Supplementary material for 'Testing High-dimensional White Noise Based On Modified Portmanteau Tests'

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The supplementary material contains additional simulation results and the proof of Theorem 1-5.

1. Simulation of white noise test for fitted residuals

In this section, we examine the finite sample performance of our proposed tests to test whether the residuals of the factor model are white noise. We consider the following dynamic factor model:

Model 11. $X_t = BY_t + E_t$, where E_t is i.i.d. $N(0, I_p)$, $B \in \mathbb{R}^{p\times 2}$ is a $p \times 2$ matrix, $B = (b_{ij})$, b_{ij} is first generate independently from uniform distribution $U(-1, 1)$, then be divided by $p^{0.25}$, $Y_t \in \mathbb{R}^2$ with $Y_t = AY_{t-1} + e_t$, $A \in \mathbb{R}^{2\times 2}$ is a 2 dimensional diagonal matrix with diagonal element

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set to be $(-0.5, -0.5)$. $e_t \overset{iid}{\sim} N(0, I_2)$ and are independent with $\{E_t\}$.

Model 11 is a dynamic factor model with 2 factors. Firstly, We generate $\{X_t\}$ from model 11. Subsequently, we fit $\{X_t\}$ with dynamic factor models, where the number of factors in the dynamic factor model is set as $1, 2, 3$. Next, we conduct white noise tests to test whether the residuals are white noise. Since the true factor number is 2, we expect that white noise tests reject the null hypothesis when the factor number is set to 1, and fail to reject the null hypothesis when the factor number is set to 2 and 3. We report the rejection rate of tests at $\alpha = 5\%$ significant level. We set $p = 20, 40, 60$, $N = 200$, and $L = 5, 10$. For each experiment, we have 500 Monte Carlo replicates.

Table 1 presents the rejection rate for residuals in Model 11 for different factor numbers r in the dynamic factor model, where $r = 1, 2, 3$. As we expect, when the factor number is correctly set ($r = 2$), our proposed tests T_1, T_2, and T_3 exhibit correct empirical sizes. The test proposed by Wang and Shao (2020) (T_{SN}), the test proposed by Chang et al. (2017) (T_C) and two tests from Wang et al. (2022) (T_{W1} and T_{W2}) can also control type I errors. All seven tests can control type I errors when the factor number is set too large ($r = 3$). When the factor number is set to $r = 1$, considering that the true factor number is $r = 2$, the residuals will not be

Table 1: Rejection rate (in %) of different test statistics at $\alpha = 5\%$ significant level for residuals in Model 11 with difference factor number r.

p	T_1	T_2	T_3	T_{SN}	T_C	T_{W1}	T_{W2}	T_1	T_2	T_3	T_{SN}	T_C	T_{W1}	T_{W2}
$r = 1$														
				L=5							L=10			
20	44.6	72.8	60.6	19.6	0	4.2	0.6	33.4	66.6	58.6	12.8	0	5.0	1.2
40	50.6	78.6	63.8	24.6	0	3.2	1.0	37.4	69.0	62.4	15.4	0	4.4	1.2
60	48.6	82.4	72.0	28.6	0	13.4	6.2	37.4	69.4	62.8	17.2	0	10.4	6.0
$r = 2$														
				L=5							L=10			
20	4.0	4.4	4.2	4.0	0	3.0	0.6	6.0	3.8	3.8	6.4	0	2.8	0.8
40	5.0	6.2	6.0	5.0	0	3.6	0.4	6.2	5.2	5.8	4.2	0	2.6	0.4
60	4.2	5.6	5.0	6.0	0	1.8	0.6	4.4	5.0	6.4	5.6	0	1.8	0.4
$r = 3$														
				L=5							L=10			
20	5.0	5.4	5.0	5.0	0	2.8	0.8	6.0	4.4	5.0	5.2	0	3.6	0.8
40	5.6	5.2	5.4	5.8	0	1.8	0.4	4.4	5.6	4.0	6.2	0	3.4	0.4
60	4.6	5.8	6.0	4.6	0	1.6	0.6	5.2	4.4	5.2	6.2	0	2.6	0.8

white noise. Our proposed tests T_1, T_2, and T_3 show satisfactory power. Notably, T_2 outperforms the others and T_3 is the second best test. T_{SN} also has nontrivial power. The rejection rates of T_{W1} and T_{W2} are relatively low, and the rejection rate of T_C is zero. This simulation indicates our proposed tests can be used to select the number of factors in the factor model.

2. Technical proofs

This section contains proofs of Theorem 1, Theorem 2, Theorem 3, Theorem 4 and Theorem 5.

We set $Y_{t,l} = \text{vec}\left(X_t X_{t+l}^\top\right)$. Given $L \in \mathbb{N}^+$, let $\mathcal{Y}_t = (\sqrt{w_1} Y_{t,1}^\top, \ldots, \sqrt{w_L} Y_{t,L}^\top)^\top$,

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then our statistics T in can be written as

$$T = \frac{1}{N} \sum_{i \neq j} \mathcal{Y}_i^\top \mathcal{Y}_j.$$

Similarly, our bootstrap statistics can be written as

$$T^* = \frac{1}{N} \sum_{i \neq j} e_i \mathcal{Y}_i^\top \mathcal{Y}_j e_j.$$

Since $\{X_t\}$ is a stationary time series with physical dependence, $\{\mathcal{Y}_t\}$ is also a stationary time series with physical dependence and can be written as:

$$\mathcal{Y}_t = g\left(\varepsilon_t, \varepsilon_{t-1}, \ldots\right).$$

Let $\mathcal{Y}_{t,j}$ be jth component of $\mathcal{Y}_t$, then $\{\mathcal{Y}_{t,j}\}$ have the form:

$$\mathcal{Y}_{t,j} = g_j\left(\varepsilon_t, \varepsilon_{t-1}, \ldots\right).$$

Let $\mathcal{Y}_{t,\{k\}} = g\left(\mathcal{F}_{t,\{k\}}\right)$, we set

$$\theta_{t,q,j} = \left\|\mathcal{Y}_{t,j} - \mathcal{Y}_{t,j,\{0\}}\right\|_q = \left\|\mathcal{Y}_{t,j} - g_j\left(\mathcal{F}_{t,\{0\}}\right)\right\|_q,$$

and

$$\Theta_{m,q,j} = \sum_{t=m}^{\infty} \theta_{t,q,j}.$$

We first show the following lemma:

Lemma 1. *Let* $\{G_t\}$ *be* p^2L*-dimensional independent Gaussian random vector,* $\mathbb{E}G_t = 0$*, and* $\mathrm{Cov}(G_t) = \mathrm{Cov}(\mathcal{Y}_t|\mathcal{F}_{t-1})$*,* $\{G_t\}$ *is independent with* $\{X_t\}$*. Define the Gaussian analog of* T *as*

$$V = \frac{1}{N}\sum_{i\neq j} G_i^\top G_j$$

Assuming condition 1 -3 hold, when $\{X_t\}$ *is white noise and* $\frac{Lp^2}{N^\delta \sigma_0} \to 0$*, we have*

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(T \leq t) - \mathbb{P}(V \leq t)| \to 0$$

proof of Lemma 1:

Proof. Define

$$g_0(u) = \left[1 - \min\{1, \max(u, 0)\}^4\right]^4 = \begin{cases} 1 & \text{if } u < 0 \\ (1-u^4)^4 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{if } u > 1 \end{cases},$$

then g_0 is a non-increasing and three times continuously differentiable function. Define $g_* = \max_u \{|g_0'(u)| + |g_0''(u)| + |g_0'''(u)|\} < \infty$.

Recall $\Sigma_0 = \mathrm{Var}(X_t)$, and $\sigma_0 = \mathrm{tr}(\Sigma_0^2)$. For any $\psi > 0$, define $g_{\psi,t}(x) = g(\psi(x-t))$, then for fix t, we can approximate the indicator function $\mathbb{I}_{x\leq t}$ by $g(\psi(x-t))$, i.e.,

$$\mathbb{I}_{x\leqslant t} \leqslant g_{\psi,t}(x) \leqslant \mathbb{I}_{x\leqslant t+\psi^{-1}}.$$

For $g_{\psi,t}(x)$, we also have $\sup_{x,t}\left|g'_{\psi,t}(x)\right| \leqslant g_*\psi, \quad \sup_{x,t}\left|g''_{\psi,t}(x)\right| \leqslant g_*\psi^2$, and $\quad \sup_{x,t}\left|g'''_{\psi,t}(x)\right| \leqslant g_*\psi^3$.

Let $H_i = \sum_{j=1}^{i-1}\mathcal{Y}_j + \sum_{j=i+1}^{n} G_j$, and

$$L_i = \frac{Q\left(\mathcal{Y}_1, \ldots, \mathcal{Y}_{i-1}, G_{i+1}, \ldots, G_n\right)}{N\sigma_0}, \quad \Delta_i = \frac{\mathcal{Y}_i^\top H_i}{N\sigma_0}, \quad \Gamma_i = \frac{G_i^\top H_i}{N\sigma_0},$$

where $Q\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N\right) = \sum_{i \neq j} x_i^\top x_j$. Let

$$\begin{aligned}
\mathrm{I} &= g'_{\psi,t}\left(L_i\right)\left(\Delta_i - \Gamma_i\right) = g'_{\psi,t}\left(L_i\right)\left(\mathcal{Y}_i - G_i\right)^\top H_i \\
\mathrm{II} &= \frac{1}{2} g''_{\psi,t}\left(L_i\right)\left(\Delta_i^2 - \Gamma_i^2\right) = \frac{1}{2} g''_{\psi,t}\left(L_i\right) H_i^\top\left(\mathcal{Y}_i\mathcal{Y}_i^\top - G_i G_i^\top\right) H_i, \\
\mathrm{III} &= \left[g_{\psi,t}\left(L_i + \Delta_i\right) - g_{\psi,t}\left(L_i + \Gamma_i\right)\right] - \mathrm{I} - \mathrm{II}.
\end{aligned}$$

For I, we have

$$\begin{aligned}
\mathbb{E}\mathrm{I} = \mathbb{E}[g'_{\psi,t}\left(L_i\right)\left(\Delta_i - \Gamma_i\right)] &= \mathbb{E}[g'_{\psi,t}\left(L_i\right)\left(\mathcal{Y}_i - G_i\right)^\top H_i] \\
&= \mathbb{E}[\mathbb{E}[g'_{\psi,t}\left(L_i\right)\left(\mathcal{Y}_i - G_i\right)^\top H_i|\mathcal{F}_{i-1}]]
\end{aligned}$$

When $\{X_t\}$ is white noise, $\{\mathcal{Y}_t\}$ is a sequence of martingale differences with respect to $\mathcal{F}_t$, i.e., $\mathbb{E}[\mathcal{Y}_t|\mathcal{F}_{t-1}] = 0$. Since $\{G_t\}$ is independent with $\{\mathcal{Y}_t\}$, L_i and H_i only contain $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_{i-1}\}$, we have

$$\mathbb{E}\mathrm{I} = \mathbb{E}[\mathbb{E}[g'_{\psi,t}\left(L_i\right)|\mathcal{F}_{i-1}]\mathbb{E}[\left(\mathcal{Y}_i - G_i\right)^\top|\mathcal{F}_{i-1}]\mathbb{E}[H_i|\mathcal{F}_{i-1}]] = 0$$

For II, similarly, we can have $\mathbb{E}[\mathrm{II}] = 0$.

For III, note that $\left|g''_{\psi,t}(u)\right| \leqslant g_*\psi^2, \left|g'''_{\psi,t}(u)\right| \leqslant g_*\psi^3$, using Taylor's expansion, we have $\mathbb{E}(\mathrm{III}) \leqslant \mathbb{E}\min\left\{g_*\psi^2\left(\left|\Delta_i\right|^2 + \left|\Gamma_i\right|^2\right), g_*\psi^3\left(\left|\Delta_i\right|^3 + \left|\Gamma_i\right|^3\right)\right\}$, let $q = 2 + \delta \in (2, 3]$, we obtain

$$
\begin{aligned}
\mathbb{E}(\mathrm{III}) &\leqslant \mathbb{E} \min \left\{ g_* \psi^2 \left(|\Delta_i|^2 + |\Gamma_i|^2 \right), g_* \psi^3 \left(|\Delta_i|^3 + |\Gamma_i|^3 \right) \right\} \\
&\leqslant C \psi^q \frac{1}{\sigma_0} \left(\mathbb{E} |\Delta_i|^q + \mathbb{E} |\Gamma_i|^q \right) \\
&\leqslant C \psi^q \frac{1}{\sigma_0} \left[\begin{array}{l} \sum_{i=1}^{N} \mathbb{E} \left| \frac{1}{N} \sum_{j \leq i-1} \mathcal{Y}_i^\top \mathcal{Y}_j \right|^q + \sum_{i=1}^{N} \mathbb{E} \left| \frac{1}{N} \sum_{j \geq i+1} \mathcal{Y}_i^\top G_j \right|^q \\ + \sum_{i=1}^{N} \mathbb{E} \left| \frac{1}{N} \sum_{j \leq i-1} \mathcal{Y}_i^\top G_j \right|^q + \sum_{i=1}^{N} \mathbb{E} \left| \frac{1}{N} \sum_{j \geq i+1} G_i^\top G_j \right|^q \end{array} \right]
\end{aligned}
$$

For $\mathcal{Y}_i^\top \sum_{j \leqslant i-1} \mathcal{Y}_j$, let $Z_j = \mathcal{Y}_i^\top \mathcal{Y}_j$, then $\mathcal{Y}_i^\top \sum_{j \leqslant i-1} \mathcal{Y}_j = \sum_{j=1}^{i-1} Z_j$. Following Theorem 1 of Liu et al. (2013), we have

$$
\max_{1 \leq i \leq N} \left\| \sum_{j=1}^{i-1} Z_j \right\|_q \leqslant C_q N^{\frac{1}{2}} (\max_j \Theta_{1,2,j} + \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_2) + C_q N^{\frac{1}{q}} (\max_j \sum_{m=1}^{+\infty} \min(m, N)^{\frac{1}{2} - \frac{1}{q}} \theta_{m,q,j} + \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_q).
$$

Condition 1 implies $\max_j \Theta_{1,2,j} < \infty$. Since $q \in (2, 3]$, when N is sufficiently large, we have

$$
\max_{1 \leq i \leq N} \left| \mathcal{Y}_i^\top \sum_{j=1}^{i-1} \mathcal{Y}_j \right|^q \leqslant C_q N (\max_j \sum_{m=1}^{+\infty} \min(m, N)^{\frac{1}{2} - \frac{1}{q}} \theta_{m,q,j} + \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_q)^q.
$$

For $\mathcal{Y}_i^\top \sum_{j \leqslant i-1} G_j$, since $\{G_t\}$ is independent, we have

$$
\max_{1 \leq i \leq N} \left| \mathcal{Y}_i^\top \sum_{j=1}^{i-1} G_j \right|^q \leqslant C_q N \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_q^q.
$$

Similarly, $\max_{1 \leq i \leq N} \left| \mathcal{Y}_i^\top \sum_{j=i+1}^{N} G_j \right|^q \leqslant C_q N \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_q^q$, and $\max_{1 \leq i \leq N} \left| G_i^\top \sum_{j=i+1}^{N} G_j \right|^q \leqslant C_q N \|\mathcal{Y}_2^\top \mathcal{Y}_1\|_q^q$.

Then

$$
\begin{aligned}
|\mathbb{E}(\mathrm{III})| &\leqslant C\psi^q\left(\mathbb{E}\left|\Delta_i\right|^q+\mathbb{E}\left|\Gamma_i\right|^q\right) \\
&\leqslant C\psi^q\frac{1}{\sigma_0}\left[\begin{array}{l}\sum_{i=1}^N\mathbb{E}\left|\frac{1}{N}\sum_{j\leq i-1}\mathcal{Y}_i^\top\mathcal{Y}_j\right|^q+\sum_{i=1}^N\mathbb{E}\left|\frac{1}{N}\sum_{j\geq i+1}\mathcal{Y}_i^\top G_j\right|^q \\
+\sum_{i=1}^N\mathbb{E}\left|\frac{1}{N}\sum_{j\leq i-1}\mathcal{Y}_i^\top G_j\right|^q+\sum_{i=1}^N\mathbb{E}\left|\frac{1}{N}\sum_{j\geq i+1}G_i^\top G_j\right|^q\end{array}\right] \\
&\leqslant\frac{C\psi^q N(\max_j\sum_{m=1}^{+\infty}\min(m,N)^{\frac{1}{2}-\frac{1}{q}}\theta_{m,q,j}+\|\mathcal{Y}_2^\top\mathcal{Y}_1\|_q)^q}{N^q\sigma_0}.
\end{aligned}
$$

Note that $\max_j\sum_{m=1}^{+\infty}\min(m,N)^{\frac{1}{2}-\frac{1}{q}}\theta_{m,q,j}=\sum_{m=1}^{+\infty}\min(m,N)^{\frac{1}{2}-\frac{1}{q}}u_{m,q}$ and $\|\mathcal{Y}_2^\top\mathcal{Y}_1\|_q=O((Lp^2)^{\frac{1}{q}})$.

Since

$$
g_{\psi,t}\left(\frac{T}{\sigma_0}\right)-g_{\psi,t}\left(\frac{V}{\sigma_0}\right)=\sum_{i=1}^n\left[g_{\psi,t}\left(L_i+\Delta_i\right)-g_{\psi,t}\left(L_i+\Gamma_i\right)\right],
$$

we have

$$
\begin{aligned}
|\mathbb{E}[g_{\psi,t}\left(\frac{T}{\sigma_0}\right)-g_{\psi,t}\left(\frac{V}{\sigma_0}\right)]| &\leqslant\sum_{i=1}^n|[g_{\psi,t}\left(L_i+\Delta_i\right)-g_{\psi,t}\left(L_i+\Gamma_i\right)]| \\
&\leqslant N|\mathbb{E}(\mathrm{III})| \\
&\leqslant\frac{C\psi^q(\sum_{m=1}^{+\infty}\min(m,N)^{\frac{1}{2}-\frac{1}{q}}u_{m,q}+(Lp^2)^{\frac{1}{q}})^q}{N^\delta\sigma_0} \\
&=L(N,p,\delta)C\psi^q.
\end{aligned}
$$

Since condition 2 hold, p and N satisfy $\frac{Lp^2}{N^\delta\sigma_0}\to 0$, we have $L(N,p,\delta)\to 0$.

Then we obtain

$$
\mathbb{P}(\frac{T}{\sigma_0}\leqslant t)\leqslant\mathbb{E}[g_{\psi,t}\left(\frac{T}{\sigma_0}\right)]\leqslant\mathbb{E}[g_{\psi,t}\left(\frac{V}{\sigma_0}\right)]+L(N,p,\delta)C\psi^q\leqslant\mathbb{P}(\frac{V}{\sigma_0}\leqslant t+\psi^{-1})+L(N,p,\delta)C\psi^q,
$$

i.e., $\mathbb{P}(\frac{T}{\sigma_0} \leqslant t) \leqslant \mathbb{P}(\frac{V}{\sigma_0} \leqslant t + \psi^{-1}) + L(N,p,\delta)C\psi^q$. Similarly, we can obtain $\mathbb{P}(\frac{V}{\sigma_0} \leqslant t - \psi^{-1}) - L(N,p,\delta)C\psi^q \leqslant \mathbb{P}(\frac{T}{\sigma_0} \leqslant t)$. Then we have

$$\mathbb{P}(\frac{V}{\sigma_0} \leqslant t-\psi^{-1})-L(N,p,\delta)C\psi^q \leqslant \mathbb{P}(\frac{T}{\sigma_0} \leqslant t) \leqslant \mathbb{P}(\frac{V}{\sigma_0} \leqslant t+\psi^{-1})+L(N,p,\delta)C\psi^q.$$

Since V is distributed as a linear combination of independent chi-squared random variables, following supplement material of Xu et al. (2019), there exists a constant $C_2 > 0$, such that

$$\mathbb{P}(\frac{V}{\sigma_0} \leqslant t + \psi^{-1}) \leqslant \mathbb{P}(\frac{V}{\sigma_0} \leqslant t) + C_2\psi^{\frac{1}{2}},$$

therefore,

$$\mathbb{P}(\frac{V}{\sigma_0} \leqslant t)-\left[C_2\psi^{\frac{1}{2}} + L(N,p,\delta)C\psi^q\right] \leqslant \mathbb{P}(\frac{T}{\sigma_0} \leqslant t) \leqslant \mathbb{P}(\frac{V}{\sigma_0} \leqslant t)+\left[C_2\psi^{\frac{1}{2}} + L(N,p,\delta)C\psi^q\right].$$

We then have

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(\frac{T}{\sigma_0} \leq t\right) - \mathbb{P}\left(\frac{V}{\sigma_0} \leq t\right)\right| \to C_2\psi^{\frac{1}{2}} + L(N,p,\delta)C\psi^q,$$

hence

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(T \leq t\right) - \mathbb{P}\left(V \leq t\right)\right| \to C_2\psi^{\frac{1}{2}} + L(N,p,\delta)C\psi^q.$$

Set $\psi = \frac{C}{C_2}L(n,\delta)^{-\frac{1}{q+\frac{1}{2}}}$, condition 1 and condition 2 indicate $\psi^{-1} \to 0$, we then have

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}\left(T \leq t\right) - \mathbb{P}\left(V \leq t\right)\right| \to 0.$$

□

We then prove Theorem 1.

proof of Theorem 1:

Proof. Since $\{G_t\}$ are p^2L-dimensional independent Guassian random vectors, $\mathbb{E}G_t = 0$, and $\mathrm{Cov}(G_t) = \mathrm{Cov}(\mathcal{Y}_t|\mathcal{F}_{t-1})$, set $\Sigma = \mathrm{Var}(\mathcal{Y}_t)$, we have

$$\frac{1}{N}\sum G_i \xrightarrow{d} N(0, \Sigma).$$

Let $\widetilde{G}_i = \mathcal{Y}_i e_i$, where $\{e_i\}$ are independent standard normal distribution, we have $\widetilde{G}_i \sim N(0, \mathcal{Y}_i\mathcal{Y}_i^\top)$, thus

$$\frac{1}{N}\sum \widetilde{G}_i \xrightarrow{d} N(0, \Sigma).$$

Therefore, under H_0, we have

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(T^* \le t) - \mathbb{P}(V \le t)| \to 0.$$

Combining with Lemma 1, we have

$$\sup_{t\in\mathbb{R}} |\mathbb{P}(T \le t) - \mathbb{P}(T^* \le t)| \to 0.$$

□

Theorem 2 naturally follows from Theorem 1.

We then prove Theorem 3

proof of Theorem 3:

Proof. Let $T_l = \frac{1}{N}\sum_{i\neq j} Y_{i,l}^\top Y_{j,l}$, then $\frac{1}{N}T$ have form

$$\frac{1}{N}T = w_1\frac{1}{N}T_1 + w_2\frac{1}{N}T_2 + \cdots + w_L\frac{1}{N}T_L.$$

Under H_1 in equation (3.6), we have for $l \geq 2$, $\mathbb{E}Y_{t,l} = \mathbb{E}\,\mathrm{vec}(X_t X_{t+l}^\top) = 0$. Notice that $\frac{1}{N}T_l = \frac{1}{N^2}\sum_{i\neq j} Y_{i,l}^\top Y_{j,l} = \mathrm{tr}\left(\hat{\Sigma}_l^\top \hat{\Sigma}_l\right) - \frac{1}{N^2}\sum_{i=1}^N Y_{i,l}^\top Y_{i,l}$, where $\hat{\Sigma}_l = \frac{1}{N}\sum X_t X_{t+l}^\top$ is the sample autocovariance matrix at lag l. Since X_t is a VMA(1) sequence under H_1 in equation (3.6), we have $\hat{\Sigma}_l \xrightarrow{p} 0$ for $l \geq 2$.

For $\frac{1}{N^2}\sum_{i=1}^N Y_{i,l}^\top Y_{i,l}$, note that the elements of $Y_{i,l}$ are of the form $X_{i,j}X_{i+l,k}$, where $X_{i,j}$ is jth element of X_i. Hence when $l \geq 2$, for an arbitrary given j and k, by applying the central limit theorem, we have $\frac{1}{\sqrt{N}}\sum_{i=1}^N X_{i,j}X_{i+l,k} \xrightarrow{d} \mathcal{N}(0,1)$. consequently, under H_1 in equation (3.6), we have $\frac{1}{N}T_l = \frac{1}{N^2}\sum_{i\neq j} Y_{i,l}^\top Y_{j,l} \xrightarrow{p} 0$ for $l \geq 2$; hence $\frac{1}{N}T$ is asymptotically equivalent to $w_1\frac{1}{N}T_1$.

For $T_1 = \frac{1}{N}\sum_{i\neq j} Y_{i,1}^\top Y_{j,1}$, we have

$$\begin{aligned}\frac{1}{N}T_1 &= \frac{1}{N^2}\sum_{i\neq j}\left(A_0 z_i + A_1 z_{i-1}\right)^\top \left(A_0 z_j + A_1 z_{j-1}\right)\left(A_0 z_{j-1} + A_1 z_{j-2}\right)^\top \left(A_0 z_{i-1} + A_1 z_{i-2}\right)\\ &= T_1(I) + T_1(II) + T_1(III)\end{aligned}$$

where

$$\begin{aligned}T_1(I) =& \frac{1}{N^2}\sum_{i\neq j}\left(z_i^\top A_0^\top A_0 z_j z_{j-1}^\top A_0^\top A_0 z_{i-1} + z_{i-1}^\top A_1^\top A_1 z_{j-1} z_{j-2}^\top A_1^\top A_1 z_{i-2}\right.\\ &\left.+ z_i^\top A_0^\top A_0 z_j z_{j-2}^\top A_1^\top A_1 z_{i-2} + z_{i-1}^\top A_1^\top A_1 z_{j-1} z_{j-1}^\top A_0^\top A_0 z_{i-1}\right),\end{aligned}$$

$$
\begin{aligned}
T_1(II) =& \frac{1}{N^2} \sum_{i \neq j} \left(z_i^\top A_0^\top A_1 z_{j-1} z_{j-1}^\top A_0^\top A_0 z_{i-1} + z_{i-1}^\top A_1^\top A_0 z_j z_{j-1}^\top A_0^\top A_0 z_{i-1} \right. \\
& + z_{i-1}^\top A_1^\top A_1 z_{j-1} z_{j-1}^\top A_0^\top A_1 z_{i-2} + z_{i-1}^\top A_1^\top A_1 z_{j-1} z_{j-2}^\top A_1^\top A_0 z_{i-1} \\
& + z_i^\top A_0^\top A_0 z_j z_{j-2}^\top A_1^\top A_0 z_{i-1} + z_i^\top A_0^\top A_0 z_j z_{j-1}^\top A_0^\top A_1 z_{i-2} \\
& \left. + z_i^\top A_0^\top A_1 z_{j-1} z_{j-2}^\top A_1^\top A_1 z_{i-2} + z_{i-1}^\top A_1^\top A_0 z_j z_{j-2}^\top A_1^\top A_1 z_{i-2} \right), \\
T_1(III) =& \frac{1}{N^2} \sum_{is \neq j} \left(z_i^\top A_0^\top A_1 z_{j-1} z_{j-1}^\top A_0^\top A_1 z_{i-2} + z_{i-1}^\top A_1^\top A_0 z_j z_{j-2}^\top A_1^\top A_0 z_{i-1} \right. \\
& \left. + z_i^\top A_0^\top A_1 z_{j-1} z_{j-2}^\top A_1^\top A_0 z_{i-1} + z_{i-1}^\top A_1^\top A_0 z_j z_{j-1}^\top A_0^\top A_1 z_{i-2} \right).
\end{aligned}
$$

We have $\mathbb{E}\{T_1(I)\} = \operatorname{tr}\left(\tilde{\Sigma}_0 \tilde{\Sigma}_1\right), \mathbb{E}\{T_1(II)\} = 0, \mathbb{E}\{T_1(III)\} = \frac{2}{N} \operatorname{tr}^2\left(\tilde{\Sigma}_{01}\right)$,

and

$$
\begin{aligned}
\operatorname{var}\{T_1(I)\} =& \frac{2}{N^2} \operatorname{tr}^2\left(\tilde{\Sigma}_0^2 + \tilde{\Sigma}_1^2\right) + \frac{6}{N^2} \operatorname{tr}^2\left(\tilde{\Sigma}_0 \tilde{\Sigma}_1\right) \\
& + \frac{4}{N}\left[2 \operatorname{tr}\left(\tilde{\Sigma}_0 \tilde{\Sigma}_1\right)^2 + (\nu_4 - 3) \operatorname{tr}\left\{D^2\left(\tilde{\Sigma}_0 \tilde{\Sigma}_1\right)\right\}\right] + R
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}\{T_1(II)\} =& \frac{8}{N^2} \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top\right) \operatorname{tr}\left(\tilde{\Sigma}_0^2 + \tilde{\Sigma}_1^2\right) + \frac{16}{N^2} \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_1\right) \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_0\right) \\
& + \frac{16}{N^2} \operatorname{tr}\left(\tilde{\Sigma}_0 + \tilde{\Sigma}_1\right)\left\{\operatorname{tr}\left(\tilde{\Sigma}_{01}^\top \tilde{\Sigma}_{01} \tilde{\Sigma}_0\right) + \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_1\right)\right\} \\
& + \frac{16}{N^2} \operatorname{tr}\left(\tilde{\Sigma}_{01}\right)\left\{\operatorname{tr}\left(\tilde{\Sigma}_0^2 \tilde{\Sigma}_{01}^\top\right) + \operatorname{tr}\left(\tilde{\Sigma}_1^2 \tilde{\Sigma}_{01}\right) + 2 \operatorname{tr}\left(\tilde{\Sigma}_1 \tilde{\Sigma}_{01} \tilde{\Sigma}_0\right)\right\} \\
& + \frac{4}{N} \operatorname{tr}\left(\tilde{\Sigma}_{01}^\top \tilde{\Sigma}_{01} \tilde{\Sigma}_0^2 + \tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_1^2 + 2 \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_1 \tilde{\Sigma}_{01} \tilde{\Sigma}_0\right) + R \\
\operatorname{var}\{T_1(III)\} =& \frac{4}{N} \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_{01}\right) + \frac{12}{N^2} \operatorname{tr}^2\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top\right) \\
& + \frac{16}{N^2} \operatorname{tr}\left(\tilde{\Sigma}_{01}\right) \operatorname{tr}\left(\tilde{\Sigma}_{01} \tilde{\Sigma}_{01}^\top \tilde{\Sigma}_{01}^\top\right) + R, \\
\operatorname{cov}\{T_1(I), T_1(III)\} =& \frac{4}{N^2} \operatorname{tr}^2\left(\tilde{\Sigma}_0 \tilde{\Sigma}_{01}\right) + \frac{4}{N^2} \operatorname{tr}^2\left(\tilde{\Sigma}_1 \tilde{\Sigma}_{01}\right) + R,
\end{aligned}
$$

$$\text{cov}\{T_1(I), T_1(II)\} = R,$$

$$\text{cov}\{T_1(II), T_1(III)\} = R,$$

where R represents the remainder terms with smaller orders than the others listed in each variance and covariance items. Using Proposition 4.1 of Li et al. (2019), we have

$$\left(\frac{1}{N}T_1 - \mu_S\right)/\sigma_{S1} \xrightarrow{d} \mathcal{N}(0,1).$$

Since $\frac{1}{N}T$ is asymptotically equivalent to $w_1 \frac{1}{N}T_1$, we have

$$\left(\frac{1}{N}T - w_1\mu_S\right)/w_1^2\sigma_{S1} \xrightarrow{d} \mathcal{N}(0,1).$$

□

We then prove Theorem 4

proof of Theorem 4:

Proof. Recall that the statistic T can be expressed as $T = \frac{1}{N}\sum_{i\neq j}\mathcal{Y}_i^\top \mathcal{Y}_j$. Under alternative hypothesis H_1 with $\mathbb{E}[\mathcal{Y}_t] = \mu \neq 0$, we have the following decomposition:

$$\begin{aligned}
\frac{1}{N}T =& \frac{1}{N^2}\sum_{i\neq j}\mathcal{Y}_i^\top \mathcal{Y}_j = \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{i}\left(\mathcal{Y}_{i+1} - \mu + \mu\right)^\top\left(\mathcal{Y}_j - \mu + \mu\right) \\
=& \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{i}\left(\mathcal{Y}_{i+1} - \mu\right)^\top\left(\mathcal{Y}_j - \mu\right) + \frac{N+1}{N}\|\mu\|_2^2 \\
&+ \frac{2}{N^2}\sum_{j=1}^{N}(N - j + 1)\left(\mathcal{Y}_j - \mu\right)^\top\mu + \frac{2}{N^2}\sum_{i=1}^{N} i\left(\mathcal{Y}_{i+1} - \mu\right)^\top\mu.
\end{aligned}$$

Let

$$\mathrm{I} = \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{i}\left(\mathcal{Y}_{i+1}-\mu\right)^{\top}\left(\mathcal{Y}_j-\mu\right),$$

$$\mathrm{II} = \frac{N+1}{N}\|\mu\|_2^2,$$

$$\mathrm{III} = \frac{2}{N^2}\sum_{j=1}^{N}(N-j+1)\left(\mathcal{Y}_j-\mu\right)^{\top}\mu,$$

$$\mathrm{IV} = \frac{2}{N^2}\sum_{i=1}^{N} i\left(\mathcal{Y}_{i+1}-\mu\right)^{\top}\mu.$$

For III, we have $\mathbb{E}\mathrm{III}=0$, and

$$\begin{aligned}
&\mathrm{Var}\left(\left(\frac{\|\mu\|\|S\|_F^{1/2}}{\sqrt{N}}\right)^{-1}\mathrm{III}\right)\\
\leq&\frac{4}{N^4}\frac{N}{\|\mu\|^2\|S\|_F}\sum_{j_1,j_2=1}^{N}N^2\left|\mathbb{E}\left[\mu^{\top}\left(\mathcal{Y}_{j_1}-\mu\right)\left(\mathcal{Y}_{j_2}-\mu\right)^{\top}\mu\right]\right|\\
\leq&\frac{4}{N\|\mu\|^2\|\Gamma\|_F}\sum_{j_1,j_2=1}^{N}\|\mu\|^2\left\|S_{|j_1-j_2|}\right\| \leq \frac{4\sum_{k=-\infty}^{\infty}\|S_k\|}{\|S\|_F}\to 0.
\end{aligned}$$

Hence, $\left(\frac{\|\mu\|\|S\|_F^{1/2}}{\sqrt{N}}\right)^{-1}\mathrm{III}\xrightarrow{P}0$. Similarly, we have $\left(\frac{\|\mu\|\|S\|_F^{1/2}}{\sqrt{N}}\right)^{-1}\mathrm{IV}\xrightarrow{P}0$. Hence, we have

$$\left(\frac{\|\mu\|\|S\|_F^{1/2}}{\sqrt{N}}\right)^{-1}\frac{1}{N}T = \left(\frac{\|\mu\|\|S\|_F^{1/2}}{\sqrt{N}}\right)^{-1}\cdot\mathrm{I}+\frac{\sqrt{N}\|\mu\|}{\|S\|_F^{1/2}}\cdot\frac{N+1}{N}+o_p(1),$$

then,

$$\frac{1}{\|S\|_F}T = \frac{1}{\|S\|_F}N\mathrm{I}+\frac{N\|\mu\|^2}{\|S\|_F}+o_p(\frac{\sqrt{N}\|\mu\|}{\|S\|_F^{\frac{1}{2}}})$$

For I, notice that $\{\mathcal{Y}_t-\mu\}$ satisfy $\mathbb{E}[\mathcal{Y}_t-\mu]=0$. Following the proof of Theorem 1, we can show that $\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left(N\cdot\mathrm{I}\leq t\right)-\mathbb{P}\left(T^*\leq t\right)\right|\to 0$. Therefore, we have $\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left(\frac{1}{\|S\|_F}N\cdot\mathrm{I}\leq t\right)-\mathbb{P}\left(\frac{1}{\|S\|_F}T^*\leq t\right)\right|\to 0$.

Hence, if $\frac{N\|\mu\|^2}{\|S\|_F} \to c$, then

$$\frac{1}{\|S\|_F} T \xrightarrow{d} \frac{1}{\|S\|_F} T^* + c,$$

we have that $\mathbb{P}(T < \hat{q}_{\frac{\alpha}{2}} \text{ or } T > \hat{q}_{1-\frac{\alpha}{2}}) \to \beta \in (\alpha, 1)$.

if $\frac{N\|\mu\|^2}{\|S\|_F} \to \infty$, then the leading term of $\frac{1}{\|S\|_F} T$ is $\frac{N\|\mu\|^2}{\|S\|_F}$, and $T \to \infty$, we have that $\mathbb{P}(T < \hat{q}_{\frac{\alpha}{2}} \text{ or } T > \hat{q}_{1-\frac{\alpha}{2}}) \to 1$. □

We then prove Theorem 5.

proof of Theorem 5:

Proof. Since we can rewrite T as $T = \sum_{l=1}^{L} w_l T_l$, where $T_l = \frac{1}{N} \sum_{i \neq j} Y_{i,l}^\top Y_{j,l}$, and $Y_{t,l} = \operatorname{vec}\left(X_t X_{t+l}^\top\right)$. We first show that $\frac{T_l}{2\|\Gamma\|_F} \xrightarrow{d} \mathcal{N}(0,1)$.

We can see that $\frac{T_l}{2\|\Gamma\|_F} = \sum_{t=1}^{N} \eta_{t+1}$, where $\eta_{t+1} = \frac{1}{N\|\Gamma\|_F} Y_{t+1,l}^\top \sum_{s=1}^{t} Y_{s,l}$. Since $\mathbb{E}[\eta_{t+1} \mid \mathcal{F}_t] = 0$, then $\{\eta_{t+1}\}$ is a martingale difference sequence with respect to $\mathcal{F}_t$. Using the martingale central limit theorem in Billingsley (2017), we need to show the following:

1. $\forall \varepsilon \geq 0, \sum_{t=1}^{N} \mathbb{E}\left[\eta_{t+1}^2 \mathbf{1}\{|\eta_{t+1}| > \varepsilon\} \mid \mathcal{F}_t\right] \xrightarrow{p} 0$,
2. $\sum_{t=1}^{N} \mathbb{E}\left[\eta_{t+1}^2 \mid \mathcal{F}_t\right] \xrightarrow{p} 1$.

Note that

$$\begin{aligned}
\mathbb{E}[\eta_{t+1}^4] =& \frac{1}{N^4\|\Gamma\|_F^4} \sum_{t=1}^{N} \mathbb{E}[(Y_{t+1,l}^\top Y_{s,l})^4] \\
\leq& \frac{1}{N^4\|\Gamma\|_F^4} \sum_{t=1}^{N} \sum_{s_1 \leq \cdots \leq s_4 = 1}^{t} \sum_{j_1,\ldots,j_4=1}^{p^2} \mathbb{E}\left[Y_{t+1,l,j_1} Y_{t+1,l,j_2} Y_{t+1,l,j_3} Y_{t+1,l,j_4} Y_{s_1,l,j_1} Y_{s_2,l,j_2} Y_{s_3,l,j_3} Y_{s_4,l,j_4}\right],
\end{aligned}$$

since $\{X_t\}$ is i.i.d. sequence, we have

$\mathbb{E}\left[Y_{t+1,l,j_1}Y_{t+1,l,j_2}Y_{t+1,l,j_3}Y_{t+1,l,j_4}Y_{s_1,l,j_1}Y_{s_2,l,j_2}Y_{s_3,l,j_3}Y_{s_4,l,j_4}\right] =$

$\mathbb{E}\left[Y_{t+1-s_4,l,j_1}Y_{t+1-s_4,l,j_2}Y_{t+1-s_4,l,j_3}Y_{t+1-s_4,l,j_4}Y_{s_1-s_4,l,j_1}Y_{s_2-s_4,l,j_2}Y_{s_3-s_4,l,j_3}Y_{0,l,j_4}\right].$

Note that $\{X_t\}$ is an i.i.d. sequence, then $Y_{t,l}$ is independent of $Y_{0,l}$ for $s \le 0$, we have

$\mathbb{E}\left[Y_{t+1-s_4,l,j_1}Y_{t+1-s_4,l,j_2}Y_{t+1-s_4,l,j_3}Y_{t+1-s_4,l,j_4}Y_{s_1-s_4,l,j_1}Y_{s_2-s_4,l,j_2}Y_{s_3-s_4,l,j_3}Y_{0,l,j_4}\right] =$

$\mathbb{E}\left[Y_{t+1-s_4,l,j_1}Y_{t+1-s_4,l,j_2}Y_{t+1-s_4,l,j_3}Y_{t+1-s_4,l,j_4}\right]\mathbb{E}\left[Y_{s_1-s_4,l,j_1}Y_{s_2-s_4,l,j_2}Y_{s_3-s_4,l,j_3}Y_{0,l,j_4}\right].$

We can obtain

$$\mathbb{E}\left[Y_{t+1-s_4,l,j_1}Y_{t+1-s_4,l,j_2}Y_{t+1-s_4,l,j_3}Y_{t+1-s_4,l,j_4}\right] =$$

$$\Gamma_{j_1,j_2}\Gamma_{j_3,j_4} + \Gamma_{j_1,j_3}\Gamma_{j_2,j_4} + \Gamma_{j_1,j_4}\Gamma_{j_2,j_3} + o(1),$$

then

$$\sum_{j_1,j_2,j_3,j_4=1}^{p^2} \mathbb{E}\left[Y_{t+1-s_4,l,j_1}Y_{t+1-s_4,l,j_2}Y_{t+1-s_4,l,j_3}Y_{t+1-s_4,l,j_4}\right] = O\left(\|\Gamma\|_F^4\right).$$

Note that

$|\mathbb{E}\left[Y_{s_1-s_4,l,j_1}Y_{s_2-s_4,l,j_2}Y_{s_3-s_4,l,j_3}Y_{0,l,j_4}\right]|$

$=|\ \Gamma_{j_1,j_2}\Gamma_{j_3,j_4}\mathbf{1}\{s_1=s_2\}\mathbf{1}\{s_3=s_4\} + \Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\mathbf{1}\{s_1=s_3\}\mathbf{1}\{s_2=s_4\}$

$+\Gamma_{j_1,j_4}\Gamma_{j_2,j_3}\mathbf{1}\{s_1=s_4\}\mathbf{1}\{s_2=s_3\}\ |$

$\le |\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}\mathbf{1}\{s_1=s_2\}\mathbf{1}\{s_3=s_4\}| + |\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\mathbf{1}\{s_1=s_3\}\mathbf{1}\{s_2=s_4\}|$

$+|\Gamma_{j_1,j_4}\Gamma_{j_2,j_3}\mathbf{1}\{s_1=s_4\}\mathbf{1}\{s_2=s_3\}|,$

by condition 4, we have $\sup_{j_1,j_2=1,\ldots,p} |\Gamma_{j_1,j_2}| \le C$, then we have

$$
\begin{aligned}
\mathbb{E}[\eta_{t+1}^4] \le & \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t=1}^{N}\sum_{s_1\le\cdots\le s_4=1}^{t}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbb{E}\left[Y_{t+1,l,j_1}Y_{t+1,l,j_2}Y_{t+1,l,j_3}Y_{t+1,l,j_4}Y_{s_1,l,j_1}Y_{s_2,l,j_2}Y_{s_3,l,j_3}Y_{s_4,l,j_4}\right] \\
& \le \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t=1}^{N}\sum_{s_1\le\cdots\le s_4=1}^{t}\sum_{j_1,\ldots,j_4=1}^{p^2}|\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}+\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}+\Gamma_{j_1,j_4}\Gamma_{j_2,j_3}| \\
& \times |\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}\mathbf{1}\{s_1=s_2\}\mathbf{1}\{s_3=s_4\}| + |\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\mathbf{1}\{s_1=s_3\}\mathbf{1}\{s_2=s_4\}| \\
& + |\Gamma_{j_1,j_4}\Gamma_{j_2,j_3}\mathbf{1}\{s_1=s_4\}\mathbf{1}\{s_2=s_3\}| \\
& \le \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t=1}^{N}\sum_{1\le s_1\le s_3\le t}\left(\sup_{j_1,j_2=1,\ldots,p^2}|\Gamma_{j_1,j_2}|\right)^2 O\left(\|\Gamma\|_F^4\right) \\
& \le O(N^{-1}) \to 0,
\end{aligned}
$$

hence $\mathbb{E}[\eta_{t+1}^4] \to 0$, then $\forall \varepsilon \ge 0, \sum_{t=1}^{N}\mathbb{E}\left[\eta_{t+1}^2\mathbf{1}\{|\eta_{t+1}|>\varepsilon\} \mid \mathcal{F}_t\right] \xrightarrow{p} 0$.

Note that

$$
\begin{aligned}
\sum_{t=1}^{N}\mathbb{E}\left[\eta_{t+1}^2 \mid \mathcal{F}_t\right] &= \sum_{t=1}^{N}\mathbb{E}\left[\frac{1}{N^2\|\Gamma\|_F^2}\left(Y_{t+1,l}^\top\sum_{s=1}^{t}Y_{s,l}\right)^2 \middle| \mathcal{F}_t\right] \\
&= \frac{1}{N^2\|\Gamma\|_F^2}\sum_{t=1}^{N}\sum_{s_1=1}^{t}\sum_{s_2=1}^{t}Y_{s_1,l}^\top\mathbb{E}\left[Y_{t+1,l}Y_{t+1,l}^\top \mid \mathcal{F}_t\right]Y_{s_2,l} \\
&= \frac{1}{N^2\|\Gamma\|_F^2}\sum_{t=1}^{N}\sum_{s_1=1}^{t}\sum_{s_2=1}^{t}Y_{s_1,l}^\top\Gamma Y_{s_2,l},
\end{aligned}
$$

set $V_N = \sum_{t=1}^{N}\mathbb{E}\left[\eta_{t+1}^2 \mid \mathcal{F}_t\right]$, we have $\mathbb{E}[V_N] \to 1$.

We can obtian

$$\mathbb{E}\left[V_N^2\right] = \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbb{E}\left[Y_{s_1,l,j_1}Y_{s_2,l,j_2}Y_{s_3,l,j_3}Y_{s_4,l,j_4}\right]\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}$$

$$= \frac{4}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbf{1}\left\{s_1=s_2\right\}\mathbf{1}\left\{s_3=s_4\right\}\Gamma_{j_1,j_2}^2\Gamma_{j_3,j_4}^2$$

$$+ \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbf{1}\left\{s_1=s_3\right\}\mathbf{1}\left\{s_2=s_4\right\}$$

$$\times\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}$$

$$+ \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{n}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbf{1}\left\{s_1=s_4\right\}\mathbf{1}\left\{s_2=s_3\right\}$$

$$\times\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}$$

$$= K_1+K_2+K_3.$$

We have $K_1 = \frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1=1}^{t_1}\sum_{s_3=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\Gamma_{j_1,j_2}^2\Gamma_{j_3,j_4}^2 \to$ 1, and

$$|K_2| = \left|\frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\sum_{j_1,\ldots,j_4=1}^{p^2}\mathbf{1}\left\{s_1=s_3\right\}\mathbf{1}\left\{s_2=s_4\right\}\right.$$

$$\times\Gamma_{j_1,j_3}\Gamma_{j_2,j_4}\Gamma_{j_1,j_2}\Gamma_{j_3,j_4}\ |$$

$$=\frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{t_1}\sum_{s_3,s_4=1}^{t_2}\mathbf{1}\left\{s_1=s_3\right\}\mathbf{1}\left\{s_2=s_4\right\}\sum_{j_1,j_2=1}^{p}\left[\left(\Gamma^2\right)_{j_1,j_2}\right]^2$$

$$\leq\frac{1}{N^4\|\Gamma\|_F^4}\sum_{t_1=1}^{N}\sum_{t_2=1}^{N}\sum_{s_1,s_2=1}^{N}\sum_{s_3,s_4=1}^{N}\mathbf{1}\left\{s_1=s_3\right\}\mathbf{1}\left\{s_2=s_4\right\}\left\|\Gamma^2\right\|_F^2$$

$$\leq\frac{1}{N^4\|\Gamma\|_F^4}N^4\|\Gamma\|_F^2\|\Gamma\|^2 = O\left(\frac{\|\Gamma\|^2}{\|\Gamma\|_F^2}\right)\to 0.$$

Similarly, we can obtain $|K_3| \to 0$. Hence $\mathbb{E}\left[V_N^2\right] \to 1$, using Chebyshev's inequality, we have $\sum_{t=1}^{N} \mathbb{E}\left[\eta_{t+1}^2 \mid \mathcal{F}_t\right] \xrightarrow{p} 1$. Hence we have $\frac{T_l}{2\|\Gamma\|_F} \xrightarrow{d} \mathcal{N}(0,1)$.

Since $\{X_t\}$ is i.i.d. sequence, for $l \neq k$, we have $\mathbb{E}[T_l T_k] = 0$, then we have $\frac{T}{2\|\Gamma\|_F} \xrightarrow{d} \mathcal{N}(0, \sum_{l=1}^{L} w_l^2)$.

$\square$

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