General Sliced Factorial Designs for Online Experiments with Multiple Platforms

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Supplementary Materials

S1 Optimal $(sN, 2^n s)$-designs among $E(d_0, \mathcal{P})$

We provide the optimal $(sN, 2^n s)$-designs among $E(d_0, \mathcal{P})$ with $N = 8, 16$ runs and the corresponding SGWLP=$(A_{3,0}, \ldots, A_{n+1,1})$. The initial design d_0 in Tables 1 and 2 are chosen to be the 2^{n-m} design given by Wu and Hamada (2021) and Mukerjee and Wu (2006), respectively. According to (4.13) in Algorithm 1, it suffices give $(\boldsymbol{p}^{opt}_{\boldsymbol{u}_1}, \ldots, \boldsymbol{p}^{opt}_{\boldsymbol{u}_m})$, denoted as $\tilde{P}$ in the tables. Then, the corresponding optimal $(sN, 2^n s)$ SFDs can be found by (4.6). Here $\boldsymbol{\omega}_*$'s are s-tuple binary vectors, where $\boldsymbol{\omega}_0 = (0, \ldots, 0)$, $\boldsymbol{\omega}_i$ as an s-tuple binary vector with $(s - i + 1)$th entry 1 and the others being 0, and

$\boldsymbol{\omega}_{1,2}$ as the elementwise sum $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ module 2, etc. We give an example to illustrate how to use the tables. Consider the case of $(5 \cdot 8, 2^7 5)$-design, in which the initial design d_0 is the 2^{7-4} MA design given by Mukerjee and Wu (2006) with four independent defining pencils $\boldsymbol{u}_1 = (1,1,0,1,0,0,0)$, $\boldsymbol{u}_2 = (1,0,1,0,1,0,0)$, $\boldsymbol{u}_3 = (0,1,1,0,0,1,0)$, $\boldsymbol{u}_4 = (1,1,1,0,0,0,1)$. Then, from (4.11), W is the identity matrix of order 4. Table 1 shows that

$$\tilde{P} = (\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,2,3,4}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, use (4.13) to obtain the optimal switch matrix

$$P = (\mathbf{0}_{5\times 3},\ \tilde{P}W^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the optimal $(5 \cdot 8, 2^7 5)$-design among $E(d_0, \mathcal{P})$ can be calculated by (4.6).

Table 1: Catalog of optimal SFDs among $E(d_0, \mathcal{P})$ with $N = 8$

d_0	s	$\tilde{P}$	SGWLP
2^{4-1*}	2	$(\boldsymbol{\omega}_1)$	$(0^3,1)$
	3	$(\boldsymbol{\omega}_1)$	$(0^2,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,2})$	$(0^3,1)$
	5	$(\boldsymbol{\omega}_{1,2})$	$(0^2,0.04,0.96)$
	6	$(\boldsymbol{\omega}_{1,2,3})$	$(0^3,1)$
	7	$(\boldsymbol{\omega}_{1,2,3})$	$(0^2,0.02,0.98)$
2^{5-2*}	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,2,1,0^3)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	$(0.22,1.78,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2})$	$(0,2,0,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2})$	$(0.08,1.92,0.04,0.96,0^2)$
	6	$(\boldsymbol{\omega}_{1,2,4}, \boldsymbol{\omega}_{1,2,3})$	$(0,2,0.11,0.89,0^2)$
	7	$(\boldsymbol{\omega}_{1,4,5}, \boldsymbol{\omega}_{1,2,3})$	$(0.04,1.96,0.02,0.98,0^2)$
2^{6-3*}	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,4,3,0^5)$
	3	$(\boldsymbol{\omega}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0.44,3.56,1.22,1.78,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,2})$	$(0,4,1,2,0^4)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2})$	$(0.16,3.84,0.44,2.56,0^4)$
	6	$(\boldsymbol{\omega}_{1,3,5}, \boldsymbol{\omega}_{1,2,4}, \boldsymbol{\omega}_{1,2,3})$	$(0,4,0.33,2.67,0^4)$
2^{7-4*}	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(3,4,3,4,0^4,1,0)$
	3	$(\boldsymbol{\omega}_2, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$	$(0.78,6.22,3.44,3.56,0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2})$	$(0,7,3,4,0^5,1)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,2,3,4})$	$(0.28,6.72,1.88,5.12,0^4,0.04,0.96)$

* The corresponding d_0 is from Table 5A.1 in Wu and Hamada (2021).

Table 2: Catalog of optimal SFD among $E(d_0, \mathcal{P})$ with $N = 16$

d_0	s	$\tilde{P}$	SGWLP(D)
5-1.1$^\diamond$	2	$(\boldsymbol{\omega}_1)$	$(0^5,1)$
	3	$(\boldsymbol{\omega}_1)$	$(0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,2})$	$(0^5,1)$
	5	$(\boldsymbol{\omega}_{1,2})$	$(0^4,0.04,0.96)$
	6	$(\boldsymbol{\omega}_{1,2,3})$	$(0^5,1)$
	7	$(\boldsymbol{\omega}_{1,2,3})$	$(0^4,0.02,0.98)$
5-1.2$^\diamond$	2	$(\boldsymbol{\omega}_1)$	$(0^3,1,0^2)$
	3	$(\boldsymbol{\omega}_1)$	$(0^2,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,2})$	$(0^3,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,2})$	$(0^2,0.04,0.96,0^2)$

d_0	s	$\tilde{P}$	SGWLP(D)
	6	$(\boldsymbol{\omega}_{1,2,3})$	$(0^3,1,0^2)$
	7	$(\boldsymbol{\omega}_{1,2,3})$	$(0^2,0.02,0.98,0^2)$
5-1.3$^\diamond$	2	$(\boldsymbol{\omega}_{1})$	$(0,1,0^4)$
	3	$(\boldsymbol{\omega}_{1})$	$(0.11,0.89,0^4)$
	4	$(\boldsymbol{\omega}_{1,2})$	$(0,1,0^4)$
	5	$(\boldsymbol{\omega}_{1,2})$	$(0.04,0.96,0^4)$
	6	$(\boldsymbol{\omega}_{1,2,3})$	$(0,1,0^4)$
	7	$(\boldsymbol{\omega}_{1,2,3})$	$(0.02,0.98,0^4)$
6-2.1$^\diamond$	2	$(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(0^2,1,2,0^4)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	$(0^2,0.33,2.67,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0^3,3,0^4)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0^2,0.12,2.88,0^4)$

d_0	s	$\tilde{P}$	SGWLP(D)
	6	$(\boldsymbol{\omega}_{2,3,5}, \boldsymbol{\omega}_{4,5})$	$(0^2,0.11,2.89,0^4)$
	7	$(\boldsymbol{\omega}_{2,3,6}, \boldsymbol{\omega}_{4,5,6})$	$(0^2,0.06,2.94,0^4)$
6-2.2$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,1,0,1,1,0^3)$
	3	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.11,0.89,0.11,0.89,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,1,0,1,0,1,0^2)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.04,0.96,0.04,0.96,0.04,0.96,0^2)$
	6	$(\boldsymbol{\omega}_{2,4,5}, \boldsymbol{\omega}_{3,4,5})$	$(0,1,0,1,0.11,0.89,0^2)$
	7	$(\boldsymbol{\omega}_{2,3,6}, \boldsymbol{\omega}_{4,5,6})$	$(0.02,0.98,0.02,0.98,0.02,0.98,0^2)$
6-2.3$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,2,0^4,1,0)$
	3	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.22,1.78,0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,2,0^5,1)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.08,1.92,0^4,0.04,0.96)$

d_0	s	$\tilde{P}$	SGWLP(D)
	6	$(\boldsymbol{\omega}_{2,4,5}, \boldsymbol{\omega}_{3,4,5})$	$(0,2,0^4,0.11,0.89)$
	7	$(\boldsymbol{\omega}_{2,3,6}, \boldsymbol{\omega}_{4,5,6})$	$(0.04,1.96,0^4,0.02,0.98)$
6-2.4$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,2,1,0^5)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	$(0.22,1.78,0.11,0.89,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,2,0,1,0^4)$
	5	$(\boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.08,1.92,0.04,0.96,0^4)$
	6	$(\boldsymbol{\omega}_{2,4,5}, \boldsymbol{\omega}_{3,4,5})$	$(0,2,0.11,0.89,0^4)$
	7	$(\boldsymbol{\omega}_{2,3,6}, \boldsymbol{\omega}_{4,5,6})$	$(0.04,1.96,0.02,0.98,0^4)$
7-3.1$^\diamond$	2	$(\boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(0^2,3,4,0^6)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_2)$	$(0^2,1.67,5.33,0^6)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3})$	$(0^2,1,6,0^6)$
	5	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,2,3,4})$	$(0^2,0.6,6.4,0^6)$

d_0	s	$\tilde{P}$	SGWLP(D)
	6	$(\boldsymbol{\omega}_{1,3,5},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{4,5})$	$(0^2,0.33,6.67,0^6)$
7-3.2$^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(0,2,1,2,2,0^5)$
	3	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2},\boldsymbol{\omega}_{2})$	$(0.22,1.78,0.33,2.67,1.11,0.89,0^4)$
	4	$(\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3})$	$(0,2,0,3,1,1,0^4)$
	5	$(\boldsymbol{\omega}_{1,3,4},\boldsymbol{\omega}_{2,4},\boldsymbol{\omega}_{3,4})$	$(0.08,1.92,0.12,2.88,0.4,1.6,0^4)$
	6	$(\boldsymbol{\omega}_{1,3,5},\boldsymbol{\omega}_{2,4,5},\boldsymbol{\omega}_{3,4,5})$	$(0,2,0.11,2.89,0.22,1.78,0^4)$
7-3.3$^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(0,3,2,0^2,1,1,0^3)$
	3	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2},\boldsymbol{\omega}_{2})$	$(0.33,2.67,0.22,1.78,0.11,0.89,1,0^3)$
	4	$(\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3})$	$(0,3,0,2,0,1,1,0^3)$
	5	$(\boldsymbol{\omega}_{1,4},\boldsymbol{\omega}_{2,3,4},\boldsymbol{\omega}_{3,4})$	$(0.12,2.88,0.08,1.92,0.04,0.96,0.36,0.64,0^2)$
	6	$(\boldsymbol{\omega}_{1,3,5},\boldsymbol{\omega}_{2,4,5},\boldsymbol{\omega}_{3,4,5})$	$(0,3,0.22,1.78,0,1,0.11,0.89,0^2)$
7-3.4$^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(0,3,3,0^6,1)$

d_0	s	$\tilde{P}$	SGWLP(D)
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	(0.33,2.67,0.33,2.67,0^4,1,0)
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	(0,3,0,3,0^4,1,0)
	5	$(\boldsymbol{\omega}_{1,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	(0.12,2.88,0.12,2.88,0^4,0.36,0.64)
	6	$(\boldsymbol{\omega}_{1,3,5}, \boldsymbol{\omega}_{2,4,5}, \boldsymbol{\omega}_{3,4,5})$	(0,3,0.33,2.67,0^5,1)
7-3.5$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	(0,4,3,0^7)
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	(0.44,3.56,1.22,1.78,0^6)
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	(0,4,1,2,0^6)
	5	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	(0.16,3.84,0.44,2.56,0^6)
	6	$(\boldsymbol{\omega}_{1,3,5}, \boldsymbol{\omega}_{2,4,5}, \boldsymbol{\omega}_{3,4,5})$	(0,4,0.33,2.67,0^6)
8-4.1$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	(0^2,6,8,0^6,1,0)
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_2)$	(0^2,3.33,10.67,0^6,1,0)
	4	$(\boldsymbol{\omega}_{1,2,3}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$	(0^2,2,12,0^6,1,0)

d_0	s	$\tilde{P}$	SGWLP(D)
	5	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0^2,1.2,12.8,0^6,1,0)$
8-4.2$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,3,3,4,4,0^4,1,0^2)$
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2}, \boldsymbol{\omega}_{2})$	$(0.33,2.67,1.67,5.33,1.33,2.67,0^2,1,0^3)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{3})$	$(0,3,3,4,0,4,0^3,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{4})$	$(0.12,2.88,0.6,6.4,0.48,3.52,0^2,1,0^3)$
8-4.3$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,4,5,0^2,4,2,0^5)$
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.44,3.56,0.55,4.45,2.22,1.78,1.11,0.89,0^4)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,4,0,5,2,2,1,1,0^4)$
	5	$(\boldsymbol{\omega}_{1,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.16,3.84,0.2,4.8,0.8,3.2,1.04,0.96,0^4)$
8-4.4$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,4,6,0^2,4,0^4,1,0)$
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.44,3.56,1.56,4.44,2.22,1.78,0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,4,1,5,2,2,0^5,1)$

d_0	s	$\tilde{P}$	SGWLP(D)
	5	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{3,4}, \boldsymbol{\omega}_{1,2,3,4})$	$(0.16,3.84,0.52,5.48,1.44,2.56,0^4,0.04,0.96)$
8-4.5$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,5,5,0^2,2,2,0^2,1,0^2)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	$(0.56,4.44,1.44,3.56,0.22,1.78,2,0,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	$(0,5,1,4,0,2,2,0^2,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4}, \boldsymbol{\omega}_{3,4})$	$(0.2,4.8,0.52,4.48,0.08,1.92,1.36,0.64,0.04,0.96,0^2)$
8-4.6$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(3,4,3,4,0^4,1,0^3)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	$(0.78,6.22,3.44,3.56,0^4,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,7,3,4,0^5,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,2,3,4})$	$(0.28,6.72,1.88,5.12,0^4,0.04,0.96,0^2)$
9-5.1$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(0,4,14,0^2,8,0^3,4,1,0^3)$
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	$(0.44,3.56,3.33,10.67,4.44,3.56,0^2,0.44,3.56,1,0^3)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,4,2,12,4,4,0^3,4,1,0^3)$

d_0	s	$\tilde{P}$	SGWLP(D)
	5	$(\boldsymbol{\omega}_{1,3,4},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{4})$	$(0.16,3.84,1.2,12.8,2.88,5.12,0^2,0.16,3.84,1,0^3)$
$9\text{-}5.2^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(0,6,9,0^2,9,6,0^6,1)$
	3	$(\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2},\boldsymbol{\omega}_{2})$	$(0.67,5.33,1.89,7.11,4.56,4.44,2.44,3.56,0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3})$	$(0,6,1,8,4,5,2,4,0^5,1)$
	5	$(\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{3,4},\boldsymbol{\omega}_{4})$	$(0.24,5.76,0.68,8.32,2.6,6.4,1.52,4.48,0^4,0.36,0.64)$
$9\text{-}5.3^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(1,5,5,5,6,2,2,2,1,1,0,1,0^2)$
	3	$(\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2},\boldsymbol{\omega}_{2})$	$(0.67,5.33,2,8,4.44,3.56,2.22,1.78,0.22,1.78,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3})$	$(0,6,1,9,4,4,2,2,0,2,0,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,4},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,4},\boldsymbol{\omega}_{3,4},\boldsymbol{\omega}_{3,4})$	$(0.24,5.76,0.72,9.28,2.88,5.12,1.44,2.56,0.08,1.92,0.04,0.96,0^2)$
$9\text{-}5.4^\diamond$	2	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1})$	$(0,7,9,0^2,6,6,0^2,3,0^4)$
	3	$(\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2},\boldsymbol{\omega}_{2})$	$(0.78,6.22,2.78,6.22,3.33,2.67,1.56,4.44,1.22,1.78,0^4)$
	4	$(\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{1,2},\boldsymbol{\omega}_{1,3},\boldsymbol{\omega}_{2,3},\boldsymbol{\omega}_{2,3})$	$(0,7,2,7,3,3,1,5,1,2,0^4)$

d_0	s	$\tilde{P}$	SGWLP(D)
	5	$(\boldsymbol{\omega}_{1,3,4}, \boldsymbol{\omega}_{1,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.28,6.72,1,8,1.2,4.8,2.48,3.52,0.44,2.56,0^4)$
9-5.5$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{0}, \boldsymbol{\omega}_{1})$	$(0,8,10,0^2,4,4,0^2,4,1,0^3)$
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.89,7.11,3.78,6.22,0.44,3.56,4,0,0.44,3.56,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,8,3,7,0,4,4,0^2,4,0,1,0^2)$
	5	$(\boldsymbol{\omega}_{1,4}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{3,4}, \boldsymbol{\omega}_{2,4}, \boldsymbol{\omega}_{3,4})$	$(0.32,7.68,2,8,0.16,3.84,2.72,1.28,0.16,3.84,0.04,0.96,0^2)$
10-6.1$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{0}, \boldsymbol{\omega}_{1})$	$(0,8,18,0,0,16,8,0,0,8,5,0^5)$
	3	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2})$	$(0.89,7.11,3.78,14.22,8.89,7.11,4.44,3.56,0.89,7.11,1.44,3.56,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	$(0,8,2,16,8,8,4,4,0,8,1,4,0^4)$
10-6.2$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(2,7,9,7,9,6,6,6,4,3,0,3,1,0^3)$
	3	$(\boldsymbol{\omega}_{2}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{2}, \boldsymbol{\omega}_{2})$	$(1,8,4.44,11.56,7,8,4.89,7.11,2.56,4.44,0.33,2.67,0.11,0.89,0^2)$
	4	$(\boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	$(0,9,3,13,6,9,4,8,2,5,0,3,0,1,0^2)$
10-6.3$^\diamond$	2	$(\boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1}, \boldsymbol{\omega}_{1})$	$(0,10,15,0^2,12,15,0^2,10,0^4,1,0)$

d_0	s	$\tilde{P}$	SGWLP(D)
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	$(1.11,8.89,6.11,8.89,1.33,10.67,10.56,4.44,1.11,8.89,0^4,0.11,0.89)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	$(0,10,5,10,0,12,10,5,0,10,0^5,1)$
10-6.4$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,10,16,0^2,12,12,0^2,10,3,0^5)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	$(1.11,8.89,5.33,10.67,6.67,5.33,3.11,8.89,2.89,7.11,1.22,1.78,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	$(0,10,4,12,6,6,2,10,2,8,1,2,0^4)$
11-7.1$^\diamond$	2	$(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(3,9,16,20,13,15,12,12,13,7,3,10,3,1,0^4)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	$(1.33,10.67,9.11,16.89,11.11,16.89,8,16,7.56,12.44,3.22,9.78,1.33,2.67,0^4)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_0, \boldsymbol{\omega}_{2,3})$	$(0,12,7,19,9,19,6,18,6,14,2,11,1,3,0^4)$
11-7.2$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1)$	$(0,13,25,0^2,25,27,0^2,23,10,0^2,3,1,0^3)$
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2)$	$(1.44,11.56,9,16,9.89,15.11,9.22,17.78,9.67,13.33,1.11,8.89,0.33,2.67,1,0^3)$
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3})$	$(0,13,7,18,8,17,7,20,8,15,0,10,0,3,1,0^3)$
11-7.3$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	$(3,10,16,10,12,12,12,12,16,10,3,10,0^4,1,0)$

d_0	s	$\tilde{P}$	SGWLP(D)
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2, \boldsymbol{\omega}_0, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$	(1.44,11.56,8.22,17.78,13.33,10.67,6.22,17.78,6.44,19.56,5.89,7.11,0^4,0.11,0.89)
	4	$(\boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_{2,3}, \boldsymbol{\omega}_0, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_{1,3}, \boldsymbol{\omega}_{2,3})$	(0,13,6,20,12,12,4,20,4,22,5,8,0^5,1)
12-8.1$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	(0,16,39,0^2,48,48,0^2,48,39,0^2,16,0^4,1,0)
	3	$(\boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2, \boldsymbol{\omega}_0, \boldsymbol{\omega}_0, \boldsymbol{\omega}_2)$	(1.78,14.22,15,24,17.78,30.22,16,32,16,32,10.56,28.44,7.11,8.89,0^4,0.11,0.89)
12-8.2$^\diamond$	2	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1)$	(0,17,38,0^2,44,52,0^2,54,33,0^2,12,4,0^2,1,0^2)
	3	$(\boldsymbol{\omega}_1, \boldsymbol{\omega}_1, \boldsymbol{\omega}_{1,2}, \boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_0, \boldsymbol{\omega}_2)$	(1.89,15.11,14.89,23.11,16.44,27.56,16.44,35.56,19.33,34.67, 9.89,23.11,4,8,0.44,3.56,1,0^3)

$^\diamond$ The corresponding d_0 has indicated in Table 3A.2 in Mukerjee and Wu (2006).

S2 Proof of Theorem 1

Equation (4.8) is straightforward.

To prove (4.9), write

$$\boldsymbol{y}_{\boldsymbol{u}} = (J_{\boldsymbol{u}}(\boldsymbol{p}_0), \ldots, J_{\boldsymbol{u}}(\boldsymbol{p}_{s-1}))^T$$

and

$$\boldsymbol{e}_v = (\chi_v^{(s)}(0), \ldots, \chi_v^{(s)}(s-1))^T,\ v = 0, \ldots, s-1. \tag{S2.1}$$

Clearly, $\boldsymbol{e}_0$ is the vector of ones, and $\langle \boldsymbol{e}_{v_1}, \boldsymbol{e}_{v_2} \rangle = s\delta_{v_1,v_2}$, where δ_{v_1,v_2} is the Kronecker delta. $\{\boldsymbol{e}_0, \ldots, \boldsymbol{e}_{s-1}\}$ is a set of s orthogonal vectors in the s-dimensional space. Therefore,

$$\begin{aligned}\sum_{v=1}^{s-1}\left(\sum_{i=0}^{s-1} J_{\boldsymbol{u}}(\boldsymbol{p}_i)\chi_v^{(s)}(i)\right)^2 &= \sum_{v=1}^{s-1}\langle \boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{e}_v\rangle^2 \\ &= s\langle \boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{y}_{\boldsymbol{u}}\rangle - \langle \boldsymbol{y}_{\boldsymbol{u}}, \boldsymbol{e}_0\rangle^2 \\ &= \sum_{i=0}^{s-1} sJ_{\boldsymbol{u}}(\boldsymbol{p}_i)^2 - J_{\boldsymbol{u}}(P)^2.\end{aligned}$$

As $J_{\boldsymbol{u}}(\boldsymbol{p}_i) = \pm 1$, obtain

$$A_{j+1,1}(D) = \frac{1}{(sN)^2}\sum_{wt(\boldsymbol{u})=j} J_{\boldsymbol{u}}(d_0)^2\big(s^2 - J_{\boldsymbol{u}}(P)^2\big) = A_j(d_0) - A_{j,0}(D).$$

S3 Proof of Proposition 1

Since $\{\boldsymbol{p}_{\boldsymbol{u}} = P\boldsymbol{u}^T : \boldsymbol{u} \in \mathcal{G}\}$ are uniquely determined by $\{\boldsymbol{p}_{\boldsymbol{u}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{u}_m}\}$. Thus $J_{\boldsymbol{u}}(P)$ is fully determined by $\{\boldsymbol{p}_{\boldsymbol{u}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{u}_m}\}$. There are $2^{m(s-1)}$ different $\boldsymbol{p}_{\boldsymbol{u}_1}, \ldots, \boldsymbol{p}_{\boldsymbol{u}_m}$, resulting in $2^{m(s-1)}$ distinct P. After excluding the P that produce equivalent SFDs through row permutations, there are at most $\binom{2^m+s-2}{s-1}$ different sequences $(B_3, \ldots, B_n)$ remaining.

Reference

Mukerjee, R. and C. F. J. Wu (2006). *"A modern theory of factorial design"*. Springer Science & Business Media, New York.

Wu, C. F. J. and M. S. Hamada (2021). *"Experiments: planning, analysis, and optimization"*. John Wiley & Sons, Hoboken.