta_n}(f)$, it suffices to construct specific functions in $\mathcal{F}_1(\beta,L)$ such that $L^*_{\Delta_n}(f) \geq Cr_n^{1\wedge\beta}$. Given that Δ_n is a closed set, let δ and δ' be two points in Δ_n such that $\|\delta-\delta'\| = r_n$. In addition, define $D_n = \{t\delta + (1-t)\delta' : 0 \leq t \leq 1\}$. Since there exists a D_n such that $D_n \subseteq \Delta_n$, hence $L^*_{\Delta_n}(f) \geq L^*_{D_n}(f)$. Without loss of generality, we assume $D_n \subseteq \{x : x_2 = x_3 = \cdots = x_d = 0\}$ and $(\delta+\delta')/2 = (1/2, 0, \ldots, 0)^\top$. Otherwise, we can construct new functions from the functions f_1 and f_2 defined below by rotations of axes and shifts of origin. Note that the rotation and transformation of a function do not change the smoothness properties of the original function. When $\beta \geq 1$, define

$$f_1(x) = L\exp(x_1 - 1), \quad x \in [0,1]^d.$$

Note that $f_1(x)$ is an infinitely differentiable function, and

$$\begin{aligned}
&\left|\frac{\partial^k f_1}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(x) - \frac{\partial^k f_1}{\partial x_1^{k_1}\cdots\partial x_d^{k_d}}(z)\right| \\
&= |L\exp(x_1-1) - L\exp(z_1-1)| \\
&\leq L\,|x_1 - z_1| \leq L\,\|x-z\| \leq L\,\|x-z\|^\alpha,
\end{aligned}$$

which verifies the conditions of $\mathcal{F}_1(\beta,L)$. Thus, $\sup_{f\in\mathcal{F}_1(\beta,L)} L^*_{\Delta_n}(f)$ is lower

bounded by

$$\sup_{f\in\mathcal{F}_1(\beta,L)} L^*_{\Delta_n}(f) \geq L^*_{D_n}(f_1) \geq C\left\{\sup_{x\in\{(1/2,0,\ldots,0)^\top - D_n\}} f_1(x) - \inf_{x\in\{(1/2,0,\ldots,0)^\top - D_n\}} f_1(x)\right\} \geq Cr_n.$$

When $0<\beta<1$, consider the function $f_2(x)=x_1^\beta$. We have

$$|f_2(x)-f_2(z)| = \left|x_1^\beta - z_1^\beta\right| \leq |x_1-z_1|^\beta \leq \|x-z\|^\beta.$$

Thus, f_2 belong the function class $\mathcal{F}_1(\beta,L)$ with $0<\beta<1$. In this case, we obtain

$$\sup_{f\in\mathcal{F}_1(\beta,L)} L^*_{\Delta_n}(f) \geq L^*_{D_n}(f_2) \geq Cr_n^\beta,$$

which completes the proof of this example.

S6 Proof of Example 2

Combining the results in Bertin (2004a,b), we can obtain

$$\inf_{\hat{f}} \sup_{f\in\mathcal{F}_2(\beta,L)} R(f,\hat{f}) \asymp \left(\frac{\log n}{n}\right)^{\frac{\bar{\beta}}{2\bar{\beta}+d}}, \tag{S6.1}$$

where $\bar{\beta} = d/(\sum_{i=1}^d 1/\beta_i)$. Therefore, it remains to determine the rate of $\sup_{f\in\mathcal{F}_2(\beta,L)} L^*_{\Delta_n}(f)$. We first construct an upper bound on $\sup_{f\in\mathcal{F}_2(\beta,L)} L^*_{\Delta_n}(f)$.

For any function f in $\mathcal{F}_2(\beta, L)$, we have

$$\begin{aligned} 2L^*_{\Delta_n}(f) &= \sup_{x' \in \mathcal{X}} \left\{ \sup_{x \in (x'-\Delta_n) \cap \mathcal{X}} f(x) - \inf_{x \in (x'-\Delta_n) \cap \mathcal{X}} f(x) \right\} \\ &\leq \sup_{x' \in \mathcal{X}} \left\{ \sup_{x,z \in x'-\Delta_n} |f(x) - f(z)| \right\} \\ &\leq \sup_{x' \in \mathcal{X}} \left\{ \sup_{x,z \in x'-\Delta_n} \left(L_1 |x_1 - z_1|^{\beta_1} + \cdots + L_d |x_d - z_d|^{\beta_d} \right) \right\} \quad \text{(S6.2)} \\ &\leq L_1 r_1^{\beta_1} + \cdots + L_d r_d^{\beta_d} \\ &\lesssim \max\{r_1^{\beta_1}, \ldots, r_d^{\beta_d}\}, \end{aligned}$$

where the third step follows from the definition of $\mathcal{F}_2(\beta, L)$. Now we derive a lower bound on $\sup_{f \in \mathcal{F}_2(\beta,L)} L^*_{\Delta_n}(f)$. We just need to construct a specific function in $\mathcal{F}_2(\beta, L)$. Define $j \triangleq \arg\max_{i \in \{1,\ldots,d\}} r_i^{\beta_i}$ and a function $f_3(x) = L_j x_j^{\beta_j}$. Obviously, we have

$$|f_3(x) - f_3(z)| = L_j \left| x_j^{\beta_j} - z_j^{\beta_j} \right| \leq L_j |x_j - z_j|^{\beta_j}.$$

Thus, we see $f_3 \in \mathcal{F}_2(\beta, L)$. And $\sup_{f \in \mathcal{F}_2(\beta,L)} L^*_{\Delta_n}(f)$ is lower bounded by

$$\sup_{f \in \mathcal{F}_2(\beta,L)} L^*_{\Delta_n}(f) \geq L^*_{\Delta_n}(f_3) \geq L_j r_j^{\beta_j} \asymp \max\{r_1^{\beta_1}, \ldots, r_d^{\beta_d}\}. \quad \text{(S6.3)}$$

Combining (S6.1)–(S6.3) with (3.10), we have

$$\inf_{\hat{f}} \sup_{f \in \mathcal{F}_2(\beta,L)} R_{\Delta_n}(f, \hat{f}) \asymp \left(\frac{\log n}{n} \right)^{\frac{\bar{\beta}}{2\bar{\beta}+d}} + \max\{r_1^{\beta_1}, \ldots, r_d^{\beta_d}\},$$

which proves the result in Example 2.

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