TESTS ON DYNAMIC RANKING

Nan Lu, Jian Shi, Xin-Yu Tian and Kai Song

Chinese Academy of Sciences, University of Chinese Academy of Sciences and University of Minnesota

Supplementary Material

The supplementary material contains additional simulation results and proofs.

S1 Simulation

S1.1 Score Variation Test

This section aims to validate the results presented in Section 3. We set the parameters as follows: $n = 100$, $m = 100$, $h = 0.003$, $M = 300$ and generate π_i^* from a uniform grid ranging from 1 to 3. The experiment is repeated 1000 times, and the test statistic is computed for each item. The histograms of the test statistics for the first ten items are depicted in Figure 1, with the line representing the density of the standard normal distribution. We observe that the empirical distribution closely aligns with the theoretical value.

Corresponding author: Jian Shi. E-mail: jshi@iss.ac.cn.

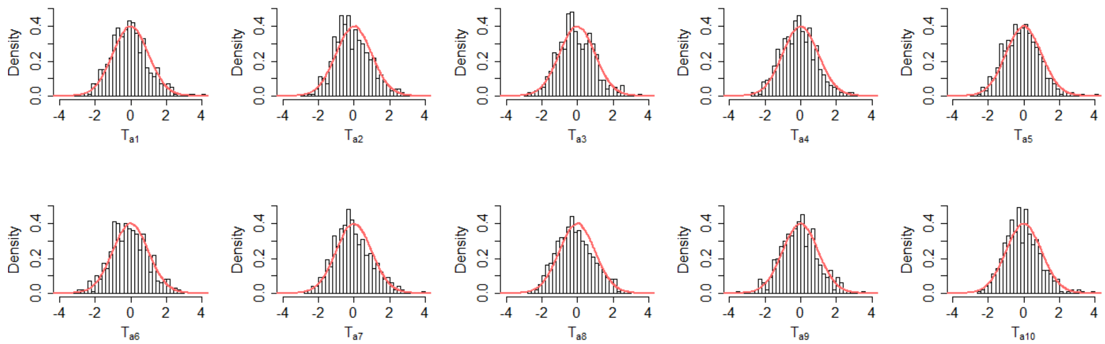

Figure 1: Empirical and theoretical density of test statistic T_a.

Then we consider the scenario where $\pi_i^*(t) \equiv a_{0i}$ for $i = 1, \dots, \frac{n}{2}$, and $\pi_i^*(t) = a_{0(i-\frac{n}{2})} + 0.5\sin(5a_{0i}t)$ for $i = \frac{n}{2}+1, \dots, n$, where a_{0i} are equidistant points sampled from the interval $[1, 3]$. We normalize the sum of score functions to $\frac{1}{2}$ at each time point for both the static and dynamic groups. We set $m = 50$, $M = 150$, $p = 1$, and vary the bandwidth h from 0.003 to 0.009. Additionally, we change the value of n from 30 to 90. The experiment is repeated 2000 times for each combination of settings. We conduct the tests for items 1 and $n/2+1$ under H_0^a and H_1^a respectively. The empirical type I errors and test powers are presented in Tables 1 and 2.

We find the empirical type I errors are close to 0.05 and the empirical test powers are close to 1. The results indicate that both the test level and power are not sensitive to the choice of h. In fact, the cross-validation algorithm proposed by Bong et al. (2020) can be directly applied here. As they point out, it is com-

Table 1: Type I error of score variation test with the change of n and h.

h	0.003	0.004	0.005	0.006	0.007	0.008	0.009
n=30	0.0850	0.0615	0.0605	0.0470	0.0495	0.0485	0.0590
n=50	0.0615	0.0640	0.0595	0.0475	0.0455	0.0490	0.0535
n=70	0.0505	0.0610	0.0615	0.0540	0.0475	0.0500	0.0585
n=90	0.0590	0.0460	0.0475	0.0535	0.0515	0.0540	0.0515

Table 2: Test power of score variation test with the change of n and h.

h	0.003	0.004	0.005	0.006	0.007	0.008	0.009
n=30	0.9720	0.9970	0.9995	1	1	1	1
n=50	1	1	1	1	1	1	1
n=70	1	1	1	1	1	1	1
n=90	1	1	1	1	1	1	1

putationally expensive to select h when n and M are large, and experiments show that h within a reasonable range yields good performance close to those obtained through cross-validation.

Then we keep $h = 0.005$ fixed and vary M and m. The results in Figure 2 indicate that n and m have minimal impact on the type I error, while the power shows an increasing trend as n, m and M grow.

We then consider the multiple hypothesis testing. For the FDR control, we still use the above score functions. We test the score variation of items 1 to k,

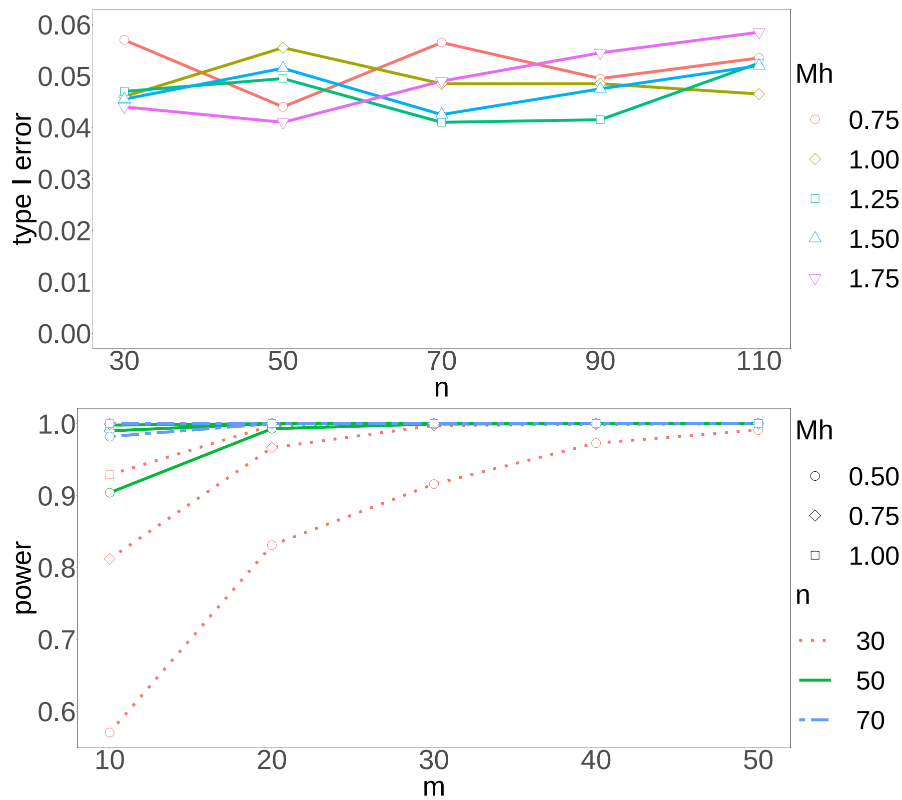

Figure 2: Results of score variation test with the change of n, m and M.

and recall that we have $k_0 = n/2$. We let $Mh = 2$, $m = 50$, $p = 1$ and set $n = 30$, 60, and 90, and vary k such that k_0/k changes from 0.5 to 1. We repeat the experiment 2000 times for each combination of settings. Table 3 presents the empirical FDR results. The results illustrate that the FDR is well controlled below the nominal α and there is an increasing trend of empirical FDR with the increase of k_0/k. Table 4 shows the empirical FDR power, which is defined as the true positive rate. The empirical FDR power is close to 1, indicating that the proposed

Table 3: Empirical FDR of score variation test with the change of n, α and k_0/k.

	α=0.05			α=0.1			α=0.2		
k_0/k	0.5	0.75	1	0.5	0.75	1	0.5	0.75	1
n=30	0.0066	0.0123	0.0280	0.0114	0.0200	0.0435	0.0186	0.0335	0.0695
n=60	0.0071	0.0135	0.0435	0.0115	0.0224	0.0620	0.0192	0.0358	0.0955
n=90	0.0071	0.0144	0.0475	0.0119	0.0223	0.0765	0.0196	0.0378	0.1230

Table 4: Empirical FDR power of score variation test with the change of n, α and k_0/k.

	α=0.05			α=0.1			α=0.2		
k_0/k	0.5	0.75	1	0.5	0.75	1	0.5	0.75	1
n=30	0.9934	0.9976	1.0000	0.9886	0.9959	1.0000	0.9814	0.9925	1.0000
n=60	0.9929	0.9967	1.0000	0.9885	0.9946	1.0000	0.9808	0.9909	1.0000
n=90	0.9929	0.9967	1.0000	0.9881	0.9945	1.0000	0.9804	0.9908	1.0000

FDR procedure is able to identify dynamic items with well-controlled FDR.

S1.2 Score Function Equality Test

Consider the functions $\pi_i^*(t) = a_{0i} + 0.5\sin(5a_{0i}t)$, $i \in [n]$, where $\{a_{0i}\}_{i\in[n]}$ are equidistant points sampled from the interval $[1, 3]$. We conduct a test under the null hypothesis H_0^b by setting $\pi_2^*(t) = \pi_1^*(t)$ and examining whether items 1 and 2 share the same score function. We set $n = 100$, $p = 1$, $m = 100$ and $h = 0.003$. We let M vary from 300 to 500, and repeat the experiment 1000 times. Figure 3 displays the density of the empirical statistic T_b in comparison to the standard normal distribution.

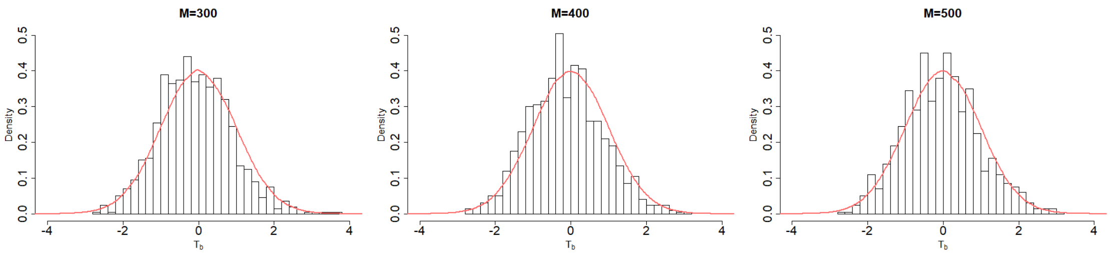

Figure 3: Empirical and theoretical density of test statistic T_b.

Then with $m = 50$ and $M = 150$, we vary n and h, repeating the experiments 2000 times. The type I error results are summarized in Table 5. The results indicate that the type I error is close to 0.05 when h lies within the range $[0.005, 0.008]$. Additionally, we conduct a test to determine whether items 1 and $[\frac{n}{4}]$ share the same score function under H_1^b. The results are shown in Table 6. We can observe that h has minimal influence on both the type I error and the test power.

To investigate further, we fix h at 0.005 and examine how the rejection proportion varies with changes of n and M. As depicted in Figure 4, the type I error remains close to 0.05, and there is a slight increasing trend as n grows. Besides, there is a significant increase in test power with larger values of n, m, and M.

S1.3 Top-K Test

In this section, we focus on the top-K test. We consider a scenario with $n = 10$, where the score functions are defined as follows: $\pi_1^*(t) = \pi_2^*(t) = 0.3+0.01\sin(5\pi t)$, $\pi_3^*(t) = 0.2+0.025\sin(5\pi t-\pi)$ and $\pi_4^*(t) = 0.15+0.025\sin(5\pi t)+\delta$. The remaining

Table 5: Type I error of score equality test with the change of n and h.

h	0.003	0.004	0.005	0.006	0.007	0.008	0.009
n=70	0.0315	0.0435	0.0390	0.0410	0.0470	0.0475	0.0620
n=90	0.0400	0.0375	0.0465	0.0430	0.0470	0.0580	0.0660
n=110	0.0355	0.0450	0.0425	0.0540	0.0475	0.0575	0.0565
n=130	0.0360	0.0495	0.0485	0.0525	0.0530	0.0505	0.0670

Table 6: Test power of score equality test with the change of n and h.

h	0.003	0.004	0.005	0.006	0.007	0.008	0.009
n=70	1	1	1	1	1	1	1
n=90	1	1	1	1	1	1	1
n=110	1	1	1	1	1	1	1
n=130	1	1	1	1	1	1	1

six items are equally partitioned to ensure that the sum of scores is 1 at each time point. Set $h = 0.05$ and $p = 1$. We employ 500 bootstrap repetitions and repeat the simulations 500 times.

We test whether item 3 ranks among the top 3 items at time point 0.1, with different distances between H_0^c and H_1^c adjusted using the parameter δ. As Theorem 5 implies, the test's difficulty is determined by $\tilde{\Delta}$. Specifically, the type I error is more likely to occur when $\pi^*_{(3)}(t_0) - \pi^*_{(4)}(t_0)$ is close to 0, since a small perturbation can lead to the wrong order of the two items. Conversely, with a large value of

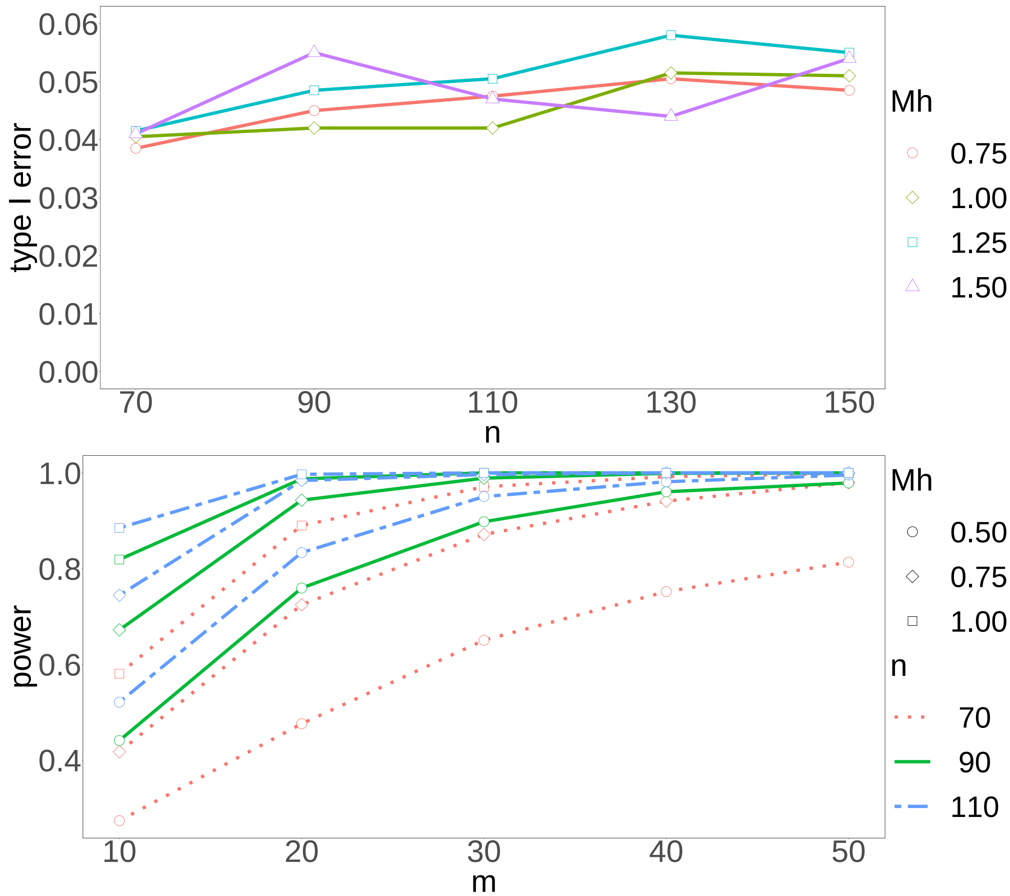

Figure 4: Results of score equality test with the change of n, M and m.

$\pi^*_{(3)}(t_0) - \pi^*_{(4)}(t_0)$, it is less prone to getting a wrong rank. To investigate the most error-prone scenario, we set $\tilde{\Delta}$ to a small value, specifically 10^{-5}. This choice of a small $\tilde{\Delta}$ allows us to examine the performance in a highly challenging situation. Additionally, we conduct the top-K test over a time interval $[0.1, 0.2]$, which is approximated by sequential points spaced by 0.01.

From the results in Tables 7 and 8, the type I error is controlled at approx-

Table 7: Rejection proportions of top-K test at time point 0.1.

	H0		H1							
$\tilde{\Delta}$	10^{-5}		0.02		0.05		0.08		0.10	
Mh	DRI	DRIS	DRI	DRIS	DRI	DRIS	DRI	DRIS	DRI	DRIS
10.0	0.000	0.012	0.000	0.062	0.006	0.330	0.052	0.866	0.192	0.992
12.5	0.000	0.024	0.000	0.110	0.004	0.462	0.100	0.928	0.324	1.000
15.0	0.000	0.026	0.002	0.096	0.020	0.530	0.164	0.974	0.512	1.000
17.5	0.002	0.062	0.004	0.172	0.030	0.634	0.292	0.990	0.700	1.000
20.0	0.000	0.040	0.002	0.162	0.038	0.700	0.384	0.996	0.778	1.000
22.5	0.000	0.038	0.002	0.222	0.046	0.762	0.462	0.996	0.890	1.000
25.0	0.000	0.050	0.004	0.194	0.092	0.846	0.622	1.000	0.940	1.000

Table 8: Rejection proportions of top-K test at time interval [0.1,0.2].

	H0		H1							
$\tilde{\Delta}$	10^{-5}		0.02		0.05		0.08		0.10	
Mh	DRI	DRIS	DRI	DRIS	DRI	DRIS	DRI	DRIS	DRI	DRIS
15.0	0.000	0.000	0.000	0.036	0.000	0.290	0.036	0.880	0.166	0.994
17.5	0.000	0.004	0.002	0.030	0.006	0.396	0.070	0.932	0.300	1.000
20.0	0.000	0.002	0.000	0.052	0.000	0.460	0.114	0.986	0.436	1.000
22.5	0.000	0.002	0.000	0.042	0.004	0.560	0.186	0.986	0.624	1.000
25.0	0.000	0.006	0.000	0.052	0.006	0.618	0.234	0.994	0.758	1.000

imately 0.05 for both the DRI and DRIS methods. Furthermore, the test power tends to 1 with growing M and $\tilde{\Delta}$. We observe that the increasing speed of DRIS

is significantly faster than that of DRI.

S2 Proof of Results in Section 3

Before presenting the proofs of Theorems, we establish the entrywise expansion result in Section S2.1, which is of independent interest. Then we analyze the main term of the test statistic in Section S2.2 and present the proofs of the theorems in the remaining part.

S2.1 Entrywise Expansion of the KRC estimator

We first present a group inverse approximation result in the Lemma S1, and then derive the entrywise expansion of the KRC estimator in Theorem S1. We introduce some notations for further discussion. Define the transition matrix $P^*(t)$ that

$$P^*_{ij}(t) = \begin{cases} \frac{1}{2np} y^*_{ij}(t) & \text{if } (i,j) \in \mathcal{E}, \\ 1 - \sum_{s \neq i} P^*_{is}(t) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{S2.1}$$

We let $A(t) = I - P^*(t)$ and $A^{\#}(t)$ be the group inverse of $A(t)$ (Cao, 1998).

Lemma S1. *Suppose that $np > c \log n$ for sufficiently large c. Letting $\tilde{A}(t)$ be the diagonal matrix such that*

$$\tilde{A}_{ii}(t) = \frac{1}{A_{ii}(t)} = \frac{2np}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)},$$

then we have

$$\max_{i\in[n]} \|\tilde{A}_{\cdot i}(t) - A^{\#}_{\cdot i}(t)\|_2 = O_p(\frac{1}{\sqrt{np}}).$$

The next theorem establishes the entrywise expansion of the KRC estimator based on the ER graph.

Theorem S1. *Let Assumptions (A1)-(A3) hold. Suppose that $np > c\log n$ for sufficiently large c. If $nMh^5 \to 0$, $\frac{\log n}{Mh} \to 0$ and $n \to \infty$, then for any fixed $i \in [n]$ and $t \in (0,1)$, we have the following expansion with probability tending to 1,*

$$\hat{\pi}_i(t) - \pi_i^*(t) = \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t) + \pi_j^*(t))\bar{\Delta}_{ij}(t) + \varepsilon_i(t),$$

where $\bar{\Delta}_{ij}(t) = \frac{\sum_{t_k\in T_{ji}}(y_{ji}(t_k) - y_{ji}^*(t_k))K_h(t,t_k)}{\sum_{t_k\in T_{ji}} K_h(t,t_k)}$. *Letting $f_i(t)$ denote the leading term* $\frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}}(\pi_i^*(t) + \pi_j^*(t))\bar{\Delta}_{ij}(t)$, *we have*

$$\sup_{t\in(0,1)} \max_i |f_i(t)| = O_p(\sqrt{\frac{\log n}{n^3pMh}}), \tag{S2.2}$$

$$\sup_{t\in(0,1)} \max_i |\varepsilon_i(t)| = O_p(\frac{\log n}{\sqrt{n^4pMh}}) + O_p(\frac{h^2}{n}). \tag{S2.3}$$

Remark S1. The conditions $\frac{\log n}{Mh} \to 0$ and $nMh^5 \to 0$ are introduced to simplify the the remainders and can be relaxed. We let $\mathcal{V}(f)$ represent the total variation of f, and impose additional assumptions that

$$\mathcal{V}(K) < \infty, \mathcal{V}(|\cdot|K) < \infty.$$

By replacing the conditions $\frac{\log n}{Mh} \to 0$ and $nMh^5 \to 0$ with $Mh \geq \mathcal{V}(K)$ and $h \log n \to 0$, Lemma 1 still holds with $\sup_{t\in(0,1)} \max_i |\varepsilon_i(t)| = O_p(\frac{h}{n}) + o_p(\sqrt{\frac{\log n}{n^3 pMh}})$.

Proof. Let $E(t) = P(t) - P^*(t)$. Define $Y_{kl}(t) = 2npP_{kl}(t)$. Let $d = 2np$. Utilizing the expansion of the group inverse, we have

$$
\begin{aligned}
&\hat{\pi}_i(t) - \pi_i^*(t) \\
&= \pi^*(t)^\top E(t) A^{\#}_{\cdot i}(t) + \hat{\pi}(t) E(t) A^{\#}(t) E(t) A^{\#}_{\cdot i}(t) \\
&= \frac{1}{d} \sum_{(k,l)\in\mathcal{E}, k<l} (\pi_k^*(t) + \pi_l^*(t))(Y_{kl}(t) - y_{kl}^*(t))(A^{\#}_{li}(t) - A^{\#}_{ki}(t)) + \hat{\pi}(t) E(t) A^{\#}(t) E(t) A^{\#}_{\cdot i}(t) \\
&= \frac{1}{d} \sum_{(k,l)\in\mathcal{E}, k<l} (\pi_k^*(t) + \pi_l^*(t))(Y_{kl}(t) - y_{kl}^*(t))(\tilde{A}_{li}(t) - \tilde{A}_{ki}(t)) \\
&\quad + \frac{1}{d} \sum_{(k,l)\in\mathcal{E}, k<l} (\pi_k^*(t) + \pi_l^*(t))(Y_{kl}(t) - y_{kl}^*(t))(A^{\#}_{li}(t) - \tilde{A}_{li}(t) + \tilde{A}_{ki}(t) - A^{\#}_{ki}(t)) \\
&\quad + \hat{\pi}(t) E(t) A^{\#}(t) E(t) A^{\#}_{\cdot i}(t). \qquad \text{(S2.4)}
\end{aligned}
$$

We use B_1, B_2 and B_3 to denote the three terms in the above equation, respectively. Utilizing the definition of $\tilde{A}$, we have

$$
\begin{aligned}
B_1 &= \frac{1}{d} \sum_{(k,l)\in\mathcal{E}, k<l} (\pi_k^*(t) + \pi_l^*(t))(Y_{kl}(t) - y_{kl}^*(t))(\tilde{A}_{li}(t) - \tilde{A}_{ki}(t)) \\
&= \frac{1}{d} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t) + \pi_j^*(t))(Y_{ji}(t) - y_{ji}^*(t)) \tilde{A}_{ii}(t) \\
&= \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t) + \pi_j^*(t))(Y_{ji}(t) - y_{ji}^*(t)). \qquad \text{(S2.5)}
\end{aligned}
$$

We further decompose (S2.5) to analyze the order.

$$
\begin{aligned}
(\text{S2.5}) &= \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t)+\pi_j^*(t))\Big(\frac{\sum_{t_k\in T_{ij}} y_{ji}(t_k)K_h(t,t_k)}{\sum_{t_k\in T_{ij}} K_h(t,t_k)} - y_{ji}^*(t)\Big)\\
&= \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t)+\pi_j^*(t))\frac{\sum_{t_k\in T_{ij}} (y_{ji}(t_k)-y_{ji}^*(t_k))K_h(t,t_k)}{\sum_{t_k\in T_{ij}} K_h(t,t_k)}\\
&\quad + \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t)+\pi_j^*(t))\frac{\sum_{t_k\in T_{ij}} (y_{ji}^*(t_k)-y_{ji}^*(t))K_h(t,t_k)}{\sum_{t_k\in T_{ij}} K_h(t,t_k)}.
\end{aligned}
\tag{S2.6}
$$

The first term in (S2.6) is $O_p(\sqrt{\frac{\log n}{n^3pMh}})$ using Hoeffding inequality. The second term is $O_p(\frac{h^2}{n})$ by the smoothness of $y_{ij}^*(t)$ and the boundedness of $\pi_i^*(t)$. Therefore, B_1 is $O_p(\sqrt{\frac{\log n}{n^3pMh}})+O_p(\frac{h^2}{n})$. We then present the order of B_2 and B_3. We have

$$
\begin{aligned}
B_2 &= \frac{1}{d}\sum_{(k,l)\in\mathcal{E},k<l} (\pi_k^*(t)+\pi_l^*(t))(Y_{kl}(t)-y_{kl}^*(t))(A_{li}^{\#}(t)-\tilde{A}_{li}(t)+\tilde{A}_{ki}(t)-A_{ki}^{\#}(t))\\
&= \frac{1}{d}\sum_{(k,l)\in\mathcal{E},k<l} (\pi_k^*(t)+\pi_l^*(t))(A_{li}^{\#}(t)-\tilde{A}_{li}(t)+\tilde{A}_{ki}(t)-A_{ki}^{\#}(t))\frac{\sum_{t_m\in T_{kl}} K_h(t-t_m)(y_{kl}(t_m)-y_{kl}^*(t_m))}{\sum_{t_m\in T_{kl}} K_h(t-t_m)}\\
&\quad + \frac{1}{d}\sum_{(k,l)\in\mathcal{E},k<l} (\pi_k^*(t)+\pi_l^*(t))(A_{li}^{\#}(t)-\tilde{A}_{li}(t)+\tilde{A}_{ki}(t)-A_{ki}^{\#}(t))\Big(\frac{\sum_{t_m\in T_{kl}} K_h(t-t_m)y_{kl}^*(t_m)}{\sum_{t_m\in T_{kl}} K_h(t-t_m)} - y_{kl}^*(t)\Big).
\end{aligned}
\tag{S2.7}
$$

The first term is $O_p(\sqrt{\frac{\log n}{n^4pMh}})$ using Hoeffding ineqaulity. The second term is $O_p(\frac{h^2}{n})$. Utilizing the similar technique in Tian et al. (2024), we can obtain that B_3 is $o_p(\frac{\log n}{\sqrt{n^4pMh}})+o_p(\frac{h^2}{n})$. Combing the above results, we have that

$$
\hat{\pi}_i(t)-\pi_i^*(t) = \frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t)+\pi_j^*(t))\frac{\sum_{t_k\in T_{ij}} (y_{ji}(t_k)-y_{ji}^*(t_k))K_h(t,t_k)}{\sum_{t_k\in T_{ij}} K_h(t,t_k)}
$$

$$+ O_p(\frac{\log n}{\sqrt{n^4 pMh}}) + O_p(\frac{h^2}{n}).$$

□

S2.2 Proof of Lemma 1

Proof. Define $v_{jkl} = \frac{K_h(t_l - t_k)}{\sum_{t_k \in T_{ji}} K_h(t_l - t_k)}(\pi_i^*(t_l) + \pi_j^*(t_l))\sqrt{y_{ji}^*(t_k)(1 - y_{ji}^*(t_k))}$ and $z_{jk} = \frac{y_{ji}(t_k) - y_{ji}^*(t_k)}{\sqrt{y_{ji}^*(t_k)(1 - y_{ji}^*(t_k))}}$ for j such that $(i,j) \in \mathcal{E}$, $k \in [M_{ji}]$. Here k is dependent on j and i, and we omit the symbols for simplicity without ambiguity. Using Theorem S1, since $m = o(\min\{\frac{1}{\sqrt{npMh^5}}, \frac{\sqrt{n}}{\log n}\})$, we have

$$\begin{aligned}
&\sum_{t \in \mathcal{S}} [\alpha_i(t)(\hat{\pi}_i(t) - \pi_i^*(t))]^2 \\
=&\sum_{l=1}^{m} \Big[\alpha_i(t_l)\Big(\frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)} \sum_{j:(i,j)\in\mathcal{E}} (\pi_i^*(t_l) + \pi_j^*(t_l))\bar{\Delta}_{ij}(t_l) + \varepsilon_i(t_l)\Big)\Big]^2 \\
=&z^\top \sum_{l=1}^{m} (\frac{\alpha_i(t_l)}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)})^2 v_l v_l^\top z + o_p(1),
\end{aligned}$$

where $v_l = (v_{j_1 1 l}, v_{j_2 1 l}, v_{j_1 2 l}, v_{j_3 1 l}, v_{j_2 2 l}, v_{j_1 3 l}, \ldots)^\top$ and $z = (z_{j_1 1}, z_{j_2 1}, z_{j_1 2}, z_{j_3 1}, z_{j_2 2}, z_{j_1 3}, \ldots)^\top$ for $\{j_1, j_2, \ldots\} = \{j \in [n] : (i,j) \in \mathcal{E}\}$. We define the matrix $W = \sum_{l=1}^{m} (\frac{\alpha_i(t_l)}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)})^2 v_l v_l^\top$ and the random variable $T_{nM} = z^\top W z$.

Then we demonstrate the asymptotic normality of T_{nM}. Let $B = \text{diag}(W)$ and $\tilde{T}_{nM} = z^\top (W - B) z$. We first consider the asymptotic distribution of $\frac{\tilde{T}_{nM}}{\sqrt{Var(\tilde{T}_{nM})}}$, and then prove the asymptotic normality of $\frac{T_{nM} - m}{\sqrt{2m}}$.

$\tilde{T}_{nM}$ is a quadratic form of independent random variables $\{z_i\}_{i=1,\ldots,d}$ with mean

0 and variance 1, where $d = \sum_{j:(i,j)\in\mathcal{E}} M_{ji}$. Note that $W - B$ is a symmetric matrix with diagonal elements equal to 0 and we have $E(T_{nM}) = tr(W) \to m$. Let and $W = (w_{ij})_{d\times d}$. We can obtain

$$\begin{aligned}
ET_{nM}^2 &= E(\sum_{1\le i,j\le d} z_i z_j w_{ij})^2 = E(\sum_{1\le i\ne j\le d} z_i z_j w_{ij})^2 + E(\sum_{i=1}^{d} z_i^2 w_{ii})^2 \\
&= E\tilde{T}_{nM}^2 + \sum_{i=1}^{d} w_{ii}^2 E z_i^4 + \sum_{1\le i\ne j\le d} w_{ii} w_{jj} E z_i^2 z_j^2 \\
&= E\tilde{T}_{nM}^2 + (\sum_{i=1}^{d} w_{ii} E z_i^2)^2 - (\sum_{i=1}^{d} w_{ii} E z_i^2)^2 + \sum_{i=1}^{d} w_{ii}^2 E z_i^4 + \sum_{1\le i\ne j\le d} w_{ii} w_{jj} E z_i^2 z_j^2 \\
&= Var(\tilde{T}_{nM}^2) + (ET_{nM})^2 + \sum_{i=1}^{d} w_{ii}^2 (E z_i^4 - (E z_i^2)^2).
\end{aligned}$$

Then we show that

$$\sigma_{\tilde{T}}^{-2} \max_{1\le i\le d} \sum_{1\le j\le d} w_{ij}^2 \to 0, \tag{S2.8}$$

where $\sigma_{\tilde{T}}^2 = Var(\tilde{T}_{nM})$, and there exists a constant c such that

$$\max_{1\le i\le d} E z_i^2 \mathbf{1}_{\{|z_i|>c\}} \to 0. \tag{S2.9}$$

Note that for $(i,j)\in\mathcal{E}$, we can obtain

$$w_{ij} = \sum_{l=1}^{m} (\frac{\alpha_i(t_l)}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)})^2 v_i v_j = O(\frac{m}{npMh}). \tag{S2.10}$$

Using $m = o(\sqrt{npMh^2})$, we have $\sigma_{\tilde{T}}^2 = Var(T_{nM}) + o(1) \to 2m$, which leads to (S2.8).

From Assumption (A1), we have that $\{z_i, i = 1, \ldots, d\}$ is uniformly bounded by $\sqrt{1+\kappa}$. Therefore, the condition (S2.9) holds for any c lager than $\sqrt{1+\kappa}$.

We use $\mu_s, s = 1, \ldots, d$ to denote the eigenvalues of $W - B$. We then show that the eigenvalues are negligible that

$$\sigma_{\tilde{T}}^{-2} \max_{1 \le i \le d} \mu_i^2 \to 0. \tag{S2.11}$$

By noticing that $(\frac{\max_{1\le i\le d}\mu_i^2}{\sigma_{\tilde{T}}^2})^2 \le \frac{\sum_{i=1}^d \mu_i^4}{\sigma_{\tilde{T}}^4}$ and $\sum_{i=1}^d \mu_i^4 = tr((W-B)^4)$, it is sufficient to prove $tr((W-B)^4) = o(m^2)$. We can obtain

$$\begin{aligned} tr((W-B)^4) =& tr(W^4) - 4tr(W^3B) + 4tr(W^2B^2) \\ &+ 2tr(WBWB) - 4tr(WB^3) + tr(B^4), \end{aligned}$$

where $tr(WB^3) = tr(B^4) = \sum_{i=1}^d w_{ii}^4 = o(1)$. Since B is a diagonal matrix with elements $o(1)$, it is sufficient to prove $tr(W^2)$, $tr(W^3)$ and $tr(W^4)$ are all o(m^2). We have

$$\begin{aligned} tr(W^2) &= tr(\sum_{l_1=1}^m (\frac{\alpha_i(t_{l_1})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_1})})^2 v_{l_1} v_{l_1}^\top \sum_{l_2=1}^m (\frac{\alpha_i(t_{l_2})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_2})})^2 v_{l_2} v_{l_2}^\top) \\ &= \sum_{l_1=1}^m \sum_{l_2=1}^m (\frac{\alpha_i(t_{l_1})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_1})})^2 (\frac{\alpha_i(t_{l_2})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_2})})^2 (\sum_{j:(i,j)\in\mathcal{E}} \sum_{k=1}^{M_{ij}} v_{jkl_1} v_{jkl_2})^2 \\ &= \sum_{l}^m (\frac{\alpha_i(t_l)}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)})^2 (\frac{\alpha_i(t_l)}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_l)})^2 (\sum_{j:(i,j)\in\mathcal{E}} \sum_{k=1}^{M_{ij}} v_{jkl} v_{jkl})^2 \\ &\quad + \sum_{l_1,l_2:l_1\ne l_2} (\frac{\alpha_i(t_{l_1})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_1})})^2 (\frac{\alpha_i(t_{l_2})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_2})})^2 (\sum_{j:(i,j)\in\mathcal{E}} \sum_{k=1}^{M_{ij}} v_{jkl_1} v_{jkl_2})^2. \end{aligned} \tag{S2.12}$$

We focus on the following term

$$(\frac{\alpha_i(t_{l_1})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_1})})^2 (\frac{\alpha_i(t_{l_2})}{\sum_{j:(i,j)\in\mathcal{E}} y_{ij}^*(t_{l_2})})^2 (\sum_{j:(i,j)\in\mathcal{E}} \sum_{k=1}^{M_{ij}} v_{jkl_1} v_{jkl_2})^2$$

$$= (\frac{h}{\int K(v)^2 dv \sqrt{\pi_i^*(t_{l_1}) \sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_j^*(t_{l_1})}{M_{ij}}} \sqrt{\pi_i^*(t_{l_2}) \sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_j^*(t_{l_2})}{M_{ij}}}} \times \frac{1}{\sum_{k=1}^M K_h(t_{l_1}-t_k)}$$
$$\times \frac{1}{\sum_{k=1}^M K_h(t_{l_2}-t_k)} \sum_{j:(i,j)\in\mathcal{E}} \sum_{k=1}^{M_{ij}} K_h(t_{l_1}-t_k) K_h(t_{l_2}-t_k)(\pi_i^*(t_{l_1})+\pi_j^*(t_{l_1}))$$
$$\times (\pi_i^*(t_{l_2})+\pi_j^*(t_{l_2}))(y_{ji}^*(t_k)(1-y_{ji}^*(t_k))))^2. \qquad \text{(S2.13)}$$

When $Mh \to \infty$, we have

$$\text{(S2.13)} \to \Big(\int K(v)^2 dv \sqrt{\pi_i^*(t_{l_1}) \sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_j^*(t_{l_1})}{M_{ij}}} \sqrt{\pi_i^*(t_{l_2}) \sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_j^*(t_{l_2})}{M_{ij}}}\Big)^{-2}$$
$$\times \Big(\sum_{j:(i,j)\in\mathcal{E}} \frac{1}{M_{ij}} \frac{\pi_i^*(t_{l_2})+\pi_j^*(t_{l_2})}{\pi_i^*(t_{l_1})+\pi_j^*(t_{l_1})} \pi_i^*(t_{l_1})\pi_j^*(t_{l_1}) \int K(v)K(v+\frac{t_{l_2}-t_{l_1}}{h})dv\Big)^2.$$
$$\text{(S2.14)}$$

We have $(S2.14) = 1$ when $l_1 = l_2$. When $l_1 \neq l_2$, we have $(S2.14) = (O((mh)^\varsigma))^2$ using Assumption (A4). Hence,

$$tr(W^2) = mO(1) + \frac{m(m-1)}{2} O((mh)^{2\varsigma}) = O(m + (m^{1+\varsigma}h^\varsigma)^2). \qquad \text{(S2.15)}$$

Similarly, we can obtain

$$tr(W^3) = O(m + (m^{1+\varsigma}h^\varsigma)^3), \qquad \text{(S2.16)}$$

$$tr(W^4) = O(m + (m^{1+\varsigma}h^\varsigma)^4). \qquad \text{(S2.17)}$$

Combing above results, we have $tr((W-B)^4) = o(m^2)$ when $m = o(\frac{1}{h^{\frac{2\varsigma}{1+2\varsigma}}})$. Therefore, we have (S2.11). Combining (S2.8), (S2.9) and (S2.11), we have

$$\frac{\tilde{T}_{nM}}{\sqrt{Var(\tilde{T}_{nM})}} \xrightarrow{\mathcal{D}} N(0,1) \qquad \text{(S2.18)}$$

using Theorem 5.2 in de Jong (1987). Further, we can obtain

$$T_{nM} - \tilde{T}_{nM} - \sum_{i=1}^{d} w_{ii} = \sum_{i=1}^{d} w_{ii} z_i^2 - \sum_{i=1}^{d} w_{ii}$$
$$= O_p(\sqrt{Var(\sum_{i=1}^{d} w_{ii} z_i^2)}) = O_p(\sqrt{\sum_{i=1}^{d} w_{ii}^2 Var(z_i^2)}) = o_p(1) \tag{S2.19}$$

and $\sum_{i=1}^{d} w_{ii} = m + o_p(1)$ using (S2.10).

Therefore, we have

$$\frac{\sum_{t\in\mathcal{S}}[\alpha_i(t)(\hat{\pi}_i(t) - \pi_i^*(t))]^2 - m}{\sqrt{2m}} = \frac{T_{nM} + o_p(1) - m}{\sqrt{2m}}$$
$$= \frac{\tilde{T}_{nM} + T_{nM} - \tilde{T}_{nM} - m + o_p(1)}{\sqrt{2m}} = \frac{\tilde{T}_{nM}}{\sqrt{2m}} + o_p(1) \xrightarrow{\mathcal{D}} N(0,1).$$

□

S2.3 Proof of Theorem 1

We need the following lemma, whose proof is provided in Section S4.2.

Lemma S2. *Let Assumptions (A1)-(A3) hold. If $h \to 0$, $Mh \to \infty$ and $n \to \infty$, then we have*

$$Var[\sum_{r=1}^{m} \alpha_i^2(t) f_i^2(t)] \to 2m.$$

Proof. We can obtain

$$\frac{\sum_{l=1}^{m}[\alpha_i(t_l)(\hat{\pi}_i(t_l) - \frac{1}{m}\sum_{k=1}^{m}\hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}}$$
$$= \frac{\sum_{l=1}^{m}[\alpha_i(t_l)(\hat{\pi}_i(t_l) - \pi_i^*(t_l) + \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^{m}\hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}}$$

$$= \frac{\sum_{t\in\mathcal{S}}[\alpha_i(t)(\hat{\pi}_i(t)-\pi_i^*(t))]^2 - m}{\sqrt{2m}} + \frac{\sum_{l=1}^m[\alpha_i(t_l)(\pi_i^*(t_l)-\frac{1}{m}\sum_{k=1}^m\hat{\pi}_i(t_k))]^2}{\sqrt{2m}}$$
$$+ \frac{\sum_{l=1}^m[2\alpha_i^2(t_l)(\hat{\pi}_i(t_l)-\pi_i^*(t_l))(\pi_i^*(t_l)-\frac{1}{m}\sum_{k=1}^m\hat{\pi}_i(t_k))]}{\sqrt{2m}}. \quad \text{(S2.20)}$$

Notice that under H_{i0}^a, $\pi_i^*(t)$ is a constant. We can obtain $\frac{1}{m}\sum_{k=1}^m\hat{\pi}_i(t_k)-\pi_i^*(t_l) = O_p(\frac{1}{\sqrt{mn^3pMh}})$, so the second term of (S2.20) is $o_p(1)$. For the last term in (S2.20), we have

$$\frac{\sum_{l=1}^m[\alpha_i^2(t_l)(\hat{\pi}_i(t_l)-\pi_i^*(t_l))(\pi_i^*(t_l)-\frac{1}{m}\sum_{k=1}^m\hat{\pi}_i(t_k))]}{\sqrt{m}}$$
$$= \frac{\frac{1}{m}\sum_{l=1}^m\sum_{k=1}^m[\alpha_i^2(t_l)(\hat{\pi}_i(t_l)-\pi_i^*)(\pi_i^*-\hat{\pi}_i(t_k))]}{\sqrt{m}}$$
$$= \frac{\frac{1}{m}\sum_{l=1}^m[\alpha_i^2(t_l)(\hat{\pi}_i(t_l)-\pi_i^*)(\pi_i^*-\hat{\pi}_i(t_l))]}{\sqrt{m}}$$
$$+ \frac{\frac{1}{m}\sum_{l=1}^m\sum_{k=1,k\neq l}^m[\alpha_i^2(t_l)(\hat{\pi}_i(t_l)-\pi_i^*)(\pi_i^*-\hat{\pi}_i(t_k))]}{\sqrt{m}}$$
$$= o_p(1) - \frac{\frac{1}{m}\sum_{l=1}^m\sum_{k=1,k\neq l}^m\alpha_i^2(t_l)f_i(t_l)f_i(t_k)}{\sqrt{m}}. \quad \text{(S2.21)}$$

We further bound the second term in (S2.21) as follows. Note that

$$Var\big(\sum_{l=1}^m\sum_{k\in[m]\backslash\{l\}}\alpha_i^2(t_l)f_i(t_l)f_i(t_k)\big)$$
$$= Cov(\sum_{l_1=1}^m\sum_{k_1\in[m]\backslash\{l_1\}}\alpha_i^2(t_{l_1})f_i(t_{l_1})f_i(t_{k_1}), \sum_{l_2=1}^m\sum_{k_2\in[m]\backslash\{l_2\}}\alpha_i^2(t_{l_2})f_i(t_{l_2})f_i(t_{k_2}))$$
$$= \sum_{l_1=1}^m\sum_{k\in[m]\backslash\{l\}}\sum_{l_2=1}^m\sum_{k_2\in[m]\backslash\{l_2\}}Cov([\alpha_i^2(t_{l_1})f_i(t_{l_1})f_i(t_{k_1})],[\alpha_i^2(t_{l_2})f_i(t_{l_2})f_i(t_{k_2})]);$$
$$\text{(S2.22)}$$

$$Cov(\alpha_i(t_s)f_i(t_s),\alpha_i(t_k)f_i(t_k)) = \alpha_i(t_s)\alpha_i(t_k)\frac{1}{(\sum_{j:(i,j)\in\mathcal{E}}y_{ij}^*(t_s))(\sum_{j:(i,j)\in\mathcal{E}}y_{ij}^*(t_k))}$$

$$\times \sum_{j:(i,j)\in\mathcal{E}} ((\pi_i^*(t_s)+\pi_j^*(t_s))(\pi_i^*(t_k)+\pi_j^*(t_k))Cov(\bar{\Delta}_{ij}(t_s),\bar{\Delta}_{ij}(t_k)), \tag{S2.23}$$

where $Cov(\bar{\Delta}_{ij}(t_s),\bar{\Delta}_{ij}(t_k)) = \frac{1}{(\sum_{l=1}^{M_{ji}} K_h(t_k,t_l))(\sum_{l=1}^{M_{ji}} K_h(t_s,t_l))} \sum_{t_l\in T_{ji}} K_h(t_k,t_l)K_h(t_s,t_l)y_{ji}^*(t_l)(1-y_{ji}^*(t_l))$. When $Mh\to\infty$, we have

$$(S2.23) \to \frac{\sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_i^*(t_s)+\pi_j^*(t_s)}{\pi_i^*(t_k)+\pi_j^*(t_k)}\pi_i^*(t_k)\pi_j^*(t_k)}{\sqrt{\pi_i^*(t_s)\sum_{j:(i,j)\in\mathcal{E}}\pi_j^*(t_s)}\sqrt{\pi_i^*(t_k)\sum_{j:(i,j)\in\mathcal{E}}\pi_j^*(t_k)}} \frac{\int K(v)K(v+\frac{t_s-t_k}{h})dv}{\int K^2(v)dv}.$$

Therefore, we have

$$\begin{aligned}
&Cov(\alpha_i(t_s)f_i(t_s),\alpha_i(t_k)f_i(t_k))\\
&=\frac{\sum_{j:(i,j)\in\mathcal{E}} \frac{\pi_i^*(t_s)+\pi_j^*(t_s)}{\pi_i^*(t_k)+\pi_j^*(t_k)}\pi_i^*(t_k)\pi_j^*(t_k)}{\sqrt{\pi_i^*(t_s)\sum_{j:(i,j)\in\mathcal{E}}\pi_j^*(t_s)}\sqrt{\pi_i^*(t_k)\sum_{j:(i,j)\in\mathcal{E}}\pi_j^*(t_k)}} \frac{\int K(v)K(v+\frac{t_s-t_k}{h})dv}{\int K^2(v)dv}+o(1)\\
&=O((mh)^{\varsigma}).
\end{aligned}$$

Hence, we can obtain that $E[\alpha_i^2(t_l)f_i(t_l)f_i(t_k)] = o(1)$. Then we consider the fourth moment of $f_i(t)$. Actually, from Lemma S2 we can obtain

$$E([\alpha_i^2(t_1)f_i(t_1)f_i(t_2)][\alpha_i^2(t_3)f_i(t_3)f_i(t_4)]) = O(\frac{1}{n^2}) + O((mh)^{2\varsigma}); \tag{S2.24}$$

$$E([\alpha_i^2(t_1)f_i(t_1)f_i(t_2)][\alpha_i^2(t_1)f_i(t_1)f_i(t_3)]) = O(\frac{1}{n^2}) + O((mh)^{\varsigma}); \tag{S2.25}$$

$$E([\alpha_i^2(t_1)f_i(t_1)f_i(t_2)][\alpha_i^2(t_1)f_i(t_1)f_i(t_2)]) = O(1). \tag{S2.26}$$

Therefore, we have

$$(S2.22) = O(m^4)\times(S2.24) + O(m^3)\times(S2.25) + O(m^2)\times(S2.26)$$

$$= O(\frac{m^4}{n^2}) + O(m^2) + O(m^4(mh)^{2\varsigma}) + O(m^3(mh)^{\varsigma}). \qquad \text{(S2.27)}$$

Thus, we can obtain $\sum_{l=1}^m \sum_{k=1,\, k\neq l}^m \alpha_i^2(t_l) f_i(t_l) f_i(t_k) = O_p(\sqrt{\frac{m^4}{n^2} + m^2 + m^4(mh)^{2\varsigma} + m^3(mh)^{\varsigma}})$.

Combining that (S2.21) is $o_p(1)$ using $m = o(n^2)$ and $m = o(\frac{1}{h^{\frac{2\varsigma}{1+2\varsigma}}})$, we have

$$\frac{\sum_{l=1}^m [\alpha_i(t_l)(\hat{\pi}_i(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}} \xrightarrow{\mathcal{D}} N(0,1).$$

By noting that $\hat{\alpha}_i(t) - \alpha_i(t) = O(n)$ and $m = o(nMh)$, we have

$$\frac{\sum_{l=1}^m [\hat{\alpha}_i(t_l)(\hat{\pi}_i(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}} \xrightarrow{\mathcal{D}} N(0,1).$$

Then we prove $P(T_{ai} > z_{1-\alpha}) \to 1$. Note that

$$\frac{\sum_{l=1}^m [\alpha_i(t_l)(\hat{\pi}_i(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}} = \frac{\sum_{t\in\mathcal{S}} [\alpha_i(t)(\hat{\pi}_i(t) - \pi_i^*(t))]^2 - m}{\sqrt{2m}}$$
$$+ \frac{\sum_{l=1}^m [\alpha_i^2(t_l)(2\hat{\pi}_i(t_l) - \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))(\pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))]}{\sqrt{2m}}. \qquad \text{(S2.28)}$$

The first term convergences to $N(0,1)$ using Lemma 1. For the second term, we have

$$\frac{\sum_{l=1}^m [\alpha_i^2(t_l)(2\hat{\pi}_i(t_l) - \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))(\pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))]}{\sqrt{2m}}$$
$$= \sqrt{\frac{m}{2}}\Big(\frac{1}{m}\sum_{l=1}^m \big(\alpha_i^2(t_l)(\pi_i^*(t_l) - \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) + 2\hat{\pi}_i(t_l) - 2\pi_i^*(t_l) + \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))$$
$$\times (\pi_i^*(t_l) - \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) + \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))\big)\Big). \qquad \text{(S2.29)}$$

When $m, n \to \infty$, we have

$$\frac{1}{m}\sum_{l=1}^m [\alpha_i^2(t_l)(\pi_i^*(t_l) - \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) + 2\hat{\pi}_i(t_l) - 2\pi_i^*(t_l) + \frac{1}{m}\sum_{l=1}^m \pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^m \hat{\pi}_i(t_k))$$

$$(\pi_i^*(t_l) - \frac{1}{m}\sum_{l=1}^{m}\pi_i^*(t_l) + \frac{1}{m}\sum_{l=1}^{m}\pi_i^*(t_l) - \frac{1}{m}\sum_{k=1}^{m}\hat{\pi}_i(t_k))] \xrightarrow{P} \int_0^1 \alpha_i^2(t)(\pi_i^*(t) - \int_0^1 \pi_i^*(s)ds)^2 dt,$$

where the right hand side is a constant of order nMh under H_{i1}^a. Therefore, we have $(S2.29) \to \infty$ in probability, which leads to

$$P\Big(\frac{\sum_{l=1}^{m}[\hat{\alpha}_i(t_l)(\hat{\pi}_i(t_l) - \frac{1}{m}\sum_{k=1}^{m}\hat{\pi}_i(t_k))]^2 - m}{\sqrt{2m}} > z_{1-\alpha}\Big) \to 1.$$

□

S2.4 Proof of Theorem 2

We need the following lemma, whose proof is provided in Section S4.3.

Lemma S3. *Let Assumptions (A1)-(A3) and H_0^b hold. If $h \to 0$, $Mh \to \infty$ and $n \to \infty$, then we have*

$$E[\sum_{s=1}^{m}(\frac{\alpha(t_s)}{\sqrt{2}}(f_i(t_s) - f_j(t_s)))^2]^2 \to m^2 + 2m.$$

Proof. We introduce some notations for further discussion. Let $z_{pqk} = \frac{y_{qp}(t_k) - y_{qp}^*(t_k)}{\sqrt{y_{qp}^*(t_k)(1-y_{qp}^*(t_k))}}$, $v_{ijkl} = \frac{4\pi_i^*(t_l)K_h(t_l-t_k)}{\sum_{t_k\in T_{ji}}K_h(t_l-t_k)}\sqrt{y_{ji}^*(t_k)(1-y_{ji}^*(t_k))}$, $v_{ipkl} = \frac{K_h(t_l-t_k)}{\sum_{t_k\in T_{pi}}K_h(t_l-t_k)}(\pi_i^*(t_l)+\pi_p^*(t_l))\sqrt{y_{pi}^*(t_k)(1-y_{pi}^*(t_k))}$ and $v_{jpkl} = -\frac{K_h(t_l-t_k)}{\sum_{t_k\in T_{pj}}K_h(t_l-t_k)}(\pi_j^*(t_l)+\pi_p^*(t_l))\sqrt{y_{pj}^*(t_k)(1-y_{pj}^*(t_k))}$ for items $i, j, p, q \in [n]$, $l \in [m]$. Here t_k is dependent on the correlated items, and we omit the symbols for simplicity.

We consider $(i, j) \in \mathcal{E}$ in the following proof. The case $(i, j) \notin \mathcal{E}$ is simpler. Since we can obtain $m\varepsilon = o_p(\sqrt{\frac{1}{n^3pMh}})$ by using $m = o(\min\{\frac{1}{\sqrt{npMh^5}}, \frac{\sqrt{n}}{\log n}\})$, we have

$$\sum_{l=1}^{m}[\frac{1}{\sqrt{2}}\alpha_i(t_l)(\hat{\pi}_i(t_l)-\hat{\pi}_j(t_l))]^2=\sum_{l=1}^{m}[\frac{1}{\sqrt{2}}\alpha_i(t_l)(f_i(t_l)-f_j(t_l))]^2+o_p(1)$$

$$=\sum_{l=1}^{m}[\frac{1}{\sqrt{2}}\alpha_i(t_l)\frac{1}{\sum_{s:(s,i)\in\mathcal{E}}y_{is}^*(t_l)}(\sum_{s:(i,s)\in\mathcal{E},s\neq j}(\pi_i^*(t_l)+\pi_s^*(t_l))\frac{\sum_{t_k\in T_{si}}(y_{si}(t_k)-y_{si}^*(t_k))K_h(t_l,t_k)}{\sum_{t_k\in T_{si}}K_h(t_l,t_k)}$$

$$-\sum_{s:(j,s)\in\mathcal{E},s\neq i}(\pi_j^*(t_l)+\pi_s^*(t_l))\frac{\sum_{t_k\in T_{sj}}(y_{sj}(t_k)-y_{sj}^*(t_k))K_h(t_l,t_k)}{\sum_{t_k\in T_{sj}}K_h(t_l,t_k)}$$

$$+4\pi_i^*(t_l)\frac{\sum_{t_k\in T_{ji}}(y_{ji}(t_k)-y_{ji}^*(t_k))K_h(t_l,t_k)}{\sum_{t_k\in T_{ji}}K_h(t_l,t_k)})]^2+o_p(1)$$

$$=z^\top\sum_{l=1}^{m}(\frac{\alpha_i(t_l)}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}}y_{is}^*(t_l)})^2v_lv_l^\top z+o_p(1),$$

where $z=(z_{ij1},z_{ii_11},z_{jj_11},\ldots)^\top$ and $v_l=(v_{ij1l},v_{ii_11l},v_{jj_11l},\ldots)^\top$ for $\{i_1,i_2,\ldots\}=\{s:(i,s)\in\mathcal{E},s\neq j\}$ and $\{j_1,j_2,\ldots\}=\{s:(j,s)\in\mathcal{E},s\neq i\}$. We use W to denote the matrix $\sum_{l=1}^{m}(\frac{\alpha_i(t_l)}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}}y_{is}^*(t_l)})^2v_lv_l^\top$ and let $T=z^\top Wz$. We can obtain $E(T)=tr(W)\to m$. Let $B=\mathrm{diag}(W)$, $\tilde{T}=z^\top(W-B)z$, $d=\sum_{l\neq i,j}M_{li}+\sum_{l\neq i,j}M_{lj}+M_{ij}$ and $\mu_s, s=1,\ldots,d$ be the eigenvalues of $W-B$. Notice that the components of z are independent random variables with mean 0 and variance 1.

We first deduce the asymptotic distribution of $\frac{\tilde{T}}{\sqrt{Var(\tilde{T})}}$, and then demonstrate the asymptotic normality of $\frac{T-m}{\sqrt{2m}}$. Note that

$$ET^2=E(\sum_{1\leq i,j\leq d}z_iz_jw_{ij})^2=E(\sum_{1\leq i\neq j\leq d}z_iz_jw_{ij})^2+E(\sum_{i=1}^{d}z_i^2w_{ii})^2$$

$$=E\tilde{T}^2+\sum_{i=1}^{d}w_{ii}^2Ez_i^4+\sum_{1\leq i\neq j\leq d}w_{ii}w_{jj}Ez_i^2z_j^2$$

$$= E\tilde{T}^2 + (\sum_{i=1}^{d} w_{ii} E z_i^2)^2 - (\sum_{i=1}^{d} w_{ii} E z_i^2)^2 + \sum_{i=1}^{d} w_{ii}^2 E z_i^4 + \sum_{1\le i\neq j\le d} w_{ii} w_{jj} E z_i^2 z_j^2$$
$$= Var(\tilde{T}^2) + (ET)^2 + \sum_{i=1}^{d} w_{ii}^2 (E z_i^4 - (E z_i^2)^2).$$

Let $\sigma_{\tilde{T}}^2$ denote $Var(\tilde{T})$. We can obtain that $w_{pq} = O(\frac{m}{npMh})$, and further we have $\sigma_{\tilde{T}}^2 \to 2m$ and

$$\sigma^{-2} \max_{1\le i\le d} \sum_{1\le j\le d} w_{ij}^2 = o(1). \tag{S2.30}$$

Note that $\{z_i, i = 1 \ldots, d\}$ is uniformly bounded by $\sqrt{1+\kappa}$ from Assumption (A1). Letting c be a constant lager than $\sqrt{1+\kappa}$, we can obtain

$$\max_{1\le i\le d} E z_i^2 \mathbf{1}_{\{|z_i|>c\}} \to 0. \tag{S2.31}$$

Following the same way of proving Theorem 1, we then show that $tr(W^2)$, $tr(W^3)$ and $tr(W^4)$ are all o(m^2).

$$tr(W^2) = tr(\sum_{l_1=1}^{m} (\frac{\alpha_i(t_{l_1})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_1})})^2 v_l v_l^\top \sum_{l_2=1}^{m} (\frac{\alpha_i(t_{l_2})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_2})})^2 v_l v_l^\top$$
$$= \sum_{l_1=1}^{m}\sum_{l_2=1}^{m} (\frac{\alpha_i(t_{l_1})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_1})})^2 (\frac{\alpha_i(t_{l_2})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_2})})^2 (v_{ij1l_1} v_{ij1l_2} + v_{i11l_1} v_{i11l_2} + \ldots)^2$$
$$= \sum_{l}^{m} (\frac{\alpha_i(t_l)}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_l)})^2 (\frac{\alpha_i(t_l)}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_l)})^2 (v_{ij1l} v_{ij1l} + v_{i11l} v_{i11l} + v_{j11l} v_{j11l} + \ldots)^2$$
$$+ \sum_{l_1,l_2:l_1\neq l_2} (\frac{\alpha_i(t_{l_1})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_1})})^2 (\frac{\alpha_i(t_{l_2})}{\sqrt{2}\sum_{s:(s,i)\in\mathcal{E}} y_{is}^*(t_{l_2})})^2 (v_{ij1l_1} v_{ij1l_2} + v_{i11l_1} v_{i11l_2} + \ldots)^2. \tag{S2.32}$$

Similar to the deduction of (S2.15), we have

$$tr(W^2) = O(m + (m^{1+\varsigma}h^{\varsigma})^2),$$

$$tr(W^3) = O(m + (m^{1+\varsigma}h^{\varsigma})^3),$$

$$tr(W^4) = O(m + (m^{1+\varsigma}h^{\varsigma})^4). \tag{S2.33}$$

In addition, we have $tr((W-B)^4) = o(m^2)$ using $m = o(\frac{1}{h^{\frac{2\varsigma}{1+2\varsigma}}})$. Combing (S2.30) and (S2.31), we have

$$\frac{\tilde{T}}{\sqrt{Var(\tilde{T})}} \xrightarrow{\mathcal{D}} N(0,1) \tag{S2.34}$$

using Theorem 5.2 in de Jong (1987).

Further, we can obtain $T - \tilde{T} - \sum_{i=1}^{d} w_{ii} = o_p(1)$ and $\sum_{i=1}^{d} w_{ii} = m + o_p(1)$ using $w_{pq} = O(\frac{m}{npMh})$, $p, q = 1, \ldots, d$. Therefore, we have

$$\frac{\sum_{l=1}^{m}[\frac{1}{\sqrt{2}}\alpha_i(t_l)(\hat{\pi}_i(t_l) - \hat{\pi}_j(t_l))]^2 - m}{\sqrt{2m}} = \frac{T + o_p(1) - m}{\sqrt{2m}} = \frac{\tilde{T} + T - \tilde{T} - m + o_p(1)}{\sqrt{2m}}$$
$$= \frac{\tilde{T}}{\sqrt{2m}} + o_p(1) \xrightarrow{\mathcal{D}} N(0,1).$$

As mentioned in Theorem 1, we have $\hat{\alpha}_i(t) - \alpha_i(t) = O(n)$ and $m = o(nMh)$. Hence, we can obtain

$$\sum_{l=1}^{m}[\frac{1}{\sqrt{2}}\alpha_i(t_l)(\hat{\pi}_i(t_l) - \hat{\pi}_j(t_l))]^2 - \sum_{t\in\mathcal{S}}[\frac{1}{\sqrt{2}}\hat{\alpha}_i(t)(\hat{\pi}_i(t) - \hat{\pi}_j(t))]^2 = o(\sqrt{m}),$$

which concludes the first part of the Theorem.

When $m, n \to \infty$, we have

$$\frac{1}{m}\sum_{t\in\mathcal{S}}[\frac{1}{\sqrt{2}}\hat{\alpha}_i(t)(\hat{\pi}_i(t) - \hat{\pi}_j(t))]^2 - 1 \to \int_0^1 [\frac{1}{\sqrt{2}}\alpha_i^*(t)(\pi_i^*(t) - \pi_j^*(t))]^2 dt + O(1),$$

the right-hand side of which is a constant of order nMh under H_1. Therefore, we can obtain

$$\frac{\sum_{t\in\mathcal{S}}[\frac{1}{\sqrt{2}}\hat{\alpha}_i(t)(\hat{\pi}_i(t)-\hat{\pi}_j(t))]^2-m}{\sqrt{2m}}\to\infty.$$

□

S3 Proof of Results in Section 4

S3.1 Proof of Theorem 3

Proof. Recall that in Theorem S1, we have

$$\hat{\pi}_i(t)-\pi_i^*(t)=\frac{1}{\sum_{j:(i,j)\in\mathcal{E}}y_{ij}^*(t)}\sum_{j:(i,j)\in\mathcal{E}}(\pi_i^*(t)+\pi_j^*(t))\bar{\Delta}_{ij}(t)+\varepsilon_i(t),$$

where $\bar{\Delta}_{ij}(t)=\frac{\sum_{t_k\in T_{ji}}(y_{ji}(t_k)-y_{ji}^*(t_k))K_h(t,t_k)}{\sum_{t_k\in T_{ji}}K_h(t,t_k)}$. Applying Hoeffding inequality, we can obtain

$$P(|\sum_{j:(i,j)\in\mathcal{E}}\sum_{t_k\in T_{ji}}\frac{(\pi_i^*(t)+\pi_j^*(t))K_h(t,t_k)(y_{ji}(t_k)-y_{ji}^*(t_k))}{\sum_{t_k\in T_{ji}}K_h(t,t_k)}|\geq x)$$
$$\leq 2\exp(-\frac{x^2}{\sum_{j:(i,j)\in\mathcal{E}}\sum_{t_k\in T_{ji}}\frac{2(\pi_i^*(t)+\pi_j^*(t))^2K_h^2(t,t_k)}{(\sum_{t_k\in T_{ji}}K_h(t,t_k))^2}}).$$

Set $x=\sqrt{b_0\sum_{j:(i,j)\in\mathcal{E}}\sum_{t_k\in T_{ji}}\frac{2(\pi_i^*(t)+\pi_j^*(t))^2K_h^2(t,t_k)}{(\sum_{t_k\in T_{ji}}K_h(t,t_k))^2}\log n}$, where b_0 is a constant. When $Mh\to\infty$ and $h\to 0$, we have

$$P\Big(|\sum_{j:(i,j)\in\mathcal{E}}\sum_{t_k\in T_{ji}}\frac{(\pi_i^*(t)+\pi_j^*(t))K_h(t,t_k)(y_{ji}(t_k)-y_{ji}^*(t_k))}{\sum_{t_k\in T_{ji}}K_h(t,t_k)}|\geq$$

$$\sqrt{\tfrac{b_0}{Mh}\sum_{j:(i,j)\in\mathcal{E}} 2(\pi_i^*(t)+\pi_j^*(t))^2 \int K^2(v)dv \log n}\Big) \quad \leq \frac{2}{n^{b_0}}.$$

Therefore, we have $P(|f_i(t)| \geq C\sqrt{\frac{\log n}{n^3pMh}}) \leq \frac{2}{n^{b_0}}$, where C is a constant independent of i. Let $T_s = \{t_1, t_2, \ldots, t_s\} = \{\frac{1}{s}, \frac{2}{s}, \ldots, 1\}$, then we can obtain

$$P(\max_{t\in T_s} |f_i(t)| \geq C\sqrt{\frac{\log n}{n^3pMh}}) \leq s \times P(|f_i(t)| \geq C\sqrt{\frac{\log n}{n^3pMh}}) \leq \frac{2s}{n^{b_0}}. \qquad \text{(S3.35)}$$

Since $|\dot{y}_{ij}^*(t)|, i, j \in [n], t \in [0,1]$ is uniformly bounded by c_1, we can deduce that $|\dot{f}_i(t)| \leq C_1\frac{1}{nh}$, where C_1 is a constant depending on c_1 and independent of i and t. Thus, we have

$$|\sup_t f_i(t) - \max_{t\in T_s} f_i(t)| \leq C_1\frac{1}{snh}. \qquad \text{(S3.36)}$$

Combining (S3.35) and (S3.36) yields

$$P(\sup_t |f_i(t)| \geq C\sqrt{\frac{\log n}{n^3pMh}} + C_1\frac{1}{snh}) \leq \frac{2s}{n^{b_0}},$$

which leads to

$$P(\max_i \sup_t |f_i(t)| \geq C\sqrt{\frac{\log n}{n^3pMh}} + C_1\frac{1}{snh}) \leq \frac{2ns}{n^{b_0}}.$$

Setting $s = \sqrt{\frac{nM}{h\log n}}$ and $b_0 = 6$, we conclude the theorem. □

S3.2 Proof of Proposition 1

Proof. Notice that $E_2 = \{\forall t \in T, \forall j \neq i, \pi_i^*(t) - \pi_j^*(t) \in [\hat{\pi}_i(t) - \hat{\pi}_j(t) - \frac{S_{1-\alpha}}{\gamma_{ij}}, \hat{\pi}_i(t) - \hat{\pi}_j(t) + \frac{S_{1-\alpha}}{\gamma_{ij}}]\}$. If there is item j such that $\hat{\pi}_i(t) - \hat{\pi}_j(t) - \frac{S_{1-\alpha}}{\gamma_{ij}} > 0$, then we have

$\pi_i^*(t) - \pi_j^*(t) > 0$. Further, we can obtain

$$\bar{r}_i(t) = n - \sum_{j\in[n]} \mathbb{1}(\pi_i^*(t) - \pi_j^*(t) > 0) \leq n - \sum_{j\in[n]} \mathbb{1}(\hat{\pi}_i(t) - \hat{\pi}_j(t) - \frac{S_{1-\alpha}}{\gamma_{ij}} > 0) = R_u(t).$$

Similarly, we can obtain $\underline{r}_i(t) \geq R_l(t)$. □

S3.3 Proof of Theorem 4

Proof. We omit the superscript in $x_{kl}^{(ij)}$ for simplicity and omit t when there is no confusion. Recall that $n_{ij} = n_i + n_j - \mathbb{1}((i,j) \in \mathcal{E})$. Define

$$S_0 = \max_{t\in\mathcal{T}} \max_{j:j\neq i} |\sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} x_{kl}|,$$

and

$$\begin{aligned} V_0 =& \max_{t\in\mathcal{T}} \max_{j:j\neq i} |\sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} x_{kl} z_{kl}| \\ =& \max_{t\in\mathcal{T}} \max_{j:j\neq i} \max\{\sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} x_{kl} z_{kl}, -\sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} x_{kl} z_{kl}\}. \end{aligned}$$

We then prove the following two statements, which are sufficient according to Theorem 2.1 in Chernozhuokov et al. (2022) and the proof of Corollary 3.1 in Chernozhukov et al. (2013).

- There exists B, which may tend to infinity, such that $c_0 \leq \frac{1}{Mn_{ij}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} E(x_{kl}^2) \leq C_1$, $\max_{k_0=1,2} \frac{1}{Mn_{ij}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} E[|x_{kl}|^{2+k_0}/B^{k_0}] + E[\exp(|x_{kl}|/B)] \leq 4$ and $\frac{B^2(\log(v(n-1)Mn_{ij}))^7}{Mn_{ij}} = o(1)$, where c_0 and C_1 are global positive constants.

- $P(|S - S_0| > \zeta_1) < \zeta_2$, $P(P(|V - V_0| > \zeta_1|y) > \zeta_2) < \zeta_2$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and $\zeta_1\sqrt{\log(v(n-1))} + \zeta_2 = o(1)$.

When n is large enough, we have

$$|x_{kl}| \leq \frac{\sqrt{\gamma_{ij}^2 n_{ij} M} K_h(t,t_k)}{\sum_{k=1}^M K_h(t,t_k)} \frac{(\pi_i^*(t) + \pi_l^*(t))}{\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t)} \lesssim \sqrt{\frac{1}{h}}.$$

Besides, we can obtain

$$\begin{aligned}
&\frac{1}{Mn_{ij}} \sum_{k=1}^M \sum_{l=1}^{n_{ij}} E(x_{kl}^2) \\
&= \frac{1}{Mn_{ij}} \sum_{k=1}^M \Big(\sum_{l:(i,l)\in\mathcal{E}, l\neq j} \frac{\gamma_{ij}^2 n_{ij} M K_h^2(t,t_k)}{(\sum_{k=1}^M K_h(t,t_k))^2} \frac{(\pi_i^*(t) + \pi_l^*(t))^2 y_{li}^*(t_k)(1 - y_{li}^*(t_k))}{(\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t))^2} \\
&\quad + \sum_{l:(j,l)\in\mathcal{E}, l\neq i} \frac{\gamma_{ij}^2 n_{ij} M K_h^2(t,t_k)}{(\sum_{k=1}^M K_h(t,t_k))^2} \frac{(\pi_j^*(t) + \pi_l^*(t))^2 y_{lj}^*(t_k)(1 - y_{lj}^*(t_k))}{(\sum_{(j,l)\in\mathcal{E}} y_{jl}^*(t))^2} \\
&\quad + \frac{\gamma_{ij}^2 n_{ij} M K_h^2(t,t_k)}{(\sum_{k=1}^M K_h(t,t_k))^2} (\pi_j^*(t) + \pi_i^*(t))^2 y_{ji}^*(t_k)(1 - y_{ji}^*(t_k)) \Big(\frac{1}{\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t)} + \frac{1}{\sum_{(j,l)\in\mathcal{E}} y_{jl}^*(t)}\Big)^2 \Big).
\end{aligned} \tag{S3.37}$$

Let B_1, B_2, B_3 represent the three terms on the right-hand side. We have

$$B_1 \asymp \frac{\gamma_{ij}^2}{Mh} \sum_{l:(i,l)\in\mathcal{E}, l\neq j} \frac{\pi_i^*(t)\pi_l^*(t)}{(\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t))^2} \int K^2(v)dv, \tag{S3.38}$$

which has uniform positive upper and lower bounds depending on κ. Similarly, B_2 has a uniform upper bound. In addition, we have

$$B_3 \asymp \frac{\gamma_{ij}^2}{Mh} \int K^2(v)dv\, \pi_i^*(t)\pi_j^*(t)\Big(\frac{1}{\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t)} + \frac{1}{\sum_{(j,l)\in\mathcal{E}} y_{jl}^*(t)}\Big)^2. \tag{S3.39}$$

Combining above results, there exist global constants $c_0, C_1 > 0$, such that

$$c_0 \leq \frac{1}{Mn_{ij}} \sum_{k=1}^M \sum_{l=1}^{n_{ij}} E(x_{kl}^2) \leq C_1.$$

Let $B = \tilde{c}\sqrt{\frac{1}{h}}$, where $\tilde{c}$ is large enough such that $\max\{2, C_1\}|x_{kl}| \leq B$ when n is large enough. Then we have $\max_{k=1,2} \frac{1}{Mn_{ij}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} E[|x_{kl}|^{2+k}/B^k] + E[\exp(|x_{kl}|/B)] \leq 4$. Since $a_1, b_1 < 1$, we have $\frac{(\log(vMn^2))^7}{npMh} \to 0$ and the condition 1 is satisfied.

Notice that

$$|V - V_0| \leq \max_{t\in\mathcal{T}} \max_{j:j\neq i} |\sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} (\hat{x}_{kl} - x_{kl}) z_{kl}|.$$

Using Theorem 3, we have $\sup_t \max_{i,l} |y^*_{il}(t) - \hat{y}_{il}(t)| \lesssim \sqrt{\frac{\log(nM)}{npMh}}$. We define $w_{tj} = \sqrt{\frac{1}{Mn_{ij}}} \sum_{k=1}^{M} \sum_{l=1}^{n_{ij}} (\hat{x}_{kl} - x_{kl}) z_{kl}$. With probability larger than $1 - \frac{1}{(nM)^5}$, we have

$$\max_{t\in\mathcal{T}} \max_{j:j\neq i} Var(|w_{tj}| \,|\, y) \lesssim \frac{\log(nM)}{npMh}.$$

Since $w_{tj}|y$ follows the normal distribution, the maximal inequality leads to

$$E[\max_{t\in\mathcal{T}} \max_{j:j\neq i} |w_{tj}| \,|\, y] \lesssim \sqrt{\frac{\log(nM)\log(vn)}{npMh}}.$$

Using Borell inequality, we can obtain

$$P(\max_{t\in\mathcal{T}} \max_{j:j\neq i} w_{tj} \gtrsim \sqrt{\frac{\log(nM)\log(vn)}{npMh}} \,|\, y) < \frac{1}{(vn)^{b_1}},$$

where b_1 can be any positive constant. Thus,

$$P(P(|V - V_0| \gtrsim \sqrt{\frac{\log(nM)\log(vn)}{npMh}} \,|\, y) > \frac{1}{(vn)^{b_1}}) < \frac{1}{(nM)^5}.$$

Using Theorem S1 and (S3.36), we have

$$|S - S_0| \lesssim \gamma_{ij}\varepsilon + \frac{\sqrt{npMh}}{hv},$$

with probability tending to 1. Therefore, we set $\zeta_1 = \max\{\frac{\log(nM)\log(vn)}{npMh}, \gamma_{ij}\varepsilon + \frac{\sqrt{npMh}}{hv}\}$, $\zeta_2 = \max\{\frac{1}{(vn)^{b_1}}, \frac{1}{(nM)^5}\}$, which leads to condition 2 and concludes the proof. □

S3.4 Proof of Proposition 2

Proof. If there exists $t_0 \in T$, such that $\hat{R}_l(t_0) \neq i$ or $\hat{R}_u(t_0) \neq i$, then there exists j, such that

$$|\hat{\pi}_j(t_0) - \pi_j^*(t_0)| + |\hat{\pi}_i(t_0) - \pi_i^*(t_0)| + \frac{V_{1-\alpha}}{\gamma_{ij}} \geq \min_{j:j\neq i} |\pi_i^*(t_0) - \pi_j^*(t_0)|$$
$$\geq \min_{j:j\neq i} \inf_{t\in T} |\pi_i^*(t) - \pi_j^*(t)|. \tag{S3.40}$$

Notice that when n is large enough, with $\alpha > 0$, we have

$$P(\sup_{t\in T} \max_{j:j\neq i} \gamma_{ij}|\hat{\pi}_i(t) - \pi_i^*(t) - \hat{\pi}_j(t) + \pi_j^*(t)| > V_{1-\alpha}) > \frac{\alpha}{2}.$$

From Theorem 3, with probability tending to 1, we have

$$\sup_{t\in T} \|\hat{\pi}(t) - \pi^*(t)\|_\infty \leq c_3\sqrt{\frac{\log(nM)}{n^3pMh}}.$$

Therefore, $\frac{V_{1-\alpha}}{\gamma_{ij}} \lesssim \sqrt{\frac{\log(nM)}{n^3pMh}}$ with probability tending to 1.

If $\min_{j:j\neq i} \inf_{t\in T} |\pi_i^*(t) - \pi_j^*(t)| \gg \sqrt{\frac{\log(nM)}{n^3pMh}}$, we can obtain that the left hand side of $(S3.40) \gg \sqrt{\frac{\log(nM)}{n^3pMh}}$ with probability tending to 1. This contradicts the uniform estimation error and the bound of $V_{1-\alpha}/\gamma_{ij}$. Therefore, we have $P(\{\text{There exists } t_0 \in T, \text{ such that } \hat{R}_l(t_0) \neq i \text{ or } \hat{R}_u(t_0) \neq i\}) \to 0$. □

S3.5 Proof of Theorem 5

Proof. We omit the symbol α in $\tilde{\Pi}_t(\alpha)$ in this proof for simplicity. Notice that the ranks that suit $R_i(t)$ are equivalent to the set of scores that satisfy certain conditions $\Pi_t := \{\pi(t) : \pi_i(t) - \pi_j(t), j \neq i$ have proper signs such that $rr(\pi(t)) \subset R_i(t)\}$. Under H_0, there exists $t_0 \in T$, such that $r^*(t_0) \notin R_i(t_0)$, and hence $\pi^*(t_0) \notin \Pi_{t_0}$. Therefore, we have

$$P(\text{reject } H_0) = P(\text{for any } t \in T, \text{ for any } \pi(t) \in \tilde{\Pi}_t, \text{ we have } \pi(t) \in \Pi_t)$$
$$\leq P(\exists t_0, \exists j \neq i, \pi_i^*(t_0) - \pi_j^*(t_0) \notin [\hat{\pi}_i(t_0) - \hat{\pi}_j(t_0) - \frac{V_{1-\alpha}}{\gamma_{ij}}, \hat{\pi}_i(t_0) - \hat{\pi}_j(t_0) + \frac{V_{1-\alpha}}{\gamma_{ij}}]),$$

whose probability is less than α when $n \to \infty$.

Under H_1, for all $t \in T$, we have $r^*(t) \in R_i(t)$. Hence,

$$\{\text{There exists } t_0 \in T, \pi \in \tilde{\Pi}_{t_0}, r \in rr(\pi), \text{ such that } r \notin R_i(t_0)\}$$
$$\subset \{\exists t_0, \|\hat{\pi}(t_0) - \pi^*(t_0)\|_\infty + \frac{V_{1-\alpha}}{\gamma_{ij}} \geq \frac{\Delta(\pi^*, R_i, t_0)}{2}\}$$
$$\subset \{\sup_{t\in T} \|\hat{\pi}(t) - \pi^*(t)\|_\infty + \frac{V_{1-\alpha}}{\gamma_{ij}} \geq \frac{\tilde{\Delta}(\pi^*, R_i, T)}{2}\}.$$

Therefore, we have

$$P(\text{accept } H_0) \leq P(\cup_{t_0\in T} \cup_{\pi\in\tilde{\Pi}_{t_0}} \cup_{r\in rr(\pi)} \{r \notin R_i(t_0)\})$$
$$\leq P(\{\sup_{t\in T} \|\hat{\pi}(t) - \pi^*(t)\|_\infty + \frac{V_{1-\alpha}}{\gamma_{ij}} \geq \frac{\tilde{\Delta}(\pi^*, R_i, T)}{2}\}). \tag{S3.41}$$

From the proof of Proposition 2, with probability tending to 1, we have

$$\frac{V_{1-\alpha}}{\gamma_{ij}} \leq 2c_3\sqrt{\frac{\log(nM)}{n^3pMh}}.$$

Hence, there exists a constant c_4 such that if $\tilde{\Delta}(\pi^*, R_i, T) > c_4\sqrt{\frac{\log(nM)}{n^3pMh}}$, we have (S3.41) holds with probability tending to 0. □

S3.6 Proof of Proposition 3

Proof. Similar to Proposition 1, we have

$$E_4 \subset \{\forall t \in T,\ \text{if } s_j(t) > 0,\ \text{then } \pi_i^*(t) - \pi_j^*(t) \in (-1, \hat{\pi}_i(t) - \hat{\pi}_j(t) + \frac{S_{1-\alpha}^{\dagger}}{\gamma_{ij}}]$$
$$\text{and } \forall t \in T,\ \text{if } s_j(t) < 0,\ \text{then } \pi_i^*(t) - \pi_j^*(t) \in [\hat{\pi}_i(t) - \hat{\pi}_j(t) - \frac{S_{1-\alpha}^{\dagger}}{\gamma_{ij}}, 1)\} \subset E_3.$$

□

S4 Proof of Lemmas in Section S2

S4.1 Proof of Lemma S1

Proof. Recall that $\pi^*(t)$ represents the latent score vector. Let e represent the $n \times 1$ vector $(1, \ldots, 1)^\top$. Utilizing the property of group inverse, we have

$$(A(t) + e\pi^*(t)^\top)A^{\#}(t) = I - e\pi^*(t)^\top. \tag{S4.42}$$

Therefore, for $i \in [n]$, we have

$$(A(t) + e\pi^*(t)^\top)(A^{\#}_{\cdot i}(t) - \tilde{A}_{\cdot i}(t)) = I_{\cdot i} - e\pi_i(t) - (A(t) + e\pi^*(t)^\top)A_{\cdot i}(t).$$

We define that $R = I_{\cdot i} - e\pi_i(t) - (A(t) + e\pi^*(t)^\top)A_{\cdot i}(t)$. Utilizing the definition of $\tilde{A}(t)$, we can obtain

$$
\begin{aligned}
&A_{i\cdot}(t)\tilde{A}_{\cdot i}(t) = A_{ii}(t)\tilde{A}_{ii}(t) = 1\\
&A_{j\cdot}(t)\tilde{A}_{\cdot i}(t) = A_{ji}(t)\tilde{A}_{ii}(t) = -\frac{1}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} y^*_{ji}(t),\\
&\pi^*(t)^\top \tilde{A}_{\cdot i}(t) = \pi^*_i(t)\tilde{A}_{ii}(t) = \pi^*_i(t)\frac{2np}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)}.
\end{aligned}
$$

Therefore, we have

$$
R_j = \begin{cases} -\pi^*_i(t)(\frac{2np}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} + 1) & j = i,\\ \pi^*_i(t)(\frac{1}{(\pi^*_i(t)+\pi^*_j(t))\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} + \frac{2np}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} - 1) & j \neq i. \end{cases}
$$

Noticing that $\frac{2np}{\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} = O(1)$ and $\frac{1}{(\pi^*_i(t)+\pi^*_j(t))\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t)} = O(1/p)$, we can obtain $\|R\|_2 = O(1/p\|e\pi_i\|_2) = O(1/(\sqrt{n}p))$. Since $np > c\log n$ for sufficiently large c, we have $1 - \max\{\lambda_2(P^*), -\lambda_n(P^*)\} \gtrsim c$ using Lemma 4 in Negahban et al. (2017). Therefore, we can obtain $\lambda_{\min}(A(t) + e\pi^*(t)^\top) = O_p(1)$. Combining (S4.42), we have

$$
\|A^{\#}_{\cdot i}(t) - \tilde{A}_{\cdot i}(t)\|_2 = O_p(\frac{1}{\sqrt{np}}). \tag{S4.43}
$$

□

S4.2 Proof of Lemma S2

Proof. Note that

$$
E[\alpha_i(t_1)f_i(t_1)\, \alpha_i(t_2)f_i(t_2)\, \alpha_i(t_3)f_i(t_3)\, \alpha_i(t_4)f_i(t_4)]
$$

$$= \frac{\alpha_i(t_1)\alpha_i(t_2)\alpha_i(t_3)\alpha_i(t_4)}{(\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t_1))(\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t_2))(\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t_3))(\sum_{j:(i,j)\in\mathcal{E}} y^*_{ij}(t_4))}$$

$$\sum_{j_1:(i,j_1)\in\mathcal{E}}\sum_{j_2:(i,j_2)\in\mathcal{E}}\sum_{j_3:(i,j_3)\in\mathcal{E}}\sum_{j_4:(i,j_4)\in\mathcal{E}} (\pi^*_i(t_1)+\pi^*_{j_1}(t_1))(\pi^*_i(t_2)+\pi^*_{j_2}(t_2))$$

$$(\pi^*_i(t_3)+\pi^*_{j_3}(t_3))(\pi^*_i(t_4)+\pi^*_{j_4}(t_4))E[\bar{\Delta}_{ij_1}(t_1)\bar{\Delta}_{ij_2}(t_2)\bar{\Delta}_{ij_3}(t_3)\bar{\Delta}_{ij_4}(t_4)].$$

We can further calculate the expectation that

$$E[\bar{\Delta}_{ij_1}(t_1)\bar{\Delta}_{ij_1}(t_2)\bar{\Delta}_{ij_2}(t_3)\bar{\Delta}_{ij_2}(t_4)] = E[\bar{\Delta}_{ij_1}(t_1)\bar{\Delta}_{ij_1}(t_2)]E[\bar{\Delta}_{ij_2}(t_3)\bar{\Delta}_{ij_2}(t_4)]$$

$$= \frac{1}{(\sum_{l=1}^{M_{ij_1}} K_h(t_1,t_l))(\sum_{l=1}^{M_{ij_1}} K_h(t_2,t_l))}\sum_{t_l\in T_{j_1 i}} K_h(t_1,t_l)K_h(t_2,t_l)y^*_{j_1 i}(t_l)(1-y^*_{j_1 i}(t_l))$$

$$\times \frac{1}{(\sum_{l=1}^{M_{ij_2}} K_h(t_3,t_l))(\sum_{l=1}^{M_{ij_2}} K_h(t_4,t_l))}\sum_{t_l\in T_{j_2 i}} K_h(t_3,t_l)K_h(t_4,t_l)y^*_{j_2 i}(t_l)(1-y^*_{j_2 i}(t_l)).$$

Letting $g_D(s_1,s_2,s_3,s_4) = K_h(t_1,s_1)K_h(t_2,s_2)K_h(t_3,s_3)K_h(t_4,s_4)$, we have

$$E[\bar{\Delta}_{ij}(t_1)\bar{\Delta}_{ij}(t_2)\bar{\Delta}_{ij}(t_3)\bar{\Delta}_{ij}(t_4)]$$

$$= \frac{1}{(\sum_{t_k\in T_{ji}} K_h(t_1,t_k))(\sum_{t_k\in T_{ji}} K_h(t_2,t_k))(\sum_{t_k\in T_{ji}} K_h(t_3,t_k))(\sum_{t_k\in T_{ji}} K_h(t_4,t_k))}$$

$$\times \{\sum_{k\in T_{ji}} K_h(t_1,t_k)K_h(t_2,t_k)K_h(t_3,t_k)K_h(t_4,t_k)[(1-y^*_{ji}(t_k))^4 y^*_{ji}(t_k) + (1-y^*_{ji}(t_k))y^*_{ji}(t_k)^4]$$

$$+ \sum_{k_1\in T_{ji}}\sum_{k_2\in T_{ji},\,k_2\neq k_1} g_D(t_{k_1},t_{k_1},t_{k_2},t_{k_2})y^*_{ji}(t_{k_1})(1-y^*_{ji}(t_{k_1}))y^*_{ji}(t_{k_2})(1-y^*_{ji}(t_{k_2}))$$

$$+ \sum_{k_1\in T_{ji}}\sum_{k_2\in T_{ji},\,k_2\neq k_1} g_D(t_{k_1},t_{k_2},t_{k_1},t_{k_2})y^*_{ji}(t_{k_1})(1-y^*_{ji}(t_{k_1}))y^*_{ji}(t_{k_2})(1-y^*_{ji}(t_{k_2}))$$

$$+ \sum_{k_1\in T_{ji}}\sum_{k_2\in T_{ji},\,k_2\neq k_1} g_D(t_{k_1},t_{k_2},t_{k_2},t_{k_1})y^*_{ji}(t_{k_1})(1-y^*_{ji}(t_{k_1}))y^*_{ji}(t_{k_2})(1-y^*_{ji}(t_{k_2}))\}.$$

All the other fourth moments of $\bar{\Delta}$ vanish. Therefore, by Letting $Mh \to \infty$, we can obtain

$$
\begin{aligned}
&E[\alpha_i(t_1)f_i(t_1)\,\alpha_i(t_2)f_i(t_2)\,\alpha_i(t_3)f_i(t_3)\,\alpha_i(t_4)f_i(t_4)] \\
&= \frac{1}{\sqrt{\pi_i^*(t_1)\sum_{j:(i,j)\in\mathcal{E}}\frac{\pi_j^*(t_1)}{M_{ij}}}\sqrt{\pi_i^*(t_2)\sum_{j:(i,j)\in\mathcal{E}}\frac{\pi_j^*(t_2)}{M_{ij}}}\sqrt{\pi_i^*(t_3)\sum_{j:(i,j)\in\mathcal{E}}\frac{\pi_j^*(t_3)}{M_{ij}}}\sqrt{\pi_i^*(t_4)\sum_{j:(i,j)\in\mathcal{E}}\frac{\pi_j^*(t_4)}{M_{ij}}}} \\
&\quad \times \frac{h^2}{(\int K(v)^2 dv)^2}\sum_{j_1:(i,j_1)\in\mathcal{E}}\ \sum_{j_2:(i,j_2)\in\mathcal{E},\, j_2\neq j_1}(g(t_1,t_2,t_3,t_4)+g(t_1,t_3,t_2,t_4)+g(t_1,t_4,t_2,t_3))+o(1),
\end{aligned}
\tag{S4.44}
$$

where

$$
\begin{aligned}
g(s_1,s_2,s_3,s_4) &= (\pi_i^*(s_1)+\pi_{j_1}^*(s_1))(\pi_i^*(s_2)+\pi_{j_1}^*(s_2))(\pi_i^*(s_3)+\pi_{j_2}^*(s_3))(\pi_i^*(s_4)+\pi_{j_2}^*(s_4)) \\
&\times \frac{y_{j_1 i}^*(s_1)y_{j_2 i}^*(s_3)(1-y_{j_1 i}^*(s_1))(1-y_{j_2 i}^*(s_3))}{M_{ij_1}M_{ij_2}h^2}\int K(v)K(v+\frac{s_2-s_1}{h})dv\int K(v)K(v+\frac{s_4-s_3}{h})dv.
\end{aligned}
$$

For different settings of $t_1, t_2, \dots, t_4$ in (S4.44), we can obtain

$$
E[\alpha_i(t)f_i(t)]^2[\alpha_i(s)f_i(s)]^2 \to \begin{cases} 3 & s=t, \\ 1 & s\neq t, \end{cases}
$$

for $t, s \in (0,1)$, which concludes the proof. □

S4.3 Proof of Lemma S3

Proof. Note that

$$
E[\alpha(t)(f_i(t)-f_j(t))]^2[\alpha(s)(f_i(s)-f_j(s))]^2 = \frac{\alpha(t)^2\alpha(s)^2}{(\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(t))^2(\sum_{l:(i,l)\in\mathcal{E}} y_{il}^*(s))^2}
$$

$$\sum_{l_1:(i,l_1)\in\mathcal{E}}\sum_{l_2:(i,l_2)\in\mathcal{E}}\sum_{l_3:(i,l_3)\in\mathcal{E}}\sum_{l_4:(i,l_4)\in\mathcal{E}}(\pi_i^*(t)+\pi_{l_1}^*(t))(\pi_i^*(t)+\pi_{l_2}^*(t)))(\pi_i^*(s)+\pi_{l_3}^*(s)(\pi_i^*(s)+\pi_{l_4}^*(s))$$
$$E(\bar{\Delta}_{il_1}(t)-\bar{\Delta}_{jl_1}(t))(\bar{\Delta}_{il_2}(t)-\bar{\Delta}_{jl_2}(t))(\bar{\Delta}_{il_3}(s)-\bar{\Delta}_{jl_3}(s))(\bar{\Delta}_{il_4}(s)-\bar{\Delta}_{jl_4}(s)), \tag{S4.45}$$

where we let $\bar{\Delta}_{jj}(t)$ denote $\bar{\Delta}_{ji}(t)$. For the notation simplicity, we assume that all the $T_{l_1l_2}, \forall l_1, l_2 \in [n]$ are same. The derivation also holds for the general case. Then we consider $E(\bar{\Delta}_{il_1}(t)-\bar{\Delta}_{jl_1}(t))(\bar{\Delta}_{il_2}(t)-\bar{\Delta}_{jl_2}(t))(\bar{\Delta}_{il_3}(s)-\bar{\Delta}_{jl_3}(s))(\bar{\Delta}_{il_4}(s)-\bar{\Delta}_{jl_4}(s))$. We discuss different settings of l_1, l_2, l_3, l_4 separately.

- If none of l_1, l_2, l_3 and l_4 is equal to j, we can obtain

$$E[(\bar{\Delta}_{il}(t)-\bar{\Delta}_{jl}(t))^2(\bar{\Delta}_{il}(s)-\bar{\Delta}_{jl}(s))^2]=\frac{1}{(\sum_{t_k\in T_{il}}K_h(t,t_k))^2(\sum_{t_k\in T_{il}}K_h(s,t_k))^2}$$
$$\sum_{t_{k_1}\in T_{il}}\sum_{t_{k_2}\in T_{il}}\sum_{t_{k_3}\in T_{il}}\sum_{t_{k_4}\in T_{il}}K_h(t,t_{k_1})K_h(t,t_{k_2})K_h(s,t_{k_3})K_h(s,t_{k_4})$$
$$E[(y_{il}(t_{k_1})-y_{jl}(t_{k_1}))(y_{il}(t_{k_2})-y_{jl}(t_{k_2}))(y_{il}(t_{k_3})-y_{jl}(t_{k_3}))(y_{il}(t_{k_4})-y_{jl}(t_{k_4}))]$$
$$=\frac{1}{(\sum_{t_k\in T_{il}}K_h(t,t_k))^2(\sum_{t_k\in T_{il}}K_h(s,t_k))^2}[\sum_{t_k\in T_{il}}K_h(t,t_k)^2K_h(s,t_k)^2(y_{il}^*(t_k)-2y_{il}^*(t_k)y_{jl}^*(t_k)$$
$$+y_{jl}^*(t_k))+\sum_{t_{k_1}\in T_{il}}\sum_{t_{k_2}\in T_{il},k_2\neq k_1}(K_h(t,t_{k_1})^2K_h(s,t_{k_2})^2+2K_h(t,t_{k_1})K_h(t,t_{k_2})K_h(s,t_{k_1})K_h(s,t_{k_2}))$$
$$\times(y_{il}^*(t_{k_1})-2y_{il}^*(t_{k_1})y_{jl}^*(t_{k_1})+y_{jl}^*(t_{k_1}))(y_{il}^*(t_{k_2})-2y_{il}^*(t_{k_2})y_{jl}^*(t_{k_2})+y_{jl}^*(t_{k_2}))].$$

  As for $l_1 \neq l_2$, we have

$$E(\bar{\Delta}_{il_1}(t_1)-\bar{\Delta}_{jl_1}(t_1))(\bar{\Delta}_{il_1}(t_2)-\bar{\Delta}_{jl_1}(t_2))(\bar{\Delta}_{il_2}(t_3)-\bar{\Delta}_{jl_2}(t_3))(\bar{\Delta}_{il_2}(t_4)-\bar{\Delta}_{jl_2}(t_4))$$

$$
\begin{aligned}
&= \frac{\sum_{t_k\in T_{il_1}} K_h(t_1,t_k)K_h(t_2,t_k)(y^*_{il_1}(t_k)-2y^*_{il_1}(t_k)y^*_{jl_1}(t_k)+y^*_{jl_1}(t_k))}{(\sum_{t_k\in T_{il_1}} K_h(t_1,t_k))(\sum_{t_k\in T_{il_1}} K_h(t_2,t_k))} \\
&\quad\times \frac{\sum_{t_k\in T_{il_2}} K_h(t_3,t_k)K_h(t_4,t_k)(y^*_{il_2}(t_k)-2y^*_{il_2}(t_k)y^*_{jl_2}(t_k)+y^*_{jl_2}(t_k))}{(\sum_{t_k\in T_{il_2}} K_h(t_3,t_k))(\sum_{t_k\in T_{il_2}} K_h(t_4,t_k))}
\end{aligned}
$$

- When some of l_1, l_2, l_3 and l_4 equal to j, we have

$$
\begin{aligned}
&E(\bar\Delta_{ij}(t_1)-\bar\Delta_{ji}(t_1))(\bar\Delta_{ij}(t_2)-\bar\Delta_{ji}(t_2))(\bar\Delta_{ij}(t_3)-\bar\Delta_{ji}(t_3))(\bar\Delta_{ij}(t_4)-\bar\Delta_{ji}(t_4)) \\
&= \frac{1}{(\sum_{t_k\in T_{ij}} K_h(t_1,t_k))(\sum_{t_k\in T_{ij}} K_h(t_2,t_k))(\sum_{t_k\in T_{ij}} K_h(t_3,t_k))(\sum_{t_k\in T_{ij}} K_h(t_4,t_k))} \\
&\quad \times [\sum_{t_k\in T_{ij}} K_h(t_1,t_k)K_h(t_2,t_k)K_h(t_3,t_k)K_h(t_4,t_k)\frac{16\pi^*_i(t_k)^4\pi^*_j(t_k)+16\pi^*_j(t_k)^4\pi^*_i(t_k)}{(\pi^*_i(t_k)+\pi^*_j(t_k))^5} \\
&\qquad + \sum_{t_{k_1}\in T_{ij}}\sum_{t_{k_2}\in T_{ij},k2\neq k_1}(K_h(t_1,t_{k_1})K_h(t_2,t_{k_1})K_h(t_3,t_{k_2})K_h(t_4,t_{k_2}) \\
&\qquad + K_h(t_1,t_{k_1})K_h(t_2,t_{k_2})K_h(t_3,t_{k_1})K_h(t_4,t_{k_2}) + K_h(t_1,t_{k_1})K_h(t_2,t_{k_2})K_h(t_3,t_{k_2})K_h(t_4,t_{k_1})) \\
&\qquad \times \frac{16\pi^*_i(t_{k_1})\pi^*_j(t_{k_1})\pi^*_i(t_{k_2})\pi^*_j(t_{k_2})}{(\pi^*_i(t_{k_1})+\pi^*_j(t_{k_1}))^2(\pi^*_i(t_{k_2})+\pi^*_j(t_{k_2}))^2}].
\end{aligned}
$$

For $l\neq i,j$, we can obtain

$$
\begin{aligned}
&E(\bar\Delta_{ij}(t_1)-\bar\Delta_{ji}(t_1))(\bar\Delta_{ij}(t_2)-\bar\Delta_{ji}(t_2))(\bar\Delta_{il}(t_3)-\bar\Delta_{jl}(t_3))(\bar\Delta_{il}(t_4)-\bar\Delta_{jl}(t_4)) \\
&= [\frac{1}{(\sum_{t_k\in T_{ij}} K_h(t_1,t_k))(\sum_{t_k\in T_{ij}} K_h(t_2,t_k))}\sum_{t_k\in T_{ij}} K_h(t_1,t_k)K_h(t_2,t_k)\frac{4\pi^*_i(t_k)\pi^*_j(t_k)}{(\pi^*_i(t_k)+\pi^*_j(t_k))^2}] \\
&\quad \times [\frac{\sum_{t_k\in T_{ij}} K_h(t_3,t_k)K_h(t_4,t_k)(y^*_{il}(t_k)-2y^*_{il}(t_k)y^*_{jl}(t_k)+y^*_{jl}(t_k))}{(\sum_{t_k\in T_{ij}} K_h(t_3,t_k))(\sum_{t_k\in T_{ij}} K_h(t_4,t_k))}].
\end{aligned}
$$

Hence, we have

$$
(S4.45) = \frac{\alpha(t)^2\alpha(s)^2}{(\sum_{l:(i,l)\in\mathcal{E}} y^*_{il}(t))^2(\sum_{l:(i,l)\in\mathcal{E}} y^*_{il}(s))^2}[\sum_{l_1:(i,l_1)\in\mathcal{E},l_1\neq j}\ \sum_{l_3:(i,l_3)\in\mathcal{E},j,l_1}(\pi^*_i(t)+\pi^*_{l_1}(t))^2(\pi^*_i(s)+\pi^*_{l_3}(s))^2
$$

$$\times \frac{\sum_{t_k \in T_{il_1}} K_h(t,t_k)^2 (y^*_{il_1}(t_k) - 2y^*_{il_1}(t_k) y^*_{jl_1}(t_k) + y^*_{jl_1}(t_k))}{(\sum_{t_k \in T_{il_1}} K_h(t,t_k))^2}$$
$$\times \frac{\sum_{t_k \in T_{il_3}} K_h(s,t_k)^2 (y^*_{il_3}(t_k) - 2y^*_{il_3}(t_k) y^*_{jl_3}(t_k) + y^*_{jl_3}(t_k))}{(\sum_{t_k \in T_{il_3}} K_h(s,t_k))^2}$$
$$+ 2 \sum_{l_1:(i,l_1)\in\mathcal{E},j} \sum_{l_2:(i,l_2)\in\mathcal{E},j,l_1} (\pi^*_i(t) + \pi^*_{l_1}(t))(\pi^*_i(t) + \pi^*_{l_2}(t))(\pi^*_i(s) + \pi^*_{l_1}(s))(\pi^*_i(s) + \pi^*_{l_2}(s))$$
$$\times \frac{\sum_{t_k \in T_{il_1}} K_h(t,t_k) K_h(s,t_k) (y^*_{il_1}(t_k) - 2y^*_{il_1}(t_k) y^*_{jl_1}(t_k) + y^*_{jl_1}(t_k))}{(\sum_{t_k \in T_{il_1}} K_h(t,t_k))(\sum_{t_k \in T_{il_1}} K_h(s,t_k))}$$
$$\times \frac{\sum_{t_k \in T_{il_2}} K_h(t,t_k) K_h(s,t_k) (y^*_{il_2}(t_k) - 2y^*_{il_2}(t_k) y^*_{jl_2}(t_k) + y^*_{jl_2}(t_k))}{(\sum_{t_k \in T_{il_2}} K_h(t,t_k))(\sum_{t_k \in T_{il_2}} K_h(s,t_k))}] + o(1).$$

Letting $n \to \infty$, we have

$$E[\alpha(t)(f_i(t) - f_j(t))]^2 [\alpha(s)(f_i(s) - f_j(s))]^2 \to \begin{cases} 12 & s = t, \\ 4 & s \neq t. \end{cases}$$

Therefore, we can obtain

$$E[\sum_{s=1}^{m} (\frac{\alpha(t_s)}{\sqrt{2}} (f_i(t_s) - f_j(t_s)))^2]^2 \to m^2 + 2m.$$

□

Bibliography

Bong, H., W. Li, S. Shrotriya, and A. Rinaldo (2020). Nonparametric estimation in the dynamic Bradley-Terry model. In *International Conference on Artificial Intelligence and Statistics*, pp. 3317–3326. PMLR.

Cao, X.-R. (1998). The maclaurin series for performance functions of markov chains. *Advances in Applied Probability 30*(3), 676–692.

Chernozhukov, V., D. Chetverikov, and K. Kato (2013). Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *Annals of Statistics 41*, 2786–2819.

Chernozhuokov, V., D. Chetverikov, K. Kato, and Y. Koike (2022). Improved central limit theorem and bootstrap approximations in high dimensions. *Annals of Statistics 50*(5), 2562–2586.

de Jong, P. (1987). A central limit theorem for generalized quadratic forms. *Probability Theory and Related Fields 75*, 261–277.

Negahban, S., S. Oh, and D. Shah (2017). Rank centrality: Ranking from pairwise comparisons. *Operations Research 65*, 266–287.

Tian, X., J. Shi, X. Shen, and K. Song (2024). A spectral approach for the dynamic bradley–terry model. *Stat 13*(3), e722.