Supplementary Material for "Distributed inference for tail risks"

Liujun Chen[1], Deyuan Li[2], and Chen Zhou[3]

[1] *University of Science and Technology of China*

[2] *Fudan University and* [3] *Erasmus University Rotterdam*

The Supplementary Material contains all the technical proofs and simulation studies.

S1 Proofs

S1.1 Tail approximation to the distribution function

In this subsection, we give a lemma regarding the tail approximation to the distribution function F.

Lemma S1. *Assume F satisfies the second order condition* (2.2) *with $\gamma \in \mathbb{R}$ and $\rho < 0$. Then,*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{D}} \left| \frac{(1+\gamma x)^{-1/\gamma}}{t(1-F(b_0(t)+xa_0(t)))} - 1 \right| = O(A(t)).$$

Remark 1. A similar, but somewhat different result has been shown in Drees et al. (2006), see also Theorem 5.1.1 in de Haan and Ferreira (2006). The 'supremum' is taken over $\mathbb{D}^* = \left\{x : (1+\gamma x)^{-1/\gamma} \leq ct^{-\delta+1}\right\}$ with $\delta > 0$ in Drees et al. (2006) and it is obvious that $\mathbb{D} \subset \mathbb{D}^*$. However, when the ‘supremum’ is taken over $\mathbb{D}^*$, the upper bound in Lemma S1 is $o(1)$ instead of a precise speed, $O(A(t))$.

Proof of Lemma S1. We define

$$y := \frac{1}{t\left\{1 - F(b_0(t) + xa_0(t))\right\}},$$

which implies that,

$$x = \frac{U(ty) - b_0(t)}{a_0(t)}.$$

First, note that for sufficiently large t, $x \in \mathbb{D}$ for some x_0, if and only if $y \geq (1+\gamma x_0)^{1/\gamma}$ for some, possibly different, x_0. So, the supremum of $x \in \mathbb{D}$ can be replaced by the supremum of $y \geq c$ for any $c > 0$.

This leads to the following expansion, where the notation $g(y) := (1 + \gamma y)^{-1/\gamma}$ and $q_t(y) := \left(U(ty) - b_0(t)\right)/a_0(t) - \left(y^\gamma - 1\right)/\gamma$ is used:

$$\begin{aligned}
&t(1 - F(b_0(t) + xa_0(t))) - (1 + \gamma x)^{-1/\gamma} \\
&\quad = -\left(1 + \gamma \frac{U(ty) - b_0(t)}{a_0(t)}\right)^{-1/\gamma} + \left(1 + \gamma \frac{y^\gamma - 1}{\gamma}\right)^{-1/\gamma} \\
&\quad = -g\left(\frac{U(ty) - b_0(t)}{a_0(t)}\right) + g\left(\frac{y^\gamma - 1}{\gamma}\right) \\
&\quad = q_t(y)\left(-g'\left(\frac{y^\gamma - 1}{\gamma}\right)\right) - \int_0^{q_t(y)} \int_0^s g''\left(\frac{y^\gamma - 1}{\gamma} + u\right) duds \\
&\quad = y^{-\gamma-1} q_t(y) - (1+\gamma) \int_0^{q_t(y)} \int_0^s \left(1 + \gamma\left(\frac{y^\gamma - 1}{\gamma} + u\right)\right)^{-1/\gamma - 2} duds.
\end{aligned}$$

The integrand function $(1 + \gamma((y^\gamma - 1)/\gamma + u))^{-1/\gamma-2}$ always lies between its value for $u = 0$ and $u = q_t(y)$, i.e., between $y^{-1-2\gamma}$ and $y^{-1-2\gamma}(1 + \gamma y^{-\gamma} q_t(y))^{-1/\gamma-2}$.

By (2.3) and recall that $\rho < 0$, we have

$$\lim_{t\to\infty} \sup_{y\geq c} y^{-\gamma} \left|q_t(y)\right| = O(A_0(t)). \tag{S1.1}$$

Hence, we have

$$\left(1+\gamma\left(\frac{y^\gamma - 1}{\gamma} + u\right)\right)^{-1/\gamma-2} \leq 2y^{-1-2\gamma}$$

for $y \geq c$ and t sufficiently large, which leads to

$$\left|t(1-F(b_0(t)+xa_0(t))) - (1+\gamma x)^{-1/\gamma} - y^{-1-\gamma}q_t(y)\right| \leq 2\left|1+\gamma\right| y^{-1-2\gamma}q_t(y)^2.$$

Combining the inequality above with (S1.1), we have

$$\lim_{t\to\infty} \sup_{y\geq c} \left|y(1+\gamma x)^{-1/\gamma} - 1\right| = O(A_0(t)).$$

□

S1.2 Proofs for Section 3

We start with proving Proposition 1. Without loss of generality, we will skip the superscript j and consider the approximation in a specific machine. We start with considering the 'simple' case where F is the standard uniform distribution.

Assume $U_1, \ldots, U_n$ are i.i.d. uniform distributed random variables. We define the uniform empirical process as

$$e_n(s) := n^{1/2}\left\{U_n(s) - s\right\}, \quad s \in [0,1],$$

where

$$U_n(s) := \frac{1}{n}\sum_{i=1}^{n} I_{\{U_i \le s\}}.$$

First, we give a weighted approximation to the uniform empirical process $e_n(s)$.

Proposition S1. *For any $v \in (0,1/2)$ and sufficiently large n, under proper Skorokhod construction, there exist a sequence of Brownian bridges $\{B_n, n \ge 1\}$ and a constant $C_2 = C_2(v) > 0$ such that for all $r > 0$,*

$$P\left(\sup_{0<s<1} \frac{|e_n(s) - B_n(s)|}{\{s(1-s)\}^{1/2-v}} > n^{-v} r \log r\right) \le C_2 r^{-\frac{1}{1/2-v}}.$$

Proof of Proposition S1. Note that, given any $r_0 > 0$, for $0 < r \le r_0$, by setting $C_2 = r_0^{\frac{1}{1/2-v}}$ such that $C_2 r^{\frac{1}{1/2-v}} \ge 1$, the statement of Proposition S1 follows. Therefore, we only need to handle the case $r > r_0$, for instance, we assume that $r > 8$. By Theorem 3.1.2 in Mason (2001), for any $v \in (0,1/2)$ and sufficiently large n, under proper Skorokhod construction, there exist constants $C_{2,1} > 0, C_{2,2} > 0$ and a sequence of Brownian bridges $\{B_n, n \ge 1\}$ such that for all $r > 0$,

$$P\left(\sup_{1/n \le s \le 1-1/n} \frac{|e_n(s) - B_n(s)|}{\{s(1-s)\}^{1/2-v}} > n^{-v} r\right) \le C_{2,1} \exp\left(-C_{2,2} r\right).$$

Therefore, the statement of Proposition S1 holds for $1/n \le s \le 1-1/n$. By symmetry, we only handle $s \in [0, 1/n]$ here.

Note that $(1-s)^{v-1/2} \to 1$ uniformly for all $0 < s \leq 1/n$, it suffices to show that for any $v \in (0, 1/2)$ and sufficiently large n, there exists a constant $C_{2,3} > 0$ such that for all $r > 8$,

$$P\left(\sup_{0<s\leq 1/n} \frac{|e_n(s) - B_n(s)|}{s^{1/2-v}} > n^{-v} r \log r\right) \leq C_{2,3} r^{-\frac{1}{1/2-v}}. \tag{S1.2}$$

Denote

$$\delta_0 := \sup_{0<s\leq 1/n} \frac{|e_n(s) - B_n(s)|}{s^{1/2-v}} \leq \sup_{0<s\leq 1/n} \frac{|e_n(s)|}{s^{1/2-v}} + \sup_{0<s\leq 1/n} \frac{|B_n(s)|}{s^{1/2-v}} =: \delta_1 + \delta_2.$$

Then, we have that $P(\delta_0 > n^{-v} r \log r) \leq P(\delta_1 > \frac{1}{2} n^{-v} r \log r) + P(\delta_2 > \frac{1}{2} n^{-v} r \log r)$.

First, we handle δ_2. Write $B_n(s) = W_n(s) - sW_n(1)$, where $\{W_n : n \geq 1\}$ is a sequence of Brownian motions. Then $\delta_2 \leq \sup_{0<s\leq 1/n} |W_n(s)|/s^{1/2-v} + \sup_{0<s\leq 1/n} |sW_n(1)|/s^{1/2-v}$. It follows that

$$\begin{aligned} P\left(\delta_2 > \frac{1}{2} n^{-v} r \log r\right) \leq & P\left(\sup_{0<s\leq 1/n} |W_n(s)|/s^{1/2-v} > \frac{1}{4} n^{-v} r \log r\right) \\ & + P\left(\sup_{0<s\leq 1/n} |sW_n(1)|/s^{1/2-v} > \frac{1}{4} n^{-v} r \log r\right). \end{aligned}$$

Since $W_n(1) \sim N(0,1)$, there exist constants $C_{2,4} > 0, C_{2,5} > 0$ such that

$$\begin{aligned} P\left\{\sup_{0<s\leq 1/n} \frac{s\,|W_n(1)|}{s^{1/2-v}} > \frac{1}{4} n^{-v} r \log r\right\} &= P\left(|W_n(1)| \geq \frac{1}{4} n^{1/2} r \log r\right) \\ &\leq C_{2,4} \exp(-C_{2,5} r^2 \log^2 r) \\ &\leq r^{-\frac{1}{1/2-v}}, \end{aligned}$$

for sufficiently large r. By replacing s with s/n and using $\sqrt{n}\left\{W_n\left(s/n\right)\right\} \stackrel{d}{=} \left\{\widetilde{W}_n(s)\right\}$ where $\left\{\widetilde{W}_n(s) : n \geq 1\right\}$ is also a sequence of Brownian motions, we get that there exist constants $C_{2,6} > 0, C_{2,7} > 0$ such that

$$P\left\{\sup_{0<s\leq 1/n} \frac{|W_n(s)|}{s^{1/2-v}} > \frac{1}{4}n^{-v} r\log r\right\} = P\left\{\sup_{0<s\leq 1} \frac{\left|\widetilde{W}_n(s)\right|}{s^{1/2-v}} > \frac{1}{4} r\log r\right\}$$

$$\leq C_{2,6}\exp(-C_{2,7}r^2\log^2 r)$$

$$\leq r^{-\frac{1}{1/2-v}},$$

for sufficently large r, where the first inequality follows by Lemma 4.2.1 in Csörgö and Horváth (1993). Combining the two parts, we obtain that, there exists a constant $C_{2,8}$ such that for all $r > 0$,

$$P\left(\delta_2 > \frac{1}{2}n^{-v} r\log r\right) \leq C_{2,8} r^{-\frac{1}{1/2-v}}.$$

Next, we handle δ_1. Note that for $r > 8$,

$$P\left(\delta_1 > \frac{1}{2}n^{-v} r\log r\right) = P\left(\delta_1 > \frac{1}{2}n^{-v} r\log r,\ U_{1,n} \leq 1/n\right),$$

since for $U_{1,n} > 1/n$, $\delta_1 = \sup_{0<s\leq 1/n} \sqrt{n}s^{1/2+v} = n^{-v} < \frac{1}{2}n^{-v} r\log r$. On the set $\{U_{1,n} \leq 1/n\}$, we have that

$$\delta_1 \leq \sup_{0<s<U_{1,n}} \sqrt{n}s^{1/2+v} + \sup_{U_{1,n}\leq s\leq 1/n} \frac{|\sqrt{n}(U_n(t)-t)|}{s^{1/2-v}}$$

$$\leq \sqrt{n}(U_{1,n})^{1/2+v} + \sup_{U_{1,n}\leq s\leq 1/n} \frac{\sqrt{n}U_n(s)}{s^{1/2-v}} + \sup_{U_{1,n}\leq s\leq 1/n} \sqrt{n}s^{1/2+v}$$

$$\leq 2n^{-v} + n^{-v} nU_n(1/n)(nU_{1,n})^{v-1/2}.$$

It follows that

$$
\begin{aligned}
&P\left(\delta_1 > \frac{1}{2}n^{-v} r\log r,\ U_{1,n} \le 1/n\right)\\
&\le P\left(2n^{-v} + n^{-v} nU_n(1/n)(nU_{1,n})^{v-1/2} > \frac{1}{2}n^{-v} r\log r\right)\\
&\le P\left(nU_n(1/n)(nU_{1,n})^{v-1/2} > \frac{1}{4} r\log r\right)\\
&\le P\left\{nU_n(1/n) > \frac{2}{1/2-v}\log r\right\} + P\left\{(nU_{1,n})^{v-1/2} > \frac{1/2-v}{2}\frac{1}{4}r\right\}\\
&= P\left\{\sum_{i=1}^n I_{\left\{U_i \le \frac{1}{n}\right\}} > \frac{4}{1-2v}\log r\right\} + P\left\{U_{1,n} < \left(\frac{16}{1-2v}\right)^{\frac{1}{1/2-v}} \frac{r^{-\frac{1}{1/2-v}}}{n}\right\}\\
&\le 2\exp\left\{-\left(\frac{2}{1-2v}\log r - \frac{1}{2}\right) + \frac{1}{4}\right\} + 1 - \left(1 - \left(\frac{16}{1-2v}\right)^{\frac{1}{1/2-v}} \frac{r^{-\frac{1}{1/2-v}}}{n}\right)^n,
\end{aligned}
$$

where the last inequality follows by applying Lemma S2 with $x = \frac{4}{1-2v}\log r - 1$ and $t = 1/2$. Thus, there exists a constant $C_{2,9} > 0$ such that

$$P\left(\delta_1 > \frac{1}{2}n^{-v} r\log r\right) \le C_{2,9} r^{-\frac{1}{1/2-v}}.$$

By combining the result for δ_1 and δ_2, the proposition is proved. □

Next, we give a weighted approximation to the tail empirical process of the i.i.d. uniform random variables, which is defined as

$$w_n(s) = \sqrt{k}\left\{\frac{n}{k}U_n\left(\frac{ks}{n}\right) - s\right\}, \quad s > 0.$$

Proposition S2. *For any $0 < v < 1/2$ and $c > 0$, under proper Skorokhod construction, there exist a sequence of Brownian motions $\{W_n : n \ge 1\}$ and a constant $C_3 = C_3(v, c) > 0$ such that for sufficiently large n and t satis-*

fying condition (3.6),

$$P\left(\sup_{0<s\le c} s^{-v}\left|w_n(s)-W_n(s)\right|\ge t\right)\le C_3 r^{-\frac{1}{1/2-v}},$$

where $r=r(t,k)$ *is defined by* $k^{-v} r\log r=t$.

Proof of Proposition S2. Replace s by ks/n in Proposition S1 and note that $(1-ks/n)^{v-1/2}\to 1$ uniformly for $0<s\le c$. After some arrangement, we obtain that there exists a constant $C'_{3,1}>0$ such that,

$$P\left(\sup_{0<s\le c} s^{v-1/2}\left|w_n(s)-\left(\frac{n}{k}\right)^{1/2} B_n\left(\frac{ks}{n}\right)\right|\ge t\right)\le C'_{3,1} r^{-\frac{1}{1/2-v}},$$

where $t=k^{-v} r\log r$. Note that we can construct a sequence of Brownian motions $\{W_n^*: n\ge 1\}$ such that $B_n(t)=W_n^*(t)-tW_n^*(1), 0\le t\le 1$. Define

$$\delta_0=\sup_{0<s\le c} s^{v-1/2}\left|w_n(s)-\left(\frac{n}{k}\right)^{1/2} W_n^*\left(\frac{ks}{n}\right)\right|,$$

$$\delta_1=\sup_{0<s\le c} s^{v-1/2}\left|w_n(s)-\left(\frac{n}{k}\right)^{1/2} B_n\left(\frac{ks}{n}\right)\right|,$$

$$\delta_2=\sup_{0<s\le c} s^{v+1/2}\left(\frac{k}{n}\right)^{1/2}\left|W_n^*(1)\right|.$$

Then,

$$\delta_0\le\delta_1+\delta_2.$$

Note that $1-\Phi(x)\le 1/2\exp(-x^2/2)$, where Φ is the cumulative distribution function of a standard normal distribution. Then we obtain that

$$\begin{aligned}P\left(\delta_2\ge\frac{1}{2}t\right)&=P\left(\left|W_n^*(1)\right|\ge\frac{1}{2}c^{-v-1/2}k^{-1/2}n^{1/2}t\right)\\&\le\exp\left(-\frac{c^{-2v-1}}{8}\frac{n}{k}t^2\right).\end{aligned}$$

Without loss of generality, we assume that $r > 1$, otherwise Proposition S2 holds by choosing $C_3 = 1$. Since $t = k^{-v} r \log r \to 0$, we have $\log k / \log r \to \infty$ as $n \to \infty$. Combining with condition (3.6), we have that, as $n \to \infty$,

$$\frac{\frac{n}{k} t^2}{\log^2 r} = \frac{nt^2}{k \log k^2} \frac{\log^2 k}{\log^2 r} \to \infty,$$

Thus, for sufficiently large r,

$$P\left(\delta_2 \geq \frac{1}{2} t\right) \leq \exp\left(-\frac{c^{-2v-1}}{8} \log^2 r\right) \leq r^{-\frac{1}{1/2-v}}.$$

The proposition is proved by combining δ_1 and δ_2 and defining a new Brownian motion $W_n(s) = (n/k)^{1/2} W_n^* (ks/n)$ for $0 < s \leq c$. □

Now, we are ready to prove Proposition 1.

Proof of Proposition 1. Define

$$Y_n(x) = \frac{n}{k} \bar{F}_n \left\{ b_0 \left(\frac{n}{k}\right) + x a_0 \left(\frac{n}{k}\right) \right\}, \quad z_n(x) = \frac{n}{k} \bar{F} \left\{ b_0 \left(\frac{n}{k}\right) + x a_0 \left(\frac{n}{k}\right) \right\}.$$

With replacing s by $z_n(x)$ in Proposition S2, and using the same Skorokhod construction as in Proposition S2, we have that there exists a constant C_3' such that for sufficiently large n, and for t satisfying condition (3.6),

$$P\left(\sup_{0<z_n(x)\leq c} \{z_n(x)\}^{v-1/2} \left| \sqrt{k} \left(Y_n(x) - z_n(x)\right) - W_n\{z_n(x)\} \right| \geq \frac{1}{3} t\right) \leq C_3' r^{-\frac{1}{1/2-v}}, \tag{S1.3}$$

where $r = r(t, k)$ is defined by $k^{-v} r \log r = t$. Note that $\mathbb{D}_1 := \{x : 0 < z_n(x) \leq c\}$ is equivalent to

$$\left\{x : \frac{U\left(\frac{1}{c}\frac{n}{k}\right) - b_0\left(\frac{n}{k}\right)}{a_0(n/k)} \leq x < \frac{1}{(-\gamma) \vee 0}\right\}.$$

Since as $n \to \infty$,

$$\frac{U\left(\frac{1}{c}\frac{n}{k}\right) - b_0\left(\frac{n}{k}\right)}{a_0(n/k)} \to \frac{c^{-\gamma} - 1}{\gamma},$$

by choosing a sufficiently large c and a sufficiently large n_1, we get $\frac{U(n/(ck)) - b_0(n/k)}{a_0(n/k)} < x_0$ for $n \geq n_1$. Thus, we can replace the supremum over $\mathbb{D}_1$ by the supremum over $\mathbb{D}$ and use the bound in Lemma S1.

Our goal is to replace the three $z_n(x)$ terms in (S1.3) by its limit $z(x) = (1 + \gamma x)^{-1/\gamma}$, and derive a probability inequality for

$$\delta_0 = \sup_{x \in \mathbb{D}} \{z(x)\}^{v-1/2} \left| \sqrt{k}\left(Y_n(x) - z(x)\right) - W_n\{z(x)\} \right.$$
$$\left. - \sqrt{k} A_0(n/k) \{z(x)\}^{1+\gamma} \Psi_{\gamma,\rho}\{1/z(x)\} \right|.$$

First, Lemma S1 allows us to replace the factor $\{z_n(x)\}^{v-1/2}$ by $\{z(x)\}^{v-1/2}$. Then, we obtain that there exists a constant $C_{1,1} > 0$ such that

$$P\left(\sup_{x \in \mathbb{D}} \{z(x)\}^{v-1/2} \left| \sqrt{k}\left(Y_n(x) - z_n(x)\right) - W_n\{z_n(x)\} \right| \geq \frac{1}{3} t\right) \leq C_{1,1} r^{-\frac{1}{1/2 - v}}. \tag{S1.4}$$

Define

$$\delta_1 = \sup_{x\in\mathbb{D}} \{z(x)\}^{v-1/2} \left|\sqrt{k}\left(Y_n(x) - z_n(x)\right) - W_n\left\{z_n(x)\right\}\right|,$$

$$\delta_2 = \sup_{x\in\mathbb{D}} \{z(x)\}^{v-1/2} \sqrt{k}\left|z(x) + A_0(n/k)\left\{z(x)\right\}^{1+\gamma} \Psi_{\gamma,\rho}\left\{1/z(x)\right\} - z_n(x)\right|,$$

$$\delta_3 = \sup_{x\in\mathbb{D}} \{z(x)\}^{v-1/2} \left|W_n\left\{z(x)\right\} - W_n\left\{z_n(x)\right\}\right|,$$

Then,

$$\delta_0 \le \delta_1 + \delta_2 + \delta_3.$$

Since δ_1 has been handled in (S1.4), the statement of Proposition 1 follows if we can prove that there exists a $n_0 > 0$ such that for all $n \ge n_0$,

(a) $\delta_2 < \frac{1}{3}t$,

(b) $\delta_3 < \frac{1}{3}t$ *a.s.*.

By Proposition 3.1 in Drees et al. (2006), we have that $\delta_2 = o(1)\sqrt{k}A_0(n/k)$. Then (a) holds from the condition (3.6). For δ_3, by the modulus of continuity of Brownian motions and Lemma S1, we have that for any $\tilde{\varepsilon} > 0$, $\delta_3 = O\{A(n/k)\}^{1/2-\tilde{\varepsilon}}$ *a.s.*. Then (b) holds from the condition (3.8). □

Finally, we prove Theorem 1.

Proof of Theorem 1. For each given N, we can construct an enlarged probability space to accommodate all $X_1, X_2, \ldots, X_N$ and $\left\{W_n^{(j)}\right\}_{j=1}^m$ such that $X_1, X_2, \ldots, X_N$ are i.i.d. and $\left\{W_n^{(j)}\right\}_{j=1}^m$ are independent acorss $1 \le j \le m$.

Next define $\Delta_N := \sqrt{m} \max_{1 \le j \le m} \delta_n^{(j)}$. We are going to show that for any constant $C > 0$, as $N \to \infty$,

$$P(\Delta_N > C) \to 0.$$

Take $t = C/\sqrt{m}$ and define $r = r(t,k)$ by $k^{-v} r \log r = t$. Since $v > (2+\eta)^{-1}$, by using the condition (A2), we have that $r \to \infty$ and $\log r / \log k \to 0$ as $N \to \infty$. First, we verify that this choice of t satisfies conditions (3.6)-(3.8). For condition (3.6),

$$k^{1/2} n^{-1/2} \log k / t = \left(\frac{km}{n}\right)^{1/2} \frac{1}{C} \log k = O(1),$$

as $N \to \infty$, from the condition (A3). For condition (3.6), by using the condition (A1), we have that,

$$k^{1/2} A_0(n/k)/t = C^{-1} \sqrt{km} A_0(n/k) = O(1),$$

as $N \to \infty$. Condition (3.8) holds with $\tilde{\varepsilon} = \eta/(4+4\eta)$, since, as $N \to \infty$,

$$\begin{aligned}
\{A_0(n/k)\}^{1/2-\tilde{\varepsilon}}/t &= C^{-1} m^{1/2} \left(A_0(n/k)\right)^{(2+\eta)/(4+4\eta)} \\
&= C^{-1} \left(\frac{m}{(km)^{(2+\eta)/(4+4\eta)}}\right)^{1/2} \left\{\sqrt{km} A_0(n/k)\right\}^{(2+\eta)/(4+4\eta)} \\
&= O(1) \left(\frac{m^{2+3\eta}}{k^{2+\eta}}\right)^{1/(8+8\eta)} = o(1),
\end{aligned}$$

where the last step follows from the condition (A2).

Then, by Theorem 1, we have that,

$$P\left(\delta_n^{(j)} \ge C/\sqrt{m}\right) \le C_1 r^{-\frac{1}{1/2-v}}.$$

By construction, $\delta_n^{(j)}, j = 1, 2, \ldots, m$ are independent. Hence, we have that,

$$P\left(\Delta_N > C\right) \leq P\left(\max_{1\leq j\leq m} \delta_n^{(j)} > C/\sqrt{m}\right) \leq 1 - \left(1 - Cr^{-\frac{1}{1/2-v}}\right)^m .$$

So, to prove $\Delta_N = o_P(1)$, as $N \to \infty$, it suffices to show that,

$$mr^{-\frac{1}{1/2-v}} \to 0. \tag{S1.5}$$

Recall that $r \log r = Ck^v/\sqrt{m}$, we have that

$$\begin{aligned}
mr^{-\frac{1}{1/2-v}} &= m\left(r\log r\right)^{-\frac{1}{1/2-v}} \left(\log r\right)^{\frac{1}{1/2-v}} \\
&= o(1)m\left(k^v/\sqrt{m}\right)^{-\frac{1}{1/2-v}} \left(\log k\right)^{\frac{1}{1/2-v}} \\
&= o(1)m^{1+\frac{1}{1-2v}} k^{-\frac{v}{1/2-v}} \left(\log k\right)^{\frac{1}{1/2-v}} \\
&= o(1)k^{\left(1+\frac{1}{1-2v}\right)\left(\frac{1}{1+\eta}\right)-\frac{v}{1/2-v}} \left(\log k\right)^{\frac{1}{1/2-v}} ,
\end{aligned}$$

where the last equality follows from the condition (A2). Since $(2+\eta)^{-1} < v < 1/2$, we get that,

$$\left(1 + \frac{1}{1-2v}\right)\left(\frac{1}{1+\eta}\right) - \frac{v}{1/2-v} < 0.$$

Hence (S1.5) holds, which implies that $\Delta_N = o_P(1)$ as $N \to \infty$.

□

S1.3 Proofs for Section 4

To prove Theorem 3, we need some preliminary lemmas.

Lemma S2. [Gut (2013), Theorem 1.2 (ii) in Chapter 3 with $b = 1$] *Let $X_1, X_2, \ldots, X_n$ be independent random variables with mean 0. Suppose that $P(|X_k| \le 1) = 1$ for all $k \ge 1$, and set $\sigma_k^2 = VarX_k$. Then for $0 < t < 1$ and $x > 0$,*

$$P(|S_n| > x) \le 2\exp\left(-tx + t^2\sum_{k=1}^{n}\sigma_k^2\right).$$

Lemma S3. *Assume condition (A3). As $N \to \infty$,*

$$P\left(Q_n^{(j)}(s) \in \mathbb{D},\ j = 1, 2, \ldots, m,\ 1/k \le s \le 1\right) \to 1.$$

Proof of Lemma S3. By checking the expression of b_0 and a_0 (see Corollary 2.3.7 in de Haan and Ferreira (2006)), we have that for $\gamma < 0$,

$$\max_{1\le j\le m} Q_n^{(j)}(s) \le \frac{X_{N,N} - b_0(\frac{n}{k})}{a_0(\frac{n}{k})} < \frac{U(\infty) - b_0(\frac{n}{k})}{a_0(\frac{n}{k})} = -\frac{1}{\gamma}\ a.s..$$

Since $Q_n^{(j)}(s)$ is a decreasing function of s and $Q_n^{(j)}(s),\ j = 1, 2, \ldots, m$ are independent, it suffices to show that as $N \to \infty$,

$$P^m\left(Q_n^{(1)}(1) > x_0\right) \to 1. \tag{S1.6}$$

Denote $x_n = x_0 a_0(n/k) + b_0(n/k)$. Note that,

$$\begin{aligned} P\left(Q_n^{(1)}(1) > x_0\right) &= P\left(\frac{X_{n-k,n}^{(1)} - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)} > x_0\right) \\ &= P\left(\sum_{i=1}^{n} I_{\left\{X_i^{(1)} > x_n\right\}} > k\right) \\ &= 1 - P\left(\sum_{i=1}^{n} I_{\left\{X_i^{(1)} > x_n\right\}} - n\bar{F}(x_n) < k - n\bar{F}(x_n)\right). \end{aligned}$$

Note that, $n\bar{F}(x_n) \sim k(1+\gamma x_0)^{-1/\gamma}$, as $N \to \infty$, and that $(1+\gamma x_0)^{-1/\gamma} > 1$ since $x_0 < 0$. Then, for sufficiently large N, $k - n\bar{F}(x_n) < 0$. We can thus apply Lemma S2 with $t = 2^{-1}t_0/(1+t_0)$ and $x = n\bar{F}(x_n) - k$, where $t_0 = (1+\gamma x_0)^{-1/\gamma} - 1$, to obtain that,

$$\begin{aligned}
&\log P\left(\sum_{i=1}^{n} I_{\left\{X_i^{(1)} > x_n\right\}} - n\bar{F}(x_n) < k - n\bar{F}(x_n)\right) \\
&\leq \log 2 - \frac{1}{2}\frac{t_0}{1+t_0}\left(n\bar{F}(x_n) - k\right) + \frac{1}{4}\frac{t_0^2}{(1+t_0)^2}n\bar{F}(x_n) \\
&= \log 2 - \frac{1}{2}\frac{t_0}{1+t_0}kt_0\{1+o(1)\} + \frac{1}{4}\frac{t_0^2}{(1+t_0)^2}k(t_0+1)\{1+o(1)\} \\
&= \log 2 - \frac{1}{4}\frac{t_0^2}{1+t_0}k + o(k).
\end{aligned}$$

Thus, (S1.6) holds, which yields the statement in Lemma S3.

□

Lemma S4. *Assume that conditions (A1) and (A2) hold. Then, as $N \to \infty$,*

$$\sqrt{km}\max_{1\leq j\leq m}\sup_{k^{-1}\leq s\leq 1} s^{v+1/2}\left\{\frac{t_n^{(j)}(s)}{s} - 1\right\}^2 = o_P(1),$$

where

$$t_n^{(j)}(s) = \left(1 + \gamma\frac{X_{n-[ks],n}^{(j)} - b_0\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)}\right)^{-1/\gamma}.$$

Proof of Lemma S4. Define $s_i = i/k$ for $1 \leq i \leq k$. Then

$$\sup_{k^{-1}\leq s\leq 1} s^{v+1/2}\left\{\frac{t_n^{(j)}(s)}{s} - 1\right\}^2 = \max_{1\leq i\leq k-1}\sup_{s_i\leq s< s_{i+1}} s^{v+1/2}\left\{\frac{t_n^{(j)}(s)}{s} - 1\right\}^2.$$

For any $s \in [s_i, s_{i+1})$, we have that $t_n^{(j)}(s) = t_n^{(j)}(s_i)$ and hence, $\frac{t_n^{(j)}(s_i)}{s_{i+1}} \leq \frac{t_n^{(j)}(s)}{s} \leq \frac{t_n^{(j)}(s_i)}{s_i}$. Since $(x-1)^2$ is a convex function of x, we obtain that,

$$\sup_{k^{-1} \leq s \leq 1} s^{v+1/2} \left(\frac{t_n^{(j)}(s)}{s} - 1 \right)^2 \leq \max_{1 \leq i \leq k-1} s_{i+1}^{v+1/2} \max \left\{ \left(\frac{t_n^{(j)}(s_i)}{s_i} - 1 \right)^2, \left(\frac{t_n^{(j)}(s_i)}{s_{i+1}} - 1 \right)^2 \right\}.$$

Thus,

$$\begin{aligned}
&\sqrt{km} \max_{1 \leq j \leq m} \sup_{k^{-1} \leq s \leq 1} s^{v+1/2} \left\{ \frac{t_n^{(j)}(s)}{s} - 1 \right\}^2 \\
&\leq \max_{1 \leq i \leq k-1} \sqrt{km} s_{i+1}^{v+1/2} \max_{1 \leq j \leq m} \max \left\{ \left(\frac{t_n^{(j)}(s_i)}{s_i} - 1 \right)^2, \left(\frac{t_n^{(j)}(s_i)}{s_{i+1}} - 1 \right)^2 \right\},
\end{aligned}$$

and hence for any constant $C > 0$,

$$\begin{aligned}
&P \left\{ \sqrt{km} \max_{1 \leq j \leq m} \sup_{k^{-1} \leq s \leq 1} s^{v+1/2} \left\{ \frac{t_n^{(j)}(s)}{s} - 1 \right\}^2 > C \right\} \\
&\leq \sum_{i=1}^{k-1} P \left\{ \sqrt{km} s_{i+1}^{v+1/2} \max_{1 \leq j \leq m} \left(\frac{t_n^{(j)}(s_i)}{s_i} - 1 \right)^2 > C \right\} \\
&\qquad + \sum_{i=1}^{k-1} P \left\{ \sqrt{km} s_{i+1}^{v+1/2} \max_{1 \leq j \leq m} \left(\frac{t_n^{(j)}(s_i)}{s_{i+1}} - 1 \right)^2 > C \right\} \\
&=: \sum_{i=1}^{k-1} I_{i,1} + \sum_{i=1}^{k-1} I_{i,2}.
\end{aligned}$$

We start with $I_{i,1}$,

$$
\begin{aligned}
I_{i,1} &= 1 - P^m \left\{ \left(\frac{t_n^{(j)}(s_i)}{s_i} - 1 \right)^2 < \frac{C}{s_{i+1}^{v+1/2}\sqrt{km}} \right\} \\
&= 1 - P^m \left\{ 1 - \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} < \frac{t_n^{(j)}(s_i)}{s_i} < 1 + \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} \right\} \\
&= 1 - \left[1 - P \left\{ \frac{t_n^{(j)}(s_i)}{s_i} > 1 + \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} \right\} \right. \\
&\qquad \left. - P \left\{ \frac{t_n^{(j)}(s_i)}{s_i} < 1 - \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} \right\} \right]^m \\
&=: 1 - (1 - I_{i,1,a} - I_{i,1,b})^m .
\end{aligned}
$$

We first handle $I_{i,1,a}$. Define

$$
x_n = \frac{\left\{ s_i \left(1 + \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} \right) \right\}^{-\gamma} - 1}{\gamma} a_0 \left(\frac{n}{k} \right) + b_0 \left(\frac{n}{k} \right).
$$

Then, we have that,

$$
\begin{aligned}
I_{i,1,a} &= P \left\{ X_{n-ks_i,n}^{(j)} < \frac{\left\{ s_i \left(1 + \frac{C^{1/2}}{s_{i+1}^{v/2+1/4}(km)^{1/4}} \right) \right\}^{-\gamma} - 1}{\gamma} a_0 \left(\frac{n}{k} \right) + b_0 \left(\frac{n}{k} \right) \right\} \\
&= P \left(\sum_{i=1}^{n} I_{\left\{ X_i^{(j)} \geq x_n \right\}} < k s_i \right) \\
&= P \left(\sum_{i=1}^{n} I_{\left\{ X_i^{(j)} \geq x_n \right\}} - n\bar{F}(x_n) < k s_i - n\bar{F}(x_n) \right).
\end{aligned}
$$

We intend to show that

$$
\limsup_{N \to \infty} \frac{\log I_{i,1,a}}{k^{v/2 - \frac{1}{4(1+\eta)}}} = -\infty. \tag{S1.7}
$$

By Lemma S1 and using the condition (A1), we have that, as $N \to \infty$,

$$n\bar{F}(x_n) = k\frac{n}{k}\bar{F}(x_n) = ks_i\left[1 + C^{1/2}s_{i+1}^{-v/2-1/4}(km)^{-1/4}\left\{1+o(1)\right\}\right].$$

Thus, for sufficiently large N, $ks_i - n\bar{F}(x_n) < 0$. We can therefore apply Lemma S2 with $t = 2^{-1}t_0/(1+t_0)$ and $x = n\bar{F}(x_n) - ks_i$, where $t_0 = C^{1/2}s_{i+1}^{-v/2-1/4}(km)^{-1/4}$, to obtain that,

$$\begin{aligned}\log\left(I_{i,1,a}/2\right) &\leq -\frac{1}{2}\frac{t_0}{1+t_0}(n\bar{F}(x_n) - ks_i) + \frac{1}{4}\frac{t_0^2}{(1+t_0)^2}n\bar{F}(x_n)\\ &= -\frac{1}{2}\frac{t_0^2}{1+t_0}ks_i\left\{1+o(1)\right\} + \frac{1}{4}\frac{t_0^2}{(1+t_0)^2}ks_i\left[1+t_0\left\{1+o(1)\right\}\right]\\ &= -\frac{1}{4}\frac{t_0^2}{1+t_0}ks_i\left\{1+o(1)\right\} + \frac{1}{4}\frac{t_0^2}{(1+t_0)^2}ks_i o(1)\\ &= -\frac{1}{4}\frac{t_0^2}{1+t_0}ks_i\left\{1+o(1)\right\}.\end{aligned}$$

Note that, $\frac{t_0^2}{1+t_0}ks_i > 2^{-1}ks_it_0$ if $t_0 > 1$ and that $\frac{t_0^2}{1+t_0}ks_i \geq 2^{-1}ks_it_0^2$ if $t_0 \leq 1$. Hence, $\frac{t_0^2}{1+t_0}ks_i \geq 2^{-1}\min\left\{ks_it_0, ks_it_0^2\right\}$. Thus, by using the condition (A2), we get that, as $N \to \infty$,

$$\frac{ks_it_0^2}{k^{v/2-\frac{1}{4(1+\eta)}}} = \frac{Ck^{1/2}m^{-1/2}s_is_{i+1}^{-v-1/2}}{k^{v/2-\frac{1}{4(1+\eta)}}} \geq Ck^{1/2-v/2+\frac{1}{4(1+\eta)}}m^{-1/2}s_1^{1/2-v} \to \infty,$$

and

$$\frac{ks_it_0}{k^{v/2-\frac{1}{4(1+\eta)}}} = C^{1/2}\frac{k^{3/4}m^{-1/4}s_is_{i+1}^{-1/4-v/2}}{k^{v/2-\frac{1}{4(1+\eta)}}} \geq C^{1/2}\frac{k^{3/4}m^{-1/4}s_1^{3/4-v/2}}{k^{v/2-\frac{1}{4(1+\eta)}}} \to \infty.$$

Hence (S1.7) holds, which leads to $I_{i,1,a} = o(1)\exp\left(-k^{v/2-\frac{1}{4(1+\eta)}}\right)$, where the $o(1)$ term is uniform for $1 \leq i \leq k-1$. The term $I_{i,1,b}$ can be handled in a similar way as that for $I_{i,1,a}$ and we obtain that, $I_{i,1,b} = o(1)\exp\left(-k^{v/2-\frac{1}{4(1+\eta)}}\right)$. Hence, as $N \to \infty$, $m(I_{i,1,a} + I_{i,1,b}) \to 0$, which implies that

$$I_{i,1} = 1 - (1 - I_{i,1,a} - I_{i,1,b})^m = o(1) m \exp\left(-k^{v/2-\frac{1}{4(1+\eta)}}\right),$$

where the $o(1)$ is uniform for $1 \le i \le k$. By using the condition (A2), we conclude that, as $N \to \infty$, $\sum_{i=1}^{k-1} I_{i,1} = o(1)$. The terms $I_{i,2}$ can be handled in a similar way as that for $I_{i,1}$. Thus, the statement in Lemma S4 follows.

□

Lemma S5. *Assume that conditions (A1) and (A2) hold. Let* $v \in ((2+\eta)^{-1}, 2^{-1})$. *Then, for any* $\delta \in (0,1)$, *as* $N \to \infty$,

$$\max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} \left| \frac{t_n^{(j)}(s)}{s} - 1 \right| = o_P(1).$$

Proof of Lemma S5. Define $i_0 = [k^\delta]$ and $s_i = i/k$ for $i_0 \le i \le k$. Similar to the proof of Lemma S4, we have that, for any constant $C > 0$,

$$\begin{aligned} & P\left\{ \max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} \left| \frac{t_n^{(j)}(s)}{s} - 1 \right| > C \right\} \\ & \le \sum_{i=i_0}^{k} P\left\{ \max_{1 \le j \le m} \left| \frac{t_n^{(j)}(s_i)}{s_i} - 1 \right| > C \right\} + \sum_{i=i_0}^{k} P\left\{ \max_{1 \le j \le m} \left| \frac{t_n^{(j)}(s_i)}{s_{i+1}} - 1 \right| > C \right\} \\ & =: \sum_{i=i_0}^{k} I_{i,1} + \sum_{i=i_0}^{k} I_{i,2}. \end{aligned}$$

We start with $I_{i,1}$,

$$\begin{aligned} I_{i,1} &= 1 - P^m\left(\left|\frac{t_n^{(j)}(s_i)}{s_i} - 1\right| < C\right) \\ &= 1 - \left\{1 - P\left(\frac{t_n^{(j)}(s_i)}{s_i} > 1 + C\right) - P\left(\frac{t_n^{(j)}(s_i)}{s_i} < 1 - C\right)\right\}^m \\ &= 1 - \left(1 - I_{i,1,a} - I_{i,1,b}\right)^m . \end{aligned}$$

We first handle $I_{i,1,a}$. Define

$$x_n = \frac{\{s_i(1+C)\}^\gamma - 1}{\gamma} a_0\left(\frac{n}{k}\right) + b_0\left(\frac{n}{k}\right).$$

Then, we have that,

$$\begin{aligned} I_{i,1,a} &= P\left(X_{n-ks_i,n}^{(j)} < x_n\right) \\ &= P\left(\sum_{i=1}^n I_{\left\{X_i^{(j)} \geq x_n\right\}} < ks_i\right) \\ &= P\left(\sum_{i=1}^n I_{\left\{X_i^{(j)} \geq x_n\right\}} - n\bar{F}(x_n) < ks_i - n\bar{F}(x_n)\right). \end{aligned}$$

We intend to show that

$$\limsup_{N\to\infty} \frac{\log I_{i,1,a}}{k^{\delta/2}} = -\infty. \tag{S1.8}$$

By Lemma S1 and condition (A1), we have that, as $N \to \infty$,

$$n\bar{F}(x_n) = k\frac{n}{k}\bar{F}(x_n) = ks_i\left(1 + C\right)\left\{1 + o(1)\right\}.$$

Thus, for sufficiently large N, $ks_i - n\bar{F}(x_n) < 0$. Applying Lemma S2 with

$t = \frac{1}{2}\frac{C}{1+C}$, we obtain that,

$$\begin{aligned}\log(I_{i,1,a}/2) &\le -\frac{1}{2}\frac{C}{1+C}\left(n\bar{F}(x_n) - ks_i\right) + \frac{1}{4}\frac{C^2}{(1+C)^2}n\bar{F}(x_n)\\ &= -\frac{1}{2}\frac{C}{1+C}ks_iC\left\{1+o(1)\right\} + \frac{1}{4}\frac{C^2}{(1+C)^2}ks_i(1+C)\left\{1+o(1)\right\}\\ &= -\frac{1}{4}\frac{C^2}{1+C}ks_i\left\{1+o(1)\right\}.\end{aligned}$$

Thus, (S1.8) holds since $ks_i \ge k^\delta$ for all $i_0 \le i \le k$. The rest of the proofs are similar to that in Lemma S4. □

Now, we are able to give the proof of Theorem 3.

Proof of Theorem 3. Lemma S3 ensures that we can replace $x, x \in \mathbb{D}$ by $Q_n^{(j)}(s), s \in [k^{-1+\delta}, 1]$ in Proposition 1, and obtain that, as $N \to \infty$,

$$\sqrt{km}\max_{1\le j\le m}\sup_{k^{-1+\delta}\le s\le 1}\left\{t_n^{(j)}(s)\right\}^{v-1/2}\Bigg| s - t_n^{(j)}(s) - \frac{1}{\sqrt{k}}W_n^{(j)}\left\{t_n^{(j)}(s)\right\} - A_0(n/k)\left\{t_n^{(j)}(s)\right\}^{1+\gamma}\Psi\left\{1/t_n^{(j)}(s)\right\}\Bigg| = o_P(1). \tag{S1.9}$$

Our goal is to replace the three $t_n^{(j)}(s)$ terms in (S1.9) by its limit s and show that, as $N \to \infty$,

$$\delta_0 := \sqrt{km}\max_{1\le j\le m}\sup_{k^{-1+\delta}\le s\le 1}s^{v-1/2}\left| s - t_n^{(j)}(s) - \frac{1}{\sqrt{k}}W_n^{(j)}(s) - A_0(n/k)s^{1+\gamma}\Psi(s^{-1})\right| = o_P(1).$$

Note that, Lemma S5 allows us to replace the first factor $\left\{t_n^{(j)}(s)\right\}^{v-1/2}$

by $s^{v-1/2}$. So, as $N \to \infty$,

$$\delta_1 := \sqrt{km} \max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} s^{v-1/2} \Bigg| s - t_n^{(j)}(s) - \frac{1}{\sqrt{k}} W_n^{(j)} \left\{ t_n^{(j)}(s) \right\} - A_0(n/k) \left\{ t_n^{(j)}(s) \right\}^{1+\gamma} \Psi \left\{ 1/t_n^{(j)}(s) \right\} \Bigg| = o_P(1). \tag{S1.10}$$

Define

$$\delta_2 = \sqrt{m} \max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} s^{v-1/2} \left| W_n^{(j)} \left\{ t_n^{(j)}(s) \right\} - W_n^{(j)}(s) \right|,$$

$$\delta_3 = \sqrt{km} A_0(n/k) \max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} s^{v-1/2} \left| \left\{ t_n^{(j)}(s) \right\}^{1+\gamma} \Psi \left\{ 1/t_n^{(j)}(s) \right\} - s^{1+\gamma} \Psi(s^{-1}) \right|.$$

Obviously, $\delta_0 \le \delta_1 + \delta_2 + \delta_3$. Since δ_1 has been handled in (S1.10), we only need to show $\delta_2 = o_P(1)$ and $\delta_3 = o_P(1)$, as $N \to \infty$.

Firstly, we handle δ_2. By the modulus of continuity of Brownian motions and Lemma S5, we have that, for any $\tilde{\varepsilon} > 0$,

$$\delta_2 \le \sqrt{m} \max_{1 \le j \le m} \sup_{k^{-1+\delta} \le s \le 1} s^{v-1/2} \left\{ t_n^{(j)}(s) - s \right\}^{1/2-\tilde{\varepsilon}} \ a.s.,$$

which yields $\delta_2 = o_P(1)$ by Lemma S4.

Next, we handle δ_3. We only consider the case $\gamma + \rho \ne 0$. The proof for $\gamma + \rho = 0$ is similar. By checking the definition of Ψ, we have that,

$$\left\{ t_n^{(j)}(s) \right\}^{1+\gamma} \Psi \left\{ 1/t_n^{(j)}(s) \right\} - s^{1+\gamma} \Psi(s^{-1}) = \frac{1}{\gamma + \rho} s^{1-\rho} \left[\left\{ t_n^{(j)}(s)/s \right\}^{1-\rho} - 1 \right].$$

By Lemma S4, we have that, as $N \to \infty$,

$$\frac{t_n^{(j)}(s)}{s} = 1 + o_P(1) s^{-v/2-1/4} (km)^{-1/4},$$

where the $o_P(1)$ term is uniform for $k^{-1} \le s \le 1$ and $1 \le j \le m$. Thus, as $N \to \infty$,

$$\max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} s^{v-1/2+1-\rho} \left[\left\{ t_n^{(j)}(s)/s \right\}^{1-\rho} - 1 \right] = o_P(1).$$

Combining with the condition $\sqrt{km}A(n/k) = O(1)$ as $N \to \infty$, we have that, $\delta_3 = o_P(1)$. Combining δ_1, δ_2 and δ_3, we conclude that, $\delta_0 = o_P(1)$ as $N \to \infty$.

By applying Taylor's expansion to the function $f(x) =: \frac{x^{-\gamma}-1}{\gamma}$ around $x = s$, and noting that $f^{'}(x) = -x^{-\gamma-1}$ and $f^{''}(x) = (\gamma+1)x^{-\gamma-2}$, we have that,

$$Q_n^{(j)}(s) = f(t_n^{(j)}(s)) = \frac{s^{-\gamma}-1}{\gamma} + s^{-\gamma-1}\left\{ s - t_n^{(j)}(s) \right\} + \frac{\gamma+1}{2}\left\{ u_n^{(j)} \right\}^{-\gamma-2}\left\{ t_n^{(j)}(s) - s \right\}^2,$$

where $u_n^{(j)}$ is a random value between s and $t_n^{(j)}(s)$. It follows that,

$$\sqrt{km} \sup_{k^{-1+\delta}\le s\le 1} s^{v+\gamma+1/2} \max_{1\le j\le m} \left| Q_n^{(j)}(s) - \frac{s^{-\gamma}-1}{\gamma} - \frac{1}{\sqrt{k}} s^{-\gamma-1} W_n^{(j)}(s) - A_0(n/k)\Psi(s^{-1}) \right|$$
$$\le \delta_0 + \sqrt{km} \max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} s^{v+\gamma+1/2} \frac{\gamma+1}{2} \left| \left\{ u_n^{(j)}(s) \right\}^{-\gamma-2} \left\{ t_n^{(j)}(s) - s \right\}^2 \right|.$$

Note that

$$\sqrt{km} \max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} s^{v+\gamma+1/2} \left| \left\{ u_n^{(j)}(s) \right\}^{-\gamma-2} \left\{ t_n^{(j)}(s) - s \right\}^2 \right|$$
$$\le \left[\max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} \left| \frac{u_n^{(j)}(s)}{s} \right|^{-\gamma-2} \right] \left[\max_{1\le j\le k} \sup_{k^{-1+\delta}\le s\le 1} \sqrt{km} s^{v+1/2} \left(\frac{t_n^{(j)}(s)}{s} - 1 \right)^2 \right]$$
$$=: I_1 \cdot I_2.$$

Obviously, $I_1 \leq \max\left\{1, \max_{1\leq j\leq m}\sup_{k^{-1+\delta}\leq s\leq 1}\left|\frac{t_n^{(j)}(s)}{s}\right|^{-\gamma-2}\right\}$. By Lemma S5, we have that $I_1 = O_P(1)$ as $N \to \infty$. By Lemma S4 , we have that as $N \to \infty$, $I_2 = o_P(1)$. Then, the proof is completed.

□

Recall that $U = \{1/(1-F)\}^{\leftarrow}$. Write $X = U(Y)$, where Y follows the Pareto (1) distribution with distribution function $1 - 1/y$. Let $\{Y_1, Y_2, \ldots, Y_N\}$ be a random sample of Y. Then we can regard our observations as $\{U(Y_1), U(Y_2), \ldots, U(Y_N)\}$, which are stored in m machines with n observations each. And let $Y_{n,n}^{(j)} \geq \cdots \geq Y_{1,n}^{(j)}$ denote the order statistics of the n Pareto (1) random variable corresponding to the observations in machine j. Then, $X_{n-[ks],n}^{(j)} = U(Y_{n-[ks],n}^{(j)})$.

Proof of Proposition 2. First, we show that, as $N \to \infty$.

$$\sqrt{km}\max_{1\leq j\leq m}\int_0^{k^{-1+\delta}} g(s)\left|Q_n^{(j)}(s) - \frac{s^{-\gamma}-1}{\gamma}\right| ds = o_P(1). \tag{S1.11}$$

We intend to apply (2.3) with $t = n/k$ and $tx = Y_{n-[ks],n}^{(j)}$. For this purpose, we introduce the set $\Omega_1 = \left\{Y_{n-k,n}^{(j)} \geq t_0, \text{ for all } 1 \leq j \leq m\right\}$. By Lemma S.2 in the supplementary material of Chen et al. (2022), we have that, under condition (A2), $\lim_{N\to\infty} P(\Omega_1) = 1$ for any $t_0 > 0$. Consequently, on the set Ω_1, we can replace t and tx by n/k and $Y_{n-[ks],n}^{(j)}$ in (2.3), respectively, for $j = 1, 2, \ldots, m$ and obtain that,

$$
\begin{aligned}
&Q_n^{(j)}(s) - \frac{s^{-\gamma}-1}{\gamma} \\
&\le \frac{\left(kY^{(j)}_{n-[ks],n}/n\right)^{\gamma} - s^{-\gamma}}{\gamma} + \left|A_0\left(\frac{n}{k}\right)\right| \Psi\left(kY^{(j)}_{n-[ks],n}/n\right) \\
&\qquad\qquad + \varepsilon \left|A_0\left(\frac{n}{k}\right)\right| \left(kY^{(j)}_{n-[ks],n}/n\right)^{\gamma+\rho\pm\delta} \\
&=: I_1^{(j)}(s) + I_2^{(j)}(s) + I_3^{(j)}(s).
\end{aligned}
$$

We are going to prove that, as $N \to \infty$,

$$
\sqrt{km} \max_{1\le j\le m} \int_0^{k^{-1+\delta}} g(s) I_i^{(j)}(s) ds = o_P(1), \quad \text{for} \quad i = 1, 2, 3.
$$

We start with $I_1^{(j)}(s)$. We consider the cases (i) $\gamma > 0$, (ii) $\gamma < 0$ and (iii) $\gamma = 0$ separately.

Case (i): $\gamma > 0$. First, from the condition $\beta > \gamma - \frac{\eta}{2(1+\eta)}$ and condition (A2), we have that, as $N \to \infty$, $\sqrt{km} \int_0^{k^{-1+\delta}} g(s) s^{-\gamma} ds = o(1)$. Next, note that, for $\gamma > 0$,

$$\begin{aligned}
&\sqrt{km} \max_{1 \le j \le m} \int_0^{k^{-1+\delta}} g(s) \left(kY^{(j)}_{n-[ks],n}/n \right)^{\gamma} ds \\
&\le \sqrt{km} \max_{1 \le j \le m} \int_0^{k^{-1+\delta}} g(s) \left(kY^{(j)}_{n,n}/n \right)^{\gamma} ds \\
&= \sqrt{km} \max_{1 \le j \le m} \left(kY^{(j)}_{n,n}/n \right)^{\gamma} \int_0^{k^{-1+\delta}} g(s) ds \\
&\le \sqrt{km} \left(kY_{N,N}/n \right)^{\gamma} \int_0^{k^{-1+\delta}} g(s) ds \\
&= O_P(1) (km)^{\gamma+1/2} k^{-(1-\delta)(\beta+1)} \\
&= o_P(1) k^{(\gamma+1/2)(1+\frac{1}{1+\eta})} k^{-(1-\delta)(\beta+1)} \\
&= o_P(1),
\end{aligned}$$

where the last equality follows from the condition that $\beta > \frac{2+\eta}{1+\eta}\gamma - \frac{\eta}{2(1+\eta)}$.

Case (ii) $\gamma < 0$. Similar to the proof for the case $\gamma > 0$, we have that, as $N \to \infty$, $\sqrt{km} \int_0^{k^{-1+\delta}} g(s) s^{-\gamma} ds = o(1)$. Since $\gamma < 0$, we have that, $(kY^{(j)}_{n-[ks],n}/n)^{\gamma} \le (kY^{(j)}_{n-[k^{\delta}],n}/n)^{\gamma}$. Thus, it suffies to show that, as $N \to \infty$,

$$\sqrt{km} \max_{1 \le j \le m} \int_0^{k^{-1+\delta}} g(s) \left(kY^{(j)}_{n-[k^{\delta}],n}/n \right)^{\gamma} ds = o_P(1). \tag{S1.12}$$

First, we introduce the set $\Omega_2 = \left\{ kY^{(j)}_{n-[k^{\delta}],n}/n > 2^{-1} k^{1-\delta}, \text{ for all } 1 \le j \le m \right\}$ and show that, as $N \to \infty$, $P(\Omega_2) \to 1$. Note that,

$$\begin{aligned}
P(\Omega_2) &= P^m \left(kY^{(j)}_{n-[k^{\delta}],n}/n > 2^{-1} k^{1-\delta} \right) \\
&= P^m \left(\sum_{i=1}^{n} I_{\left\{ Y_i^{(j)} \ge 2^{-1} n/k^{\delta} \right\}} > [k^{\delta}] \right) \\
&= \left\{ 1 - P \left(\sum_{i=1}^{n} I_{\left\{ Y_i^{(j)} \ge 2^{-1} n/k^{\delta} \right\}} - 2k^{\delta} < [k^{\delta}] - 2k^{\delta} \right) \right\}^m.
\end{aligned}$$

We intend to prove $\lim_{N\to\infty} P(\Omega_2) = 1$ by showing that

$$\limsup_{N\to\infty} \frac{\log P\left\{\sum_{i=1}^n I_{\left\{Y_i^{(j)} \geq 2^{-1}n/k^\delta\right\}} - 2k^\delta < [k^\delta] - 2k^\delta\right\}}{k^{\delta/2}} = -\infty.$$

Applying Lemma S2 with $t = 1/4$ and $x = 2k^\delta - [k^\delta]$, we have that,

$$\log P\left\{\sum_{i=1}^n I_{\left\{Y_i^{(j)} \geq 2^{-1}n/k^\delta\right\}} - 2k^\delta < [k^\delta] - 2k^\delta\right\} \leq \log 2 - \frac{1}{4}k^\delta + \frac{1}{8}k^\delta = \log 2 - \frac{1}{8}k^\delta.$$

Thus, as $N \to \infty$, $P(\Omega_2) \to 1$.

On the set Ω_2, we have that,

$$\begin{aligned}\sqrt{km} \max_{1\leq j\leq m} \int_0^{k^{-1+\delta}} g(s)\left(kY_{n-[k^\delta],n}^{(j)}/n\right)^\gamma ds &\leq \sqrt{km} \max_{1\leq j\leq m} \int_0^{k^{-1+\delta}} g(s)\left(k^{1-\delta}/2\right)^\gamma ds \\ &= O(1)\sqrt{km}k^{\gamma(1-\delta)}k^{-(1-\delta)(\beta+1)}.\end{aligned}$$

From the condition that $\beta > \gamma - \frac{\eta}{2(1+\eta)}$, we have that, $\sqrt{km}k^{\gamma(1-\delta)}k^{-(1-\delta)(\beta+1)} \to 0$ as $N \to \infty$ and hence (S1.12) holds.

Case (iii) $\gamma = 0$. For $\gamma = 0$, the term $I_1^{(j)}(s)$ is interpreted as $\log\left(kY_{n-[ks],n}^{(j)}/n\right) - \log s$. Since $\log(x)$ is an increasing function of x, the case $\gamma = 0$ can be handled in a similar way as that for the case $\gamma > 0$.

Thus, we conclude that, as $N \to \infty$,

$$\sqrt{km} \max_{1\leq j\leq m} \int_0^{k^{-1+\delta}} g(s) I_1^{(j)}(s) ds = o_P(1).$$

The terms $I_2^{(j)}(s)$ and $I_3^{(j)}(s)$ can be handled in a similar way as that for $I_1^{(j)}(s)$. Hence, (S1.11) follows.

By using the condition $\beta > \gamma - \frac{\eta}{2(1+\eta)}$ and checking the definition of Ψ, we get that, as $N \to \infty$,

$$\sqrt{km} A_0 \left(\frac{n}{k}\right) \int_0^{k^{-1+\delta}} g(s)\Psi(s^{-1})ds = o(1).$$

Thus, the statement in Proposition 2 holds provided that, as $N \to \infty$,

$$\sqrt{km} \max_{1 \le j \le m} \int_0^{k^{-1+\delta}} g(s) \left| \frac{1}{\sqrt{k}} s^{-\gamma-1} W_n^{(j)}(s) \right| ds = o_P(1).$$

By the modulus of continuity of Brownian motions, we have that, for any constant $\varepsilon > 0$, $\left|W_n^{(j)}(s)\right| \le s^{1/2-\varepsilon} a.s.$ for all $0 < s < k^{-1+\delta}$ and sufficiently large k. Thus, we have that, as $N \to \infty$,

$$\begin{aligned} \sqrt{km} \max_{1 \le j \le m} \int_0^{k^{-1+\delta}} g(s) \left| \frac{1}{\sqrt{k}} s^{-\gamma-1} W_n^{(j)}(s) \right| ds &\le \sqrt{m} \int_0^{k^{-1+\delta}} g(s) s^{-\gamma-1/2-\varepsilon} ds, \\ &= o(1) k^{\frac{1}{2(1+\eta)}} k^{-(1-\delta)(\beta-\gamma+1/2-\varepsilon)} \\ &= o(1), \quad a.s. \end{aligned}$$

where the last equality follows from the condition that $\beta > \eta - \frac{\eta}{2(1+\eta)}$. Proposition 2 is thus proved. □

S1.4 Proofs for Section 5

Proof of Corollary 2. By applying the same techniques used in proving the asymptotic normality of the oracle Hill estimator (cf. Example 5.1.5 in de Haan and Ferreira (2006)), we have that, as $N \to \infty$,

$$\begin{aligned}\widehat{\gamma}_H^D - \gamma =& \frac{1}{m}\sum_{j=1}^m \int_{X_{n-k,n}^{(j)}/U(n/k)}^1 s^{-1/\gamma}\frac{ds}{s} \\ &+ \frac{1}{m}\sum_{j=1}^m \int_{X_{n-k,n}^{(j)}/U(n/k)}^1 \left[\frac{n}{k}\left\{1-F_n^{(j)}\left(sU\left(\frac{n}{k}\right)\right)\right\} - s^{-1/\gamma}\right]\frac{ds}{s} \\ &+ \frac{1}{m}\sum_{j=1}^m \int_1^\infty \left[\frac{n}{k}\left\{1-F_n^{(j)}\left(sU\left(\frac{n}{k}\right)\right)\right\} - s^{-1/\gamma}\right]\frac{ds}{s} \\ =:& \, I_1 + I_2 + I_3.\end{aligned}$$

For I_1, note that, as $N \to \infty$,

$$I_1 = \frac{1}{m}\sum_{j=1}^m \left\{\gamma\left(X_{n-k,n}^{(j)}/U(n/k)\right)^{-1/\gamma} - \gamma\right\}.$$

By taking $s = 1$ in Theorem 4, we get that, as $N \to \infty$,

$$\sqrt{m}\max_{1\le j\le m}\left|\sqrt{k}\left(\frac{X_{n-k,n}^{(j)}}{U(n/k)} - 1\right) - \gamma W_n^{(j)}(1)\right| = o_P(1). \tag{S1.13}$$

Thus, as $N \to \infty$,

$$I_1 = -\gamma(km)^{-1/2}\frac{1}{\sqrt{m}}\sum_{j=1}^m W_n^{(j)}(1) + (km)^{-1/2}o_P(1).$$

For I_2, the uniform convergence in (S1.13) and Theorem 1 imply that as $N \to \infty$, $I_2 = (km)^{-1/2}o_P(1)$.

For I_3, since $F_N = m^{-1}\sum_{j=1}^m F_n^{(j)}$, we obtain that,

$$I_3 = \int_1^\infty \left[\frac{n}{k}\left\{1-F_N\left(sU\left(\frac{n}{k}\right)\right)\right\} - s^{-1/\gamma}\right]\frac{ds}{s}.$$

We can handle $\widehat{\gamma}_H^{Oracle}$ in a similar way and get that,

$$\widehat{\gamma}_H^{Oracle} - \gamma = -\gamma\frac{1}{\sqrt{km}}W_N(1) + (km)^{-1/2}o_P(1) + I_3.$$

The Corollary is proved by noting that $W_N = m^{-1/2}\sum_{j=1}^m W_n^{(j)}$. □

Proof of Corollary 3. For a continuous function $f : [0,1] \to \mathbb{R}$, define an operator

$$L(f) = (1-\gamma)(2-\gamma)\int_0^1 \{(1-4s)-\gamma(1-2s)\}\{f(s)-f(1)\}\,ds.$$

It is obvious that L is a linear operator.

Note that, for the oracle PWM estimator using top km exceedances, we have that, as $N \to \infty$,

$$\widehat{\gamma}_{PWM}^{Oracle} - \gamma = \frac{1}{\sqrt{km}} L\left(s^{-\gamma-1}W_N(s)\right) + A_0\left(\frac{n}{k}\right) L\left(\Psi(s^{-1})\right) + \frac{1}{\sqrt{km}} o_P(1),$$

see e.g. Section 3.6.1 in de Haan and Ferreira (2006). By using similar techniques, we obtain that, as $N \to \infty$,

$$\begin{aligned}
&\frac{1}{m}\sum_{j=1}^m \widehat{\gamma}_{PMW}^{(j)} - \gamma \\
&= \frac{1}{m}\sum_{j=1}^m \frac{1}{\sqrt{k}} L\left(s^{-\gamma-1}W_n^{(j)}(s)\right) + A_0\left(\frac{n}{k}\right) L\left(\Psi(s^{-1})\right) \\
&\quad + O_P(1)\max_{1\le j\le m}\int_0^1 \left|f_n^{(j)}(s)\right| ds + O_P(1)\max_{1\le j\le m}\int_0^1 s\left|f_n^{(j)}(s)\right| ds.
\end{aligned}$$

Recall that L is a linear operator and $W_N = m^{-1/2}\sum_{j=1}^m W_n^{(j)}$, we get that

$$\frac{1}{\sqrt{km}} L\left(s^{-\gamma-1}W_N(s)\right) = \frac{1}{m}\sum_{j=1}^m \frac{1}{\sqrt{k}} L\left(s^{-\gamma-1}W_n^{(j)}(s)\right).$$

The Corollary is proved provided that, as $N \to \infty$, $I_1 := \max_{1\le j\le m}\int_0^1 \left|f_n^{(j)}(s)\right| ds = (km)^{-1/2}o_P(1)$ and $I_2 := \max_{1\le j\le m}\int_0^1 s\left|f_n^{(j)}(s)\right| ds = (km)^{-1/2}o_P(1)$.

For handling I_1, we divide $[0,1]$ into $[k^{-1+\delta},1]$ and $[0,k^{-1+\delta}]$. Thus,

$$\begin{aligned}I_1 &\leq \max_{1\leq j\leq m}\int_0^{k^{-1+\delta}} \left|f_n^{(j)}(s)\right| ds + \max_{1\leq j\leq m}\int_{k^{-1+\delta}}^1 \left|f_n^{(j)}(s)\right| ds \\ &:= I_{1,1} + I_{1,2}.\end{aligned}$$

We first handle $I_{1,2}$. Note that, for $\eta > \frac{2\gamma}{1/2-\gamma}$, we can always find a $v > \frac{1}{2+\eta}$ such that $v+\gamma<1/2$. Then, by Theorem 3, as $N\to\infty$,

$$I_{1,2} = o_P(1)(km)^{-1/2}\int_{k^{-1+\delta}}^1 s^{-v-1/2-\gamma}ds = (km)^{-1/2}o_P(1).$$

The term $I_{1,1}$ can be handled by Proposition 2 as follows. Choose $g(s)=1$. Since $\gamma<1/2$ and $\eta>\max\left\{0,\frac{2\gamma}{1/2-\gamma}\right\}$, the conditions in Proposition 2 hold. The proposition yields that $I_{1,1}=(km)^{-1/2}o_P(1)$. Hence, we obtain $I_1=(km)^{-1/2}o_P(1)$ as $N\to\infty$. The term I_2 can be handled in a similar way with choosing $g(s)=s$. □

Next, we prove Corollary 4, the oracle property of the maximum likelihood estimator (MLE). A detailed proof will be given only for $\gamma>0$. The cases $\gamma=0$ or $-1/2<\gamma<0$ can be handled in a similar way.

Let $U_1,U_2,\ldots,U_N$ be i.i.d. uniformly distributed random variables and denote $X_i = U\left(\frac{1}{U_i}\right), i=1,2,\ldots,N$. Assume that the N observations are stored in m machines with n observations in each machine. Let $U_{n,n}^{(j)} \geq U_{n-1,n}^{(j)} \geq \cdots \geq U_{1,n}^{(j)}$ be the order statistics of uniform random variables corresponding to the observations in machine j. Then,

$X^{(j)}_{n-[ks],n} = U\left(\frac{1}{U^{(j)}_{[ks]+1,n}}\right)$.

Lemma S6. *Assume that condition (A2) holds. Then, as $N \to \infty$,*

$$\max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} \frac{nU^{(j)}_{[ks]+1,n}}{ks} = O_P(1).$$

Proof. Define $i_0 = [k^\delta]$ and $s_i = i/k$ for $i_0 \le i \le k$. For any $s \in [s_i, s_{i+1})$, we have that, $U^{(j)}_{[ks]+1,n} = U^{(j)}_{[ks_i]+1,n}$ and hence

$$\frac{nU^{(j)}_{[ks]+1,n}}{ks} \le \frac{nU^{(j)}_{[ks_i]+1,n}}{ks_i}.$$

Thus, we have that,

$$\max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} \frac{nU^{(j)}_{[ks]+1,n}}{ks} \le \max_{1\le j\le n} \max_{i_0\le i\le k} \frac{nU^{(j)}_{[ks_i]+1,n}}{ks_i}.$$

Hence, for any constant $C > 0$,

$$\begin{aligned} P\left\{\max_{1\le j\le m} \sup_{k^{-1+\delta}\le s\le 1} \frac{nU^{(j)}_{[ks]+1,n}}{ks} > C\right\} &\le \sum_{i=i_0}^{k} P\left\{\max_{1\le j\le m} \frac{nU^{(j)}_{[ks_i]+1,n}}{ks_i} > C\right\} \\ &= \sum_{i=i_0}^{k} \left\{1 - P^m\left(\frac{nU^{(1)}_{[ks_i]+1,n}}{ks_i} \le C\right)\right\} \\ &=: \sum_{i=i_0}^{k} \left(1 - I_i^m\right). \end{aligned}$$

Note that,

$$\begin{aligned} I_i &= P\left\{U^{(1)}_{[ks_i]+1,n} > Cks_i/n\right\} \\ &= P\left\{\sum_{i=1}^{n} I\left(U_i^{(1)} < Cks_i/n\right) \le [ks_i]\right\} \\ &\le P\left\{\sum_{i=1}^{n} I\left(U_i^{(1)} < Cks_i/n\right) - Cks_i < (1-C)ks_i\right\} \end{aligned}$$

Applying Lemma S2 with $t = 1/2$, we obtain that,

$$\begin{aligned}\log I_i &\leq -\frac{1}{2}(C-1)ks_i + \frac{1}{4}ks_i\left(1 - \frac{Cks_i}{n}\right)\\ &= -\frac{1}{4}\left(2C-3\right)ks_i\left\{1+o(1)\right\}.\end{aligned}$$

Since $ks_i \leq k^{\delta}$ for all $i_0 \leq i \leq k$, by choosing $C > 3/2$, we have that,

$$\limsup_{N\to\infty} \frac{\log I_i}{k^{\delta/2}} = -\infty.$$

The rest of the proofs are similar to that of Lemma S4. □

Define

$$Z_n^{(j)}(s) = k^{1/2}\left(Q_n^{(j)}(s) - Q_n^{(j)}(1) - \frac{s^{-\gamma}-1}{\gamma}\right).$$

We rewrite the equation (10) as

$$\begin{aligned}&\int_0^1 \left\{\frac{1}{\left(\tilde{\gamma}^{(j)}\right)^2}\log\left(1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)\right)\right.\\ &\qquad\left. - \left(\frac{1}{\tilde{\gamma}^{(j)}}+1\right)\frac{\left(1/\tilde{\sigma}_0^{(j)}\right)\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)}{1 + \left(\tilde{\gamma}^{(j)}/\tilde{\sigma}_0^{(j)}\right)\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)}\right\} ds = 0, \qquad \text{(S1.14)}\\ &\qquad\int_0^1 \left(\frac{1}{\tilde{\gamma}^{(j)}}+1\right)\frac{\left(\tilde{\gamma}_0^{(j)}/\tilde{\sigma}^{(j)}\right)\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)}{1 + \left(\tilde{\gamma}^{(j)}/\tilde{\sigma}_0^{(j)}\right)\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)} ds = 1,\end{aligned}$$

where $\tilde{\sigma}_0^{(j)} = \tilde{\sigma}^{(j)}/a_0\left(n/k\right)$.

Lemma S7. *Assume the same conditions as in Corollary 4 and $\gamma > 0$. Let $(\tilde{\gamma}^{(j)}, \tilde{\sigma}^{(j)})$ be such that*

$$|\tilde{\gamma}^{(j)}/\tilde{\sigma}^{(j)} - \gamma| = O_p(k^{-1/2})$$

uniformly for $1 \leq j \leq m$. *Then,*

$$P\left\{1+\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s)-Q_n^{(j)}(1)\right)\geq C_N s^{-\gamma}, s\in[k^{-1+\delta},1], j=1,2,\ldots,m\right\}\to 1.$$

Proof. By using the second order condition and Lemma S6, we obtain that, for all $\delta>0$,

$$\max_{1\leq j\leq m}\sup_{s\in[k^{-1+\delta},1]} s^{\gamma+\rho+\delta}\left|\frac{\frac{U\left(\frac{1}{U^{(j)}_{[ks]+1,n}}\right)-U\left(\frac{n}{k}\right)}{a_0\left(\frac{n}{k}\right)}-\frac{\left(\frac{k}{nU^{(j)}_{[ks]+1,n}}\right)^{\gamma}-1}{\gamma}}{A_0\left(\frac{n}{k}\right)}-\Psi\left(\frac{k}{nU^{(j)}_{[ks]+1,n}}\right)\right|$$

$$=o_P(1).$$

We use this approximation simultaneously for $s\in[k^{-1+\delta},1]$ and $s=1$. Then, we have that,

$$\begin{aligned}\frac{X^{(j)}_{n-[ks],n}-X^{(j)}_{n-k,n}}{a_0\left(\frac{n}{k}\right)}&=\frac{U\left(\frac{1}{U^{(j)}_{[ks]+1,n}}\right)-U\left(\frac{1}{U^{(j)}_{k+1,n}}\right)}{a_0\left(\frac{n}{k}\right)}\\&=\frac{1}{\gamma}\left(\frac{k}{nU^{(j)}_{[ks]+1,n}}^{\gamma}\right)-\frac{1}{\gamma}\left(\frac{k}{nU^{(j)}_{k+1,n}}\right)^{\gamma}\\&+A_0\left(\frac{n}{k}\right)\Psi\left(\frac{k}{nU^{(j)}_{[ks]+1,n}}\right)-A_0\left(\frac{n}{k}\right)\Psi\left(\frac{k}{nU^{(j)}_{k+1,n}}\right)\\&+o_P(1)A_0\left(\frac{n}{k}\right)\left(s^{-\gamma-\rho-\delta}-1\right).\end{aligned}$$

Hence,

$$
\begin{aligned}
&1+\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\frac{X_{n-[ks],n}^{(j)}-X_{n-k,n}^{(j)}}{a_0\left(\frac{n}{k}\right)}\\
&\stackrel{d}{=}\left\{1-\left(\frac{k}{nU_{k+1,n}^{(j)}}\right)^{\gamma}\right\}-\left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}-\gamma\right)\frac{1}{\gamma}\left(\frac{k}{nU_{k+1,n}^{(j)}}\right)^{\gamma}\\
&+\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\frac{1}{\gamma}\left(\frac{k}{nU_{[ks]+1,n}^{(j)}}\right)^{\gamma}+\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}A_0\left(\frac{n}{k}\right)\Psi\left(\frac{k}{nU_{[ks]+1,n}^{(j)}}\right)\\
&-\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}A_0\left(\frac{n}{k}\right)\Psi\left(\frac{k}{nU_{k+1,n}^{(j)}}\right)+o_P(1)A_0\left(\frac{n}{k}\right)\left(s^{-\gamma-\rho-\delta}-1\right)\\
&=: I_j+II_j+III_j+IV_j+V_j+VI_j.
\end{aligned}
$$

By Lemma S6, we have that, $s^{\gamma}III_j$ is bounded away from zero uniformly for $s\in[k^{-1+\delta},1]$ and $1\leq j\leq m$. We can show that all the other terms tend to 0 uniformly for $s\in[k^{-1+\delta},1], 1\leq j\leq m$ when multiplied by s^{γ}, so Lemma S7 follows with $C_N:=\min_{1\leq j\leq m}\inf_{s\in[k^{-1+\delta},1]}s^{\gamma}III_j-\varepsilon_N$, for a suitable sequence $\varepsilon_N\downarrow 0$.

□

Proof of Corollary 4. If $\gamma\neq 0$, the equation (S1.14) can be simplified to

$$
\begin{aligned}
&\int_0^1\log\left(1+\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s)-Q_n^{(j)}(1)\right)\right)ds=\tilde{\gamma}^{(j)},\\
&\int_0^1\frac{1}{1+(\tilde{\gamma}^{(j)}/\tilde{\sigma}_0^{(j)})\left(Q_n^{(j)}(s)-Q_n^{(j)}(1)\right)}ds=\frac{1}{\tilde{\gamma}^{(j)}+1}.
\end{aligned}
\tag{S1.15}
$$

We will find expansions for the left-hand side of both equations uniformly for $1\leq j\leq m$.

Rewrite the first one as

$$\int_0^{k^{-1+\delta}} \log\left(1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)\right) ds + \int_{k^{-1+\delta}}^1 \log(s^{-\gamma}) ds$$

$$+ \int_{k^{-1+\delta}}^1 \log\left\{ s^{\gamma}\left(1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)\right)\right\} ds$$

$$= I_1^{(j)} + \gamma\left(1 - O(1)k^{-1+\delta}\log k\right) + I_2^{(j)}.$$

We start from handling $I_1^{(j)}$. Note that, as $N \to \infty$,

$$I_1^{(j)} \le k^{-1+\delta} \log\left(1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(0) - Q_n^{(j)}(1)\right)\right).$$

and

$$1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(0) - Q_n^{(j)}(1)\right) = O_P(N^{\gamma}),$$

uniformly for $1 \le j \le m$.

Thus, we have that,

$$I_1^{(j)} = O_P(1)k^{-1+2\delta} = (km)^{-1/2} o_P(1),$$

uniformly for $1 \le j \le m$.

Next, we handle $I_2^{(j)}$. Define

$$K^{(j)}(s) = s^{\gamma}\left(1 + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}}\left(Q_n^{(j)}(s) - Q_n^{(j)}(1)\right)\right) - 1.$$

By the mean value theorem, we have that,

$$I_2^{(j)} = \int_{k^{-1+\delta}}^1 K^{(j)}(s) ds + \int_{k^{-1+\delta}}^1 \frac{-1}{(1 + \xi^{(j)}(s))^2}\left(K^{(j)}(s)\right)^2 ds,$$

$$=: I_3^{(j)} + I_4^{(j)}.$$

where $\xi^{(j)}(s)$ is a random value between 0 and $K^{(j)}(s)$.

For $I_3^{(j)}$, note that,

$$K^{(j)}(s) = \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right)\frac{1-s^\gamma}{\gamma} + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} s^\gamma k^{-1/2} Z_n^{(j)}(s).$$

Thus,

$$\begin{aligned}
I_3^{(j)} &= \int_{k^{-1+\delta}}^1 \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right)\frac{1-s^\gamma}{\gamma} ds + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} k^{-1/2} \int_{k^{-1+\delta}}^1 s^\gamma Z_n^{(j)}(s) ds \\
&= \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right)\frac{1}{\gamma+1} + \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right) O(k^{-1+\delta}) + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} k^{-1/2} \int_{k^{-1+\delta}}^1 s^\gamma Z_n^{(j)}(s) ds.
\end{aligned}$$

Choose $g(s) = s^\gamma$. Since $\eta > 2\gamma$, the condition of Proposition 2 holds. By applying Proposition 2, we obtain that,

$$\int_0^{k^{-1+\delta}} s^\gamma Z_n^{(j)}(s) = o_P(1) m^{-1/2},$$

uniformly for $1 \le j \le m$. Thus, we conclude that,

$$I_3^{(j)} = \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right)\frac{1}{\gamma+1} + \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma\right) O(k^{-1+\delta}) + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} k^{-1/2} \int_0^1 s^\gamma Z_n^{(j)}(s) ds + o_P(1)(km)^{-1/2}.$$

For $I_4^{(j)}$, by Lemma S6, we obtain that,

$$\max_{1\le j\le m} \sup_{s\in[k^{-1+\delta},1]} \frac{1}{(1+\xi^{(j)}(s))^2} = O_P(1).$$

Thus,

$$I_4^{(j)} = O_P(1) \int_{k^{-1+\delta}}^1 \left(K^{(j)}(s)\right)^2 ds = o_P(1)(km)^{-1/2},$$

uniformly for $1 \le j \le m$.

Combining $I_3^{(j)}$ and $I_4^{(j)}$, we conclude that, as $N \to \infty$,

$$\tilde{\gamma}^{(j)} = \gamma + \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma \right) \frac{1}{\gamma + 1} + \frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} k^{-1/2} \int_0^1 s^{\gamma} Z_n^{(j)}(s) ds + o_P(1)(km)^{-1/2},$$

uniformly for $1 \leq j \leq m$. Similarly, we can find the asymptotic expansions for the second equation of (S1.15):

$$\frac{1}{\gamma+1} - \left(\frac{\tilde{\gamma}^{(j)}}{\tilde{\sigma}_0^{(j)}} - \gamma \right) \frac{1}{(\gamma+1)(2\gamma+1)} - \gamma k^{-1/2} \int_0^1 s^{2\gamma} Z_n^{(j)}(t) dt + o_P(1)(km)^{-1/2} = \frac{1}{\gamma+1},$$

uniformly for $1 \leq j \leq m$.

By Proposition 3.1 of Drees et al. (2004), we can obtain similar expansions for $(\hat{\gamma}_{mle}^{Oracle}, \hat{\sigma}_{mle}^{Oracle})$ and thus the oracle property holds for the MLE for the extreme value index and the scale parameter.

□

S2 Simulation

In this section, we conduct a simulation study to demonstrate the finite sample performance of the distributed estimator. We consider four distributions: the Fréchet distribution ($F(x) = \exp(-x^{-3}), x \geq 0$), the Gumbel distribution ($F(x) = \exp(-e^{-x}), x \in \mathbb{R}$), the standard normal distribution and the reversed Burr distribution ($F(x) = 1 - (1 + (4-x)^{-4})^{-5/4}, x \leq 4$). The first and second order indices of the four distributions are list in Table S1. The normal distribution is not under our framework as we assume that

$\rho < 0$. Note that the distributed Hill estimator in Chen et al. (2022) is only applicable for estimating a positive extreme value index. As a result, the distributed Hill estimator will fail for distributions with a negative or zero extreme value index, such as the Gumbel distribution, normal distribution, and reversed Burr distribution in our simulation.

Table S1: Distributions for simulation.

Parameters	Fréchet	Gumbel	Normal	Reversed Burr
γ	1/3	0	0	-0.2
ρ	-1	-1	0	-0.8

S2.1 Distributed PWM estimation for extreme value index

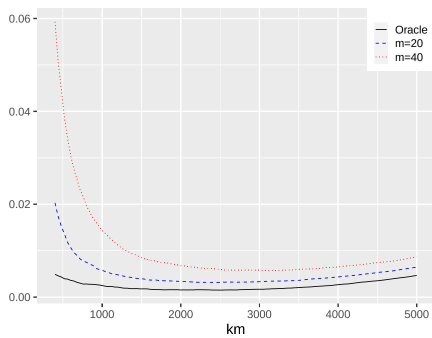
We compare the finite sample performances of the distributed PWM estimators for the extreme value index $\widehat{\gamma}_{PWM}^{D}$ with different levels of m for various levels of km. Recall that, $\widehat{\gamma}_{PWM}^{D}$ involves km top exceedances: with k from each machine. In the calculation of the PWM estimator, we take the suggestion in Hosking and Wallis (1987) to modify $Q_n^{(j)}$ as

$$Q_n^{(j)} := \frac{1}{k}\sum_{i=0}^{k-1} \frac{i+0.35}{k}\left(X_{n-i,n}^{(j)} - X_{n-k,n}^{(j)}\right).$$

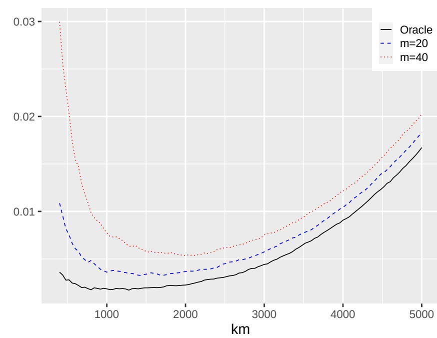
We generate $r = 200$ samples with sample size $N = 10000$. We assume that the $N = 10000$ observations are stored in $m = 1, 20, 40$ machines with $n = N/m$ observations in each machine. Note that the case $m = 1$ corresponds to applying the statistical procedure to the oracle sample directly. Thus, the corresponding estimator is the oracle estimator.

In Figure S1, we plot the mean squared errors (MSE) of the estimates of γ against different levels of km for different distributions. We observe that, for large m, when km is low, the distributed PWM estimator fails. This is in line with the condition that m should be much smaller than k. As km increases, the distributed PWM estimators and the oracle PWM estimator ($m = 1$) have similar MSEs. Although the oracle PWM estimator outperforms the distributed PWM estimator, it may be impractical in scenarios where data is stored in a distributed manner due to privacy, storage, or memory constraints, as highlighted at the beginning of the introduction. The distributed PWM estimator can overcome these challenges by employing a divide-and-conquer algorithm, achieving performance comparable to that of the oracle PWM estimator.

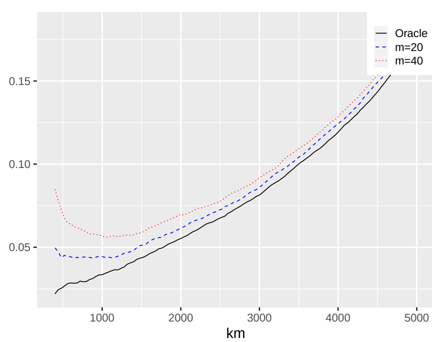
We then evaluate the performance of the distributed PWM estimators for different values of N. Let $\widetilde{N}$ be chosen from $\{1000, 2000, \ldots, 20000\}$. We fix $n = [\widetilde{N}^{0.75}]$, $m = [\widetilde{N}/n]$ and $k = [n^{0.55}]$. Denote $N = nm$. For each value of $\widetilde{N}$, we generate $B = 50000$ samples with sample size $\widetilde{N}$. For each sample i, we calculate the distributed PWM estimator $\widehat{\gamma}_{PWM}^{D,i}$ and the



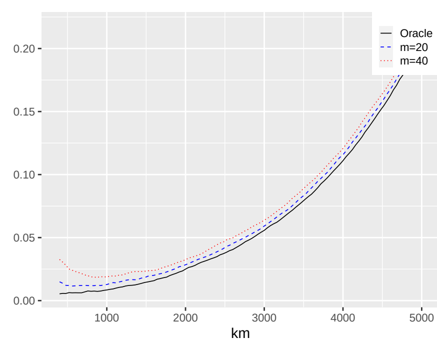
(a) Fréchet Distribution



(b) Gumbel Distribution



(c) Normal Distribution



(d) Reversed Burr Distribution

Figure S1: MSE of γ for 200 samples with sample size $N = 10000$.

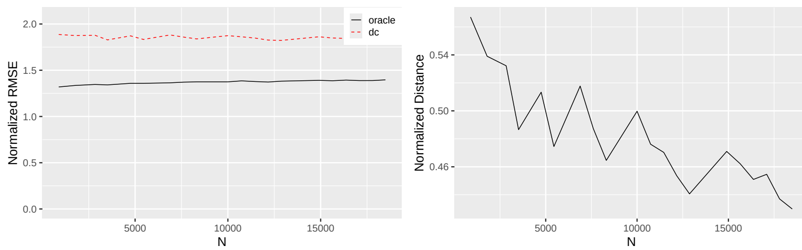
oracle PWM estimator $\widehat{\gamma}_{PWM}^{O,i}$. We then calculate the normalized root mean squared error (RMSE) for the distributed PWM estimator and oracle PWM estimator as

$$\sqrt{km}\sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\gamma}_{PWM}^{D,i}-\gamma\right)^2}, \quad \sqrt{km}\sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\gamma}_{PWM}^{Oracle,i}-\gamma\right)^2},$$

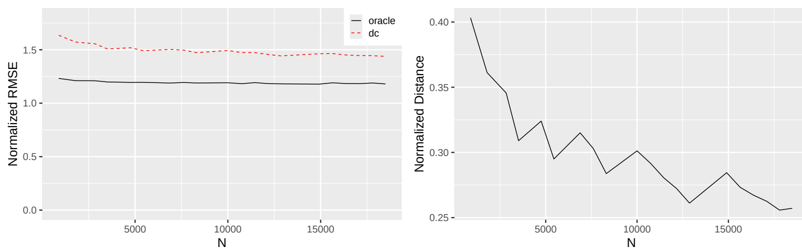
respectively. Moreover, we calculate the normalized distance between the two estimators as

$$\sqrt{km}\sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\gamma}_{PWM}^{D,i}-\widehat{\gamma}_{PWM}^{Oracle,i}\right)^2}.$$

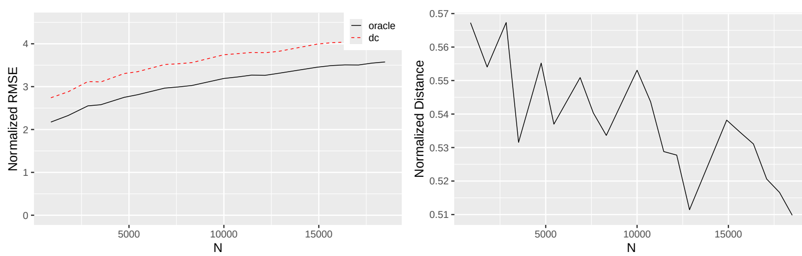
The normalized RMSE and normalized distance across various values of N for the four distributions are shown in Figure S2. We observe that, the normalized RMSE remains almost constant, consistent with the fact that $\sqrt{km}\left(\widehat{\gamma}_{PWM}^{D}-\gamma\right) = O_P(1)$ and $\sqrt{km}\left(\widehat{\gamma}_{PWM}^{Oracle}-\gamma\right) = O_P(1)$ as $n \to \infty$. Furthermore, the normalized distance tends to decrease as N increases, aligning with Corollary 3, which states that $\sqrt{km}\left(\widehat{\gamma}_{PWM}^{D}-\widehat{\gamma}_{PWM}^{Oracle}\right) = o_P(1)$ as $N \to \infty$. The observed fluctuations in the normalized distance in Figure S2 arise due to the relative size of m (the number of machines) and k (the number of observations per machine). Condition (A2) states that $k/m \to \infty$ as $N \to \infty$. Due to the finite sample setup and the integer function, as $\widetilde{N}$ varies from 1000 to 20000, the ratio k/m may either increase or decrease, leading to the observed fluctuations in Figure S2.



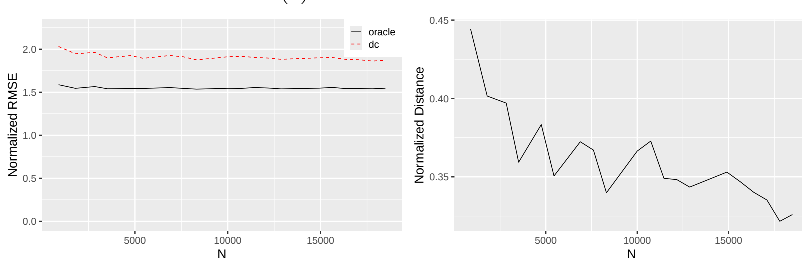
(a) Fréchet Distribution



(b) Gumbel Distribution



(c) Normal Distribution



(d) Reversed Burr Distribution

Figure S2: Normalized RMSE and normalized distance between the distributed PWM estimator and oracle PWM estimator.

S2.2 Distributed PWM estimation for high quantile

Recall that the estimation procedures for high quantile proposed in Section 5 first compute the distributed estimation for the extreme value index, scale parameter and location parameter. Then the estimation for high quantile can be obtained. Nevertheless, one can also directly apply the DC algorithm to the estimation of the high quantile. The detailed procedures are given as follows:

- On each machine j, we calculate $\widehat{\gamma}_{PWM}^{(j)}, \widehat{a}^{(j)}\left(\frac{n}{k}\right), X_{n-k,n}^{(j)}$.
- On each machine j, we estimate the high quantile by

$$\widehat{x}^{(j)}(p_N) = X_{n-k,n}^{(j)} + \widehat{a}^{(j)}\left(\frac{n}{k}\right)\frac{\left(\frac{k}{np_N}\right)^{\widehat{\gamma}_{PWM}^{(j)}} - 1}{\widehat{\gamma}_{PWM}^{(j)}},$$

  and transmit $\widehat{x}^{(j)}(p_N)$ to the central machine.

- On the central machine, we take the average of these estimates by

$$\widehat{x}^{D,2}(p_N) = \frac{1}{m}\sum_{j=1}^{m}\widehat{x}^{(j)}(p_N).$$

By using similar techniques as in proving the oracle property of the PWM estimator for the extreme value index, we can also establish the oracle property of $\widehat{x}^{D,2}(p_N)$.

We compare the finite sample performances of the two estimators $\widehat{x}^{D}(p_N)$ and $\widehat{x}^{D,2}(p_N)$ of the $1-p$ quantile with $p = 1/5000$. We also generate

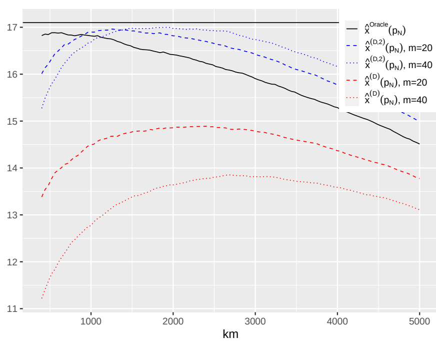
$r = 200$ samples with sample size $N = 10000$. Figures S3 and S4 demonstrate the performances of the high quantile estimators in terms of the average of the estimates (bias) and the MSE. We have three main observations from these figures. First, the performance of distributed high quantile estimator is similar to that of the oracle estimator both in the bias and the MSE, and the gap between them widens as m increases. This is in line with the asymptotic theory.

Second, the average of $\widehat{x}^{D,2}(p_N)$ is overall higher than that of $\widehat{x}^{D}(p_N)$ for the same m, which can be explained by the fact that $(a^x - 1)/x$ is a convex function of x, for large a.

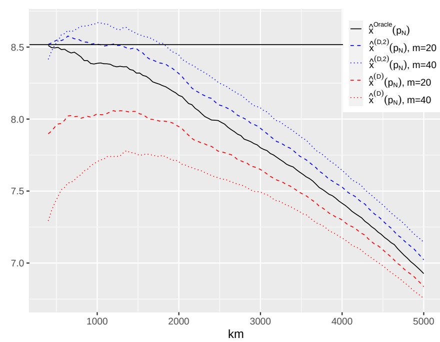
Third, we observe that, the average of $\widehat{x}^{D,2}(p_N)$ is generally higher than that of the oracle estimate. Since the oracle PWM estimator tends to underestimate the high quantile, $\widehat{x}^{D,2}(p_N)$ performs better than the oracle estimator in terms of the MSE.

S2.3 Distributed MLE for the extreme value index

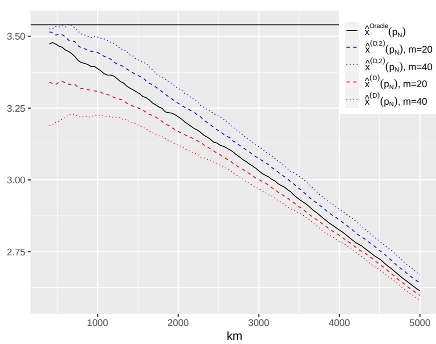
We compare the MSE and the computation cost of the distributed MLE and the oracle MLE. We generate $r = 100$ samples with sample size $N = 5 \times 10^7$. Fix $m = 1000$ and $n = 50000$. We consider three different choices of $k = 1000, 1500, 2500$.



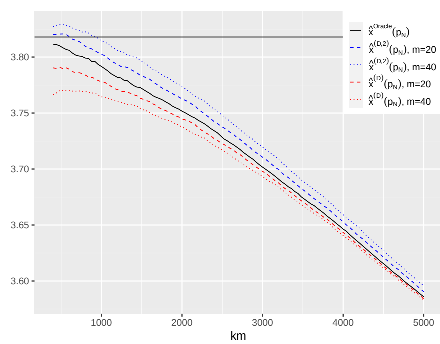
(a) Fréchet Distribution



(b) Gumbel Distribution

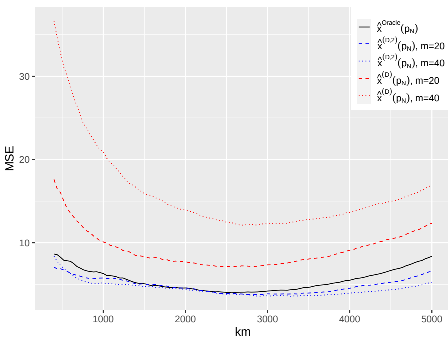


(c) Normal Distribution

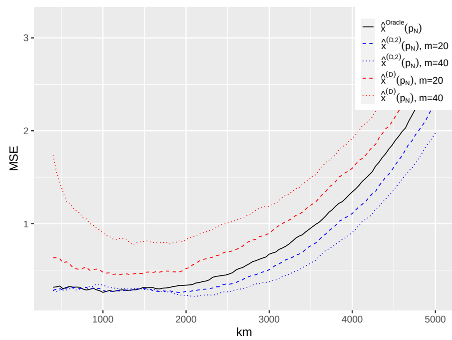


(d) Reversed Burr Distribution

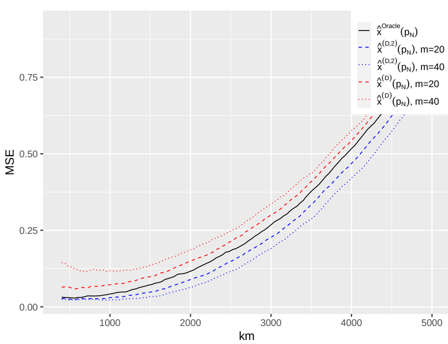
Figure S3: Average of quantile estimation for 200 samples with sample size $N = 10000$. Horizontal lines indicate the value of x_p with $p = 1/5000$.



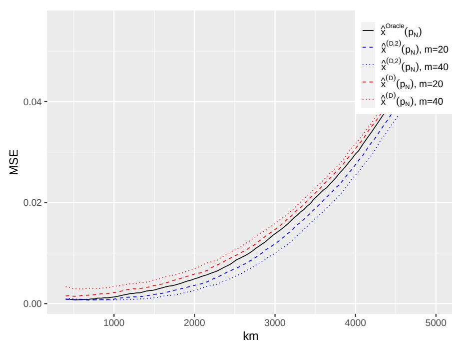
(a) Fréchet Distribution



(b) Gumbel Distribution



(c) Normal Distribution



(d) Reversed Burr Distribution

Figure S4: MSE of quantile estimation ($p = 1/5000$) for 200 samples with sample size $N = 10000$.

Table S2 summarizes the MSE of the distributed MLE and the oracle MLE for different levels of k. The values in parentheses are the MSE of the oracle MLE. For the Gumbel distribution, the distributed MLE achieves better performance compared to the oracle MLE. For the Fréchet, Normal and Reversed Burr distributions, the distributed MLE has higher MSE compared to the oracle MLE for all levels of k. Nevertheless, the MSE of these two estimators are still comparable.

We further compare the computation time of the distributed MLE and the oracle MLE. For the calculation of MLE, we use the *gpdFit()* function from R package "fExtremes" (Wuertz et al. (2017)).

First, assume that no parallel computing techniques can be used, then the distributed MLE must be calculated in sequence. In particular, we split the datasets into m subsets. For $j = 1, 2, \ldots, m$, we record the local computation time $t^{(j)}$. The total computation time for the distributed MLE is $\sum_{j=1}^{m} t^{(j)}$. Table S3 summarizes the computation time of these two estimators. The values in parentheses are the computation time of the oracle MLE. The distributed MLE performs faster for the Fréchet, Normal and Reversed Burr distributions. For the Gumbel distribution, the oracle MLE perfroms slightly faster while the computation time of these two estimators

The comparison is conducted in R on an AMD R5 4600U CPU.

are still comparable.

Table S2: The MSE of the distributed MLE (and oracle MLE)

	$k = 1000$	$k = 1500$	$k = 2500$
Fréchet	$1.8 \times 10^{-5}(5.0 \times 10^{-6})$	$2 \times 10^{-5}(9.3 \times 10^{-6})$	$3.4 \times 10^{-5}(2.5 \times 10^{-5})$
Gumbel	$3 \times 10^{-5}(0.01)$	$3.3 \times 10^{-5}(0.011)$	$5.81 \times 10^{-5}(0.012)$
Normal	0.011(0.0099)	0.013(0.012)	0.015(0.013)
Reversed Burr	$3 \times 10^{-4}(1.5 \times 10^{-4})$	$4.3 \times 10^{-4}(2.7 \times 10^{-4})$	$8 \times 10^{-4}(5.4 \times 10^{-4})$

Next, we consider the computation time of the distributed MLE if we can utilize parallel computing. We take the maximum of the local computation time $\max_{1 \le j \le m} t^{(j)}$ as an estimated time for applying the distributed MLE. The total computational time for all the 100 samples is shown in Table S4. The computational efficiency can be significantly improved by using the distributed MLE and parallel computing.

Table S3: Total computational time (in seconds) of the distributed MLE (and oracle MLE) without parallel computing.

	$k = 1000$	$k = 1500$	$k = 2500$
Fréchet	804.6(1001.5)	874(1014.2)	1052.5(1322)
Gumbel	811.7(785.4)	906.3(791.3)	1086.9(980.1)
Normal	865.2(984.8)	974.5(1057.9)	1181.5(1361.3)
Reversed Burr	810.5(1062.3)	903(1206.5)	1085.1(1531.8)

Table S4: Total computational time (in seconds) of the distributed MLE with parallel computing.

	$k = 1000$	$k = 1500$	$k = 2500$
Fréchet	5.201	6.12	6.521
Gumbel	3.076	6.295	7.373
Normal	2.699	6.556	7.689
Reversed Burr	2.607	3.189	4.845

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