Supplement to "On runs tests for directional data and their local and asymptotic optimality properties"

Maxime Boucher[†], Christian Francq[*], Yuichi Goto[**] and Thomas Verdebout[†]

Kyushu University[**], Université de Lille and CREST (Paris)[*] and Université libre de Bruxelles (ULB)[†]

S1. Proofs

Proof of Proposition ??. Since under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},g}$, $\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_1), \ldots, \mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_n)$ are i.i.d. uniformly distributed over $\mathcal{S}^{p-2}$, we have that

$$\mathrm{E}[\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_i)] = \mathbf{0} \quad \text{and} \quad \mathrm{E}[\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_i)\mathbf{S}'_{\boldsymbol{\theta}}(\mathbf{X}_i)] = \frac{1}{p-1}\,\mathbf{I}_{p-1}. \tag{S1.1}$$

Therefore, we easily see that $\mathrm{E}[R^{(n)}_{1d}(\boldsymbol{\theta})] = 0$ and that

$$\begin{aligned}\mathrm{E}[(R^{(n)}_{1d}(\boldsymbol{\theta}))^2] &= \frac{1}{n-1}\sum_{t,s=2}^{n}\mathrm{E}[\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_t)'\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{t-1})\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_s)'\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{s-1})] \\ &= \frac{1}{n-1}\{(n-1)(p-1)^{-1}\} = (p-1)^{-1}.\end{aligned}$$

The central limit theorem for 2-dependent stationary processes then entails that $s_n^{-1/2}(\boldsymbol{\theta})R^{(n)}_d(\boldsymbol{\theta})$ converges weakly to a standard Gaussian random variable. Finally, the law of large numbers entails that

$$\begin{aligned} s_n(\boldsymbol{\theta}) &= \mathrm{tr}[(n^{-1}\sum_{i=1}^{n}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_t)')^2] \\ &= \mathrm{tr}[((p-1)^{-1}\mathbf{I}_{p-1})^2] + o_{\mathrm{P}}(1) = (p-1)^{-1} + o_{\mathrm{P}}(1) \end{aligned}$$

as $n \to \infty$. The Slutsky Lemma concludes the proof. □

Proof of Theorem ??. Consider first the case with $\boldsymbol{\theta} = \boldsymbol{\theta}_0 := (1, 0, \ldots, 0)' \in \mathbb{R}^p$. Clearly, $\mathbf{X}_i = (V_i, (1 - V_i^2)^{1/2}\mathbf{S}_i')'$, with $V_i := \mathbf{X}_i'\boldsymbol{\theta}_0 = X_{i1}$ and $\mathbf{S}_i := \mathbf{S}_{\boldsymbol{\theta}_0}(\mathbf{X}_i) = (X_{i2}, \ldots, X_{ip})'/\sqrt{1 - X_{i1}^2}$, where we used the notation introduced previously. The vectors $\mathbf{S}_1, \ldots, \mathbf{S}_n$ take their values in $\mathcal{S}^{p-2}$, and have joint density

$$(\mathbf{s}_1, \ldots, \mathbf{s}_n) \to c_\lambda \exp(\lambda(\sum_{t=2}^{n}\mathbf{s}_t'\mathbf{s}_{t-1}))$$

with respect to the surface area measure. Therefore, conditionally on $V_1 = v_1, \ldots, V_n = v_n$,

$$(1 - V_1^2)^{1/2}\mathbf{S}_1, \ldots, (1 - V_n^2)^{1/2}\mathbf{S}_n,$$

take their values on the hyperspheres $\mathcal{S}^{p-2}(r_{v_1}), \ldots, \mathcal{S}^{p-2}(r_{v_n})$ with radii $r_{v_1} := (1 - v_1^2)^{1/2}, \ldots, r_{v_n} := (1 - v_n^2)^{1/2}$. Their joint density with respect to the product of surface area measures on $\mathcal{S}^{p-2}(r_{v_1}) \times \ldots \times \mathcal{S}^{p-2}(r_{v_n})$ is (recall that the V_i's and the $\mathbf{S}_i$'s are independent)

$$(\mathbf{w}_1, \ldots, \mathbf{w}_n) \to c_\lambda \exp(\lambda(\sum_{t=2}^{n}\frac{\mathbf{w}_t'\mathbf{w}_{t-1}}{r_{v_t}r_{v_{t-1}}}))\prod_{t=1}^{n} r_{v_t}^{-(p-2)},$$

where $r^{-(p-2)}$ is the Jacobian of the radial projection of $\mathcal{S}^{p-2}(r)$ onto $\mathcal{S}^{p-2}$. Since, letting $\sigma_{p-2,r}$ be the surface area measure over the hypersphere with radius r, $d\sigma_{p-2,r} = r^{p-2}d\sigma_{p-2}$, the joint density of $(\mathbf{X}_1,\ldots,\mathbf{X}_n)$ with respect to the product measure $(\mu\times\sigma_{p-2})^n$ (where μ stands for the Lebesgue measure on $[-1,1]$) is

$$(\mathbf{x},\ldots,\mathbf{x}_n)\mapsto c_\lambda\exp(\lambda(\sum_{t=2}^{n}\mathbf{S}'_{\boldsymbol{\theta}_0}(\mathbf{x}_t)\mathbf{S}_{\boldsymbol{\theta}_0}(\mathbf{x}_{t-1})))\, c^n_{p,g}\prod_{t=1}^{n}(1-v^2_{\boldsymbol{\theta}_0}(\mathbf{x}_t))^{(p-3)/2}g(v_{\boldsymbol{\theta}_0}(\mathbf{x}_t)).$$

The result for $\boldsymbol{\theta}=\boldsymbol{\theta}_0$ then follows from the fact that (see, e.g., page 44 of Watson (1983))

$$\frac{d(\mu\times\sigma_{p-2})}{d\sigma_{p-1}}(\mathbf{x}) = (1-v^2_{\boldsymbol{\theta}_0}(\mathbf{x}))^{(p-3)/2}.$$

To obtain the result for an arbitrary value of $\boldsymbol{\theta}$, let $(\mathbf{X}_1,\ldots,\mathbf{X}_n)\sim \mathrm{P}^{(n)}_{\boldsymbol{\theta},\lambda,g}$ (i.i.d. with density (**??**)) and pick a $p\times p$ orthogonal matrix $\mathbf{O}$ such that $\mathbf{O}\boldsymbol{\theta}=\boldsymbol{\theta}_0$. Since $\mathbf{O}\boldsymbol{\Gamma}_{\boldsymbol{\theta}}=\boldsymbol{\Gamma}_{\boldsymbol{\theta}_0}$, we have that $(\mathbf{O}\mathbf{X}_1,\ldots,\mathbf{O}\mathbf{X}_n)\sim \mathrm{P}^{(n)}_{\boldsymbol{\theta}_0,g,\lambda}$. Therefore, the result for $\boldsymbol{\theta}=\boldsymbol{\theta}_0$ implies that the density of $\mathbf{X}$ with respect to σ_{p-1} is

$$\begin{aligned}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\mapsto |\det\mathbf{O}|\, c_\lambda\exp(\lambda(\sum_{t=2}^{n}\mathbf{S}'_{\boldsymbol{\theta}_0}(\mathbf{O}\mathbf{x}_t)\mathbf{S}_{\boldsymbol{\theta}_0}(\mathbf{O}\mathbf{x}_{t-1})))c^n_{p,g}\prod_{t=1}^{n}g(v_{\boldsymbol{\theta}_0}(\mathbf{O}\mathbf{x}_t))\\ = c_\lambda\exp(\lambda(\sum_{t=2}^{n}\mathbf{S}'_{\boldsymbol{\theta}}(\mathbf{x}_t)\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{x}_{t-1})))c^n_{p,g}\prod_{t=1}^{n}g(v_{\boldsymbol{\theta}}(\mathbf{x}_t)),\end{aligned}$$

as was to be proved. $\square$

Proof of Theorem ??. Letting $\boldsymbol{\theta}_n := \boldsymbol{\theta} + n^{-1/2}\boldsymbol{\tau}_n$, first note that

$$\begin{aligned}\Lambda^{(n)} &= \log \frac{d\mathrm{P}^{(n)}_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau}_n, n^{-1/2}\ell_n, g}}{d\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}} \\ &= \log \frac{d\mathrm{P}^{(n)}_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau}_n, n^{-1/2}\ell_n, g}}{d\mathrm{P}^{(n)}_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau}_n,0,g}} + \log \frac{d\mathrm{P}^{(n)}_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau}_n,0,g}}{d\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}},\end{aligned}$$

so that following Ley et al. (2013), we have that ($\boldsymbol{\Gamma}_{\boldsymbol{\theta}} := \mathcal{J}_p(g)\mathbf{I}_{p-1}$)

$$\Lambda^{(n)} = n(\log(c_{n^{-1/2}\ell_n}) - \log(c_0)) + n^{-1/2}\ell_n \sum_{t=2}^{n} \mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1}) + \boldsymbol{\tau}'_n\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}} - \frac{1}{2}\boldsymbol{\tau}'_n\boldsymbol{\Gamma}_{\boldsymbol{\theta}}\boldsymbol{\tau}_n + o_{\mathrm{P}}(1),$$

where

$$\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}} = n^{-1/2}\sum_{t=1}^{n} \varphi_g(v_{\boldsymbol{\theta}}(\mathbf{X}_t))(1 - v^2_{\boldsymbol{\theta}}(\mathbf{X}_t))^{1/2}\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_t)$$

is the central sequence for the location parameter obtained in Ley et al. (2013). Now it follows directly from Cutting et al. (2017) that

$$n(\log(c_{n^{-1/2}\ell_n}) - \log(c_0)) = -\frac{1}{2(p-1)} + o_{\mathrm{P}}(1)$$

as $n \to \infty$. Therefore to obtain the result, we have to show that

$$S_n := n^{-1/2}\sum_{t=2}^{n} \mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1}) - \mathbf{S}'_{\boldsymbol{\theta}}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{t-1}) \qquad \text{(S1.2)}$$

is $o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}$. Note that under $\mathrm{P}^{(n)}_{\boldsymbol{\theta}_n,0,g}$, we have that

$$(n^{-1/2}\sum_{t=2}^{n} \mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1}), \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\theta}_n})'$$

is asymptotically normal with mean zero and covariance matrix $\mathrm{diag}(\frac{1}{p-1}, \boldsymbol{\Gamma}_{\boldsymbol{\theta}})$.

The Le Cam third Lemma therefore entails that under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}$, $n^{-1/2}\sum_{t=2}^{n} \mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})$

is asymptotically normal with mean zero and variance $\frac{1}{p-1}$. On the other hand, the central limit theorem entails that under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}$,

$$n^{-1/2}\sum_{t=2}^{n}\mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})-\mathrm{E}[\mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})]$$

is asymptotically normal with mean zero and variance $\frac{1}{p-1}$. Therefore, under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}$,

$$n^{-1/2}\sum_{t=2}^{n}\mathrm{E}[\mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})]=o(1) \tag{S1.3}$$

as $n\to\infty$. Now, letting

$$T_n:=n^{-1/2}\sum_{t=2}^{n}\mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})-\mathbf{S}'_{\boldsymbol{\theta}}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_{t-1})-\mathrm{E}[\mathbf{S}'_{\boldsymbol{\theta}_n}(\mathbf{X}_t)\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_{t-1})]=:n^{-1/2}\sum_{t=2}^{n}S_t^{(n)},$$

we have that under $\mathrm{P}^{(n)}_{\boldsymbol{\theta},0,g}$

$$\begin{aligned}\mathrm{E}[T_n^2] &= n^{-1}\sum_{t,s=2}^{n}\mathrm{E}[S_t^{(n)}S_s^{(n)}]\\ &= n^{-1}\sum_{t=2}^{n}\mathrm{E}[(S_t^{(n)})^2]+2n^{-1}\sum_{t=2}^{n-1}\mathrm{E}[S_t^{(n)}S_{t+1}^{(n)}]\\ &= \frac{(n-1)}{n}\mathrm{E}[(S_1^{(n)})^2]+\frac{2(n-2)}{n}\mathrm{E}[S_2^{(n)}S_3^{(n)}]\end{aligned} \tag{S1.4}$$

Now writing $\mathbf{s}_t^{(n)}:=\mathbf{S}_{\boldsymbol{\theta}_n}(\mathbf{X}_t)$ and $\mathbf{s}_t:=\mathbf{S}_{\boldsymbol{\theta}}(\mathbf{X}_t)$, the Cauchy-Schwarz inequality entails that

$$\begin{aligned}
\mathrm{E}[(S_1^{(n)})^2] &= \mathrm{Var}[S_1^{(n)}] \\
&= \mathrm{Var}[(\mathbf{s}_2^{(n)})'\mathbf{s}_1^{(n)} - \mathbf{s}_2'\mathbf{s}_1] \\
&\leq \mathrm{E}[((\mathbf{s}_2^{(n)})'\mathbf{s}_1^{(n)} - \mathbf{s}_2'\mathbf{s}_1)^2] \\
&= \mathrm{E}[((\mathbf{s}_2^{(n)} - \mathbf{s}_2)'\mathbf{s}_1^{(n)} - \mathbf{s}_2'(\mathbf{s}_1 - \mathbf{s}_1^{(n)}))^2] \\
&\leq 2\{\mathrm{E}[((\mathbf{s}_2^{(n)} - \mathbf{s}_2)'\mathbf{s}_1^{(n)})^2] + \mathrm{E}[(\mathbf{s}_2'(\mathbf{s}_1 - \mathbf{s}_1^{(n)}))^2]\} \\
&\leq 2(\mathrm{E}[\|\mathbf{s}_2^{(n)} - \mathbf{s}_2\|^2] + \mathrm{E}[\|\mathbf{s}_1 - \mathbf{s}_1^{(n)}\|^2]) \\
&= 4\mathrm{E}[\|\mathbf{s}_1^{(n)} - \mathbf{s}_1\|^2],
\end{aligned}$$

which is o(1) as $n \to \infty$ using the same arguments as the ones used in the proof of Lemma C2 in García-Portugués et al. (2020). Now using (S1.3) we have that

$$\begin{aligned}
\mathrm{E}[S_2^{(n)}S_3^{(n)}] &= \mathrm{E}[((\mathbf{s}_1^{(n)})'\mathbf{s}_2^{(n)} - \mathrm{E}[(\mathbf{s}_1^{(n)})'\mathbf{s}_2^{(n)}])(((\mathbf{s}_3^{(n)})'\mathbf{s}_2^{(n)} - \mathrm{E}[(\mathbf{s}_3^{(n)})'\mathbf{s}_2^{(n)}]))] \\
&= \mathrm{E}[(\mathbf{s}_1^{(n)})'\mathbf{s}_2^{(n)}(\mathbf{s}_2^{(n)})'\mathbf{s}_3^{(n)}] + o(1) \\
&= \mathrm{E}[(\mathbf{s}_1^{(n)} - \mathbf{s}_1)'\mathbf{s}_2^{(n)}(\mathbf{s}_2^{(n)})'(\mathbf{s}_3^{(n)} - \mathbf{s}_3)] + o(1) \\
&\leq \mathrm{E}^2[\|(\mathbf{s}_1^{(n)} - \mathbf{s}_1)\|] + o(1),
\end{aligned}$$

which is $o(1)$ as $n \to \infty$ using the same arguments as the ones used in the proof of Lemma C2 in García-Portugués et al. (2020). It therefore follows from (S1.4) that T_n is $o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathrm{P}_{\boldsymbol{\theta},0,g}^{(n)}$. Combining this with

(S1.3), we obtain that S_n in (S1.2) is $o_{\mathrm{P}}(1)$ as $n \to \infty$. The result follows.

$\square$

Proof of Proposition ??. First note that letting

$$\mathbf{T}_H^{(n)} := (p-1)^{1/2}(R_{1d}^{(n)}(\boldsymbol{\theta}), R_{2d}^{(n)}(\boldsymbol{\theta}), \ldots, R_{Hd}^{(n)}(\boldsymbol{\theta})),$$

we directly have from the central limit theorem that $\mathbf{T}_H^{(n)}$ converges weakly under $\mathrm{P}_{\boldsymbol{\theta},g}^{(n)}$ to a Gaussian random vector with mean zero and covariance matrix $\mathbf{I}_H$. Point (i) follows. Theorem **??** can be used to apply the third Le Cam Lemma and directly obtain that $\mathbf{T}_H^{(n)}$ converges weakly under $\mathrm{P}_{\boldsymbol{\theta},n^{-1/2}\ell^{(n)},g}^{(n)}$ to a Gaussian random vector with mean

$$((p-1)^{-1/2}\ell, 0, \ldots, 0)'$$

and covariance matrix $\mathbf{I}_H$. The result then follows readily using (**??**). $\square$

S2. Complements to the real data illustration

In Figure 10 we provide plots of the sunspots locations for solar cycles 16 to 24.

In Figures 11 and 12 we provide the partial autocorrelation functions of (i) the absolute values $|\mathbf{X}_{i1}'\boldsymbol{\theta}|, \ldots, |\mathbf{X}_{in_i}'\boldsymbol{\theta}|$ of the latitudes for various solar cycles; (ii) the latitudes $\mathbf{X}_{i1}'\boldsymbol{\theta}, \ldots, \mathbf{X}_{in_i}'\boldsymbol{\theta}$ for various solar cycles and (iii) the

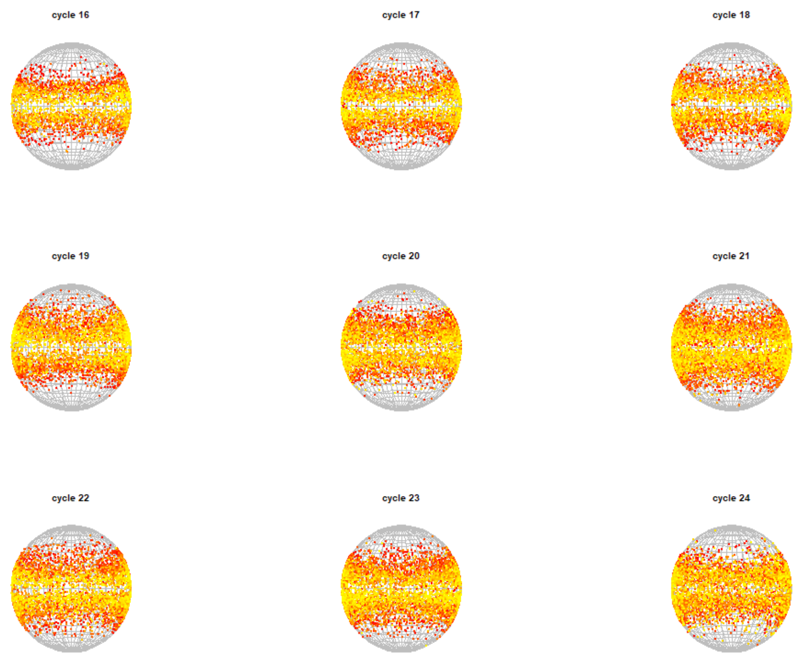

Figure 10: Plots of the locations of sunspots for solar cycles 16 to 24. The locations are colored with a red-yellow gradient according to the relative position of the sunspot appearance date within the solar cycle in order to visualize the Spörer's law.

angle associated with the longitudes for various solar cycles (each longitude or meridian is a bivariate unit vector and is therefore characterized by an angle).

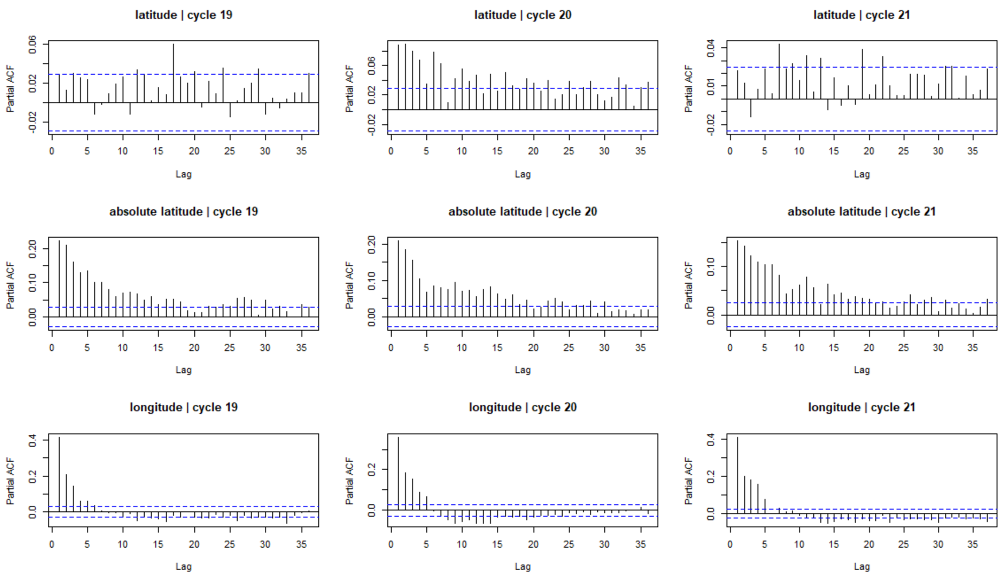

Figure 11: Partial autocorrelation functions of (i) the latitudes, (ii) the absolute values of the latitudes and (iii) the angles associated with the longitudes for various solar cycles (cycles 19 to 21).

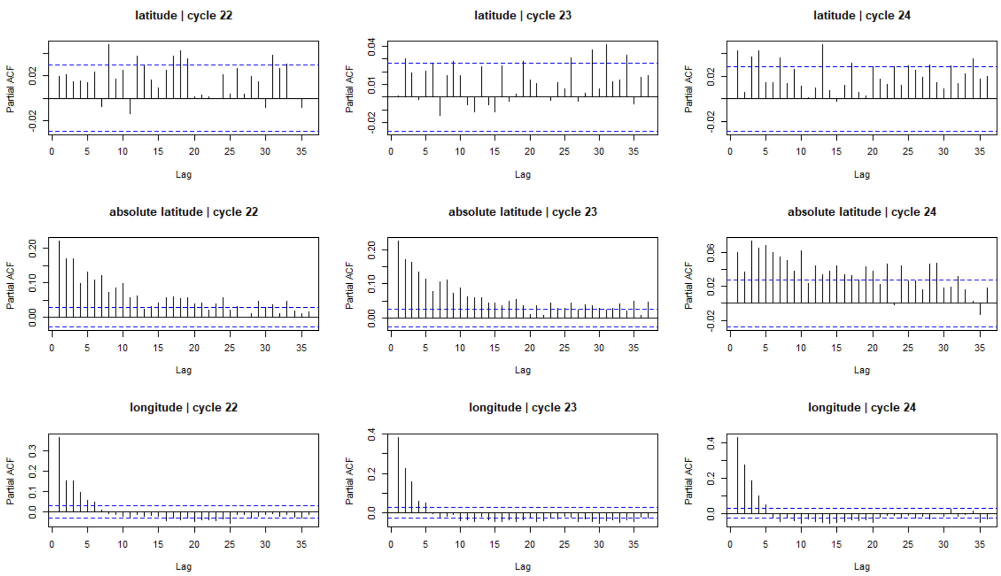

Figure 12: Partial autocorrelation functions of (i) the latitudes, (ii) the absolute values of the latitudes and (iii) the angles associated with the longitudes for various solar cycles (cycles 22 to 24).

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Maxime Boucher

Département de Mathématique and ECARES

Université libre de Bruxelles (ULB)

Campus Plaine, Boulevard du Triomphe, CP210

B-1050 Brussels, Belgium

E-mail: maxime.boucher@ulb.be

Christian Francq CREST and University of Lille, France

Laboratoire Finance-Assurance

5 avenue Henry Le Chatelier

91764 Cedex, France

E-mail: christian.francq@ensae.fr

Yuichi Goto Faculty of Mathematics, Kyushu University

744, Motooka, Nishiku

Fukuoka, Japan 819-0395

E-mail: goto.yuichi.436@m.kyushu-u.ac.jp

Thomas Verdebout

Département de Mathématique and ECARES

Université libre de Bruxelles (ULB)

Campus Plaine, Boulevard du Triomphe, CP210

B-1050 Brussels, Belgium

E-mail: thomas.verdebout@ulb.be