Multiple Testing of One-Sided Hypotheses under General Dependence

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Supplementary Material

Supplementary material contains the proofs of the main theorems, details of simulation results and the heatmaps from the case study.

S1 Proofs

S1.1 Proof of Theorem 1

Proof of Theorem 1. For a fixed $\mathbf{w} = (w_1, \ldots, w_K)^\top$, let $M = \max_{j \in \mathcal{H}_0} |g_j(\mathbf{w})|$ and

$$X_j = \frac{g_j(\mathbf{w})}{M}\Big[I(\lambda < P_j \le \tau | \mathbf{W} = \mathbf{w}) - \Pr(\lambda < P_j \le \tau | \mathbf{W} = \mathbf{w})\Big].$$

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Note that $|X_j| \leq 1$ almost surely and $\mathbb{E}X_j = 0$. Now, we consider $\mathrm{Var}\left(\frac{1}{p_0}\sum_{j\in\mathcal{H}_0} X_j\right)$. First, note that

$$\begin{aligned}
&\mathrm{Var}\left(\frac{1}{p_0}\sum_{j\in\mathcal{H}_0} X_j\right) = \mathrm{Var}\left(\frac{1}{p_0}\sum_{j\in\mathcal{H}_0} \frac{g_j(\mathbf{w})}{M} I(\lambda < P_j \leq \tau | \mathbf{W} = \mathbf{w})\right) \\
&= p_0^{-2}\sum_{j\in\mathcal{H}_0} \frac{g_j(\mathbf{w})^2}{M^2} \mathrm{Var}\left(I(\lambda < P_j \leq \tau|\mathbf{W}=\mathbf{w})\right) \\
&\quad + p_0^{-2}\sum_{j_1,j_2\in\mathcal{H}_0} \frac{g_{j_1}(\mathbf{w})g_{j_2}(\mathbf{w})}{M^2} \mathrm{Cov}\left(I(\lambda < P_{j_1} \leq \tau|\mathbf{W}=\mathbf{w}), I(\lambda < P_{j_2} \leq \tau|\mathbf{W}=\mathbf{w})\right).
\end{aligned} \tag{S1.1}$$

The first term in (**??**) is $O(p_0^{-1})$, since $\mathrm{Var}\left(I(\lambda < P_j \leq \tau|\mathbf{W}=\mathbf{w})\right) \leq 1/4$. As in the proof of Proposition 2 of **?**, we can show that

$$\begin{aligned}
&\mathrm{Cov}\left(I(\lambda < P_{j_1} \leq \tau|\mathbf{W}=\mathbf{w}), I(\lambda < P_{j_2} \leq \tau|\mathbf{W}=\mathbf{w})\right) \\
&= (\phi(c_{1,j_1}) - \phi(c_{2,j_1}))(\phi(c_{1,j_2}) - \phi(c_{2,j_2}))a_{j_1}a_{j_2}C_{j_1j_2} + O(|C_{j_1j_2}|^{3/2})
\end{aligned}$$

where $c_{1,j} = -a_j(\mu_j^* + z_\lambda + \mathbf{b}_j^\top\mathbf{w})$, $c_{2,j} = -a_j(\mu_j + z_\tau + \mathbf{b}_j^\top\mathbf{w})$, and $C_{j_1j_2} = \mathbf{\Sigma}_{u,j_1j_2}$. Hence, we obtain that

$$\mathrm{Var}\left(\frac{1}{p_0}\sum_{j\in\mathcal{H}_0} X_j\right) = O(p^{-\delta}) \quad \text{a.s.}$$

Here, we utilize the following lemma, which is used in **?** and has the formal statement and proof in **?**.

Lemma 1 (Strong law of large numbers for weakly correlated variables). *Let $\{X_n\}_{n=1}^\infty$ be a sequence of real-valued random variables. If $|X_n| \leq 1$ almost surely and*

$$\sum_{N\geq 1} \frac{1}{N}\mathbb{E}\left|\frac{1}{N}\sum_{n=1}^{N} X_n\right|^2 < \infty,$$

then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} X_n = 0$ almost surely.

By Lemma **??**, for fixed $\mathbf{w} = (w_1, \ldots, w_K)^\top$,

$$\lim_{p\to\infty} \left[p_0^{-1} \sum_{j\in\mathcal{H}_0} g_j(\mathbf{w}) \Big[I(\lambda < P_j \leq \tau | \mathbf{W} = \mathbf{w}) - \Pr(\lambda < P_j \leq \tau | \mathbf{W} = \mathbf{w}) \Big] \right] = 0 \quad \text{a.s.}$$

This implies that for almost every $\omega \in \Omega$,

$$\lim_{p\to\infty} \left[p_0^{-1} \sum_{j\in\mathcal{H}_0} g_j(\mathbf{W}(\omega)) \Big[I(\lambda < P_j(\omega) \leq \tau) - \Pr(\lambda < P_j \leq \tau | \mathbf{W}(\omega)) \Big] \right] = 0,$$

which is the definition for

$$\lim_{p\to\infty} \left[p_0^{-1} \sum_{j\in\mathcal{H}_0} g_j(\mathbf{W}) \Big[I(\lambda < P_j \leq \tau) - \Pr(\lambda < P_j \leq \tau | \mathbf{W}) \Big] \right] = 0 \quad \text{a.s.}$$

This completes the proof. □

S1.2 Proof of Theorem 2

Proof of Theorem 2. We first note that bounding makes $V_{\text{orc}}^{\text{DAB}}(t; \lambda, \tau, \epsilon)$ smaller than $V_{\text{orc}}^{\text{DA}}(t; \lambda, \tau)$. We observe

$$\begin{aligned}
\mathbb{E}\left(V_{\text{orc}}^{\text{DA}}(t; \lambda, \tau) - V_{\text{orc}}^{\text{DAB}}(t; \lambda, \tau, \epsilon) | \mathbf{W}\right) &= \sum_{j\in\mathcal{H}_0} \Phi(a_j(\mu_j + z_t + \mathbf{b}_j^\top \mathbf{W})) D_j \left(\frac{1}{D_j} - \frac{1}{D_j \vee \epsilon} \right) \\
&\leq \sum_{j\in\mathcal{H}_0} \left(1 - \frac{D_j}{\epsilon} \right) I(D_j < \epsilon) \\
&\leq \sum_{j\in\mathcal{H}_0} I(D_j < \epsilon)
\end{aligned}$$

where $D_j = \Phi(a_j(\mu_j + z_\tau + \mathbf{b}_j^\top \mathbf{W})) - \Phi(a_j(\mu_j + z_\lambda + \mathbf{b}_j^\top \mathbf{W}))$. Hence, we obtain

$$\mathbb{E}\left(V_{\text{orc}}^{\text{AD}}(t; \lambda, \tau) - V_{\text{orc}}^{\text{ADB}}(t; \lambda, \tau, \epsilon)\right) \leq \sum_{j\in\mathcal{H}_0} \Pr(D_j < \epsilon).$$

By the mean value theorem, there exists $z_j \in (z_\lambda, z_\tau)$ such that $D_j = \phi(a_j(\mu_j + z_j + \mathbf{b}_j^\top \mathbf{W}))a_j(z_\tau - z_\lambda)$. Note that for an index j with $\lambda < P_j \le \tau$, we have

$$\begin{aligned}
\Pr(D_j < \epsilon) &\le \Pr\left[\exp\left(-a_j^2(\mu_j + z_j + \mathbf{b}_j^\top \mathbf{W})^2/2\right) < \frac{\sqrt{2\pi}\epsilon}{a_j(z_\tau - z_\lambda)} \le C_1\epsilon\right] \\
&\le \Pr\left[|\mu_j + z_j + \mathbf{b}_j^\top \mathbf{W}| > \frac{1}{a_j}\sqrt{\log\left(\frac{1}{C_1^2\epsilon^2}\right)}\right] \\
&\le \Pr\left[C_2 + |\mathbf{b}_j^\top \mathbf{W}| \ge \frac{1}{C_a}\sqrt{\log\left(\frac{1}{C_1^2\epsilon^2}\right)}\right] \\
&\le \Pr\left[|Z| \ge \frac{1}{C_a}\sqrt{\log\left(\frac{1}{C_1^2\epsilon^2}\right)} - C_2\right] \\
&\le \Pr\left[|Z| \ge \frac{C_3}{C_a}\sqrt{\log\left(\frac{1}{C_1^2\epsilon^2}\right)}\right] \\
&\le \exp\left[-\frac{C_3}{2C_a^2}\log\left(\frac{1}{C_1^2\epsilon^2}\right)\right] = O(\epsilon^\beta) = O(p^{-\alpha\beta}),
\end{aligned}$$

for some positive constants $C_1, C_2 > 0$, $0 < C_3, \beta < 1$, and $Z \sim N(0,1)$. This implies

$$\mathbb{E}\left(p_0^{-1}|V_{\text{orc}}^{\text{DAB}}(t;\lambda,\tau) - V_{\text{orc}}^{\text{DA}}(t;\lambda,\tau,\epsilon)|\right) = O(p^{-\alpha\beta}),$$

and by Markov's inequality, the desired result is obtained. □

S1.3 Proof of Lemma 2

Proof of Lemma 2. It is equivalent to showing that

$$\frac{\Phi(x+C)}{\Phi(C)} \le \frac{\Phi(x+A) - \Phi(x+B)}{\Phi(A) - \Phi(B)}$$

for any $x \le 0$ and $A > B > C$. By Cauchy's mean value theorem, there exists $D = D(x) \in (B, A)$ such that

$$\frac{\Phi(x+A) - \Phi(x+B)}{\Phi(A) - \Phi(B)} = \frac{\phi(x+D)}{\phi(D)}$$

where ϕ is the density function of a standard normal distribution. Now, it is sufficient to show that

$$\frac{\Phi(x+C)}{\Phi(C)} \leq \frac{\phi(x+D)}{\phi(D)} \tag{S1.2}$$

for any $x \leq 0$ and $C < D$. To show the above inequality, for given constants C, D with $C < D$, we define a real-valued function f by

$$f(x) = \frac{\phi(x+D)}{\Phi(x+C)}.$$

Then, the inequality (**??**) can be rewritten as $f(0) \leq f(x)$. With a simple calculation, we observe

$$f'(x) = -\frac{\phi(x+D)}{\Phi(x+C)^2}\left((x+D)\Phi(x+C) + \phi(x+C)\right) \leq 0$$

for any $x \leq 0$. Here, we use the assumption $C < D$ and the fact that for any $t \in \mathbb{R}$,

$$t\Phi(t) = \int_{-\infty}^{t} t\phi(u)du \geq \int_{-\infty}^{t} u\phi(u)du = -\phi(t).$$

Therefore, it holds that $f(x) \geq f(0)$ for all $x \leq 0$, and thus the desired result holds. □

S1.4 Proof of Theorem 3

Proof of Theorem 3. The proof of Theorem 3 is analogous to the proof of Theorem 1 in **?**. For brevity, we omit the superscript DAB. Define $\tilde{\mathbf{W}} = (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{Z}$. Then,

$$\mathbf{B}\tilde{\mathbf{W}} = \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{Z} = \left(\sum_{k=1}^{K} \boldsymbol{\gamma}_k\boldsymbol{\gamma}_k^\top\right)\mathbf{Z},$$

and similarly

$$\widehat{\mathbf{B}}\widehat{\mathbf{W}} = \widehat{\mathbf{B}}(\widehat{\mathbf{B}}^\top\widehat{\mathbf{B}})^{-1}\widehat{\mathbf{B}}^\top\mathbf{Z} = \left(\sum_{k=1}^{K} \widehat{\boldsymbol{\gamma}}_k\widehat{\boldsymbol{\gamma}}_k^\top\right)\mathbf{Z}.$$

Define

$$V_{\mathrm{T}}(t;\lambda,\tau,\epsilon) = \sum_{j=1}^{p} \frac{\Phi(a_j(z_t+\tilde{\eta}_j))I(\lambda < P_j \leq \tau)}{(\Phi(a_j(z_\tau+\tilde{\eta}_j)) - \Phi(a_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon}.$$

in which $\tilde{\eta}_j = \mathbf{b}_j^\top \tilde{\mathbf{W}}$. Then,

$$\widehat{\mathrm{FDP}}(t) - \mathrm{FDP}_{\mathrm{A}}(t) = \left\{\widehat{\mathrm{FDP}}(t) - \mathrm{FDP}_{\mathrm{T}}(t)\right\} + \left\{\mathrm{FDP}_{\mathrm{T}}(t) - \mathrm{FDP}_{\mathrm{U}}(t)\right\}$$

where $\mathrm{FDP}_{\mathrm{T}}(t;\lambda,\tau,\epsilon) = V_{\mathrm{T}}(t;\lambda,\tau,\epsilon)/(R(t)\vee 1)$. We rewrite each term by

$$\widehat{\mathrm{FDP}}(t) - \mathrm{FDP}_{\mathrm{T}}(t) = \Delta_1/(R(t)\vee 1),$$

$$\mathrm{FDP}_{\mathrm{T}}(t) - \mathrm{FDP}_{\mathrm{U}}(t) = \Delta_2/(R(t)\vee 1),$$

where

$$\Delta_1 = \sum_{j=1}^{p}\left[\frac{\Phi(\widehat{a}_j(z_t+\widehat{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(\widehat{a}_j(z_\tau+\widehat{\eta}_j))-\Phi(\widehat{a}_j(z_\lambda+\widehat{\eta}_j)))\vee\epsilon} - \frac{\Phi(a_j(z_t+\tilde{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(a_j(z_\tau+\tilde{\eta}_j))-\Phi(a_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon}\right],$$

$$\Delta_2 = \sum_{j=1}^{p}\left[\frac{\Phi(a_j(z_t+\tilde{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(a_j(z_\tau+\tilde{\eta}_j))-\Phi(a_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon} - \frac{\Phi(a_j(z_t+\eta_j))I(\lambda<P_j\le\tau)}{(\Phi(a_j(z_\tau+\eta_j))-\Phi(a_j(z_\lambda+\eta_j)))\vee\epsilon}\right].$$

First, we address the term $\Delta_1 = \sum_{j=1}^{p}\Delta_{1j}$, in which

$$\begin{aligned}\Delta_{1j} &= \left\{\frac{\Phi(\widehat{a}_j(z_t+\widehat{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(\widehat{a}_j(z_\tau+\widehat{\eta}_j))-\Phi(\widehat{a}_j(z_\lambda+\widehat{\eta}_j)))\vee\epsilon} - \frac{\Phi(\widehat{a}_j(z_t+\tilde{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(\widehat{a}_j(z_\tau+\tilde{\eta}_j))-\Phi(\widehat{a}_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon}\right\}\\ &\quad+\left\{\frac{\Phi(\widehat{a}_j(z_t+\tilde{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(\widehat{a}_j(z_\tau+\tilde{\eta}_j))-\Phi(\widehat{a}_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon} - \frac{\Phi(a_j(z_t+\tilde{\eta}_j))I(\lambda<P_j\le\tau)}{(\Phi(a_j(z_\tau+\tilde{\eta}_j))-\Phi(a_j(z_\lambda+\tilde{\eta}_j)))\vee\epsilon}\right\}\\ &\quad= \Delta_{11j}+\Delta_{12j}.\end{aligned}$$

By the mean value theorem, there exists η_j between $\widehat{\eta}_j = \widehat{\mathbf{b}}_j^\top\widehat{\mathbf{W}}$ and $\tilde{\eta}_j = \mathbf{b}_j^\top\tilde{\mathbf{W}}$ such that

$$\Delta_{11j} = \frac{\partial}{\partial\eta}g(a,\eta)\Big|_{a=\widehat{a}_j,\eta=\eta_j}(\widehat{\eta}_j-\tilde{\eta}_j)I(\lambda<P_j\le\tau)$$

where

$$g(a,\eta) = \frac{\Phi(a(z_t+\eta))}{(\Phi(a(z_\tau+\eta))-\Phi(a(z_\lambda+\eta)))\vee\epsilon}.$$

For $(a, \eta) = (\widehat{a}_j, \eta_j)$ satisfying $\Phi(a(z_\tau + \eta)) - \Phi(a(z_\lambda + \eta)) > \epsilon$, we observe

$$\begin{aligned}\left|\frac{\partial}{\partial \eta} g(a, \eta)\right| &= a \left| \frac{\phi(a(z_t + \eta))}{\Phi(a(z_\tau + \eta)) - \Phi(a(z_\lambda + \eta))} - \frac{\Phi(a(z_t + \eta))(\phi(a(z_\tau + \eta)) - \phi(a(z_\lambda + \eta)))}{(\Phi(a(z_\tau + \eta)) - \Phi(a(z_\lambda + \eta)))^2} \right| \\ &\leq a \left(\frac{1}{\sqrt{2\pi}\epsilon} + \frac{|a(z + \eta)|}{\epsilon} \right) \\ &= O(\epsilon^{-1}),\end{aligned}$$

for some $z \in (z_\lambda, z_\tau)$. The second equation is due to Cauchy's mean value theorem. For $(a, \eta) = (\widehat{a}_j, \eta_j)$ satisfying $\Phi(a(z_\tau + \eta)) - \Phi(a(z_\lambda + \eta)) \leq \epsilon$, we have

$$\frac{\partial}{\partial \eta} g(a, \eta) = \frac{a\phi(a(z_t + \eta))}{\epsilon},$$

which implies $\left|\frac{\partial}{\partial \eta} g(a, \eta)\right| = O(\epsilon^{-1})$. Additionally, we observe

$$\begin{aligned}\sum_{j=1}^{p} |\widehat{\eta}_j - \tilde{\eta}_j| = \|\widehat{\mathbf{B}}\widehat{\mathbf{W}} - \mathbf{B}\tilde{\mathbf{W}}\|_1 &\leq \sqrt{p} \left\| \left(\sum_{k=1}^{K} \left(\widehat{\boldsymbol{\gamma}}_k \widehat{\boldsymbol{\gamma}}_k^\top - \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \right) \mathbf{Z} \right\| \\ &\leq \sqrt{p} \left\| \sum_{k=1}^{K} \left(\widehat{\boldsymbol{\gamma}}_k \widehat{\boldsymbol{\gamma}}_k^\top - \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \right\| \|\mathbf{Z}\|.\end{aligned}$$

Since we have

$$\begin{aligned}\left\| \sum_{k=1}^{K} \left(\widehat{\boldsymbol{\gamma}}_k \widehat{\boldsymbol{\gamma}}_k^\top - \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \right\| &\leq \left\| \sum_{k=1}^{K} \left(\widehat{\boldsymbol{\gamma}}_k \widehat{\boldsymbol{\gamma}}_k^\top - \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \right\|_F \\ &\leq \sum_{k=1}^{K} \|\widehat{\boldsymbol{\gamma}}_k (\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k)^\top + (\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k) \boldsymbol{\gamma}_k^\top\|_F \\ &\leq 2 \sum_{k=1}^{K} \|\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k\| = O_p(Kp^{-\nu_1})\end{aligned}$$

and

$$\mathbb{E}\|\mathbf{Z}\|^2 \leq 2\|\boldsymbol{\mu}\|^2 + 2p,$$

we obtain

$$\left| \sum_{j=1}^{p} \Delta_{11j} \right| = O_p(K\epsilon^{-1} p^{1/2-\nu_1} (\|\boldsymbol{\mu}\| + p^{1/2})). \tag{S1.3}$$

Next, we address the term $\sum_{j=1}^{p} \Delta_{12j}$. The mean value theorem implies that there exists a_j between $\widehat{a}_j$ and a_j such that $\Delta_{12j} = \frac{\partial}{\partial a} g(a,\eta)\big|_{a=a_j,\eta=\tilde{\eta}_j} (\widehat{a}_j - a_j) I(\lambda < P_j \leq \tau)$. For $(a,\eta) = (a_j, \tilde{\eta}_j)$ satisfying $\Phi(a(z_\tau+\eta)) - \Phi(a(z_\lambda+\eta)) \leq \epsilon$,

$$\frac{\partial}{\partial a} g(a,\eta) = \frac{(z_t+\eta)\phi(a(z_t+\eta))}{\epsilon}.$$

Since $|z|\phi(z)$ is bounded (*i.e.* $|z|\phi(z) \leq C$ for a positive constant $C > 0$), we obtain $\left|\frac{\partial}{\partial a} g(a,\eta)\right| = O(\epsilon^{-1})$ in this case. For $(a,\eta) = (a_j, \tilde{\eta}_j)$ satisfying $\Phi(a(z_\tau+\eta)) - \Phi(a(z_\lambda+\eta)) > \epsilon$, we observe

$$\begin{aligned}
\left|\frac{\partial}{\partial a} g(a,\eta)\right| &= \Bigg| \frac{(z_t+\eta)\phi(a(z_t+\eta))}{\Phi(a(z_\tau+\eta)) - \Phi(a(z_\lambda+\eta))} \\
&\qquad - \frac{\Phi(a(z_t+\eta))\left\{(z_\tau+\eta)\phi(a(z_\tau+\eta)) - (z_\lambda+\eta)\phi(a(z_\lambda+\eta))\right\}}{\left(\Phi(a(z_\tau+\eta)) - \Phi(a(z_\lambda+\eta))\right)^2} \Bigg| \\
&\leq \frac{C}{a\epsilon} + \frac{1 + a^2(z+\eta)^2}{a\epsilon} \\
&= O(\epsilon^{-1}),
\end{aligned}$$

for some $z \in (z_\lambda, z_\tau)$. Here, Cauchy's mean value theorem is used. Thus, we have $\left|\frac{\partial}{\partial a} g(a,\eta)\right| = O(\epsilon^{-1})$. Using the definitions of $\widehat{a}_j$ and a_j, we obtain

$$\begin{aligned}
|\widehat{a}_j - a_j| &= |(1 - \|\widehat{\mathbf{b}}_j\|^2)^{-1/2} - (1 - \|\mathbf{b}_j\|^2)^{-1/2}| \\
&\leq C|\|\widehat{\mathbf{b}}_j\|^2 - \|\mathbf{b}_j\|^2|,
\end{aligned}$$

for a constant $C > 0$ where the mean value theorem and assumption (C2)$'$ are used to show the above inequality. Hence,

$$\begin{aligned}
\sum_{j=1}^{p} |\widehat{a}_j - a_j| &\leq C \sum_{j=1}^{p} |\|\widehat{\mathbf{b}}_j\|^2 - \|\mathbf{b}_j\|^2| \\
&= C \sum_{j=1}^{p} \left| \sum_{k=1}^{K} (\widehat{\lambda}_k - \lambda_k)\widehat{\gamma}_{kj}^2 + \sum_{k=1}^{K} \lambda_k (\widehat{\gamma}_{kj}^2 - \gamma_{kj}^2) \right| \\
&\leq \sum_{k=1}^{K} |\widehat{\lambda}_k - \lambda_k| + \sum_{k=1}^{K} \lambda_k \sum_{j=1}^{p} |\widehat{\gamma}_{kj}^2 - \gamma_{kj}^2|.
\end{aligned}$$

Note that the second term is bounded as

$$\sum_{j=1}^{p} |\widehat{\gamma}_{kj}^2 - \gamma_{kj}^2| \le \left(\sum_{j=1}^{p} |\widehat{\gamma}_{kj} - \gamma_{kj}|^2 \sum_{j=1}^{p} |\widehat{\gamma}_{kj} + \gamma_{kj}|^2 \right)^{1/2}$$
$$\le \|\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k\| \left(2 \sum_{j=1}^{p} (\widehat{\gamma}_{kj}^2 + \gamma_{kj}^2) \right) = 2\|\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k\|.$$

Thus, we obtain

$$\left| \sum_{j=1}^{p} \Delta_{12j} \right| = O_p(\epsilon^{-1} p (p^{-\nu_1} + p^{-\nu_2})). \tag{S1.4}$$

Finally, we address the term $\Delta_2 = \sum_{j=1}^{p} \Delta_{2j}$. Calculating the convergence rate of Δ_2 is similar to calculating that of $\sum_{j=1}^{p} \Delta_{11j}$. By the mean value theorem, there exists η_j between $\tilde{\eta}_j = \mathbf{b}_j^\top \tilde{\mathbf{W}}$ and $\eta_j = \mathbf{b}_j^\top \mathbf{W}$ such that

$$\Delta_{2j} = \frac{\partial}{\partial \eta} g(a, \eta) \bigg|_{a=a_j, \eta=\eta_j} (\tilde{\eta}_j - \eta_j) I(\lambda < P_j \le \tau).$$

Similar to the preceding arguments, we can show that

$$\frac{\partial}{\partial \eta} g(a, \eta) \bigg|_{a=a_j, \eta=\eta_j} = O(\epsilon^{-1}).$$

We note that

$$\mathbf{B}\tilde{\mathbf{W}} = \mathbf{B}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{B}^\top (\boldsymbol{\mu} + \mathbf{B}\mathbf{W} + \mathbf{u}) = \left(\sum_{k=1}^{K} \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \boldsymbol{\mu} + \mathbf{B}\mathbf{W}$$

since $\mathbf{B}$ and $\mathrm{Var}(\mathbf{u}) = \boldsymbol{\Sigma}_{\mathbf{u}}$ are orthogonal. Hence, we obtain

$$\sum_{j=1}^{p} |\tilde{\eta}_j - \eta_j| = \|\mathbf{B}\tilde{\mathbf{W}} - \mathbf{B}\mathbf{W}\|_1 \le \sqrt{p} \left\| \left(\sum_{k=1}^{K} \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right) \boldsymbol{\mu} \right\|$$
$$\le \sqrt{p} \left\| \sum_{k=1}^{K} \boldsymbol{\gamma}_k \boldsymbol{\gamma}_k^\top \right\| \|\boldsymbol{\mu}\| = \sqrt{p} \|\boldsymbol{\mu}\|.$$

Therefore, we have

$$|\Delta_2| = O_p(\epsilon^{-1} p^{1/2} \|\boldsymbol{\mu}\|). \tag{S1.5}$$

Combining all the results we have obtained, we conclude that

$$|\widehat{\mathrm{FDP}}(t) - \mathrm{FDP}_{\mathrm{U}}(t)| = O_p \left\{ \epsilon^{-1} p^{\theta} (K p^{-\nu_1} + p^{-\nu_2} + \|\boldsymbol{\mu}\| p^{-1/2}) \right\}, \tag{S1.6}$$

which is the desired result. □

S1.5 Proof of Theorem 4

Proof of Theorem 4. The proof of Theorem 4 is nearly the same as that of Theorem 3 in **?**. For completeness, we repeat the proof. We note that on the event $\{\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}\| = O_p(d_p p^{-\nu})\}$,

$$|\lambda_j - \widehat{\lambda}_{j+1}| \geq |\lambda_j - \lambda_{j+1}| - |\lambda_{j+1} - \widehat{\lambda}_{j+1}|$$
$$\geq d_p - \|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}\| \geq d_p/2.$$

In the first inequality, the triangular inequality is used, and the second inequality is due to Lemma 2 and the assumption $\min_{1\leq k\leq K}\{\lambda_k - \lambda_{k+1}\} \geq d_p$. Similarly, we have $|\widehat{\lambda}_{j-1} - \lambda_j| \geq d_p/2$. Then, by Lemma 2, we obtain $\|\widehat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_j\| = O_p(p^{-\nu})$, which means condition (C6) holds with $\nu_1 = \nu$. Using Lemma 2 again, we have

$$\sum_{k=1}^{K} |\widehat{\lambda}_k - \lambda_k| \leq K\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}\| = O_p(Kd_p p^{-\nu}).$$

Thus, condition (C7) holds with $p^{-\nu_2} = Kd_p p^{-\nu}/p$. Therefore, from the result of Theorem 3, it holds that

$$|\widehat{\mathrm{FDP}}^{\mathrm{DAB}}(t;\lambda,\tau,\epsilon) - \mathrm{FDP}_{\mathrm{U}}^{\mathrm{DAB}}(t;\lambda,\tau,\epsilon)| = O_p\left\{p^{\alpha+\theta}\left(K(d_p/p+1)p^{-\nu} + \|\boldsymbol{\mu}\|p^{-1/2}\right)\right\}. \quad \text{(S1.7)}$$

□

S2 Tables and Figures

Table S1: The averages of the FDP values over 200 repetition are calculated with their standard error in parentheses for Model 1, 2 and 3.

			Method					
Model	μ_N	π_1	P-PFA	DAB-P-PFA	S-PFA	DAB-S-PFA	EB	BH
M1	0	0.1	0.095 (0.038)	0.107 (0.046)	0.092 (0.037)	0.104 (0.045)	0.022 (0.052)	0.073 (0.091)
		0.3	0.072 (0.019)	0.104 (0.030)	0.071 (0.019)	0.103 (0.030)	0.019 (0.027)	0.062 (0.053)
		0.5	0.052 (0.014)	0.103 (0.023)	0.052 (0.013)	0.102 (0.023)	0.016 (0.020)	0.046 (0.032)
	-0.1	0.1	0.068 (0.031)	0.077 (0.034)	0.066 (0.029)	0.074 (0.033)	0.015 (0.041)	0.054 (0.076)
		0.3	0.056 (0.018)	0.080 (0.023)	0.054 (0.018)	0.078 (0.023)	0.013 (0.023)	0.048 (0.046)
		0.5	0.041 (0.012)	0.085 (0.020)	0.040 (0.012)	0.084 (0.019)	0.012 (0.015)	0.037 (0.028)
	-0.2	0.1	0.050 (0.026)	0.056 (0.028)	0.049 (0.026)	0.053 (0.028)	0.011 (0.036)	0.041 (0.064)
		0.3	0.042 (0.015)	0.061 (0.022)	0.041 (0.015)	0.060 (0.022)	0.009 (0.018)	0.037 (0.041)
		0.5	0.032 (0.010)	0.070 (0.018)	0.031 (0.010)	0.069 (0.018)	0.008 (0.013)	0.029 (0.025)
M2	0	0.1	0.102 (0.072)	0.136 (0.115)	0.100 (0.071)	0.134 (0.115)	0.040 (0.117)	0.086 (0.158)
		0.3	0.075 (0.040)	0.117 (0.075)	0.074 (0.039)	0.116 (0.075)	0.030 (0.071)	0.076 (0.097)
		0.5	0.052 (0.024)	0.103 (0.048)	0.052 (0.024)	0.103 (0.048)	0.023 (0.040)	0.055 (0.058)
	-0.1	0.1	0.071 (0.059)	0.082 (0.078)	0.070 (0.058)	0.081 (0.078)	0.032 (0.105)	0.066 (0.141)
		0.3	0.055 (0.033)	0.078 (0.056)	0.054 (0.033)	0.077 (0.056)	0.020 (0.054)	0.060 (0.087)
		0.5	0.039 (0.020)	0.077 (0.041)	0.039 (0.020)	0.076 (0.041)	0.016 (0.034)	0.045 (0.052)
	-0.2	0.1	0.048 (0.046)	0.053 (0.061)	0.048 (0.046)	0.053 (0.061)	0.023 (0.091)	0.052 (0.125)
		0.3	0.039 (0.026)	0.054 (0.045)	0.039 (0.026)	0.053 (0.045)	0.015 (0.047)	0.047 (0.077)
		0.5	0.029 (0.017)	0.057 (0.034)	0.029 (0.017)	0.056 (0.034)	0.012 (0.028)	0.037 (0.048)
M3	0	0.1	0.102 (0.036)	0.113 (0.044)	0.098 (0.036)	0.110 (0.043)	0.027 (0.067)	0.084 (0.103)
		0.3	0.074 (0.021)	0.105 (0.034)	0.073 (0.021)	0.103 (0.034)	0.022 (0.032)	0.066 (0.058)
		0.5	0.052 (0.014)	0.098 (0.027)	0.051 (0.014)	0.097 (0.027)	0.018 (0.021)	0.049 (0.034)
	-0.1	0.1	0.076 (0.034)	0.079 (0.036)	0.074 (0.034)	0.075 (0.035)	0.020 (0.052)	0.062 (0.088)
		0.3	0.056 (0.018)	0.077 (0.028)	0.055 (0.018)	0.075 (0.028)	0.016 (0.026)	0.052 (0.051)
		0.5	0.041 (0.012)	0.077 (0.024)	0.040 (0.012)	0.077 (0.024)	0.013 (0.017)	0.040 (0.031)
	-0.2	0.1	0.054 (0.029)	0.053 (0.032)	0.052 (0.029)	0.051 (0.031)	0.013 (0.042)	0.047 (0.076)
		0.3	0.042 (0.016)	0.056 (0.023)	0.041 (0.015)	0.055 (0.023)	0.011 (0.021)	0.041 (0.044)
		0.5	0.031 (0.011)	0.060 (0.023)	0.031 (0.011)	0.059 (0.023)	0.009 (0.014)	0.031 (0.027)

Table S2: The averages of the FDP values over 200 repetition are calculated with their standard error in parentheses for Model 4, 5 and 6.

			Method					
Model	μ_{N}	π_1	P-PFA	DAB-P-PFA	S-PFA	DAB-S-PFA	EB	BH
M4	0	0.1	0.098 (0.045)	0.113 (0.061)	0.096 (0.045)	0.110 (0.061)	0.033 (0.098)	0.084 (0.134)
		0.3	0.074 (0.024)	0.105 (0.041)	0.073 (0.024)	0.104 (0.040)	0.023 (0.046)	0.072 (0.074)
		0.5	0.050 (0.017)	0.095 (0.033)	0.050 (0.017)	0.095 (0.033)	0.020 (0.030)	0.052 (0.043)
	-0.1	0.1	0.067 (0.035)	0.071 (0.042)	0.066 (0.034)	0.069 (0.041)	0.023 (0.083)	0.064 (0.119)
		0.3	0.054 (0.020)	0.072 (0.032)	0.053 (0.020)	0.071 (0.031)	0.017 (0.039)	0.057 (0.066)
		0.5	0.039 (0.015)	0.071 (0.030)	0.038 (0.014)	0.071 (0.030)	0.015 (0.026)	0.042 (0.039)
	-0.2	0.1	0.046 (0.026)	0.044 (0.031)	0.045 (0.026)	0.043 (0.030)	0.015 (0.065)	0.049 (0.104)
		0.3	0.040 (0.017)	0.050 (0.027)	0.039 (0.016)	0.050 (0.027)	0.012 (0.034)	0.045 (0.060)
		0.5	0.029 (0.013)	0.053 (0.029)	0.029 (0.012)	0.052 (0.028)	0.011 (0.021)	0.034 (0.036)
M5	0	0.1	0.093 (0.041)	0.107 (0.047)	0.092 (0.041)	0.105 (0.047)	0.029 (0.039)	0.093 (0.074)
		0.3	0.073 (0.018)	0.104 (0.025)	0.072 (0.018)	0.103 (0.025)	0.023 (0.023)	0.072 (0.038)
		0.5	0.053 (0.011)	0.099 (0.018)	0.052 (0.011)	0.098 (0.018)	0.019 (0.015)	0.053 (0.023)
	-0.1	0.1	0.070 (0.035)	0.083 (0.040)	0.069 (0.035)	0.081 (0.039)	0.019 (0.030)	0.071 (0.060)
		0.3	0.057 (0.017)	0.086 (0.023)	0.057 (0.017)	0.085 (0.023)	0.017 (0.019)	0.058 (0.034)
		0.5	0.042 (0.010)	0.085 (0.016)	0.042 (0.010)	0.084 (0.017)	0.015 (0.013)	0.043 (0.021)
	-0.2	0.1	0.054 (0.031)	0.065 (0.034)	0.053 (0.030)	0.064 (0.034)	0.012 (0.025)	0.053 (0.052)
		0.3	0.045 (0.015)	0.071 (0.020)	0.045 (0.015)	0.070 (0.020)	0.012 (0.014)	0.046 (0.029)
		0.5	0.034 (0.009)	0.073 (0.015)	0.033 (0.009)	0.073 (0.016)	0.010 (0.010)	0.034 (0.019)
M6	0	0.1	0.092 (0.036)	0.103 (0.040)	0.091 (0.036)	0.102 (0.040)	0.029 (0.055)	0.084 (0.079)
		0.3	0.070 (0.016)	0.100 (0.028)	0.069 (0.016)	0.100 (0.028)	0.021 (0.026)	0.064 (0.041)
		0.5	0.049 (0.011)	0.094 (0.019)	0.049 (0.011)	0.094 (0.019)	0.016 (0.015)	0.047 (0.023)
	-0.1	0.1	0.069 (0.030)	0.081 (0.033)	0.068 (0.030)	0.080 (0.033)	0.019 (0.046)	0.064 (0.070)
		0.3	0.055 (0.015)	0.083 (0.022)	0.054 (0.015)	0.082 (0.022)	0.015 (0.021)	0.050 (0.037)
		0.5	0.040 (0.010)	0.081 (0.016)	0.040 (0.009)	0.080 (0.016)	0.012 (0.013)	0.037 (0.021)
	-0.2	0.1	0.052 (0.026)	0.064 (0.029)	0.051 (0.026)	0.064 (0.029)	0.011 (0.034)	0.047 (0.058)
		0.3	0.042 (0.013)	0.068 (0.019)	0.042 (0.013)	0.067 (0.019)	0.011 (0.018)	0.040 (0.033)
		0.5	0.031 (0.009)	0.069 (0.015)	0.031 (0.009)	0.068 (0.015)	0.009 (0.011)	0.030 (0.019)

Table S3: The averages of the TPP values over 200 repetition are calculated with their standard error in parentheses for Model 1, 2 and 3.

			Method					
Model	μ_{N}	π_1	P-PFA	DAB-P-PFA	S-PFA	DAB-S-PFA	EB	BH
M1	0	0.1	0.786 (0.134)	0.797 (0.129)	0.781 (0.134)	0.793 (0.130)	0.519 (0.088)	0.711 (0.064)
		0.3	0.874 (0.107)	0.903 (0.091)	0.872 (0.107)	0.901 (0.092)	0.698 (0.064)	0.860 (0.051)
		0.5	0.909 (0.089)	0.953 (0.056)	0.908 (0.089)	0.952 (0.057)	0.791 (0.060)	0.911 (0.043)
	-0.1	0.1	0.782 (0.134)	0.791 (0.139)	0.779 (0.135)	0.786 (0.140)	0.437 (0.134)	0.707 (0.066)
		0.3	0.872 (0.107)	0.897 (0.100)	0.870 (0.108)	0.896 (0.101)	0.679 (0.053)	0.858 (0.052)
		0.5	0.909 (0.089)	0.949 (0.065)	0.908 (0.089)	0.948 (0.065)	0.777 (0.053)	0.909 (0.044)
	-0.2	0.1	0.781 (0.135)	0.787 (0.146)	0.777 (0.136)	0.783 (0.147)	0.423 (0.120)	0.705 (0.066)
		0.3	0.871 (0.108)	0.892 (0.108)	0.869 (0.109)	0.891 (0.109)	0.671 (0.056)	0.856 (0.053)
		0.5	0.908 (0.089)	0.945 (0.072)	0.907 (0.090)	0.945 (0.073)	0.771 (0.053)	0.909 (0.044)
M2	0	0.1	0.801 (0.174)	0.814 (0.165)	0.800 (0.174)	0.813 (0.165)	0.549 (0.160)	0.713 (0.094)
		0.3	0.858 (0.151)	0.884 (0.137)	0.858 (0.151)	0.884 (0.137)	0.708 (0.115)	0.852 (0.085)
		0.5	0.883 (0.140)	0.923 (0.118)	0.882 (0.140)	0.923 (0.118)	0.793 (0.098)	0.902 (0.076)
	-0.1	0.1	0.799 (0.174)	0.801 (0.175)	0.798 (0.175)	0.800 (0.175)	0.508 (0.169)	0.709 (0.096)
		0.3	0.857 (0.152)	0.869 (0.149)	0.857 (0.152)	0.869 (0.149)	0.686 (0.118)	0.849 (0.087)
		0.5	0.882 (0.140)	0.909 (0.129)	0.882 (0.140)	0.909 (0.128)	0.779 (0.096)	0.900 (0.077)
	-0.2	0.1	0.798 (0.175)	0.793 (0.182)	0.797 (0.175)	0.792 (0.182)	0.446 (0.181)	0.706 (0.098)
		0.3	0.856 (0.152)	0.860 (0.158)	0.856 (0.152)	0.859 (0.158)	0.665 (0.112)	0.847 (0.088)
		0.5	0.882 (0.140)	0.900 (0.138)	0.881 (0.141)	0.900 (0.138)	0.768 (0.091)	0.900 (0.078)
M3	0	0.1	0.780 (0.157)	0.789 (0.156)	0.775 (0.158)	0.785 (0.156)	0.522 (0.085)	0.707 (0.065)
		0.3	0.862 (0.123)	0.889 (0.113)	0.860 (0.124)	0.888 (0.113)	0.701 (0.064)	0.858 (0.054)
		0.5	0.896 (0.103)	0.938 (0.085)	0.895 (0.104)	0.938 (0.085)	0.788 (0.059)	0.908 (0.048)
	-0.1	0.1	0.777 (0.158)	0.777 (0.166)	0.772 (0.159)	0.774 (0.167)	0.452 (0.140)	0.704 (0.066)
		0.3	0.861 (0.124)	0.878 (0.126)	0.859 (0.125)	0.877 (0.126)	0.681 (0.060)	0.856 (0.055)
		0.5	0.895 (0.103)	0.930 (0.096)	0.894 (0.104)	0.929 (0.096)	0.777 (0.055)	0.907 (0.048)
	-0.2	0.1	0.773 (0.160)	0.765 (0.177)	0.769 (0.161)	0.762 (0.177)	0.423 (0.123)	0.701 (0.068)
		0.3	0.860 (0.125)	0.868 (0.137)	0.858 (0.125)	0.866 (0.137)	0.667 (0.067)	0.854 (0.056)
		0.5	0.895 (0.104)	0.920 (0.109)	0.894 (0.105)	0.920 (0.109)	0.768 (0.054)	0.905 (0.049)

Table S4: The averages of the TPP values over 200 repetition are calculated with their standard error in parentheses for Model 4, 5 and 6.

			Method					
Model	μ_N	π_1	P-PFA	DAB-P-PFA	S-PFA	DAB-S-PFA	EB	BH
M4	0	0.1	0.778 (0.170)	0.787 (0.166)	0.776 (0.171)	0.786 (0.166)	0.534 (0.100)	0.705 (0.080)
		0.3	0.854 (0.139)	0.877 (0.129)	0.853 (0.139)	0.876 (0.128)	0.700 (0.084)	0.856 (0.063)
		0.5	0.884 (0.122)	0.924 (0.102)	0.883 (0.122)	0.925 (0.101)	0.789 (0.071)	0.903 (0.056)
	-0.1	0.1	0.776 (0.171)	0.773 (0.176)	0.774 (0.171)	0.772 (0.177)	0.470 (0.128)	0.700 (0.081)
		0.3	0.852 (0.140)	0.861 (0.143)	0.851 (0.140)	0.861 (0.142)	0.683 (0.075)	0.854 (0.064)
		0.5	0.883 (0.122)	0.911 (0.117)	0.882 (0.123)	0.910 (0.116)	0.776 (0.067)	0.902 (0.057)
	-0.2	0.1	0.775 (0.171)	0.763 (0.184)	0.773 (0.171)	0.761 (0.185)	0.409 (0.138)	0.698 (0.084)
		0.3	0.851 (0.140)	0.849 (0.153)	0.850 (0.140)	0.849 (0.153)	0.646 (0.086)	0.852 (0.065)
		0.5	0.882 (0.123)	0.897 (0.131)	0.882 (0.123)	0.897 (0.131)	0.764 (0.062)	0.901 (0.057)
M5	0	0.1	0.734 (0.091)	0.750 (0.088)	0.733 (0.091)	0.748 (0.089)	0.509 (0.067)	0.710 (0.059)
		0.3	0.860 (0.076)	0.894 (0.062)	0.858 (0.077)	0.893 (0.063)	0.688 (0.042)	0.859 (0.035)
		0.5	0.901 (0.065)	0.951 (0.037)	0.900 (0.066)	0.951 (0.037)	0.780 (0.037)	0.905 (0.029)
	-0.1	0.1	0.732 (0.090)	0.752 (0.092)	0.730 (0.091)	0.750 (0.093)	0.500 (0.062)	0.706 (0.059)
		0.3	0.858 (0.076)	0.893 (0.070)	0.857 (0.077)	0.892 (0.071)	0.681 (0.043)	0.857 (0.036)
		0.5	0.900 (0.066)	0.950 (0.043)	0.899 (0.066)	0.949 (0.043)	0.770 (0.036)	0.904 (0.030)
	-0.2	0.1	0.730 (0.090)	0.755 (0.096)	0.728 (0.091)	0.753 (0.097)	0.446 (0.090)	0.704 (0.060)
		0.3	0.857 (0.076)	0.894 (0.076)	0.855 (0.078)	0.892 (0.078)	0.677 (0.043)	0.855 (0.036)
		0.5	0.899 (0.066)	0.950 (0.047)	0.898 (0.067)	0.949 (0.048)	0.764 (0.040)	0.903 (0.030)
M6	0	0.1	0.761 (0.091)	0.776 (0.088)	0.759 (0.092)	0.774 (0.089)	0.515 (0.068)	0.721 (0.058)
		0.3	0.874 (0.078)	0.906 (0.061)	0.873 (0.079)	0.905 (0.061)	0.695 (0.047)	0.864 (0.037)
		0.5	0.911 (0.070)	0.956 (0.042)	0.911 (0.070)	0.956 (0.042)	0.786 (0.040)	0.909 (0.031)
	-0.1	0.1	0.758 (0.091)	0.778 (0.092)	0.756 (0.092)	0.776 (0.093)	0.497 (0.072)	0.718 (0.059)
		0.3	0.873 (0.079)	0.907 (0.067)	0.871 (0.080)	0.906 (0.067)	0.684 (0.045)	0.862 (0.038)
		0.5	0.911 (0.070)	0.955 (0.049)	0.910 (0.070)	0.955 (0.049)	0.777 (0.038)	0.909 (0.032)
	-0.2	0.1	0.755 (0.091)	0.782 (0.096)	0.754 (0.093)	0.780 (0.096)	0.427 (0.101)	0.715 (0.060)
		0.3	0.871 (0.079)	0.908 (0.072)	0.870 (0.080)	0.907 (0.072)	0.681 (0.046)	0.861 (0.038)
		0.5	0.910 (0.070)	0.955 (0.053)	0.909 (0.071)	0.955 (0.053)	0.771 (0.039)	0.908 (0.032)

Table S5: For each $\zeta = \lambda/\tau \in \{0.1, 0.2, 0.3\}$ with fixed $(\tau, \epsilon) = (0.5, 0.01)$, the average of the FDP values is calculated with their standard error in parentheses. Simulation data are generated from Model 1.

		FDP			TPP		
		ζ			ζ		
μ_N	π_1	0.1	0.2	0.3	0.1	0.2	0.3
0	0.1	0.106 (0.044)	0.107 (0.046)	0.107 (0.047)	0.796 (0.133)	0.797 (0.129)	0.796 (0.131)
	0.3	0.103 (0.025)	0.104 (0.030)	0.104 (0.032)	0.901 (0.094)	0.903 (0.091)	0.901 (0.093)
	0.5	0.100 (0.019)	0.103 (0.023)	0.103 (0.025)	0.952 (0.058)	0.953 (0.056)	0.951 (0.060)
-0.1	0.1	0.078 (0.033)	0.077 (0.034)	0.076 (0.038)	0.793 (0.138)	0.791 (0.139)	0.789 (0.138)
	0.3	0.081 (0.022)	0.080 (0.023)	0.078 (0.026)	0.899 (0.099)	0.897 (0.100)	0.894 (0.102)
	0.5	0.084 (0.017)	0.085 (0.020)	0.083 (0.023)	0.949 (0.064)	0.949 (0.065)	0.946 (0.068)
-0.2	0.1	0.058 (0.029)	0.056 (0.028)	0.053 (0.029)	0.792 (0.143)	0.787 (0.146)	0.783 (0.146)
	0.3	0.064 (0.020)	0.061 (0.022)	0.059 (0.023)	0.897 (0.103)	0.892 (0.108)	0.889 (0.108)
	0.5	0.071 (0.015)	0.070 (0.018)	0.067 (0.020)	0.948 (0.067)	0.945 (0.072)	0.942 (0.074)

Table S6: For each $\tau \in \{0.4, 0.5, 0.6\}$ with fixed $(\zeta, \epsilon) = (0.2, 0.01)$, the average of the FDP values is calculated with their standard error in parentheses. Simulation data are generated from Model 1.

		FDP			TPP		
		τ			τ		
μ_N	π_1	0.4	0.5	0.6	0.4	0.5	0.6
0	0.1	0.107 (0.044)	0.107 (0.046)	0.107 (0.045)	0.797 (0.130)	0.797 (0.129)	0.796 (0.131)
	0.3	0.104 (0.028)	0.104 (0.030)	0.104 (0.031)	0.902 (0.093)	0.903 (0.091)	0.902 (0.093)
	0.5	0.101 (0.021)	0.103 (0.023)	0.103 (0.024)	0.952 (0.057)	0.953 (0.056)	0.952 (0.058)
-0.1	0.1	0.079 (0.033)	0.077 (0.034)	0.076 (0.036)	0.793 (0.139)	0.791 (0.139)	0.788 (0.138)
	0.3	0.082 (0.023)	0.080 (0.023)	0.078 (0.025)	0.898 (0.100)	0.897 (0.100)	0.896 (0.100)
	0.5	0.085 (0.018)	0.085 (0.020)	0.084 (0.022)	0.950 (0.064)	0.949 (0.065)	0.947 (0.065)
-0.2	0.1	0.059 (0.029)	0.056 (0.028)	0.053 (0.028)	0.792 (0.145)	0.787 (0.146)	0.782 (0.146)
	0.3	0.065 (0.021)	0.061 (0.022)	0.058 (0.021)	0.896 (0.105)	0.892 (0.108)	0.889 (0.110)
	0.5	0.073 (0.018)	0.070 (0.018)	0.067 (0.019)	0.949 (0.067)	0.945 (0.072)	0.943 (0.075)

Table S7: For each $\epsilon \in \{0.001, 0.01, 0.1\}$ with fixed $(\zeta, \tau) = (0.2, 0.5)$, the average of the FDP values is calculated with their standard error in parentheses. Simulation data are generated from Model 1.

		FDP			TPP		
		ϵ			ϵ		
μ_{N}	π_1	0.001	0.01	0.1	0.001	0.01	0.1
0	0.1	0.107 (0.046)	0.107 (0.046)	0.108 (0.046)	0.797 (0.129)	0.797 (0.129)	0.797 (0.128)
	0.3	0.104 (0.030)	0.104 (0.030)	0.104 (0.030)	0.903 (0.091)	0.903 (0.091)	0.903 (0.091)
	0.5	0.103 (0.023)	0.103 (0.023)	0.103 (0.023)	0.953 (0.056)	0.953 (0.056)	0.953 (0.056)
-0.1	0.1	0.077 (0.034)	0.077 (0.034)	0.078 (0.036)	0.791 (0.139)	0.791 (0.139)	0.791 (0.137)
	0.3	0.080 (0.023)	0.080 (0.023)	0.080 (0.025)	0.897 (0.100)	0.897 (0.100)	0.898 (0.098)
	0.5	0.085 (0.020)	0.085 (0.020)	0.085 (0.020)	0.949 (0.065)	0.949 (0.065)	0.949 (0.063)
-0.2	0.1	0.056 (0.028)	0.056 (0.028)	0.056 (0.028)	0.787 (0.146)	0.787 (0.146)	0.787 (0.144)
	0.3	0.061 (0.022)	0.061 (0.022)	0.062 (0.022)	0.892 (0.108)	0.892 (0.108)	0.893 (0.106)
	0.5	0.070 (0.018)	0.070 (0.018)	0.070 (0.019)	0.945 (0.072)	0.945 (0.072)	0.946 (0.070)

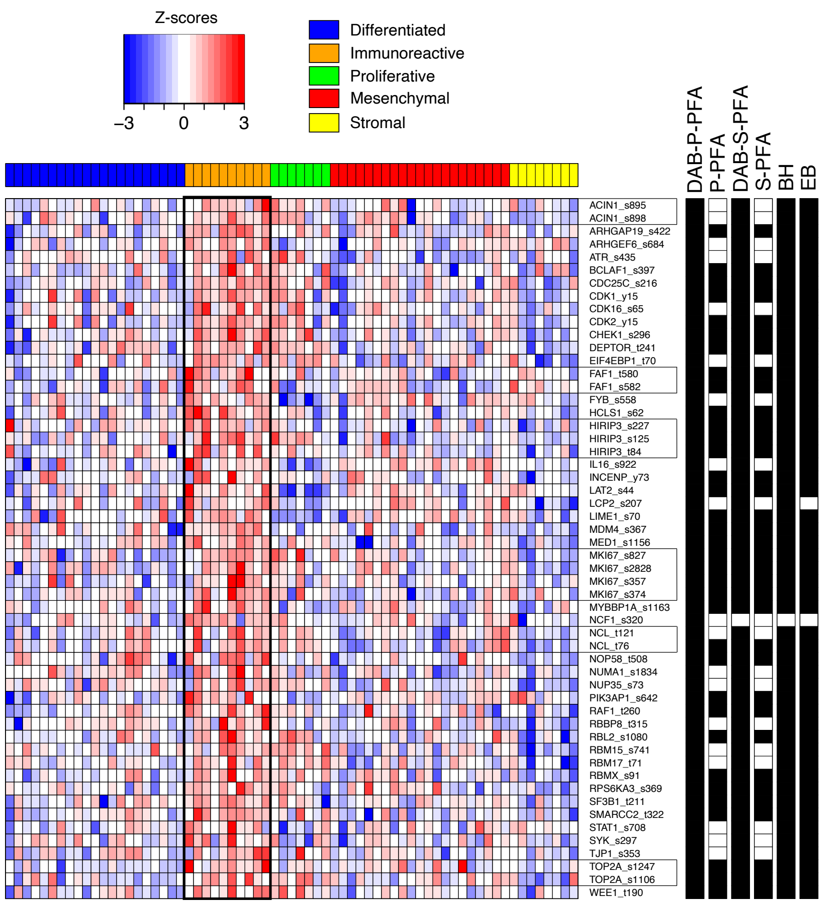

Figure S1: Heatmap of relative phosphorylation levels at high confidence phosphorylation sites with validated evidence of signaling by three or more kinases. Phosphorylation is uniquely elevated in these sites for the samples of the immunoreactive subtype. The black and white color bar to the far right side shows whether each phosphorylation site has been above the threshold or not by each method (black = null hypothesis rejected, and white = not rejected).

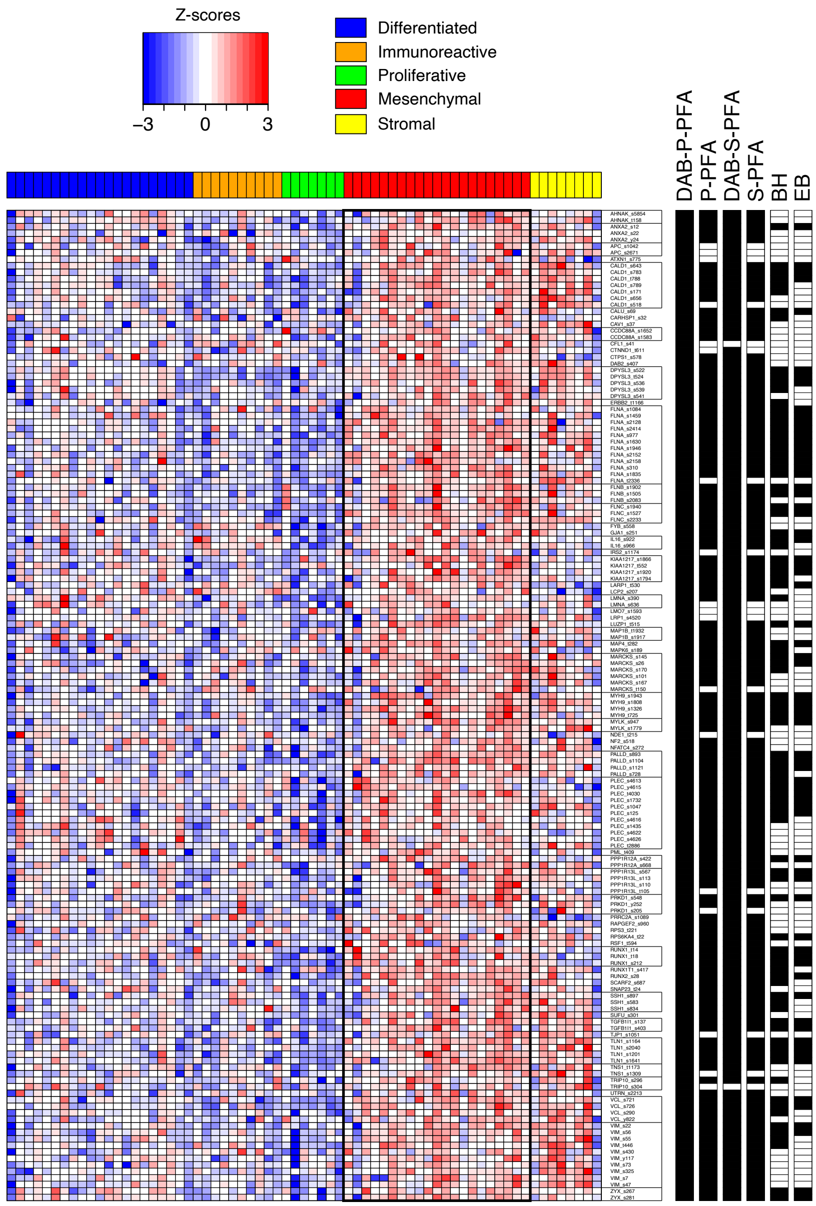

Figure S2: Heatmap of relative phosphorylation levels at high confidence phosphorylation sites with validated evidence of signaling by three or more kinases. Phosphorylation is uniquely elevated in these sites for the samples of mesenchymal subtype. The black and white color bar to the far right side shows whether each phosphorylation site has been above the threshold or not by each method (black = null hypothesis rejected, and white = not rejected).

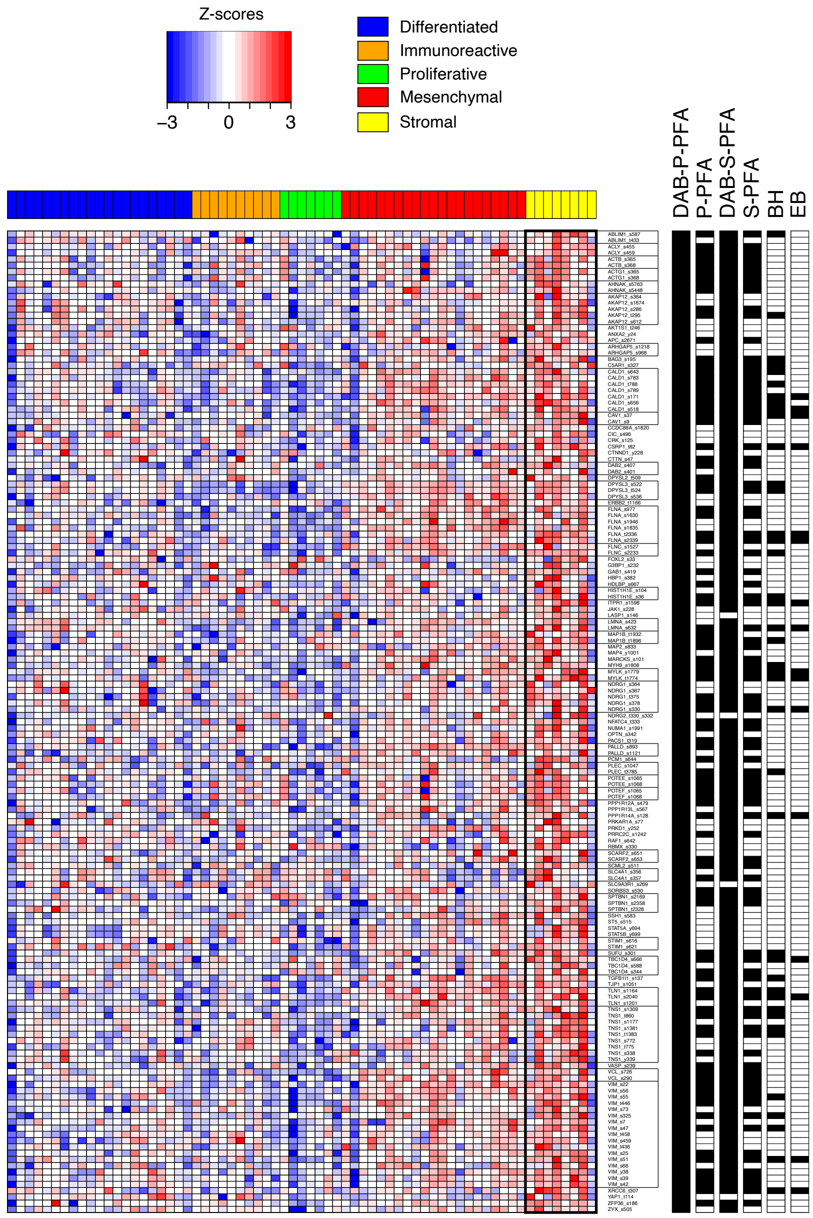

Figure S3: Heatmap of relative phosphorylation levels at high confidence phosphorylation sites with validated evidence of signaling by three or more kinases. Phosphorylation is uniquely elevated in these sites for the samples of Stromal subtype. The black and white color bar to the far right side shows whether each phosphorylation site was above the threshold by each method (black = null hypothesis rejected, and white = not rejected).