Supplementary Material

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This supplement contains technical proofs for the manuscript "The Method of Limits and Its Application to The Analysis of Count Data in Genome-wide Association Studies".

Overview

There are many theoretical results in this work, and some of the proofs have significant similarities. Therefore, we only present the proofs that introduce new ideas or methods. The rest of the proofs can be given by applying the proved results or similar arguments.

Specifically, the presented proofs include those of Lemma 2, Theorem 2 and Lemma 4.

Throughout the supplement, the manuscript, "The method of limits and its application to the analysis of count data in genome-wide association studies", is referred to as MS.

A.1 Proof of Lemma 2

Let us first assume that the entries of Z are $N(0,1)$; later we relax this assumption. It can be shown that, given α, $(y_i, z_i), i = 1, \dots, n$ are conditionally independent. We have $T_1 = n^{-2}\sum_{i_1\neq i_2} z_{i_1}' z_{i_2} y_{i_1} y_{i_2} = n^{-2}\sum_{i_1\neq i_2} u_{i_1}' u_{i_2}$, where $u_i = y_i z_i$. Write

$$u_{i_1}' u_{i_2} = \mathrm{E}(u_{i_1}|\alpha)'\mathrm{E}(u_{i_2}|\alpha) + \mathrm{E}(u_{i_2}|\alpha)'\delta_{i_1} + \mathrm{E}(u_{i_1}|\alpha)'\delta_{i_2} + \delta_{i_2}'\delta_{i_1},$$

where $\delta_i = u_i - \mathrm{E}(u_i|\alpha)$. Then, we have

$$\begin{aligned} T_1 &= \frac{1}{n^2}\sum_{i_1\neq i_2} \mathrm{E}(u_{i_1}|\alpha)'\mathrm{E}(u_{i_2}|\alpha) + \frac{2}{n^2}\sum_{i_1\neq i_2} \mathrm{E}(u_{i_2}|\alpha)'\delta_{i_1} + \frac{1}{n^2}\sum_{i_1\neq i_2} \delta_{i_2}'\delta_{i_1} \\ &= T_{11} + 2T_{12} + T_{13}, \end{aligned} \tag{A.1}$$

with the terms defined in obvious ways. It can be shown that

$$\mathrm{E}(u_i|\alpha) = e^{(\sigma_0^2 + p^{-1}\alpha'\alpha)/2}\left(\frac{\alpha}{\sqrt{p}}\right) N_i. \tag{A.2}$$

It follows that

$$T_{11} = e^{\sigma_0^2 + p^{-1}\alpha'\alpha} \left(\frac{\alpha'\alpha}{p}\right) \left(\bar{N}^2 - \frac{1}{n^2}\sum_{i=1}^{n} N_i^2\right) \xrightarrow{\text{P}} N^2\omega\sigma_1^2 e^{\sigma_0^2 + \omega\sigma_1^2}. \tag{A.3}$$

Next, we have $T_{12} = e^{(\sigma_0^2 + p^{-1}\alpha'\alpha)/2}(\alpha/\sqrt{p})' n^{-2}\sum_{i_1=1}^{n}(\sum_{i_2\neq i_1} N_{i_2})\delta_{i_1}$. It follows that

$$\text{E}(T_{12}^2|\alpha) = \frac{1}{n^4} e^{\sigma_0^2 + p^{-1}\alpha'\alpha} \left(\frac{\alpha}{\sqrt{p}}\right)' \left\{\sum_{i_1=1}^{n}\left(\sum_{i_2\neq i_1} N_{i_2}\right)^2 \text{Var}(u_{i_1}|\alpha)\right\}\left(\frac{\alpha}{\sqrt{p}}\right).$$

Using the fact that for $\nu \sim N(0,1)$, we have $\text{E}(\nu^2 e^{a\nu}) = (1 + a^2)e^{a^2/2}$ for any constant a, it can be shown that $\text{E}(e^{\lambda\gamma_i} z_i z_i'|\alpha) = e^{(\lambda^2/2)p^{-1}\alpha'\alpha}(I_p + \lambda^2 p^{-1}\alpha\alpha')$. It follows that

$$\begin{aligned}
\text{Var}(u_i|\alpha) &\leq \text{E}(u_i u_i'|\alpha) \\
&= N_i e^{\sigma_0^2/2}\text{E}(e^{\gamma_i} z_i z_i'|\alpha) + N_i^2 e^{2\sigma_0^2}\text{E}(e^{2\gamma_i} z_i z_i'|\alpha) \\
&= N_i e^{(\sigma_0^2 + p^{-2}\alpha'\alpha)/2}\left(I_p + \frac{\alpha\alpha'}{p}\right) + N_i^2 e^{2(\sigma_0^2 + p^{-1}\alpha'\alpha)}\left(I_p + 4\frac{\alpha\alpha'}{p}\right) \\
&\leq 2N_i^2 e^{2(\sigma_0^2 + p^{-1}\alpha'\alpha)}\left(I_p + 4\frac{\alpha\alpha'}{p}\right)
\end{aligned} \tag{A.4}$$

(for symmetric matrices A, B, $A \leq B$ means that $B - A$ is nonnegative definite), hence

$$\left(\frac{\alpha}{\sqrt{p}}\right)' \text{Var}(u_{i_1}|\alpha)\left(\frac{\alpha}{\sqrt{p}}\right) \leq 2N_{i_1}^2 e^{2(\sigma_0^2 + p^{-1}\alpha'\alpha)}\left\{\frac{\alpha'\alpha}{p} + 4\left(\frac{\alpha'\alpha}{p}\right)^2\right\},$$

$$\text{E}(T_2^2|\alpha) \leq e^{3(\sigma_0^2 + p^{-1}\alpha'\alpha)}\left\{\frac{\alpha'\alpha}{p} + 4\left(\frac{\alpha'\alpha}{p}\right)^2\right\}\frac{2}{n^4}\sum_{i_1=1}^{n}\left(\sum_{j_2\neq i_1} N_{i_2}\right)^2 N_{i_1}^2$$

$$\leq 2e^{3(\sigma_0^2 + p^{-1}\alpha'\alpha)}\left\{\frac{\alpha'\alpha}{p} + 4\left(\frac{\alpha'\alpha}{p}\right)^2\right\}\frac{(n-1)}{n^4}\sum_{i_1\neq i_2} N_{i_1}^2 N_{i_2}^2,$$

using the Cauchy-Schwarz inequality for the last step. It is seen that $\text{E}(T_{12}^2|\alpha) = o_{\text{P}}(1)$, hence, by the dominated convergence theorem, it can be shown that

$$T_{12} = o_{\text{P}}(1). \tag{A.5}$$

Finally, we can write $\sum_{i_1\neq i_2}\delta_{i_2}'\delta_{i_1} = 2\sum_{i_1=1}^{n} d_{i_1}$, where $d_{i_1} = (\sum_{i_2<i_1}\delta_{i_2})'\delta_{i_1}$, $\mathcal{F}_{i_1} = \sigma(\alpha, u_i, i \leq i_1)$, $1 \leq i_1 \leq n$ is a sequence of martingale differences. Thus, using the

martingale property, and (A.4), we have

$$
\begin{aligned}
\mathrm{E}(T_{13}^2|\alpha) &= \frac{4}{n^4}\sum_{i_1=1}^{n}\mathrm{E}(d_{i_1}^2|\alpha) \\
&= \frac{4}{n^4}\sum_{i_1=1}^{n}\mathrm{E}\left\{\left(\sum_{i_2<i_1}\delta_{i_2}\right)'\mathrm{Var}(u_{i_1}|\alpha)\left(\sum_{i_2<i_1}\delta_{i_2}\right)\Bigg|\,\alpha\right\} \\
&= \frac{4}{n^4}\sum_{i_1=1}^{n}\mathrm{tr}\left\{\mathrm{Var}(u_{i_1}|\alpha)\sum_{i_2<i_1}\mathrm{Var}(u_{i_2}|\alpha)\right\} \\
&\leq 8e^{4(\sigma_0^2+p^{-1}\alpha'\alpha)}\left\{1+\frac{8}{p}\left(\frac{\alpha'\alpha}{p}\right)+\frac{16}{p}\left(\frac{\alpha'\alpha}{p}\right)^2\right\}\frac{p}{n^4}\sum_{i_1\neq i_2}N_{i_1}^2N_{i_2}^2.
\end{aligned}
$$

It is seen that $\mathrm{E}(T_{13}^2|\alpha) = o_{\mathrm{P}}(1)$; thus, by the dominated convergence theorem, we have

$$T_{13} = o_{\mathrm{P}}(1). \tag{A.6}$$

By (A.1), (A.3), (A.5) and (A.6), the result follows.

We now relax the normality assumption. What we do is to revisit the places in the proof, where the normality assumption was used, and make appropriate changes under the sub-Gaussian distribution. The first place is (A.2). Note that we can write $\mathrm{E}(u_i|\alpha) = N_i e^{\sigma_0^2/2}\mathrm{E}(e^{\gamma_i}z_i|\alpha)$, and the jth component of $\mathrm{E}(e^{\gamma_i}z_i|\alpha)$ is

$$\mathrm{E}(e^{\gamma_i}z_{ij}|\alpha) = \mathrm{E}\left\{\exp\left(\frac{\alpha_j}{\sqrt{p}}z_{ij}\right)z_{ij}\Bigg|\,\alpha\right\}\prod_{k\neq j}\mathrm{E}\left\{\exp\left(\frac{\alpha_k}{\sqrt{p}}z_{ik}\right)\Bigg|\,\alpha\right\}. \tag{A.7}$$

By Taylor series expansion, it can be shown that

$$\mathrm{E}\left\{\exp\left(\frac{\alpha_k}{\sqrt{p}}z_{ik}\right)\Bigg|\,\alpha\right\} = \sum_{q=0}^{\infty}\frac{1}{q!}\left(\frac{\alpha_k}{\sqrt{p}}\right)^q\mathrm{E}(z_{ik}^q).$$

By Lemma 2.3 of Jiang *et al.* (2016), there is a constant $K > 0$ such that $\mathrm{E}(|z_{ik}|^q) \leq (K\sqrt{q})^q$, $q = 1, 2, \ldots$. Also, by Stirling's approximation, there is a constant $c > 0$ such that $(K\sqrt{q})^q/q! \le c(1/2)^q$, $q = 3, 4, \ldots$. It follows that

$$
\begin{aligned}
\mathrm{E}\left\{ \exp\left(\frac{\alpha_k}{\sqrt{p}} z_{ik} \right) \middle| \alpha \right\} &\le 1 + \frac{\alpha_k^2}{2p} + c \sum_{q=3}^{\infty} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^q \\
&= 1 + \frac{\alpha_k^2}{2p} + \frac{c}{1 - |\alpha_k|/2\sqrt{p}} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^3 \\
&\le \exp\left\{ \frac{\alpha_k^2}{2p} + \frac{c}{1 - |\alpha_k|/2\sqrt{p}} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^3 \right\}, \qquad \text{(A.8)}
\end{aligned}
$$

using $1 + x \le e^x$ for all $x \in \mathbb{R}$ for the last step. On the other hand, on the set $\mathcal{A} = \{(\max_{1 \le k \le p} |\alpha_k|)/\sqrt{p} \le 1 \wedge (2/c)\}$, we have $|\alpha_k|/2\sqrt{p} \le 1/2$, hence

$$
\left| \frac{c}{1 - |u_k|/2\sqrt{p}} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right) \right| \le 2c \left(\frac{|\alpha_k|}{2\sqrt{p}} \right) \le 2.
$$

It follows that the expression inside the exponential on the right side of (A.8) is nonnegative. Then, using the inquality $1 + x \ge e^{x - x^2/2}$ for $x \ge 0$, we have, on $\mathcal{A}$,

$$
\begin{aligned}
\mathrm{E}\left\{ \exp\left(\frac{\alpha_k}{\sqrt{p}} z_{ik} \right) \middle| \alpha \right\} &\ge 1 + \frac{\alpha_k^2}{2p} - c \sum_{q=3}^{\infty} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^q \\
&= 1 + \frac{\alpha_k^2}{2p} - \frac{c}{1 - |\alpha_k|/2\sqrt{p}} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^3 \\
&\ge \exp\left\{ \frac{\alpha_k^2}{2p} - \frac{c}{1 - |\alpha_k|/2\sqrt{p}} \left(\frac{|\alpha_k|}{2\sqrt{p}} \right)^3 - \frac{u_k^2}{2} \right\},
\end{aligned}
$$

where u_k denotes the expression before $-u_k^2/2$ inside the latest exponential. It can be shown that, on $\mathcal{A}$, we have $u_k^2 \le \alpha_k^4/4p^2 + c^2|\alpha_k|^6/16p^3$. It follows that, on $\mathcal{A}$, we have

$$
\begin{aligned}
\exp\left(\frac{\alpha_k^2}{2p} - \frac{c|\alpha_k|^3}{4p^{3/2}} - \frac{\alpha_k^4}{8p^2} - \frac{c^2\alpha_k^6}{32p^3} \right) &\le \mathrm{E}\left\{ \exp\left(\frac{\alpha_k}{\sqrt{p}} z_{ik} \right) \middle| \alpha \right\} \\
&\le \exp\left(\frac{\alpha_k^2}{2p} + \frac{c|\alpha_k|^3}{4p^{3/2}} \right)
\end{aligned}
$$

for any $k \ne j$. It follows that, on $\mathcal{A}$, we have

$$
\begin{aligned}
&\exp\left(\frac{1}{2p} \sum_{k \ne j} \alpha_k^2 - \frac{c}{4p^{3/2}} \sum_{k \ne j} |\alpha_k|^3 - \frac{1}{8p^2} \sum_{k \ne j} \alpha_k^4 - \frac{c^2}{32p^3} \sum_{k \ne j} \alpha_k^6 \right) \\
&\le \prod_{k \ne j} \mathrm{E}\left\{ \exp\left(\frac{\alpha_k}{\sqrt{p}} z_{ik} \right) \middle| \alpha \right\} \le \exp\left(\frac{1}{2p} \sum_{k \ne j} \alpha_k^2 + \frac{c}{4p^{3/2}} \sum_{k \ne j} |\alpha_k|^3 \right).
\end{aligned}
$$

Note that the left side of the above inequalities can be written as $e^{\alpha'\alpha/2p-g_j}$ while the right side can be written as $e^{\alpha'\alpha/2p+h_j}$, where

$$\begin{aligned} g_j &= \frac{c}{4p^{3/2}}\sum_{k\neq j}|\alpha_k|^3 + \frac{1}{8p^2}\sum_{k\neq j}\alpha_k^4 + \frac{c^2}{32p^3}\sum_{k\neq j}\alpha_k^6 + \frac{\alpha_j^2}{2p}, \\ h_j &= \frac{c}{4p^{3/2}}\sum_{k\neq j}|\alpha_k|^3 - \frac{\alpha_j^2}{2p}. \end{aligned}$$

Write $a_j = \prod_{k\neq j}\mathrm{E}\{e^{(\alpha_k/\sqrt{p})z_{ik}}|\alpha\}$. Then, we have, on $\mathcal{A}$,

$$|a_j - e^{\alpha'\alpha/2p}| \le e^{\alpha'\alpha/2p}\max_{1\le j\le p}\{|e^{-g_j}-1| \vee |e^{h_j}-1|\} = e^{\alpha'\alpha/2p}D_1,$$

with D_1 defined in an obvious way. It can be shown that $D_1 \le e^{D_3/\sqrt{p}}(D_4/\sqrt{p})$, where

$$D_3 = \frac{c}{4p}\sum_{k=1}^{p}|\alpha_k|^3 + \frac{D_2}{2\sqrt{p}}, \quad D_4 = D_3 + \frac{1}{8p^{3/2}}\sum_{k=1}^{p}\alpha_k^4 + \frac{c^2}{32p^{5/2}}\sum_{k=1}^{p}\alpha_k^6,$$

and $D_2 = \max_{1\le j\le p}\alpha_j^2$. On the other hand, by similar arguments, it can be shown that

$$b_j = \mathrm{E}\left\{\exp\left(\frac{\alpha_j}{\sqrt{p}}z_{ij}\right)z_{ij}\middle|\alpha\right\} = \frac{\alpha_j}{\sqrt{p}}(1+r_j)$$

with $|r_j| \le c|\alpha_j|/2\sqrt{p}$. It follows that

$$\left|b_j - \frac{\alpha_j}{\sqrt{p}}\right| \le \frac{c}{2p}\alpha_j^2 \le \frac{c}{2p}D_2.$$

Combining (A.7) and the above results, we have, on $\mathcal{A}$,

$$\begin{aligned} &\left|\mathrm{E}(e^{\gamma_i}z_{ij}|\alpha) - e^{\alpha'\alpha/2p}\frac{\alpha_j}{\sqrt{p}}\right| \\ =& \left|a_jb_j - e^{\alpha'\alpha/2p}\frac{\alpha_j}{\sqrt{p}}\right| \\ \le& |a_j - e^{\alpha'\alpha/2p}|\cdot|b_j| + e^{\alpha'\alpha/2p}\left|b_j - \frac{\alpha_j}{\sqrt{p}}\right| \\ \le& e^{\alpha'\alpha/2p}\left\{\sqrt{\frac{D_2}{p}}D_1 + \frac{c}{2p}(1+D_1)D_2\right\} \\ =& e^{\alpha'\alpha/2p}\frac{\sqrt{D_2}}{p}\left\{\frac{c}{2}\left(1+e^{D_3/\sqrt{p}}\frac{D_4}{\sqrt{p}}\right)\sqrt{D_2} + e^{D_3/\sqrt{p}}D_4\right\} \\ =& \frac{D}{p}, \quad 1\le j\le p, \end{aligned}$$

where D is defined in an obvious way. Thus, on $\mathcal{A}$ and for any $i_1 \neq i_2$, we have

$$\begin{aligned}
\left|\mathrm{E}(e^{\gamma_{i_1}} z_{i_1 j}|\alpha)\mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) - e^{\alpha'\alpha/p}\frac{\alpha_j^2}{p}\right| &\leq \left|\mathrm{E}(e^{\gamma_{i_1}} z_{i_1 j}|\alpha) - e^{\alpha'\alpha/2p}\frac{\alpha_j}{\sqrt{p}}\right| \mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) \\
&\quad + e^{\alpha'\alpha/2p}\frac{|\alpha_j|}{\sqrt{p}}\left|\mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) - e^{\alpha'\alpha/2p}\frac{\alpha_j}{\sqrt{p}}\right| \\
&\leq \frac{D}{p}\left(e^{\alpha'\alpha/2p}\frac{|\alpha_j|}{\sqrt{p}} + \frac{D}{p}\right) + e^{\alpha'\alpha/2p}\frac{|\alpha_j|}{\sqrt{p}} \cdot \frac{D}{p} \\
&\leq \frac{D}{p}\left(2e^{\alpha'\alpha/2p}\sqrt{\frac{D_2}{p}} + \frac{D}{p}\right). \qquad \text{(A.9)}
\end{aligned}$$

Combining the above results, we have, on $\mathcal{A}$, that

$$\begin{aligned}
T_{11} &= T_{110} + \frac{e^{\sigma_0^2}}{n^2}\sum_{i_1 \neq i_2} N_{i_1} N_{i_2} \sum_{j=1}^{p}\left\{\mathrm{E}(e^{\gamma_{i_1}} z_{i_1 j}|\alpha)\mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) - e^{\alpha'\alpha/p}\frac{\alpha_j^2}{p}\right\} \\
&= T_{110} + T_{111},
\end{aligned}$$

where T_{110} is the same as the left side of (A.3), and T_{111} is defined in an obvious way. According to (A.3), we have $T_{110} \stackrel{\mathrm{P}}{\longrightarrow} N^2\omega\sigma_1^2 e^{\sigma_0^2+\omega\sigma_1^2}$. Furthermore, by (A.9), we have

$$|T_{111}| \leq e^{\sigma_0^2} D\left(2e^{\alpha'\alpha/2p}\sqrt{\frac{D_2}{p}} + \frac{D}{p}\right).$$

It can be shown that $\max_{1\leq j\leq p}|\alpha_j| = O_{\mathrm{P}}(\log p)$, hence $D_2 = (\log p)^2 O_{\mathrm{P}}(1)$. It is then easy to show that $D = (\log p)^2 O_{\mathrm{P}}(1)$. It follows that $T_{111} = \{(\log p)^3/\sqrt{p}\}O_{\mathrm{P}}(1) = o_{\mathrm{P}}(1)$. It follows that T_{11} converges in probability to the right side of (A.3).

Next, write $T_{12} = 2e^{\sigma_0^2/2} n^{-2}\sum_{i_1=1}^{n} d_{i_1}$, where $d_{i_1} = \sum_{i_2\neq i_1} N_{i_2}\mathrm{E}(e^{\gamma_{i_2}} z'_{i_2}|\alpha)\}\delta_{i_1}$. For $i_1 \neq i'_1$, $\delta_{i_1}, \delta_{i'_1}$ are conditionally independent given W; thus, we have

$$\mathrm{E}(\delta_{i_1}\delta'_{i'_1}|W) = \mathrm{E}(\delta_{i_1}|W)\mathrm{E}(\delta_{i'_1}|W) = \nu_{i_1}\nu'_{i'_1},$$

where $\nu_i = \mathrm{E}(u_i|W) - \mathrm{E}(u_i|\alpha) = N_i e^{\eta_i} z_i - \mathrm{E}(u_i|\alpha)$. It is seen that, given α, $\nu_{i_1}, \nu_{i'_1}$ are conditionally independent. It follows that $\mathrm{E}(\delta_{i_1}\delta'_{i'_1}|\alpha) = \mathrm{E}(\nu_{i_1}\nu'_{i'_1}|\alpha) = \mathrm{E}(\nu_{i_1}|\alpha)\mathrm{E}(\nu'_{i'_1}|\alpha) = 0$, hence $\mathrm{E}(d_{i_1} d_{i'_1}|\alpha) = \{\sum_{i_2\neq i_1} N_{i_2}\mathrm{E}(e^{\gamma_{i_2}} z'_{i_2}|\alpha)\}\mathrm{E}(\delta_{i_1}\delta'_{i'_1}|\alpha)\{\sum_{i_2\neq i'_1} N_{i_2}\mathrm{E}(e^{\gamma_{i_2}} z_{i_2}|\alpha)\} = 0$.

It follows that $\mathrm{E}(T_{12}^2|\alpha) = 4e^{\sigma_0^2} n^{-4} \sum_{i_1=1}^{n} \mathrm{E}(d_{i_1}^2|\alpha)$

$$\begin{aligned}
&\leq \frac{4e^{\sigma_0^2}}{n^4} \sum_{i_1=1}^{n} \left\{ \sum_{i_2 \neq i_1} N_{i_2} \mathrm{E}(e^{\gamma_{i_2}} z'_{i_2}|\alpha) \right\} \mathrm{E}(u_{i_1} u'_{i_1}|\alpha) \left\{ \sum_{i_2 \neq i_1} N_{i_2} \mathrm{E}(e^{\gamma_{i_2}} z_{i_2}|\alpha) \right\} \\
&\leq \frac{8e^{3\sigma_0^2}}{n^4} \sum_{i_1=1}^{n} \left\{ \sum_{i_2 \neq i_1} N_{i_2} \mathrm{E}(e^{\gamma_{i_2}} z'_{i_2}|\alpha) \right\} N_{i_1}^2 \mathrm{E}(e^{2\gamma_{i_1}} z_{i_1} z'_{i_1}|\alpha) \left\{ \sum_{i_2 \neq i_1} N_{i_2} \mathrm{E}(e^{\gamma_{i_2}} z_{i_2}|\alpha) \right\} \\
&= \frac{8e^{3\sigma_0^2}}{n^4} \sum_{i_1=1}^{n} \sum_{i_2, i'_2 \neq i_1} N_{i_1}^2 N_{i_2} N_{i'_2} \mathrm{E}(e^{\gamma_{i_2}} z'_{i_2}|\alpha) \mathrm{E}(e^{2\gamma_{i_1}} z_{i_1} z'_{i_1}|\alpha) \mathrm{E}(e^{\gamma_{i'_2}} z_{i'_2}|\alpha) \\
&= \frac{8e^{3\sigma_0^2}}{n^4} \sum_{i_1=1}^{n} \sum_{i_2, i'_2 \neq i_1} N_{i_1}^2 N_{i_2} N_{i'_2} \sum_{j,k=1}^{p} \mathrm{E}(e^{2\gamma_{i_1}} z_{i_1 j} z_{i_1 k}|\alpha) \mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) \mathrm{E}(e^{\gamma_{i'_2}} z_{i'_2 k}|\alpha).
\end{aligned}$$

By similar arguments, it can be shown that, on $\mathcal{B} = \{\max_{1 \leq j \leq p} |\alpha_j|/\sqrt{p} \leq 1/2\}$, we have

$$\begin{aligned}
&\left| \mathrm{E}(e^{2\gamma_i} z_{ij} z_{ik}|\alpha) - \frac{4}{p} e^{2p^{-1}\alpha'\alpha} \alpha_j \alpha_k \right| \leq \left(\frac{\log p}{\sqrt{p}} \right)^3 O_{\mathrm{P}}(1), \quad j \neq k, \\
&\left| \mathrm{E}(e^{2\gamma_i} z_{ij}^2|\alpha) - e^{2p^{-1}\alpha'\alpha} \right| \leq \frac{\log p}{\sqrt{p}} O_{\mathrm{P}}(1),
\end{aligned}$$

where, and hereafter, the $O_{\mathrm{P}}(1)$s do not depend on any of the indexes. Furthermore, by earlier results, we have $|\mathrm{E}(e^{\gamma_i} z_{ij}|\alpha) - e^{\alpha'\alpha/2p}(\alpha_j/\sqrt{p})| \leq \{(\log p)^2/p\} O_{\mathrm{P}}(1)$, which implies

$$\left| \mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) \mathrm{E}(e^{\gamma_{i'_2}} z_{i'_2 k}|\alpha) - e^{p^{-1}\alpha'\alpha} \frac{\alpha_j \alpha_k}{p} \right| \leq \left(\frac{\log p}{\sqrt{p}} \right)^3 O_{\mathrm{P}}(1), \quad \forall j, k.$$

It can then be shown that, for $j \neq k$, we have

$$\left| \mathrm{E}(e^{2\gamma_{i_1}} z_{i_1 j} z_{i_1 k}) \mathrm{E}(e^{\gamma_{i_2}} z_{i_2 j}|\alpha) \mathrm{E}(e^{\gamma_{i'_2}} z_{i'_2 k}|\alpha) - \frac{4}{p^2} e^{3p^{-1}\alpha'\alpha} \alpha_j^2 \alpha_k^2 \right| \leq \left(\frac{\log p}{\sqrt{p}} \right)^5 O_{\mathrm{P}}(1),$$

and, for $j = k$, we have

$$\left| \mathrm{E}(e^{2\gamma_i} z_{ij}^2|\alpha) \mathrm{E}(e^{\gamma_2} z_{i_2 j}|\alpha) \mathrm{E}(e^{\gamma_{i'_2}} z_{i'_2 j}|\alpha) - \frac{1}{p} e^{3p^{-1}\alpha'\alpha} \alpha_j^2 \right| \leq \left(\frac{\log p}{\sqrt{p}} \right)^3 O_{\mathrm{P}}(1).$$

Combining the above results, it follows that

$$\mathrm{E}(T_{12}^2|\alpha) \leq O_{\mathrm{P}}(1) \left\{ \left(\frac{\alpha'\alpha}{p} \right)^2 + \frac{\alpha'\alpha}{p} + \frac{(\log p)^5}{\sqrt{p}} + \frac{(\log p)^3}{\sqrt{p}} \right\} \frac{(n-1)}{n^4} \sum_{i_1 \neq i_2} N_{i_1}^2 N_{i_2}^2,$$

which is $o_{\mathrm{P}}(1)$ by the following fact, which follows from the assumptions:

$$\frac{n \vee p}{n^4} \sum_{i_1 \neq i_2} N_{i_1}^2 N_{i_2}^2 \to 0 \tag{A.10}$$

$[a \vee b = \max(a, b)]$. This implies $T_{12} = o_{\mathrm{P}}(1)$.

Finally, let us consider T_{13}. It follows by an earlier result (see the beginning of the proof; note that this has nothing to do with the normality) that, given α, $u_i = y_i z_i, 1 \leq i \leq n$ are conditionally independent. It follows that $\mathrm{E}(u_i|\mathcal{F}_{i-1}) = \mathrm{E}(u_i|u_1, \ldots, u_{i-1}, \alpha) = \mathrm{E}(u_i|\alpha)$, hence $\mathrm{E}(\delta_i|\mathcal{F}_{i-1}) = 0$. Thus, $\delta_i, \mathcal{F}_i, 1 \leq i \leq n$ is still a sequence of martingale differences. Thus, by earlier arguments, it can be shown that $\mathrm{E}(T_{13}^2|\alpha) \leq$

$$\begin{aligned}
& \frac{8}{n^4} e^{4\sigma_0^2} \sum_{i_1 \neq i_2} N_{i_1}^2 N_{i_2}^2 \sum_{j,k} \mathrm{E}(e^{2\gamma_{i_1}} z_{i_1 j} z_{i_1 k}|\alpha) \mathrm{E}(e^{2\gamma_{i_2}} z_{i_2 j} z_{i_2 k}|\alpha) \\
= & \frac{8}{n^4} e^{4\sigma_0^2} \sum_{i_1 \neq i_2} N_{i_1}^2 N_{i_2}^2 \left(\sum_{j \neq k} \cdots + \sum_{j=k} \cdots \right) \\
= & \frac{8p}{n^4} O_{\mathrm{P}}(1) \sum_{i_1 \neq i_2} N_{i_1}^2 N_{i_2}^2,
\end{aligned}$$

which is $o_{\mathrm{P}}(1)$ by (A.10). This implies $T_{13} = o_{\mathrm{P}}(1)$.

The proof is complete.

A.2 Proof of Theorem 2

Part (I): Similar to the proof of Lemma 2, given α, X, N, $(y_i, z_i), i = 1, \ldots, n$ are conditionally independent. Furthermore, for any fixed vector $a = (a_j)_{1 \leq j \leq p} \in \mathbb{R}^p$, define

$$c_p(a) = \mathrm{E}(e^{a' \tilde{z}_1}) = \prod_{j=1}^{p} \mathrm{E}\{\exp(a_j z_{11}/\sqrt{p})\},$$

and $d_p(a) = [d_p(a_j)]_{1 \leq j \leq p}$ with

$$d_p(a_j) = \frac{\mathrm{E}(z_{11} \exp(a_j z_{11}/\sqrt{p})\}}{\mathrm{E}\{\exp(a_j z_{11}/\sqrt{p})\}},$$

where the expectations are taken with respect to the distribution of z_{11}. We have

$$\begin{aligned}
\mathrm{E}(y_i|\alpha, X, N) &= N_i e^{\mu+\tilde{x}_i'\tilde{\beta}+\sigma_0^2/2} c_p(\alpha) \equiv f(\tilde{x}_i, N_i, \alpha), \\
\mathrm{E}\{y_i(y_i-1)|\alpha, X, N\} &= (N_i e^{\mu+\tilde{x}_i'\tilde{\beta}+\sigma_0^2})^2 c_p(2\alpha) \equiv g(\tilde{x}_i, N_i, \alpha), \\
\mathrm{E}(y_i z_i|\alpha, X, N) &= N_i e^{\mu+\tilde{x}_i'\tilde{\beta}+\sigma_0^2/2} c_p(\alpha) d_p(\alpha) \equiv h(\tilde{x}_i, N_i, \alpha).
\end{aligned}$$

Let $\overline{y(y-1)}$ and S denote the left sides of (2.10), (2.11), and $b_s, s = 1, 2, 3, 4$ the right sides of (2.8)–(2.11) of the MS, respectively. The following expressions can be established:

$$\begin{aligned}
\bar{y} - b_1 &= \frac{b_1}{2}\bar{\Delta} + \bar{d}_1 + \bar{\delta}_1 + \frac{o_{\mathrm{P}}(1)}{\sqrt{n}}, \\
T_1 - b_2 &= \left(1 + \frac{1}{\sigma_\alpha^2}\right) b_2\bar{\Delta} + \frac{2b_2\bar{d}_2}{e^{\tau^2/2}\mathrm{E}(N_1)} + \frac{2}{n^2}\sum_{i_1=1}^{n}\left(\sum_{i_2\neq i_1} h_{i_2}\right)' \delta_{2i_1} + \frac{o_{\mathrm{P}}(1)}{\sqrt{n}}, \\
\overline{y(y-1)} - b_3 &= 2b_3\bar{\Delta} + \bar{d}_3 + \bar{\delta}_3 + \frac{o_{\mathrm{P}}(1)}{\sqrt{n}}, \\
S - b_4 &= b_4\bar{\Delta} + \mathrm{E}(f_1|\alpha)\{2\beta'\bar{d}_4 - \mathrm{E}(f_1|\alpha)(\beta'\bar{D}\beta + 2\bar{d}'\beta)\} \\
&\quad + \frac{2}{n^2}\sum_{i_1=1}^{n}\left(\sum_{i_2\neq i_1} \hat{x}_{i_1}'\hat{x}_{i_2} f_{i_2}\right)\delta_{1i_1} + \frac{o_{\mathrm{P}}(1)}{\sqrt{n}},
\end{aligned}$$

where $\Delta = p^{-1}\sum_{j=1}^{p}\Delta_j$ with $\Delta_j = \alpha_j^2 - \sigma_\alpha^2$; $\bar{d}_s = n^{-1}\sum_{i=1}^{n} d_{si}$, $s = 1, 2, 3, 4$ with $d_{1i} = f_i - \mathrm{E}(f_i|\alpha)$ and $f_i = f(\tilde{x}_i, N_i, \alpha)$, $d_{2i} = v_i - \mathrm{E}(v_i|\alpha)$ and $v_i = N_i e^{\tilde{x}_i'\tilde{\beta}}$, $d_{3i} = g_i - \mathrm{E}(g_i|\alpha)$ and $g_i = g(\tilde{x}_i, N_i, \alpha)$, $d_{4i} = t_i - \mathrm{E}(t_i|\alpha)$, $t_i = f_i(x_i - b)$, and $h_i = h(\tilde{x}_i, N_i, \alpha)$; $\bar{d} = n^{-1}\sum_{i=1}^{n} d_i$ with $d_i = x_i - b$, $\bar{D} = n^{-1}\sum_{i=1}^{n} D_i$ with $D_i = d_i d_i' - B$; $\delta_{1i} = y_i - \mathrm{E}(y_i|\alpha, X, N)$, $\delta_{2i} = y_i z_i - \mathrm{E}(y_i z_i|\alpha, X, N)$, $\delta_{3i} = y_i(y_i - 1) - \mathrm{E}\{y_i(y_i-1)|\alpha, X, N\}$, and $\bar{\delta}_s = n^{-1}\sum_{i=1}^{n}\delta_{si}$, $s = 1, 3$. Note that $\mathrm{E}(f_1|\alpha) = e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)c_p(\alpha)$.

From the first two expressions, and Taylor expansion, we have

$$\begin{aligned}
\sqrt{n}(\hat{\sigma}_\alpha^2 - \sigma_\alpha^2) &= \sqrt{n}\left(\frac{T_1}{\bar{y}^2} - \frac{b_2}{b_1^2}\right) \\
&= -\frac{2b_2}{b_1^3}\sqrt{n}(\bar{y} - b_1) + \frac{\sqrt{n}}{b_1^2}(T_1 - b_2) + o_{\mathrm{P}}(1) \\
&= M_1 + M_2 + M_3 + o_{\mathrm{P}}(1),
\end{aligned}$$

where $M_1 = \sum_{j=1}^p M_{1j}$ and $M_s = \sum_{i=1}^n M_{si}$, $s = 2, 3$ with

$$\begin{aligned} M_{1j} &= \frac{b_2\sqrt{n}}{b_1^2\sigma_\alpha^2 p}\Delta_j, \\ M_{2i} &= \frac{2b_2}{b_1^2\sqrt{n}}\left\{\frac{d_{2i}}{e^{\tau^2/2}\mathrm{E}(N_1)} - \frac{d_{1i}}{b_1}\right\}, \\ M_{3i} &= \frac{2}{b_1^2\sqrt{n}}(\bar{h}'\delta_{2i} - \sigma_\alpha^2 b_1\delta_{1i}), \end{aligned}$$

where $\bar{h} = n^{-1}\sum_{i=1}^n h_i$. It can be seen that $d_{1i} = e^{\mu+\sigma_0^2/2}c_p(\alpha)d_{2i}$. Thus, we have

$$M_2 = -\frac{2b_2\{c_p(\alpha) - e^{\sigma_\alpha^2/2}\}}{b_1^2 e^{(\sigma_\alpha^2+\tau^2)/2}\mathrm{E}(N_1)}\left(\frac{d_{2\cdot}}{\sqrt{n}}\right) = o_{\mathrm{P}}(1),$$

because, by the proof of Lemma 2, we have

$$c_p(\alpha) = e^{\sigma_\alpha^2/2} + o_{\mathrm{P}}(1), \tag{A.11}$$

and $d_{2\cdot}/\sqrt{n} = n^{-1/2}\sum_{i=1}^n d_{2i} = O_{\mathrm{P}}(1)$. Thus, we can write

$$\sqrt{n}(\hat{\sigma}_\alpha^2 - \sigma_\alpha^2) = M_1 + M_3 + o_{\mathrm{P}}(1) = \sum_{k=1}^{p+n} M_k + o_{\mathrm{P}}(1), \tag{A.12}$$

where $M_j = M_{1j}, 1 \le j \le p$ and $M_{p+i} = M_{3i}, 1 \le i \le n$. Define $\mathcal{F}_j = \sigma(\alpha_{j'}, j' \le j)$, $1 \le j \le p$ and $\mathcal{F}_{p+i} = \sigma(\alpha, X, N, y_{i'}, z_{i'}, i' \le i)$, $1 \le i \le n$. Then, $M_k, \mathcal{F}_k, 1 \le k \le p + n$ is an array of martingale differences. Note that $M_k, \mathcal{F}_k$ depends on n, p, but the latter are suppressed for notational simplicity. According to the martingale central limit theorem (Hall and Heyde 1980, p. 58), to show that $M = \sum_{k=1}^{p+n} M_k \xrightarrow{\mathrm{d}} N(0, \sigma^2)$, where σ^2 is a positive constant, one needs to verify the following three conditions:

$$\max_{1\le k\le p+n} M_k \xrightarrow{\mathrm{P}} 0, \tag{A.13}$$

$$\sum_{k=1}^{p+n} M_k^2 \xrightarrow{\mathrm{P}} \sigma^2, \tag{A.14}$$

$$\mathrm{E}\left(\max_{1\le k\le p+n} M_k^2\right) \text{ is bounded.} \tag{A.15}$$

(A.13): First, we have $\max_{1\le j\le p}|M_{1j}| \le (\sqrt{n}/p)(\log p)^2 O_{\mathrm{P}}(1) = o_{\mathrm{P}}(1)$.

Next, we have $\mathrm{E}(y_i|\alpha, X, N) = e^{\mu+\sigma_0^2/2}c_p(\alpha)v_i$, hence $\max_{1\le i\le n}|\delta_{1i}| \le \max_{1\le i\le n} y_i + e^{\mu+\sigma_0^2/2}c_p(\alpha)\max_{1\le i\le n} v_i$. For any constant $c > 0$, we have

$$y_i \le c + \frac{y_i^2}{c} \le c + \frac{1}{c}\sqrt{\sum_{i'=1}^{n} y_{i'}^4} \ \Rightarrow\ \max_{1\le i\le n}\frac{y_i}{\sqrt{n}} \le \frac{c}{\sqrt{n}} + \frac{1}{c}\sqrt{\frac{1}{n}\sum_{i'=1}^{n} y_{i'}^4} = \frac{c}{\sqrt{n}} + \frac{O_{\mathrm{P}}(1)}{c}.$$

Thus, by first choosing c sufficiently large and then letting $n \to \infty$, it can be shown that $n^{-1/2}\max_{1\le i\le n} y_i = o_{\mathrm{P}}(1)$. By similar arguments, we have $n^{-1/2}\max_{1\le i\le n} v_i = o_{\mathrm{P}}(1)$. Furthermore, write $w_p(\alpha) = c_p(\alpha)d_p(\alpha) = [w_{pj}(\alpha)]_{1\le j\le p}$. By the arguments in the proof of Lemma 2, it can be shown that we have the following expression:

$$w_{pj}(\alpha) = \left(\frac{\alpha_j}{\sqrt{p}} + r_{pj}\right)\prod_{k\ne j}\left(1 + \frac{\alpha_k^2}{2p} + s_{pk}\right) = \left(\frac{\alpha_j}{\sqrt{p}} + r_{pj}\right)R_{pj},$$

with R_{pj} defined in an obvious way, where

$$|r_{pj}| \le \frac{\alpha_j^2}{p}O_{\mathrm{P}}(1),\ \ |s_{pk}| \le \left(\frac{|\alpha_k|}{\sqrt{p}}\right)^3 O_{\mathrm{P}}(1);$$

hereafter, the $O_{\mathrm{P}}(1)$s do not depend on any indexes. It can then be shown that, with probability tending to one, one has

$$\exp\left\{\frac{\alpha'\alpha}{2p} - \frac{O_{\mathrm{P}}(1)}{\sqrt{p}}\right\} \le R_{pj} \le \exp\left\{\frac{\alpha'\alpha}{2p} + \frac{O_{\mathrm{P}}(1)}{\sqrt{p}}\right\}.$$

It follows that $|R_{pj} - e^{\alpha'\alpha/2p}| \le O_{\mathrm{P}}(1)/\sqrt{p}$. Thus, we can write

$$w_{pj}(\alpha) - \frac{\alpha_j}{\sqrt{p}}e^{\alpha'\alpha/2p} = \frac{\alpha_j}{\sqrt{p}}(R_{pj} - e^{\alpha'\alpha/2p}) + r_{pj}R_{pj}$$

to get $|w_{pj}(\alpha) - (\alpha_j/\sqrt{p})e^{\alpha'\alpha/2p}| \le p^{-1}(|\alpha_j| + \alpha_j^2)O_{\mathrm{P}}(1)$, $1 \le j \le p$. On the other hand, note that $h_i = e^{\mu+\sigma_0^2/2}v_i w_p(\alpha)$. It follows that

$$\left|\frac{1}{n^{3/2}}\left(\sum_{i'\ne i} h_{i'}\right)'\delta_{2i}\right| \le e^{\mu+\sigma_0^2/2}\bar{v}n^{-1/2}|w_p'(\alpha)\delta_{2i}| \le \frac{O_{\mathrm{P}}(1)}{\sqrt{n}}\max_{1\le i\le n} w_i,$$

where $\bar{v} = n^{-1}\sum_{i=1}^{n} v_i$ and $w_i = |w_p'(\alpha)\delta_{2i}|$. Recall $u_i = y_i z_i = (u_{ij})_{1\le j\le p}$. It can be shown that $\mathrm{E}(|u_{ij}|^3|\alpha, X, N) \leq O_{\mathrm{P}}(1)(v_i^3+1)$. Now, for any constant $\delta > 0$, we have

$$\begin{aligned}
\mathrm{P}\left(\max_{1\leq i\leq n} w_i > \delta\sqrt{n}\middle|\alpha, X, N\right) &\leq \sum_{i=1}^{n}\mathrm{P}(w_i > \delta\sqrt{n}|\alpha, X, N) \\
&\leq \frac{1}{(\delta\sqrt{n})^3}\sum_{i=1}^{n}\mathrm{E}\left(\left|\sum_{j=1}^{p} w_{pj}(\alpha)\delta_{2ij}\right|^3\middle|\alpha, X, N\right) \\
&\leq \frac{c}{\delta^3 n^{3/2}}\sum_{i=1}^{n}\mathrm{E}\left[\left\{\sum_{j=1}^{p} w_{pj}^2(\alpha)\delta_{2ij}^2\right\}^{3/2}\middle|\alpha, X, N\right],
\end{aligned}$$

using the Marcinkiewicz–Zygmund inequality [e.g., (5.71) of Jiang (2022)] for the last step, where $\delta_{2ij} = u_{ij} - \mathrm{E}(u_{ij}|\alpha, X, N)$. By an earlier result, it can be shown that $w_{pj}^2(\alpha) \leq O_{\mathrm{P}}(1)\alpha_j^2/p$, where the $O_{\mathrm{P}}(1) \in \sigma(\alpha)$. Thus, we have

$$\begin{aligned}
\mathrm{E}\left[\left\{\sum_{j=1}^{p} w_{pj}^2(\alpha)\delta_{2ij}^2\right\}^{3/2}\middle|\alpha, X, N\right] &\leq O_{\mathrm{P}}(1)\mathrm{E}\left\{\left(\frac{1}{p}\sum_{j=1}^{p}\alpha_j^2\delta_{2ij}^2\right)^{3/2}\middle|\alpha, X, N\right\} \\
&\leq O_{\mathrm{P}}(1)\mathrm{E}\left\{\frac{1}{p}\sum_{j=1}^{p}|\alpha_j|^3|\delta_{2ij}|^3\middle|\alpha, X, N\right\} \\
&\leq \frac{O_{\mathrm{P}}(1)}{p}\sum_{j=1}^{p}|\alpha_j|^3\mathrm{E}(|u_{ij}|^3|\alpha, X, N) \\
&\leq O_{\mathrm{P}}(1)(v_i^3+1),
\end{aligned}$$

using Jensen's inequality for the second step. Combining the above results, we have $\mathrm{P}(\max_{1\leq i\leq n} w_i > \delta\sqrt{n}|\alpha, X, N) \leq O_{\mathrm{P}}(1)/\delta^3\sqrt{n}$; thus, by the arbitrariness of δ, and the dominated convergence theorem, we have $n^{-1/2}\max_{1\leq i\leq n} w_i = o_{\mathrm{P}}(1)$. Combining the above results, we have $\max_{1\leq i\leq n}|M_{3i}| = o_{\mathrm{P}}(1)$. Thus, (A.13) has been verified.

(A.14): First, by the law of large numbers, it is easy to show that

$$\sum_{j=1}^{p} M_{1j}^2 = \frac{b_2^2 n}{b_1^4\sigma_\alpha^4 p^2}\sum_{j=1}^{p}\Delta_j^2 \xrightarrow{\mathrm{P}} \gamma\sigma_\alpha^4(3\psi - 1),$$

where $\psi = \omega^{-1}$. Next, some tedious derivations show that

$$\sum_{i=1}^{n}\mathrm{E}(M_{3i}^2|\mathcal{F}_{p+i-1}) \xrightarrow{\mathrm{P}} \frac{4}{b_1^2}\{\sigma_\alpha^2(b_1+b_3)+\sigma_\alpha^4 b_3\}. \tag{A.16}$$

Furthermore, it can be shown that $\sum_{i=1}^n \{M_{3i}^2 - \mathrm{E}(M_{3i}^2|\mathcal{F}_{p+i-1})\} \xrightarrow{\mathrm{P}} 0$. Thus, $\sum_{i=1}^n M_{3i}^2$ converges in probability to the same limit as the right side of (A.16).

(A.14) has now been verified with $\sigma^2 = v_1^2$, given more explicitly by (22) of MS.

(A.15): We have $\max_{1\le j\le p} M_{1j}^2 \le (b_2/b_1^2\sigma_\alpha^2)^2 (n/p^2) \sum_{j=1}^p \Delta_j^2$. It follows that

$$\mathrm{E}\left(\max_{1\le j\le p} M_{1j}^2\right) \le \left(\frac{b_2}{b_1^2\sigma_\alpha^2}\right)^2 \sigma_\alpha^4 (3\psi - 1)\frac{n}{p},$$

which is bounded. Similarly, we have

$$\mathrm{E}\left(\max_{1\le i\le n} M_{3i}^2\right) \le \frac{c}{b_1^4 n}\sum_{i=1}^n \{\mathrm{E}(\delta_{1i}^2) + \mathrm{E}(\bar{h}'\delta_{2i})^2\}$$

for some constant c. It can be shown that $\mathrm{E}\{c_p(\alpha)\} \le (1 - c_1\sigma_1^2/p)^{-p/2} \to e^{c_1\sigma_1^2/2}$ for some contant $c_1 > 0$, and similarly $\mathrm{E}\{c_p(2\alpha)\} \le (1 - 4c_1\sigma_1^2/p)^{-p/2} \to e^{2c_1\sigma_1^2}$. Furthermore, it can be shown that $\mathrm{E}(\delta_{1i}^2) \le f_i + e^{2(\mu+\sigma_0^2)} v_i^2 c_p(2\alpha)$. It follows that

$$\mathrm{E}(\delta_{1i}^2) \le e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)\mathrm{E}\{c_p(\alpha)\} + e^{2(\mu+\sigma_0^2+\tau^2)}\mathrm{E}(N_1^2)\mathrm{E}\{c_p(2\alpha)\},$$

which is bounded, hence $n^{-1}\sum_{i=1}^n \mathrm{E}(\delta_{1i}^2)$ is bounded. Next, we have

$$\mathrm{E}\{(\bar{h}'\delta_{2i})^2|\alpha, X, N\} = \bar{h}'\mathrm{Var}(u_i|\alpha, X, N)\bar{h} \le \bar{h}'\mathrm{E}(u_i u_i'|\alpha, X, N)\bar{h}.$$

It can be further shown that

$$\begin{aligned}
\bar{h}'\mathrm{E}(u_i u_i'|\alpha, X, N)\bar{h} &\le e^{3(\mu+\sigma_0^2/2)}(\bar{v})^2 v_i w_p'(\alpha)\mathrm{E}(e^{\alpha'\tilde{z}_i} z_i z_i'|\alpha, X, N) w_p(\alpha) \\
&\quad + e^{4\mu+3\sigma_0^2}(\bar{v})^2 v_i^2 w_p'(\alpha)\mathrm{E}(e^{2\alpha'\tilde{z}_i} z_i z_i'|\alpha, X, N) w_p(\alpha) \\
&= e^{3(\mu+\sigma_0^2/2)}(\bar{v})^2 v_i I_1 + e^{4\mu+3\sigma_0^2}(\bar{v})^2 v_i^2 I_2,
\end{aligned}$$

with I_1, I_2 defined in obvious ways. Note that

$$I_1 = \sum_{j,k=1}^p \mathrm{E}(e^{\alpha'\tilde{z}_i} z_{ij} z_{ik}|\alpha, X, N) w_{pj}(\alpha) w_{pk}(\alpha).$$

Recall $w_{pj} = \mathrm{E}(z_{11} e^{\alpha_j z_{ii}/\sqrt{p}}) \prod_{l\ne j} \mathrm{E}(e^{\alpha_l z_{ii}/\sqrt{p}})$; similarly, we have

$$\mathrm{E}(e^{\alpha'\tilde{z}_i} z_{ij} z_{ik}|\alpha, X, N) = \mathrm{E}(z_{11} e^{\alpha_j z_{11}/\sqrt{p}})\mathrm{E}(z_{11} e^{\alpha_k z_{11}/\sqrt{p}}) \prod_{l\ne j,k} \mathrm{E}(e^{\alpha_l z_{11}/\sqrt{p}}),$$

where the expectations are taken with respect to z_{11}. It follows that

$$\mathrm{E}(e^{\alpha'\tilde{z}_i} z_{ij} z_{ik} | \alpha, X, N) w_{pj}(\alpha) w_{pk}(\alpha) = \lambda_p(\alpha_j)\lambda_p(\alpha_k) \prod_{l \neq j,k} \mu_p(\alpha_l),$$

where $\lambda_p(a) = \{\mathrm{E}(z_{11} e^{az_{11}/\sqrt{p}})\}^2 \mathrm{E}(e^{az_{11}/\sqrt{p}})$ and $\mu_p(a) = \{\mathrm{E}(e^{az_{11}/\sqrt{p}})\}^3$. Thus, we have

$$\mathrm{E}\{\mathrm{E}(e^{\alpha'\tilde{z}_i} z_{ij} z_{ik} | \alpha, X, N) w_{pj}(\alpha) w_{pk}(\alpha)\} = [\mathrm{E}\{\lambda_p(\alpha_1)\}]^2 [\mathrm{E}\{\mu_p(\alpha_1)\}]^{p-2}.$$

Using Jensen's inequality, properties of normal and sub-Gaussian [e.g., Lemma 2.3 of Jiang *et al.* (2016)] distributions, and Stirling's approximation, we have

$$\begin{aligned}
&\mathrm{E}\{\mu_p(\alpha_1)\} \leq \mathrm{E}(e^{3\alpha_1 z_{11}/\sqrt{n}}) = \mathrm{E}\{\mathrm{E}(e^{3b_1\xi_1 z_{11}/\sqrt{p}} | b_1, z_{11})\} = \mathrm{E}\{e^{(9/2p)\sigma_1^2 b_1 z_{11}^2}\} \\
&\leq \mathrm{E}\{e^{(9\sigma_1^2/2p) z_{11}^2}\} = \mathrm{E}\left\{1 + \sum_{q=1}^{\infty} \left(\frac{9\sigma_1^2}{2p}\right)^q \frac{z_{11}^{2q}}{q!}\right\} = 1 + \sum_{q=1}^{\infty} \left(\frac{9\sigma_1^2}{2p}\right)^q \frac{\mathrm{E}(z_{11}^{2q})}{q!} \\
&\leq 1 + c\sum_{q=1}^{\infty} \left(\frac{9\sigma_1^2}{2p}\right)^q \frac{(K_2\sqrt{2q})^{2q}}{\sqrt{2\pi q}(q/e)^q} = 1 + c\sum_{q=1}^{\infty} \left(\frac{9}{p} e\sigma_1^2 K_2^2\right)^q \leq 1 + \frac{c}{p}
\end{aligned}$$

for large p, where K_2 is a positive constant, and c is a generic constant, whose value may be different at different places (same hereafter). Thus, we have

$$[\mathrm{E}\{\mu_p(\alpha_1)\}]^{p-2} \leq \left(1 + \frac{c}{p}\right)^{p-2} \to e^c.$$

On the other hand, it is easy to show that

$$\mathrm{E}(z_{11} e^{\alpha_1 z_{11}/\sqrt{p}}) = \frac{\alpha_1}{\sqrt{p}} + \sum_{q=2}^{\infty} \left(\frac{\alpha_1}{\sqrt{p}}\right)^q \frac{\mathrm{E}(z_{11}^{q+1})}{q!} = \frac{\alpha_1}{\sqrt{p}} + r(\alpha_1),$$

with $r(\alpha_1)$ defined in an obvious way. Furthermore, we have

$$|r(\alpha_1)| \leq \frac{\alpha_1^2}{p} \mathrm{E}\left\{|z_{11}|^3 \sum_{q=2}^{\infty} \frac{1}{q!} \left(\frac{|\alpha_1 z_{11}|}{\sqrt{p}}\right)^{q-2}\right\} \leq \frac{|\alpha_1|^3}{2p} \mathrm{E}(|z_{11}|^3 e^{|\alpha_1 z_{11}|/\sqrt{p}}).$$

Similarly, we have $\mathrm{E}(e^{\alpha_1 z_{11}/\sqrt{p}}) = 1 + s(\alpha_1)$ with $|s(\alpha_1)| \leq (\alpha_1^2/2p)\mathrm{E}(z_{11}^2 e^{|\alpha_1 z_{11}|/\sqrt{p}})$. It follows that $\lambda_p(\alpha_1) = \{\alpha_1/\sqrt{p} + r(\alpha_1)\}^2\{1 + s(\alpha_1)\} = \alpha_1^2/p + R(\alpha_1)$, and, by the Cauchy-Schwarz inequality, it can be shown that

$$|R(\alpha_1)| \leq \frac{c}{p^{3/2}}(\alpha_1^8 \vee 1)\{\mathrm{E}(e^{2|\alpha_1 z_{11}|/\sqrt{p}})\}^{3/2} \leq \frac{c}{p^{3/2}}(\alpha_1^8 \vee 1)\mathrm{E}(e^{3|\alpha_1 z_{11}|/\sqrt{p}}),$$

using Jensen's inequality for the last step. Thus, by similar arguments, we have

$$\mathrm{E}\{\lambda_p(\alpha_1)\} \le \frac{\sigma_\alpha^2}{p} + \frac{c}{p^{3/2}}\{\mathrm{E}(e^{6|\alpha_1 z_{11}|/\sqrt{p}})\}^{1/2},$$

where the last expectation is with respect to both α_1 and z_{11}. Furthermore, we have

$$\mathrm{E}(e^{6|\alpha_1 z_{11}|/\sqrt{p}}) = 1 + \sum_{q=1}^{\infty}\left(\frac{6}{\sqrt{p}}\right)^q \frac{\mathrm{E}(|\alpha_1|^q)\mathrm{E}(|z_{11}|^q)}{q!}$$
$$\le 1 + c\sum_{q=1}^{\infty}\left(\frac{6}{\sqrt{p}}\right)^q \frac{(K_1\sqrt{q})^q(K_2\sqrt{q})^q}{\sqrt{2\pi q}(q/e)^q} \le 1 + c\sum_{q=1}^{\infty}\left(\frac{6eK_1K_2}{\sqrt{p}}\right)^q \le 1 + \frac{c}{\sqrt{p}},$$

where K_1, K_2 are positive constants.

Combining the above results, we have $\mathrm{E}(I_1) \le p^2(c/p^2) = c$. Similarly, it can be shown that $\mathrm{E}(I_2) \le c$. Therefore, we have

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\{(\bar{h}'\delta_{2i})^2\} \le e^{3(\mu+\sigma_0^2)}\mathrm{E}(\bar{v})^3\mathrm{E}(I_1) + e^{4\mu+3\sigma_0^2}\mathrm{E}\{(\bar{v})^2\overline{v^2}\}\mathrm{E}(I_2) \le c.$$

Note that it can be seen that I_1, I_2 depend only on α, hence are independent with the v_is. Also, it is easy to show that $\mathrm{E}(\bar{v})^3 \le e^{9\tau^2/2}\mathrm{E}(N_1^3)$ and $\mathrm{E}\{(\bar{v})^2\overline{v^2}\} \le e^{8\tau^2}\mathrm{E}(N_1^4)$.

(A.15) has now been verified, hence the proof for part (I) is complete.

Part (II): Because $S_x \xrightarrow{\mathrm{P}} B > 0$ (positive definite), there is a constant $a > 0$ such that $\lambda_{\min}(S_x) \ge a$ with probability tending to one. By similar arguments, it can be shown that

$$\sqrt{n}(\hat{\sigma}_0^2 - \sigma_0^2) = M_2 + M_3 + o_{\mathrm{P}}(1) = \sum_{k=1}^{2n} M_k + o_{\mathrm{P}}(1), \tag{A.17}$$

where $M_2 = \sum_{i=1}^n M_{2i}$, $M_3 = \sum_{i=1}^n M_{3i}$ with

$$\begin{aligned} M_{2i} &= \frac{2}{\sqrt{n}}\left\{a_2 d_{2i} + \frac{d_{3i}}{2b_3} - \left(\frac{a_1}{b_1}\right)\beta' d_{4i} + a_1^2 d_i'\beta + \frac{a_1^2}{2}\beta' D_i\beta \right. \\ &\quad \left. + \frac{N_i - \mathrm{E}(N_1)}{\mathrm{E}(N_1)} - \frac{N_i^2 - \mathrm{E}(N_1^2)}{2\mathrm{E}(N_1^2)}\right\}, \\ M_{3i} &= \frac{2}{\sqrt{n}} 1_{[\lambda_{\min}(S_x)\ge a]}\left\{\left(\sigma_\alpha^2 + \tau^2 - 1 - \frac{\bar{w}'\hat{x}_i}{b_1}\right)\frac{\delta_{1i}}{b_1} - \frac{\bar{h}'\delta_{2i}}{b_1^2} + \frac{\delta_{3i}}{2b_3}\right\}, \end{aligned}$$

where $a_1 = \mathrm{E}(f_1|\alpha)/b_1$, $a_2 = \{(\sigma_\alpha^2 + \tau^2 - 1)c_p(\alpha) - \sigma_\alpha^2 e^{\sigma_\alpha^2/2}\}/\{e^{(\sigma_\alpha^2+\tau^2)/2}\mathrm{E}(N_1)\}$, $\bar{w} = n^{-1}\sum_{i=1}^n w_i$ and $w_i = f_i\hat{x}_i$; and $M_i = M_{2i}$, $M_{n+i} = M_{3i}$, $1 \le i \le n$. Note that the

M notations are re-defined [after proving Part (I)] to avoid notation complexity. Similarly, re-define $\mathcal{F}_i = \sigma(\alpha, x_{i'}, N_{i'}, i' \leq i)$, $\mathcal{F}_{n+i} = \sigma(\alpha, X, N, y_{i'}, z_{i'}, i' \leq i)$, $1 \leq i \leq n$. Then, $M_k, \mathcal{F}_k, 1 \leq k \leq 2n$ is an array of martingale differences. Thus, again, to show $\sum_{k=1}^{2n} M_k \xrightarrow{\mathrm{d}} N(0, \sigma^2)$ we need to verify (A.13)–(A.15) with $p+n$ replaced by $2n$.

(A.13) can be shown similarly using the results or arguments from the proof of part (I). It follows, by the dominated convergence theorem (e.g., Jiang 2022, Theorem 2.16), that the convergence in probability also holds conditional on $\mathcal{X} = \sigma(x_i, i = 1, 2, \dots)$.

(A.14): First consider $\sum_{i=1}^n M_{2i}^2$. It is easy to show that $\sum_{i=1}^n \{M_{2i}^2 - \mathrm{E}(M_{2i}^2|\mathcal{F}_{i-1}\} = o_{\mathrm{P}}(1)$. Next, write $U = a_2 d_{2i} + d_{3i}/2b_3 - (a_1/b_1)\beta' d_{4i}$, $V = a_1^2 d_i' \beta + (a_1^2/2)\beta' D_i \beta$, and $W = \{N_i - \mathrm{E}(N_1)\}/\mathrm{E}(N_1) - \{N_i^2 - \mathrm{E}(N_1^2)\}/2\mathrm{E}(N_1^2)$. Then, we have

$$\mathrm{E}\{(U+V+W)^2|\alpha) = \mathrm{E}(U^2|\alpha) + 2\{\mathrm{E}(UV|\alpha) + \mathrm{E}(UW|\alpha)\} + \mathrm{E}(V^2|\alpha) + \mathrm{E}(W^2).$$

We have $\mathrm{E}(W^2) = \mathrm{var}[(N_1)_*] - \mathrm{cov}[(N_1)_*, (N_1^2)_*] + \mathrm{var}[(N_1^2)_*]/4$, where for any random variable ζ with finite, nonzero mean, ζ_* is defined as $\zeta/\mathrm{E}(\zeta)$. Also, it can be shown that $\mathrm{E}(V^2|\alpha) = (\tau^2/2)(\tau^2+2)a_1^4$. Furthermore, some tedious derivations show that

$$\begin{aligned} \mathrm{E}(UV|\alpha) &= \frac{\tau^2}{2}(\tau^2+2)e^{\tau^2/2}\mathrm{E}(N_1)a_1^2 a_2 + \tau^2(\tau^2+1)e^{2(\mu+\sigma_0^2+\tau^2)}\mathrm{E}(N_1^2)c_p(2\alpha)\frac{a_1^2}{b_3} \\ &\quad -\frac{\tau^2}{2}(\tau^4+4\tau^2+2)e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)c_p(\alpha)\frac{a_1^3}{b_1}; \end{aligned}$$

$$
\begin{aligned}
\mathrm{E}(UW|\alpha) &= e^{\tau^2/2}\left\{a_2-\left(\frac{a_1}{b_1}\right)\tau^2 e^{\mu+\sigma_0^2/2}c_p(\alpha)\right\}\left\{\frac{\mathrm{var}(N_1)}{\mathrm{E}(N_1)}-\frac{\mathrm{cov}(N_1,N_1^2)}{2\mathrm{E}(N_1^2)}\right\} \\
&\quad +\frac{c_p(2\alpha)}{2b_3}e^{2(\mu+\sigma_0^2+\tau^2)}\left\{\frac{\mathrm{cov}(N_1,N_1^2)}{\mathrm{E}(N_1)}-\frac{\mathrm{var}(N_1^2)}{2\mathrm{E}(N_1^2)}\right\}; \\
\mathrm{E}(U^2|\alpha) &= e^{\tau^2}\left[e^{\tau^2}\mathrm{E}(N_1^2)-\{\mathrm{E}(N_1)\}^2\right]a_2^2 \\
&\quad -\frac{2}{b_1}a_1a_2\tau^2 e^{\mu+\sigma_0^2/2+\tau^2}c_p(\alpha)\left[2e^{\tau^2}\mathrm{E}(N_1^2)-\{\mathrm{E}(N_1)\}^2\right] \\
&\quad +\left(\frac{a_1}{b_1}\right)^2\tau^2 e^{2\mu+\sigma_0^2+\tau^2}c_p^2(\alpha)\left[(4\tau^2+1)e^{\tau^2}\mathrm{E}(N_1^2)-\tau^2\{\mathrm{E}(N_1)\}^2\right] \\
&\quad +\frac{a_2}{b_3}e^{2(\mu+\sigma_0^2)+5\tau^2/2}c_p(2\alpha)\left\{e^{2\tau^2}\mathrm{E}(N_1^3)-\mathrm{E}(N_1)\mathrm{E}(N_1^2)\right\} \\
&\quad -\frac{a_1\tau^2}{b_1b_3}e^{3\mu+(5/2)(\sigma_0^2+\tau^2)}c_p(\alpha)c_p(2\alpha)\left\{3e^{2\tau^2}\mathrm{E}(N_1^3)-\mathrm{E}(N_1)\mathrm{E}(N_1^2)\right\} \\
&\quad +\frac{c_p^2(2\alpha)}{4e^{4\sigma_\alpha^2}}\left\{e^{4\tau^2}\mathrm{E}(N_1^2)_*^2-1\right\}.
\end{aligned}
$$

It is seen that all of the above expectations do not depend on any index. Furthermore, it can be shown that $-\mathrm{E}(UV|\alpha)$ and $\mathrm{E}(V^2|\alpha)$ converge in probability to $\tau^2(\tau^2+2)/2$; $-\mathrm{E}(UW|\alpha)\stackrel{\mathrm{P}}{\longrightarrow}\mathrm{E}(W^2)$, whose expression is given above; and

$$
\begin{aligned}
\mathrm{E}(U^2|\alpha) &\stackrel{\mathrm{P}}{\longrightarrow} (\tau^4+3\tau^2+1)e^{\tau^2}\mathrm{E}\{(N_1)_*^2\}-(2\tau^2+1)e^{2\tau^2}\mathrm{E}\{(N_1)_*(N_1^2)_*\} \\
&\quad +\frac{1}{4}\left[e^{4\tau^2}\mathrm{E}\{(N_1^2)_*^2\}-1\right].
\end{aligned}
$$

It can then be shown that $\sum_{i=1}^n \mathrm{E}(M_{2i}^2|\alpha)$ converges in probability to

$$
\begin{aligned}
v_{01}^2 &= 4\{(\tau^4+3\tau^2+1)e^{\tau^2}-1\}\mathrm{E}\{(N_1)_*^2\}-4\{(2\tau^2+1)e^{2\tau^2}-1\}\mathrm{E}\{(N_1)_*(N_1^2)_*\} \\
&\quad +(e^{4\tau^2}-1)\mathrm{E}\{(N_1^2)_*^2\} \\
&= \mathrm{E}\left[2(N_1)_*\{(1-\tau^2)(e^{Y_3})_*+\tau^2(Y_3e^{Y_3})_*-1\}-(N_1^2)_*\{(e^{2Y_3})_*-1\}\right]^2, \quad \text{(A.18)}
\end{aligned}
$$

where Y_3 is defined above (2.19) of MS. Combining the results, we have $\sum_{i=1}^n M_{2i}^2 \stackrel{\mathrm{P}}{\longrightarrow} v_{01}^2$.

More tedious derivation shows that $\sum_{i=1}^n M_{3i}^2 \stackrel{\mathrm{P}}{\longrightarrow} v_{02}^2$, where

$$\begin{aligned} v_{02}^2 &= 4\left[\{(\sigma_\alpha^2+\tau^2+1)^2+\sigma_\alpha^2+\tau^2\}e^{\sigma^2}-(\tau^4+3\tau^2+1)e^{\tau^2}\right]\mathrm{E}\{(N_1)_*^2\} \\ &\quad -4\left\{(2\sigma_\alpha^2+2\tau^2+1)e^{2\sigma^2}-(2\tau^2+1)e^{2\tau^2}\right\}\mathrm{E}\{(N_1)_*(N_1^2)_*\} \\ &\quad +\left(e^{4\sigma^2}-e^{4\tau^2}\right)\mathrm{E}\{(N_1^2)_*^2\} \\ &\quad +\frac{4}{b_1}\left[e^{\sigma^2}\frac{\mathrm{E}(N_1)\mathrm{E}(N_1^3)}{\{\mathrm{E}(N_1^2)\}^2}-(\sigma_\alpha^2+\tau^2+1)\right]+\frac{2}{b_3} \\ &= \mathrm{E}\left[\frac{2}{b_1}(\sigma_\alpha^2+\tau^2-1-Y_2-Y_3)Y\right. \\ &\quad \left.+\frac{Y(Y-1)}{b_3}+2\{(1-\tau^2)(e^{Y_3})_*+\tau^2(Y_3e^{Y_3})_*\}-(e^{2Y_3})_*(N_1^2)_*\right]^2, \qquad \text{(A.19)} \end{aligned}$$

where, in addition to Y_3, Y_1, Y_2, Y are defined above (2.19) of MS. Note that $v_{01}^2+v_{02}^2=v_0^2$, which is equal to the right side of (2.18) as well as (2.19) of MS, because the random variables inside the expected squares on the right sides of (A.18) and (A.19) are orthogonal to each other (i.e., the mean of their product is zero; hence, v_0^2 is equal to the mean squared difference of those two random variables).

Combining the above results, (A.14) has been verified with $\sigma^2=v_0^2$.

The verification of (A.15) is similar to that in Part (I). The indicator $1_{[\lambda_{\min}(S_x)\geq a]}$ plays an important role to make sure that (A.15) holds. The proof of part (II) is complete.

A.3 Proof of Lemma 4

First introduce some notation. Denote $\underline{i}=(i_1,i_2,i_3,i_4)$. For any quantity indexed by i, say, q_i, $q_{\underline{i}}$ is defined as $q_{i_1}\cdots q_{i_4}$; similarly, for a quantity indexed by i,j, say, q_{ij}, $q_{\underline{i}j}=q_{i_1j}\cdots q_{i_4j}$. Let $\mathcal{I}=\{\underline{i}:1\leq i_r\leq n, r=1,2,3,4 \text{ and } i_1,i_2,i_3,i_4 \text{ distinct}\}$. For any $\underline{i},\underline{i}'\in\mathcal{I}$, $\underline{i}\cap\underline{i}'$, $\underline{i}\setminus\underline{i}'$, and $\underline{i}'\setminus\underline{i}$ denote the subsets of indexes that appear in both $\underline{i}$ and $\underline{i}'$, in $\underline{i}$ but not in $\underline{i}'$, and in $\underline{i}'$ but not in $\underline{i}$, respectively. Recall $u_{ij}=z_{ij}y_i$. Let $\mu_i=\mathrm{E}(y_i|W,X,N)=e^\mu N_i e^{\tilde{x}_i'\tilde{\beta}+\gamma_i+\epsilon_i}$ with $\gamma_i=\alpha'\tilde{z}_i$, and $\mu_{ij}=z_{ij}\mu_i$.

First, it can be shown that $\mathrm{E}(T_2|\alpha)=e^{4\mu+2(\sigma_0^2+p^{-1}\alpha'\alpha+\tau^2)}\{\mathrm{E}(N_1)\}^4p^{-1}\sum_{j=1}^p\alpha_j^4+$

$\{(\log p)^5/\sqrt{p}\}O_{\mathrm{P}}(1) \xrightarrow{\mathrm{P}} 3\psi b_2^2$. Thus, it suffices to show that $\mathrm{var}(T_2|\alpha) = o_{\mathrm{P}}(1)$.

Let $\zeta = (n-1)(n-2)(n-3)T_2 = \sum_{j=1}^{p}\sum_{\underline{i}\in\mathcal{I}} u_{\underline{i}j}$. We have

$$\mathrm{var}(\zeta|\alpha) = \sum_{j,k=1}^{p}\sum_{\underline{i},\underline{i}'\in\mathcal{I}} \mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha). \tag{A.20}$$

Note that $\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha) = \mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|\alpha) - \mathrm{E}(\mu_{\underline{i}j}|\alpha)\mathrm{E}(\mu_{\underline{i}'k}|\alpha)$. If $\underline{i}\cap\underline{i}' = \emptyset$, it is easy to see that $\mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|\alpha) = \mathrm{E}(\mu_{\underline{i}j}|\alpha)\mathrm{E}(\mu_{\underline{i}'k}|\alpha)$, hence $\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha) = 0$ for all j, k. Now suppose $\underline{i}\cap\underline{i}' \neq \emptyset$. By earlier results (see the proof of Lemma 2), we have

$$\begin{aligned}\mathrm{E}(\mu_{\underline{i}j}|\alpha) &= e^{4\mu+2(\sigma_0^2+\tau^2)}\{\mathrm{E}(N_1)\}^4\{\mathrm{E}(z_{1j}e^{\gamma_1}|\alpha)\}^4 \\ &= e^{4\mu+2(\sigma_0^2+p^{-1}\alpha'\alpha+\tau^2)}\{\mathrm{E}(N_1)\}^4\frac{\alpha_j^4}{p^2} + \left(\frac{\log p}{\sqrt{p}}\right)^5 O_{\mathrm{P}}(1).\end{aligned}$$

Furthermore, we have $\mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|\alpha) = \mathrm{E}\{\mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|W, X, N)|\alpha\}$ and

$$\mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|W, X, N) = \prod_{r\in\underline{i}\cap\underline{i}'} z_{rj}z_{rk}(\mu_r + \mu_r^2)\prod_{r\in\underline{i}\setminus\underline{i}'}\mu_{rj}\prod_{r\in\underline{i}'\setminus\underline{i}}\mu_{rk}.$$

Thus, we have $\mathrm{E}(u_{\underline{i}j}u_{\underline{i}'k}|\alpha) =$

$$\begin{aligned}&\prod_{r\in\underline{i}\cap\underline{i}'}\mathrm{E}\{z_{rj}z_{rk}(\mu_r+\mu_r^2)|\alpha\}\prod_{r\in\underline{i}\setminus\underline{i}'}\mathrm{E}(\mu_{rj}|\alpha)\prod_{r\in\underline{i}'\setminus\underline{i}}\mathrm{E}(\mu_{rk}|\alpha) \\ =\ &\left\{e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)\mathrm{E}(z_{1j}z_{1k}e^{\gamma_1}|\alpha) + e^{2(\mu+\sigma_0^2+\tau^2)}\mathrm{E}(N_1^2)\mathrm{E}(z_{1j}z_{1k}e^{2\gamma_1}|\alpha)\right\}^{|\underline{i}\cap\underline{i}'|} \\ &\times\left\{e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)\right\}^{8-2|\underline{i}\cap\underline{i}'|}\{\mathrm{E}(z_{1j}e^{\gamma_1}|\alpha)\}^{|\underline{i}\setminus\underline{i}'|}\{\mathrm{E}(z_{1k}e^{\gamma_1}|\alpha)\}^{|\underline{i}'\setminus\underline{i}|}.\end{aligned}$$

By earlier results, we have $\mathrm{E}(z_{1j}e^{\gamma_1}|\alpha) = e^{p^{-1}\alpha'\alpha/2}(\alpha_j/\sqrt{p}) + (\log p/\sqrt{p})^2 O_{\mathrm{P}}(1)$, hence

$$\begin{aligned}&\{\mathrm{E}(z_{1j}e^{\gamma_1}|\alpha)\}^{|\underline{i}\setminus\underline{i}'|}\{\mathrm{E}(z_{1k}e^{\gamma_1}|\alpha)\}^{|\underline{i}'\setminus\underline{i}|} \\ &= O_{\mathrm{P}}(1)\left(\frac{\alpha_j}{\sqrt{p}}\right)^{|\underline{i}\setminus\underline{i}'|}\left(\frac{\alpha_k}{\sqrt{p}}\right)^{|\underline{i}'\setminus\underline{i}|} + O_{\mathrm{P}}(1)\left(\frac{\log p}{\sqrt{p}}\right)^{9-2|\underline{i}\cap\underline{i}'|}.\end{aligned}$$

On the other hand, if $j \neq k$, we have, again by earlier results,

$$\begin{aligned}&e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)\mathrm{E}(z_{1j}z_{1k}e^{\gamma_1}|\alpha) + e^{2(\mu+\sigma_0^2+\tau^2)}\mathrm{E}(N_1^2)\mathrm{E}(z_{1j}z_{1k}e^{2\gamma_1}|\alpha) \\ &= O_{\mathrm{P}}(1)\left\{\left(\frac{\alpha_j\alpha_k}{p}\right) + \left(\frac{\log p}{\sqrt{p}}\right)^3\right\}.\end{aligned}$$

It follows that, when $j \neq k$, $\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha)$ is bounded in absolute value by

$$O_{\mathrm{P}}(1)\left\{\left(\frac{\alpha_j\alpha_k}{p}\right)^4 + \left(\frac{\log p}{\sqrt{p}}\right)^9\right\};$$

therefore, $\sum_{j\neq k}\sum_{\underline{i}\cap\underline{i}'\neq\emptyset}|\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha)|$ is bounded by

$$O_{\mathrm{P}}(1)n^7\left\{\frac{1}{p^2}\left(\frac{1}{p}\sum_{j=1}^{p}\alpha_j^4\right)^2 + \frac{(\log p)^9}{p^{5/2}}\right\} = O_{\mathrm{P}}(1)n^5.$$

If $j = k$, then, by earlier result, we have

$$e^{\mu+(\sigma_0^2+\tau^2)/2}\mathrm{E}(N_1)\mathrm{E}(z_{1j}^2e^{\gamma_1}|\alpha) + e^{2(\mu+\sigma_0^2+\tau^2)}\mathrm{E}(N_1^2)\mathrm{E}(z_{1j}^2e^{2\gamma_1}|\alpha) = O_{\mathrm{P}}(1).$$

Thus, considering $|\underline{i}\cap\underline{i}'| = s$ $(1 \leq s \leq 4)$, it is seen that $\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'k}|\alpha)$ is bounded in absolute value by $O_{\mathrm{P}}(1)\{(\alpha_j/\sqrt{p})^{8-2s} + (\log p/\sqrt{p})^{9-2s}\}$; therefore, we have

$$\sum_{j=1}^{p}\sum_{|\underline{i}\cap\underline{i}'|=s}|\mathrm{cov}(u_{\underline{i}j}, u_{\underline{i}'j}|\alpha)| \leq O_{\mathrm{P}}(1)n^{8-s}\left\{\frac{1}{p^{4-s}}\sum_{j=1}^{p}\alpha_j^{8-2s} + \frac{(\log p)^{9-2s}}{p^{7/2-s}}\right\}$$
$$= pn^4\left(\frac{n}{p}\right)^{4-s}\left\{\frac{1}{p}\sum_{j=1}^{p}\alpha_j^{8-2s} + \frac{(\log p)^{9-2s}}{\sqrt{p}}\right\} = O_{\mathrm{P}}(1)n^4p.$$

In conclusion, we have shown that the right side of (A.20) is bounded $O_{\mathrm{P}}(1)n^4(n+p)$. Thus, we have $\mathrm{var}(T_2|\alpha) \leq O(1)n^{-6}\mathrm{var}(\zeta|\alpha) = n^{-1}O_{\mathrm{P}}(1)$. The proof is complete.

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