TAIL GINI FUNCTIONAL UNDER ASYMPTOTIC INDEPENDENCE

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Supplementary Material

In the supplementary material, we present the proofs of four auxiliary lemmas in Section S1, the proof of Proposition 1 in Section S2, an additional simulation study based on real value cases in SectionS3 and then show some additional figures for the application in Section S4.

S1 Auxiliary Lemmas

To obtain the limit result in Proposition 1, we need Lemmas 1 and 2 below, which are important auxiliary results on the tail empirical processes and the tail empirical copula processes . They are analogous to Lemmas 1 and

2 in Cai and Musta (2020). The difference lies in the range of y. Here we take $y \in [0,1]$ instead of $y \in [1/2,2]$ in Cai and Musta (2020).

Lemma 1. *(i) The function $y \mapsto \int_0^\infty \tau(x,y)dx^{-\gamma_1}$ is Lipschitz, that is, there exists $C_1 > 0$ such that, for each $y_1, y_2 \in [0,1]$,*

$$\left|\int_0^\infty \tau\left(x, y_1\right) dx^{-\gamma_1} - \int_0^\infty \tau\left(x, y_2\right) dx^{-\gamma_1}\right| \leq C_1 \left|y_1 - y_2\right|.$$

(ii) Let $M_n = \sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}$. Assumptions 1, 2, 4 and 5 imply that

$$\sup_{0\leq y\leq 1} M_n \left|\int_0^\infty \tau(x,y)dx^{-\gamma_1} - \int_0^\infty \tau\left(s_{\frac{k}{n}}(x), y\right) dx^{-\gamma_1}\right| \to 0.$$

(iii) Assumptions 3 and 4 imply that for $\rho = 1, 2, 2+\delta$,

$$\sup_{0\leq y\leq 1} \left|\int_0^\infty \tau_{\frac{k}{n}}\left(x, y\right) dx^{-\rho\gamma_1} - \int_0^\infty \tau(x,y)dx^{-\rho\gamma_1}\right| \to 0.$$

(iv) Assumptions 3, 4 and 5 imply that

$$\sup_{0\leq y\leq 1} M_n \left|\int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right) dx^{-\gamma_1} - \int_0^\infty \tau\left(s_{\frac{k}{n}}(x), y\right) dx^{-\gamma_1}\right| \to 0.$$

Proof of Lemma 1. (i) By the homogeneity of $\tau(x,y)$, we have that, for $y_1, y_2 \in [0,1]$,

$$\begin{aligned}
&\left|\int_0^\infty \tau\left(x, y_1\right) dx^{-\gamma_1} - \int_0^\infty \tau\left(x, y_2\right) dx^{-\gamma_1}\right| \\
=&\left|\int_0^\infty y_1^{1/\eta}\tau\left(\frac{x}{y_1}, 1\right) dx^{-\gamma_1} - \int_0^\infty y_2^{1/\eta}\tau\left(\frac{x}{y_2}, 1\right) dx^{-\gamma_1}\right| \\
=&\left|y_1^{1/\eta-\gamma_1}\int_0^\infty \tau(x,1)dx^{-\gamma_1} - y_2^{1/\eta-\gamma_1}\int_0^\infty \tau(x,1)dx^{-\gamma_1}\right| \\
\leq& C_1 \left|y_1 - y_2\right|,
\end{aligned}$$

where $C_1 > 0$ is finite and the last inequality follows by $0 < 1/\eta - \gamma_1 < 1$.

(ii) For sufficiently small $\epsilon > 0$, let $l_n = \left(\frac{k}{n}\right)^{1-\epsilon}$. We write

$$\begin{aligned} &\sup_{0\le y\le 1} M_n \left| \int_0^\infty \tau(x,y)dx^{-\gamma_1} - \int_0^\infty \tau\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right| \\ &\le \sup_{0\le y\le 1} M_n \left| \int_0^{l_n} \left[\tau(x,y) - \tau\left(s_{\frac{k}{n}}(x),y\right)\right]dx^{-\gamma_1}\right| \\ &\quad + \sup_{0\le y\le 1} M_n \left\{ \left| \int_{l_n}^\infty \tau(x,y)dx^{-\gamma_1}\right| + \left| \int_{l_n}^\infty \tau\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right| \right\}. \qquad \text{(A1)} \end{aligned}$$

For the first term on the right-hand side of (A1), by the homogeneity and monotonicity of $\tau(x,y)$, we have that, for $x_1, x_2 > 0, y \in [0,1]$,

$$|\tau(x_1,y) - \tau(x_2,y)| \le \left| \left(\frac{x_2}{x_1}\right)^{1/\eta} - 1\right| \tau(x_1,y),$$

and hence

$$\begin{aligned} &\sup_{0\le y\le 1} M_n \left| \int_0^{l_n} \left[\tau(x,y) - \tau\left(s_{\frac{k}{n}}(x),y\right)\right]dx^{-\gamma_1}\right| \\ &\le \sup_{0\le y\le 1} M_n \left| \int_0^{l_n} \left| \left(\frac{s_{\frac{k}{n}}(x)}{x}\right)^{1/\eta} - 1\right| \tau(x,y)dx^{-\gamma_1}\right|. \end{aligned}$$

Note that, for any $\epsilon_0 > 0$, for sufficiently large n and $x < l_n$ (see Cai, 2012, p.85),

$$\left| \frac{s_{\frac{k}{n}}(x)/x - 1}{A_1(n/k)} - \frac{x^{-\rho_1} - 1}{\gamma_1\rho_1}\right| \le x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right).$$

This implies that for $\epsilon_0 < -\rho_1(1-\lambda)/\lambda$ and $x < l_n$,

$$\left| \frac{s_{\frac{k}{n}}(x)}{x} - 1\right| \le |A_1(n/k)| \left\{ \left| \frac{x^{-\rho_1} - 1}{\gamma_1\rho_1}\right| + x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right)\right\} = o(1).$$

By a Taylor expansion, we obtain

$$\left|\left(\frac{s_{\frac{k}{n}}(x)}{x}\right)^{1/\eta}-1\right| = |A_1(n/k)|\left\{\left|\frac{x^{-\rho_1}-1}{\gamma_1\rho_1}\right| + x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right)\right\}$$
$$+ o\left(A_1(n/k)\left\{\left|\frac{x^{-\rho_1}-1}{\gamma_1\rho_1}\right| + x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right)\right\}\right).$$

Consequently,

$$\sup_{0\le y\le 1} M_n\left|\int_0^{l_n}\left|\left(\frac{s_{\frac{k}{n}}(x)}{x}\right)^{1/\eta}-1\right|\tau(x,y)dx^{-\gamma_1}\right|$$
$$\le C_2 \sup_{0\le y\le 1} M_n\left|A_1(n/k)\right|\left|\int_0^{l_n} x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right)\tau(x,y)dx^{-\gamma_1}\right|, \quad \text{(A2)}$$

where $C_2 > 0$ is finite.

Furthermore, using the triangular inequality and Cauchy-Schwarz inequality, we have

$$\left|\int_0^{l_n} x^{-\rho_1}\max\left(x^{\epsilon_0}, x^{-\epsilon_0}\right)\tau(x,y)dx^{-\gamma_1}\right|$$
$$\le\int_0^1 x^{-\rho_1-\epsilon_0}\tau(x,y)dx^{-\gamma_1} + \left|\int_1^\infty \tau(x,y)^2 dx^{-\gamma_1}\right|^{1/2}\left|\int_1^{l_n} x^{-2\rho_1+2\epsilon_0}dx^{-\gamma_1}\right|^{1/2}$$
$$=O\left(l_n^{-\rho_1+\epsilon_0-\frac{\gamma_1}{2}}\right),$$

where the last equality follows from Assumption 2. Going back to (A2), we obtain

$$\sup_{0\le y\le 1} M_n\left|\int_0^{l_n}\left|\left(\frac{s_{\frac{k}{n}}(x)}{x}\right)^{1/\eta}-1\right|\tau(x,y)dx^{-\gamma_1}\right| = O\left(\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}-\frac{\gamma_1(1-\epsilon)}{2}}\right)\to 0,$$

because of Assumption 5.

Next, we deal with the second term in the right-hand side of (A1). By

Cauchy-Schwarz inequality and Assumption 2, we obtain

$$\begin{aligned}\left|\int_{l_n}^{\infty}\tau(x,y)dx^{-\gamma_1}\right| &\le \gamma_1\left(\int_{l_n}^{\infty}x^{-\gamma_1-1}dx\right)^{1/2}\left(\int_{1}^{\infty}\tau(x,y)^2x^{-\gamma_1-1}dx\right)^{1/2}\\ &\le C_3 l_n^{-\gamma_1/2}\end{aligned}$$

for some constant $C_3 > 0$. Moreover, by Assumption 5 and the fact that

$$M_n l_n^{\gamma_1/2} = \sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}-\frac{\gamma_1(1-\epsilon)}{2}},$$

it follows that

$$\sup_{0\le y\le 1} M_n\left|\int_{l_n}^{\infty}\tau(x,y)dx^{-\gamma_1}\right| \to 0.$$

Again, the triangular inequality yields

$$\begin{aligned}&\left|\int_{l_n}^{\infty}\tau\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right|\\ \le&\left|\int_{l_n}^{\infty}\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right| + \left|\int_{l_n}^{\infty}\left[\tau\left(s_{\frac{k}{n}}(x),y\right)-\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)\right]dx^{-\gamma_1}\right|\\ \le&\left|\int_{l_n}^{\infty}\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right| + \sup_{\substack{1<x<\infty\\0\le y\le 1}}\frac{\left|\tau_{\frac{k}{n}}(x,y)-\tau(x,y)\right|}{x^{\beta_2}}\int_{l_n}^{\infty}\left(s_{\frac{k}{n}}(x)\right)^{\beta_2}dx^{-\gamma_1}.\end{aligned}$$

Note that, by Assumption 4,

$$\sup_{\substack{1<x<\infty\\0\le y\le 1}}\frac{\left|\tau_{\frac{k}{n}}(x,y)-\tau(x,y)\right|}{x^{\beta_2}} = O\left(\left(\frac{n}{k}\right)^{-\xi}\right).$$

Then, by Jensen inequality and a change of variable, we obtain

$$\begin{aligned}\int_1^\infty s_{\frac{k}{n}}(x)^{\beta_2}dx^{-\gamma_1} &= \int_1^\infty \left\{\frac{n}{k}\mathbf{P}\left(X > Q_1(1-k/n)x^{-\gamma_1}\right)\right\}^{\beta_2} dx^{-\gamma_1} \\ &= \left(\frac{n}{k}\right)^{\beta_2}\int_0^1 \left\{\mathbf{P}\left(X > Q_1(1-k/n)x\right)\right\}^{\beta_2} dx \\ &\leq \left(\frac{n}{k}\right)^{\beta_2}\left\{\int_0^1 \mathbf{P}\left(X > Q_1(1-k/n)x\right) dx\right\}^{\beta_2} \\ &= \left(\frac{n}{k}\right)^{\beta_2}\left\{\frac{1}{Q_1(1-k/n)}\int_0^{Q_1(1-k/n)} \mathbf{P}(X > x)dx\right\}^{\beta_2} \\ &\leq \left(\frac{n}{k}\right)^{\beta_2(1-\gamma_1)}(\mathbb{E}[X])^{\beta_2}. \end{aligned} \tag{A3}$$

Hence, $\int_{l_n}^\infty s_{\frac{k}{n}}(x)^{\beta_2}dx^{-\gamma_1} \leq \int_1^\infty s_{\frac{k}{n}}(x)^{\beta_2}dx^{-\gamma_1} = O\left((\frac{n}{k})^{\beta_2(1-\gamma_1)}\right)$. By Assumption 5, we have

$$\sup_{0\leq y\leq 1} M_n\left|\int_{l_n}^\infty \left[\tau\left(s_{\frac{k}{n}}(x), y\right) - \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right)\right] dx^{-\gamma_1}\right| = O\left(\sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}-\xi+\beta_2(1-\gamma_1)}\right) \to 0.$$

On the other hand, using the definition of $s_{\frac{k}{n}}$, we get

$$\begin{aligned}&\left|\int_{l_n}^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right) dx^{-\gamma_1}\right| \\ =&\left|\int_{l_n}^\infty \left(\frac{n}{k}\right)^{1/\eta}\mathbf{P}\left[X > Q_1\left(1-\frac{ks_{\frac{k}{n}}(x)}{n}\right), Y > Q_2\left(1-\frac{ky}{n}\right)\right] dx^{-\gamma_1}\right| \\ \leq&\gamma_1\frac{ky}{n}\left(\frac{n}{k}\right)^{1/\eta} l_n^{-\gamma_1}.\end{aligned}$$

As a result, by Assumption 5 we obtain

$$\sup_{0\leq y\leq 1} M_n\left|\int_{l_n}^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right) dx^{-\gamma_1}\right| \leq C\sqrt{k}\left(\frac{n}{k}\right)^{\frac{1}{2\eta}-\frac{1}{2}-(1-\epsilon)\gamma_1} \to 0.$$

(iii) We write

$$\begin{aligned}
&\sup_{0\le y\le 1}\left|\int_0^\infty \tau_{\frac{k}{n}}(x,y)\,dx^{-\rho\gamma_1}-\int_0^\infty \tau(x,y)dx^{-\rho\gamma_1}\right|\\
\le&\sup_{\substack{0<x<\infty\\0\le y\le 1}}\frac{\left|\tau_{\frac{k}{n}}(x,y)-\tau(x,y)\right|}{x^{\beta_1}\wedge x^{\beta_2}}\left|\int_0^\infty x^{\beta_1}\wedge x^{\beta_2}dx^{-\rho\gamma_1}\right|\\
=&O\left(\left(\frac{n}{k}\right)^{-\xi}\right)\left(\int_0^1 x^{\beta_1}dx^{-\rho\gamma_1}+\int_1^\infty x^{\beta_2}dx^{-\rho\gamma_1}\right)\\
=&o(1),
\end{aligned}$$

by Assumptions 3 and 4.

(iv) By Assumptions 3 and 4, we have

$$\begin{aligned}
&\sup_{0\le y\le 1}M_n\left|\int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)-\tau\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}\right|\\
\le&M_n\sup_{\substack{0<x<\infty\\0\le y\le 1}}\frac{\left|\tau_{\frac{k}{n}}(x,y)-\tau(x,y)\right|}{x^{\beta_1}\wedge x^{\beta_2}}\int_0^\infty\left(s_{\frac{k}{n}}(x)^{\beta_1}\wedge s_{\frac{k}{n}}(x)^{\beta_2}\right)dx^{-\gamma_1}\\
=&O\left(M_n\left(\frac{n}{k}\right)^{-\xi}\right)\int_0^\infty\left(s_{\frac{k}{n}}(x)^{\beta_1}\wedge s_{\frac{k}{n}}(x)^{\beta_2}\right)dx^{-\gamma_1}.
\end{aligned}$$

Next, we obtain an upper bound for the integral in the last equality. Because $s_{\frac{k}{n}}(x)$ is monotone and $s_{\frac{k}{n}}(1)=1$, we get the following bound for the integral from zero to one:

$$\int_0^1 s_{\frac{k}{n}}(x)^{\beta_1}dx^{-\gamma_1}<\int_{\mathbb{R}}\left(s_{\frac{k}{n}}(x)^{\beta_1}\wedge 1\right)dx^{-\gamma_1},$$

which is shown to be $O(1)$ in Cai, Einmahl, de Haan and Zhou (2015).

Recall that in (A3) we show

$$\int_1^\infty s_{\frac{k}{n}}(x)^{\beta_2} dx^{-\gamma_1} = O\left(\frac{n}{k}\right)^{\beta_2(1-\gamma_1)}.$$

Finally by Assumption 5, we get

$$O\left(M_n\left(\frac{n}{k}\right)^{-\xi+\beta_2(1-\gamma_1)}\right) = \sqrt{k}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}-\xi+\beta_2(1-\gamma_1)} \to 0.$$

□

Lemma 2. *Suppose Assumptions 2, 3 and 4 hold. For* $y \in [0,1]$ *and* $\rho \in \{1, 2, 2+\delta\}$*, define*

$$A_n(y,\rho) = \left(\frac{n}{k}\right)^{1/\eta}\left(-\int_0^\infty \mathbf{1}_{\left\{1-F_1(X_1)<\frac{k}{n}x, 1-F_2(Y_1)<\frac{ky}{n}\right\}} dx^{-\gamma_1}\right)^\rho.$$

Then,

$$\mathbb{E}\left[A_n(y,\rho)\right] \to -\int_0^\infty \tau(x,y) dx^{-\rho\gamma_1}.$$

Proof of Lemma 2. Let $U_i = 1 - F_1(X_i), V_i = 1 - F_2(Y_i), i = 1, \cdots, n$, then we can write the integral as

$$\int_0^\infty \mathbf{1}_{\left\{U_1<\frac{k}{n}x, V_1<\frac{ky}{n}\right\}} dx^{-\gamma_1} = -\mathbf{1}_{\left\{V_1<\frac{ky}{n}\right\}}\left(U_1\frac{n}{k}\right)^{-\gamma_1}.$$

By a change of variable, we obtain

$$\begin{aligned}\mathbb{E}\left[A_n(y,\rho)\right] &= \left(\frac{n}{k}\right)^{1/\eta}\mathbb{E}\left[\mathbf{1}_{\left\{V_1<\frac{ky}{n}\right\}}\left(\frac{n}{k}U_1\right)^{-\rho\gamma_1}\right] \\ &= \left(\frac{n}{k}\right)^{1/\eta}\int_0^\infty \mathbf{P}\left(U_1<\frac{k}{n}x^{-\frac{1}{\rho\gamma_1}}, V_1<\frac{ky}{n}\right)dx \\ &= -\left(\frac{n}{k}\right)^{1/\eta}\int_0^\infty \mathbf{P}\left(U_1<\frac{k}{n}x, V_1<\frac{ky}{n}\right)dx^{-\rho\gamma_1} \\ &= -\int_0^\infty \tau_{\frac{k}{n}}(x,y)\,dx^{-\rho\gamma_1}.\end{aligned}$$

The statement follows from (iii) in Lemma 1. □

Recall $s_{\frac{k}{n}}(x) = n\bar{F}_1\left(Q_1(1-k/n)x^{-\gamma_1}\right)/k, x\in(0,\infty)$. In Lemma 3 below we study the processes

$$Z^*_{n,i}(y) = -\frac{1}{M_n}\int_0^\infty \mathbf{1}_{\left\{U_i<\frac{k}{n}s_{\frac{k}{n}}(x), V_i<\frac{k}{n}y\right\}}dx^{-\gamma_1}, i=1,\cdots,n.$$

Lemma 3. *Suppose that Assumptions 1 to 5 hold. Then as $n\to\infty$,*

$$\left\{\sum_{i=1}^n\left(Z^*_{n,i}(y)-\mathbb{E}\left[Z^*_{n,i}(y)\right]\right)\right\}_{y\in[0,1]}\xrightarrow{d}\{W(y)\}_{y\in[0,1]},$$

where $W(\cdot)$ is a mean zero Gaussian process on $[0,1]$ with covariance structure

$$\mathbf{E}\left[W(y_1)W(y_2)\right] = -\int_0^\infty \tau(x, y_1\wedge y_2)\,dx^{-2\gamma_1}, \quad y_1,y_2\in[0,1].$$

Proof of Lemma 3. Given that we have $\lim_{n\to\infty} s_{\frac{k}{n}}(x) = x$ by the regular variation of $1-F_1$. We shall study a simpler process obtained by replacing

$s_{\frac{k}{n}}(x)$ with x :

$$Z_{n,i}(y) = -\frac{S_n}{k} \int_0^\infty \mathbf{1}_{\left\{U_i < \frac{k}{n}x, V_i < \frac{k}{n}y\right\}} dx^{-\gamma_1}.$$

To prove Lemma 3 , it suffices to show that

$$\sup_{y\in[0,1]} n\mathbb{E}\left[\left|Z_{n,1}^*(y) - Z_{n,1}(y)\right|\right] \to 0, \tag{A4}$$

and

$$\left\{\sum_{i=1}^{n} \left(Z_{n,i}(y) - \mathbb{E}\left[Z_{n,i}(y)\right]\right)\right\}_{y\in[0,1]} \xrightarrow{d} \left\{W(y)\right\}_{y\in[0,1]}. \tag{A5}$$

(A4) implies that

$$\sup_{y\in[0,1]} \sum_{i=1}^{n} \left(Z_{n,i}^*(y) - Z_{n,i}(y)\right) \xrightarrow{\mathbf{P}} 0$$

and

$$\sup_{y\in[0,1]} \sum_{i=1}^{n} \left(\mathbb{E}\left[Z_{n,i}^*(y)\right] - \mathbb{E}\left[Z_{n,i}(y)\right]\right) \xrightarrow{\mathbf{P}} 0.$$

Step 1: Proof of (A4)

Using the triangular inequality, we write

$$\begin{aligned}
&n\mathbb{E}\left[\left|Z_{n,1}^*(y) - Z_{n,1}(y)\right|\right] \\
&= -\left(\frac{n}{k}\right)^{\frac{1}{2\eta}-\frac{1}{2}} \frac{n}{\sqrt{k}} \int_0^\infty \mathbf{P}\left(\frac{k}{n}\left(x \wedge s_{\frac{k}{n}}(x)\right) < U_1 < \frac{k}{n}\left(x \vee s_{\frac{k}{n}}(x)\right), V_1 < \frac{k}{n}y\right) dx^{-\gamma_1} \\
&= -M_n \int_0^\infty \left(\tau_{\frac{k}{n}}\left(x \vee s_{\frac{k}{n}}(x), y\right) - \tau_{\frac{k}{n}}\left(x \wedge s_{\frac{k}{n}}(x), y\right)\right) dx^{-\gamma_1} \\
&\le -M_n \int_0^\infty \left|\tau(x,y) - \tau\left(s_{\frac{k}{n}}(x), y\right)\right| dx^{-\gamma_1} \\
&\quad - M_n \int_0^\infty \left|\tau_{\frac{k}{n}}(x,y) - \tau(x,y)\right| dx^{-\gamma_1} \\
&\quad - M_n \int_0^\infty \left|\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right) - \tau\left(s_{\frac{k}{n}}(x), y\right)\right| dx^{-\gamma_1}.
\end{aligned}$$

All three terms on the right-hand side converge to zero by (ii) and (iii) in Lemma 1.

Step 2: Proof of (A5)

We aim to apply Theorem 2.11.9 in van der Vaart and Wellner (1996). In order to do this, we need to check the four conditions in that theorem are satisfied. Let $(\mathcal{F}, \rho) = \{[0,1], \rho(y_1, y_2) = |y_1 - y_2|\}$, and $\|Z\|_{\mathcal{F}} = \sup_{y \in \mathcal{F}} |Z(y)|$.

(a) Fix $\epsilon > 0$. By the fact that $\|Z_{n,1}\|_{\mathcal{F}} \leq Z_{n,1}(1)$, we have that, with the notation δ in Assumption 2,

$$\begin{aligned} n\mathbb{E}\left[\|Z_{n,1}\|_{\mathcal{F}} \mathbf{1}_{\left\{\|Z_{n,1}\|_{\mathcal{F}} > \epsilon\right\}}\right] &\leq n\mathbb{E}\left[Z_{n,1}(1)\mathbf{1}_{\{Z_{n,1}(1) > \epsilon\}}\right] \\ &\leq \frac{n}{\epsilon^{1+\delta}}\mathbb{E}\left[Z_{n,1}^{2+\delta}(1)\right] \\ &= \frac{1}{\epsilon^{1+\delta}} M_n^{-\delta}\mathbb{E}\left[\left(\frac{n}{k}\right)^{1/\eta}\left(-\int_0^\infty \mathbf{1}_{\left\{U_i < \frac{k}{n}x, V_i < \frac{k}{n}\right\}} dx^{-\gamma_1}\right)^{2+\delta}\right] \\ &\to 0. \end{aligned} \tag{A6}$$

The last convergence follows from $M_n \to \infty$ and Lemma 2 .

(b) Take a positive sequence $\delta_n \to 0$. Then we have

$$\begin{aligned}
&\sup_{|y_1-y_2|<\delta_n} \sum_{i=1}^{n} \mathbb{E}\left[\left(Z_{n,i}(y_1) - Z_{n,i}(y_2)\right)^2\right] \\
&= \sup_{|y_1-y_2|<\delta_n} \sum_{i=1}^{n} \left(\frac{n}{k}\right)^{\frac{1}{\eta}-1} \frac{1}{k}\mathbb{E}\left[\left(\int_0^\infty \mathbf{1}_{\left\{U_i<\frac{k}{n}x, \frac{k}{n}y_2<V_i<\frac{k}{n}y_1\right\}} dx^{-\gamma_1}\right)^2\right] \\
&= \sup_{|y_1-y_2|<\delta_n} \left(\frac{n}{k}\right)^{\frac{1}{\eta}} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{k}{n}y_1<V_1<\frac{k}{n}y_2\right\}} \left(\frac{n}{k}U_1\right)^{-2\gamma_1}\right].
\end{aligned}$$

Moreover, by triangular inequality and by (i) and (iii) in Lemma 1, we get

$$\begin{aligned}
&\left(\frac{n}{k}\right)^{\frac{1}{\eta}} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{k}{n}y<V_1<\frac{k}{n}(y+\delta_n)\right\}} \left(\frac{n}{k}U_1\right)^{-2\gamma_1}\right] \\
&= -\left(\frac{n}{k}\right)^{\frac{1}{\eta}} \int_0^\infty \mathbf{P}\left(U_1 < \frac{k}{n}x, \frac{k}{n}y < V_1 < \frac{k}{n}(y+\delta_n)\right) dx^{-2\gamma_1} \\
&= \left|\int_0^\infty \tau_{\frac{k}{n}}(x, y+\delta_n)\, dx^{-2\gamma_1} - \int_0^\infty \tau_{\frac{k}{n}}(x, y)\, dx^{-2\gamma_1}\right| \to 0. \qquad \text{(A7)}
\end{aligned}$$

(c) Let $N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)$ be the minimal number of sets N_ϵ in a partition $[0,1] = \cup_{j=1}^{N_\epsilon} I_{n,j}^\epsilon$ such that

$$\sum_{i=1}^{n} \mathbb{E}\left[\sup_{y_1,y_2\in I_{n,j}^\epsilon} \left|Z_{n,i}(y_1) - Z_{n,i}(y_2)\right|^2\right] \le \epsilon^2, \quad \forall j = 1, \ldots, N_\epsilon.$$

Consider the partition given by $I_{n,j}^\epsilon = [(j-1)\Delta_n, j\Delta_n]$. Then, $N_\epsilon = 1/\Delta_n$.

We aim to find Δ_n such that, for every sequence $\delta_n \to 0$, it follows that

$$\int_0^{\delta_n} \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)} d\epsilon \to 0.$$

Notice that

$$n\mathbb{E}\left[\sup_{y_1,y_2\in I_{n,j}^\epsilon} \left|Z_{n,1}(y_1) - Z_{n,1}(y_2)\right|^2\right] = \sup_{y_1,y_2\in I_{n,j}^\epsilon} \left(\frac{n}{k}\right)^{\frac{1}{\eta}} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{k}{n}y_1<V_1<\frac{k}{n}y_2\right\}} \left(\frac{n}{k}U_1\right)^{-2\gamma_1}\right] =: B_n.$$

Let $\bar{y}_1 = (j-1)\Delta_n$ and $\bar{y}_2 = j\Delta_n$. Next, we will derive two different upper bounds for B_n. Let $q = (2+\delta)/2$ and p such that $1/p + 1/q = 1$. By Hölder inequality, we obtain

$$\begin{aligned}
B_n &\leq \left(\frac{n}{k}\right)^{\frac{1}{\eta}} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{k}{n}\bar{y}_1 < V_1 < \frac{k}{n}\bar{y}_2\right\}} \left(U_1 \frac{n}{k}\right)^{-2\gamma_1}\right] \\
&\leq \left(\frac{n}{k}\right)^{\frac{1}{\eta}} \mathbb{E}\left[\mathbf{1}_{\left\{\frac{k}{n}\bar{y}_1 < V_1 < \frac{k}{n}\bar{y}_2\right\}}\right]^{1/p} \mathbb{E}\left[\mathbf{1}_{\left\{V_1 < \frac{k}{n}\bar{y}_2\right\}} \left(U_1 \frac{n}{k}\right)^{-2q\gamma_1}\right]^{1/q} \\
&\leq \left(\frac{n}{k}\right)^{\frac{1}{\eta} - \frac{1}{p} - \frac{1}{nq}} \left|\bar{y}_1 - \bar{y}_2\right|^{\frac{1}{p}} \mathbb{E}\left[A_n\left(\bar{y}_2, 2q\right)\right] \\
&= K_1 \left(\frac{n}{k}\right)^{\frac{1}{\eta} - \frac{1}{p} - \frac{1}{nq}} \Delta_n^{\frac{1}{p}},
\end{aligned}$$

for some constant $K_1 > 0$, where the last equality is obtained by applying Lemma 2.

On the other hand, by (i) and (iii) in Lemma 1 and the triangular inequality, we get a second bound for B_n :

$$\begin{aligned}
B_n \leq & -2\int_0^\infty \left(\tau_{\frac{k}{n}}\left(x, \bar{y}_2\right) - \tau_{\frac{k}{n}}\left(x, \bar{y}_1\right)\right) dx^{-2\gamma_1} \\
= & -2\int_0^\infty \left(\tau\left(x, \bar{y}_2\right) - \tau\left(x, \bar{y}_1\right)\right) dx^{-2\gamma_1} \\
& -2\int_0^\infty \left(\tau_{\frac{k}{n}}\left(x, \bar{y}_2\right) - \tau\left(x, \bar{y}_2\right)\right) dx^{-2\gamma_1} \\
& -2\int_0^\infty \left(\tau\left(x, \bar{y}_1\right) - \tau_{\frac{k}{n}}\left(x, \bar{y}_1\right)\right) dx^{-2\gamma_1} \\
\leq & K_2\Delta_n + K_3\left(\frac{k}{n}\right)^{\xi}
\end{aligned}$$

for some positive constants K_2 and K_3.

If $\epsilon^2 < \left(\frac{k}{n}\right)^{\xi^*}$ for some $\xi^* \in (0, \xi)$, we use the first bound for B_n. By choosing

$$\Delta_n = (K_1)^{-p} \left(\frac{n}{k}\right)^{-\frac{p}{\eta}+1+\frac{p}{\eta q}} \epsilon^{2p},$$

we get $B_n \leq K_1 \left(\frac{n}{k}\right)^{\frac{1}{\eta}-\frac{1}{p}-\frac{1}{\eta q}} \Delta_n^{\frac{1}{p}} \leq \epsilon^2$. Hence,

$$N_\epsilon \leq \frac{(K_1)^p}{\epsilon^{2p}} \left(\frac{n}{k}\right)^{\frac{p}{\eta}-1-\frac{p}{\eta q}}.$$

Otherwise, if $\epsilon^2 > \left(\frac{k}{n}\right)^{\xi^*}$, for sufficiently large n,

$$K_3 \left(\frac{k}{n}\right)^{\xi} < \frac{1}{2}\left(\frac{k}{n}\right)^{\xi^*} < \frac{1}{2}\epsilon^2,$$

and we use the second bound for B_n with $\Delta_n = \epsilon^2/(2K_2)$, which means

$$B_n \leq K_2\Delta_n + K_3 \left(\frac{k}{n}\right)^{\xi} \leq \epsilon^2$$

Hence, in this case,

$$N_\epsilon \leq \frac{2K_2}{\epsilon^2}.$$

Now, we distinguish between two cases. If $\delta_n\sqrt{\log(n/k)} \to 0$, using the inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, for $a, b > 0$ and the inequality $\log(x) \leq x$ for large x, we get

$$\begin{aligned}\int_0^{\delta_n} \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)} d\epsilon &\leq \int_0^{\delta_n} \sqrt{\left(\frac{p}{\eta} - 1 - \frac{p}{\eta q}\right)\log(n/k) + 2p\log\epsilon^{-1} + \log\left(K_1\right)^p} d\epsilon \\ &\leq K_4 \left(\int_0^{\delta_n} \sqrt{\log(n/k)} d\epsilon + \int_0^{\delta_n} \sqrt{\epsilon^{-1}} d\epsilon\right),\end{aligned}$$

for some positive constants K_4, and the left-hand side converges to zero as $\delta_n \to 0$.

On the other hand, if $\delta_n\sqrt{\log(n/k)} \nrightarrow 0$, we take $\delta_n^* = (k/n)^{\xi^*}$. Note that $\delta_n^*\sqrt{\log(n/k)} \to 0$. Hence, we write

$$\begin{aligned}\int_0^{\delta_n} \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)} d\epsilon &= \int_0^{\delta_n^*} \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)} d\epsilon + \int_{\delta_n^*}^{\delta_n} \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2^n\right)} d\epsilon \\ &\leq o(1) + \int_{\delta_n^*}^{\delta_n} \sqrt{\log\left(2K_2/\epsilon^2\right)} d\epsilon \\ &\leq o(1) + \sqrt{2}\int_0^{\delta_n} \sqrt{\epsilon^{-1}} d\epsilon \to 0.\end{aligned}$$

(d) We will show marginal convergence, that is, for each positive integer M and for each $y_1, \ldots, y_M \in [0,1]$, the random vector

$$\left(\sum_{i=1}^n \left(Z_{n,i}\left(y_1\right) - \mathbb{E}\left[Z_{n,i}\left(y_1\right)\right]\right), \ldots, \sum_{i=1}^n \left(Z_{n,i}\left(y_M\right) - \mathbb{E}\left[Z_{n,i}\left(y_M\right)\right]\right)\right)$$

converges to a multivariate normal distribution. It suffices to show that, for each $a_1, \ldots, a_M \in \mathbb{R}$,

$$\sum_{j=1}^M a_j \left[\sum_{i=1}^n \left(Z_{n,i}\left(y_j\right) - \mathbb{E}\left[Z_{n,i}\left(y_j\right)\right]\right)\right] =: \sum_{i=1}^n \left(N_{n,i} - \mathbb{E}\left[N_{n,i}\right]\right)$$

converges to a normal distribution, where $N_{n,i} = \sum_{j=1}^M a_j Z_{n,i}\left(y_j\right)$. This will follow from the Lindeberg-Feller central limit theorem (see, e.g., Proposition 2.27 in van der Vaart (1998)), once we show that, for each $\epsilon > 0$,

$$\sum_{i=1}^n \mathbb{E}\left[\left|N_{n,i}\right|^2 \mathbf{1}_{\{|N_{n,i}|>\epsilon\}}\right] \to 0 \tag{A8}$$

and

$$\sum_{i=1}^n \operatorname{Var}\left(N_{n,i}\right) \to \sigma_N^2. \tag{A9}$$

We first proceed with (A8). Note that

$$\sum_{i=1}^{n}\mathbb{E}\left[\left|N_{n,i}\right|^{2}\mathbf{1}_{\{|N_{n,i}|>\epsilon\}}\right]=n\mathbb{E}\left[\left|N_{n,1}\right|^{2}\mathbf{1}_{\{|N_{n,1}|>\epsilon\}}\right]$$
$$\leq\frac{n\mathbb{E}\left[\left|N_{n,1}\right|^{2+\delta}\right]}{\epsilon^{\delta}}\leq K_5 n\sum_{j=1}^{M}\left|a_j\right|^{2+\delta}\frac{\mathbb{E}\left[\left|Z_{n,1}(1)\right|^{2+\delta}\right]}{\epsilon^{\delta}},$$

for some positive constants K_5, and converges to zero by (A6). For (A9), we write

$$\sum_{i=1}^{n}\mathrm{Var}\left(N_{n,i}\right)=n\left\{\mathbb{E}\left[\left(\sum_{j=1}^{M}a_jZ_{n,1}\left(y_j\right)\right)^{2}\right]-\left(\mathbb{E}\left[\sum_{j=1}^{M}a_jZ_{n,1}\left(y_j\right)\right]\right)^{2}\right\}$$
$$=n\mathbb{E}\left[\sum_{j=1}^{M}\sum_{k=1}^{M}a_ja_kZ_{n,1}\left(y_j\right)Z_{n,1}\left(y_k\right)\right]-\left(\sqrt{n}\sum_{j=1}^{M}a_j\mathbb{E}\left[Z_{n,1}\left(y_j\right)\right]\right)^{2}$$
$$=n\sum_{j=1}^{M}\sum_{k=1}^{M}a_ja_k\mathbb{E}\left[Z_{n,1}\left(y_j\right)Z_{n,1}\left(y_k\right)\right]+o(1),$$

because it is easy to check that $\sqrt{n}\mathbb{E}\left[Z_{n,1}\left(y_j\right)\right]\to 0$, for $j=1,\ldots,M$. Observe that

$$n\mathbb{E}\left[Z_{n,1}\left(y_j\right)Z_{n,1}\left(y_k\right)\right]=\left(\frac{n}{k}\right)^{\frac{1}{n}}\mathbb{E}\left[\left(\int_0^{\infty}\mathbf{1}_{\left\{U_1<\frac{k}{n}x,V_1<\frac{k}{n}\left(y_j\wedge y_k\right)\right\}}dx^{-\gamma_1}\right)^{2}\right]$$
$$=\mathbb{E}\left[A_n\left(y_j\wedge y_k,1\right)\right].$$

Thus, by Lemma 2, it follows that (A9) holds with

$$\sigma_N^2=-\sum_{j=1}^{M}\sum_{k=1}^{M}a_ja_k\int_0^{\infty}\tau\left(x,y_j\wedge y_k\right)dx^{-2\gamma_1}.$$

We have verified the four conditions required by Theorem 2.11.9 in van der Vaart and Wellner (1996), which means we have the conclusion that

$\sum_{i=1}^{n}\left(Z_{n,i}-\mathbb{E}\left[Z_{n,i}\right]\right)$ converges weakly to a mean-zero Gaussian process W.

Now, we compute the covariance structure of the limit process. For each $y_1, y_2 \in [0,1]$, by independence, we have

$$\begin{aligned}
\mathbb{E}\left[W\left(y_1\right)W\left(y_2\right)\right] &= \lim_{n\to\infty}\operatorname{Cov}\left(\sum_{i=1}^{n} Z_{n,i}\left(y_1\right), \sum_{i=1}^{n} Z_{n,i}\left(y_2\right)\right)\\
&= \lim_{n\to\infty} n\operatorname{Cov}\left(Z_{n,1}\left(y_1\right), Z_{n,1}\left(y_2\right)\right)\\
&= \lim_{n\to\infty}\left(n\mathbb{E}\left[Z_{n,1}\left(y_1\right)Z_{n,1}\left(y_2\right)\right]-n\mathbb{E}\left[Z_{n,1}\left(y_1\right)\right]\mathbb{E}\left[Z_{n,1}\left(y_2\right)\right]\right)\\
&= -\int_0^\infty \tau\left(x, y_1\wedge y_2\right)dx^{-2\gamma_1}\\
&= \int_0^\infty \tau\left(x^{-\frac{1}{2\gamma_1}}, y_1\wedge y_2\right)dx.
\end{aligned}$$

□

In order to prove Proposition 2, we also need the Gaussian approximation to the tail empirical process for the marginal distribution of Y. Lemma 4 below is derived from Proposition 3.1 in Einmahl de Haan and Li (2006).

Lemma 4. *There exists a sequence of mean zero Gaussian processes* $\{\widetilde{W}_n(y)\}_{y\in[0,1]}$ *with covariance structure*

$$\mathbb{E}\left[\widetilde{W}_n\left(y_1\right)\widetilde{W}_n\left(y_2\right)\right] = y_1\wedge y_2, \quad y_1, y_2\in[0,1],$$

such that for any $q\in[0,1/2)$*, we have*

$$\sup_{y\in(0,1]} y^{-q}\left|\sqrt{n\alpha}\left(\frac{1}{n\alpha}\sum_{i=1}^{n}\mathbf{1}_{\{V_i<\alpha y\}}-y\right)-\widetilde{W}_n(y)\right| \xrightarrow{\mathbf{P}} 0.$$

S2 Proof of Proposition 1

Let $\hat{V}_i = 1 - F_{n2}(Y_i)$, where $F_{n2}(y) = \frac{1}{n+1}\sum_{i=1}^n \mathbf{1}_{\{Y_i \le y\}}$ is the empirical distribution function of F_2. Recall that $\alpha = k/n$. Denote

$$\omega_n = 2\mathbb{E}\left((X_1 - X_2)(V_2 - V_1)\mathbf{1}_{\{V_1<\alpha, V_2<\alpha\}}\right),$$

$$h(x_1, v_1, x_2, v_2) = 2(x_1 - x_2)(v_2 - v_1)\mathbf{1}_{\{V_1<\alpha, V_2<\alpha\}} - \omega_n,$$

$$h_1(x_1, v_1) = 2\mathbb{E}\left((x_1 - X_2)(V_2 - v_1)\mathbf{1}_{\{V_1<\alpha, V_2<\alpha\}}\right) - \omega_n.$$

Therefore, $\omega_n = \alpha^3 \operatorname{TG}_{\frac{k}{n}}(X;Y)$ and

$$\begin{aligned}\hat{\theta}_{\frac{k}{n}} =& \frac{2n}{k^2(k-1)}\sum_{1\le i<j\le n}\left[\omega_n + h\left(X_i, \hat{V}_i, X_j, \hat{V}_j\right)\right] \\ =& \frac{n^2(n-1)}{k^2(k-1)}\omega_n + \frac{2n}{k^2(k-1)}\sum_{1\le i<j\le n} h\left(X_i, \hat{V}_i, X_j, \hat{V}_j\right).\end{aligned}$$

By Hoeffding's Decomposition, we have that

$$\left(\frac{\sqrt{k}}{2n(n-1)}\sum_{1\le i<j\le n}\frac{h\left(X_i, \hat{V}_i, X_j, \hat{V}_j\right)}{\alpha^3 Q_1(1-\alpha)}\right) = \sqrt{\frac{\alpha}{n}}\sum_{i=1}^n \frac{h_1\left(X_i, \hat{V}_i\right)}{2\alpha^3 Q_1(1-\alpha)} + o_{\mathbf{P}}(1).$$

Note that

$$
\begin{aligned}
h_1(x_1,v_1) = & -2x_1v_1I(v_1<\alpha)\mathbf{P}(V_2<\alpha)+2x_1I(v_1<\alpha)\mathbb{E}V_2I(V_2<\alpha) \\
& +2v_1I(v_1<\alpha)\mathbb{E}X_2I(V_2<\alpha)-2I(v_1<\alpha)\mathbb{E}X_2V_2I(V_2<\alpha)-\omega_n \\
= & 2\alpha(-x_1v_1I(v_1<\alpha)+\mathbb{E}X_1V_1I(V_1<\alpha)) \\
& +\alpha^2(x_1I(v_2<\alpha)-\mathbb{E}X_1I(V_1<\alpha)) \\
& +2(v_1I(v_1<\alpha)-\mathbb{E}V_1I(V_1<\alpha)\mathbb{E}X_1I(V_1<\alpha)) \\
& -2(I(v_1<\alpha)-\alpha)\mathbb{E}X_1V_1I(V_1<\alpha).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& (\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}}\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}\frac{h_1\left(X_i,\hat{V}_i\right)}{2\alpha^3Q_1(1-\alpha)} \\
=& (\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}}\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}\left(\frac{-X_i\hat{V}_iI\left(\hat{V}_i<\alpha\right)+\mathbb{E}X_1V_1I(V_1<\alpha)}{\alpha^2Q_1(1-\alpha)}\right) \\
& +(\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}}\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}\left(\frac{X_iI\left(\hat{V}_i<\alpha\right)-\mathbb{E}X_1I(V_1<\alpha)}{2\alpha Q_1(1-\alpha)}\right) \\
& +(\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}}\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}\frac{\left(\hat{V}_iI\left(\hat{V}_i<\alpha\right)-\mathbb{E}V_1I(V_1<\alpha)\right)\mathbb{E}X_1I(V_1<\alpha)}{\alpha^3Q_1(1-\alpha)} \\
& -(\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}}\sqrt{\frac{\alpha}{n}}\sum_{i=1}^{n}\frac{\left(I\left(\hat{V}_i<\alpha\right)-\alpha\right)\mathbb{E}X_1V_1I(V_1<\alpha)}{\alpha^3Q_1(1-\alpha)} \\
=:& J_1+J_2+J_3+J_4.
\end{aligned}
$$

Let $e_n=\bar{F}_2(Y_{n-k,n})/\alpha$. By Skorohod's representation and Lemma 3, there exists a sequence of mean zero Gaussian processes $\{W_n(y)\}_{y\in[0,1]}$ with covariance structure

$$\mathbf{E}\left[W_n\left(y_1\right)W_n\left(y_2\right)\right]=-\int_0^{\infty}\tau\left(x,y_1\wedge y_2\right)dx^{-2\gamma_1},\quad y_1,y_2\in[0,1]$$

such that

$$\sup_{y\in[0,1]}\left|\sum_{i=1}^{n}\left(Z_{n,i}^*(y)-\mathbb{E}\left[Z_{n,i}^*(y)\right]\right)-W_n(y)\right|=o_{\mathbf{P}}(1).$$

For J_1, by Fubini's Theorem and a change of variables, we have that

$$\begin{aligned}
J_1=&\int_0^1\sum_{i=1}^{n}\left(\mathbb{E}\left[Z_{n,i}^*(y)\right]-e_n^2Z_{n,i}^*(e_ny)\right)dy\\
=&-e_n^2\int_0^1\left(\sum_{i=1}^{n}\left(Z_{n,i}^*(e_ny)-\mathbb{E}\left[Z_{n,i}^*(e_ny)\right]\right)-W_n(e_ny)\right)dy\\
&-e_n^2\int_0^1\sum_{i=1}^{n}\left(\mathbb{E}\left[Z_{n,i}^*(e_ny)-Z_{n,i}^*(y)\right]\right)dy\\
&-(e_n^2-1)\int_0^1\sum_{i=1}^{n}\mathbb{E}\left[Z_{n,i}^*(y)\right]dy\\
&-e_n^2\int_0^1W_n(e_ny)dy\\
=:&\ J_{11}+J_{12}+J_{13}+J_{14}.
\end{aligned}$$

First, we show that $J_{11}=o_{\mathbf{P}}(1)$. Define $Q_n(y)=\sum_{i=1}^{n}\left(Z_{n,i}^*(y)-\mathbb{E}\left[Z_{n,i}^*(y)\right]\right)-W_n(y)$. By Lemma 4, we have $\sqrt{k}\left(e_n-1\right)=-\widetilde{W}_n(1)+o_{\mathbf{P}}(1)$, and thus

$$\lim_{n\to\infty}\mathbf{P}(|e_n-1|>k^{-1/4})=0.$$

Corollary 1.11 in Adler (1990) implies that as $n\to\infty$,

$$\sup_{y\in(0,1],|e_n-1|<k^{-1/4}}\left|Q_n\left(e_ny\right)-Q_n(y)\right|=o(1)\text{ a.s.},$$

which leads to

$$J_{11} = -e_n^2 \int_0^1 Q_n(e_n y) dy = o_{\mathbf{P}}(1).$$

Second, for J_{12}, using triangular inequality, we write

$$\begin{aligned}
J_{12} =& - e_n^2 \int_0^1 n\mathbb{E}\big[\big|Z_{n,1}^*(e_n y) - Z_{n,1}^*(y)\big|\big] dy \\
=& e_n^2 \left(\frac{n}{k}\right)^{\frac{1}{2\eta}-\frac{1}{2}} \frac{n}{\sqrt{k}} \int_0^1 \int_0^\infty \left[\mathbf{P}\left(U_1 < \frac{k}{n} s_{\frac{k}{n}}(x), V_1 < \frac{k}{n} e_n y\right) - \mathbf{P}\left(U_1 < \frac{k}{n} s_{\frac{k}{n}}(x), V_1 < \frac{k}{n} y\right)\right] dx^{-\gamma_1} dy \\
=& e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), e_n y\right) - \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right)\right] dx^{-\gamma_1} dy \\
\leq& e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), e_n y\right) - \tau\left(s_{\frac{k}{n}}(x), e_n y\right)\right] dx^{-\gamma_1} dy \\
&- e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x), y\right) - \tau\left(s_{\frac{k}{n}}(x), y\right)\right] dx^{-\gamma_1} dy \\
&+ e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau\left(s_{\frac{k}{n}}(x), e_n y\right) - \tau\left(x, e_n y\right)\right] dx^{-\gamma_1} dy \\
&- e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau\left(s_{\frac{k}{n}}(x), y\right) - \tau\left(x, y\right)\right] dx^{-\gamma_1} dy \\
&+ e_n^2 \int_0^1 M_n \int_0^\infty \left[\tau\left(x, e_n y\right) - \tau\left(x, y\right)\right] dx^{-\gamma_1} dy \\
=:& J_{121} - J_{122} + J_{123} - J_{124} + J_{125}.
\end{aligned}$$

We have $J_{121} = o_{\mathbf{P}}(1), J_{122} = o_{\mathbf{P}}(1), J_{123} = o_{\mathbf{P}}(1), J_{124} = o_{\mathbf{P}}(1)$ by (iii) and (iv) in Lemma 1. For J_{125}, by the homogeneity of function τ, we have

$$J_{125} = M_n(e_n^{\frac{1}{\eta}-\gamma_1} - 1)e_n^2 \int_0^1 \int_0^\infty \tau\left(x, y\right) dx^{-\gamma_1} dy.$$

Note that

$$M_n(e_n^{\frac{1}{\eta}-\gamma_1} - 1) = (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \sqrt{k}(e_n^{\frac{1}{\eta}-\gamma_1} - 1) \xrightarrow{\mathbf{P}} 0$$

because $\sqrt{k}(e_n^{\frac{1}{\eta}-\gamma_1}-1)=O_{\mathbf{P}}(1)$. Consequently, $J_{125}=o_{\mathbf{P}}(1)$ and hence $J_{12}=o_{\mathbf{P}}(1)$.

Third, $\sqrt{k}(e_n^2-1)=O_{\mathbf{P}}(1)$ and

$$\int_0^1\int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}dy=\int_0^1\int_0^\infty \tau\left(x,y\right)dx^{-\gamma_1}dy+o_{\mathbf{P}}(M_n^{-1})$$

lead to

$$J_{13}=\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}}\sqrt{k}(e_n^2-1)\int_0^1\int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),y\right)dx^{-\gamma_1}dy\xrightarrow{\mathbf{P}}0.$$

Fourth, again by Corollary 1.11 in Adler (1990), the continuity of W_n implies that as $n\to\infty$,

$$\sup_{y\in(0,1],|e_n-1|<k^{-1/4}}|W_n\left(e_ny\right)-W_n(y)|=o(1)\text{ a.s.},$$

which leads to that $J_{14}=-\int_0^1 W_n(y)dy+o_{\mathbf{P}}(1)$. Thus, we have $J_1=-\int_0^1 W_n(y)dy+o_{\mathbf{P}}(1)$.

Similarly, for J_2 we have that

$$\begin{aligned}
J_2 =& \frac{1}{2}\sum_{i=1}^{n}\left(\mathbb{E}\left[Z_{n,i}^*(1)\right] - e_n Z_{n,i}^*(e_n)\right)\\
=& -\frac{e_n}{2}\sum_{i=1}^{n}\left(Z_{n,i}^*(e_n) - \mathbb{E}\left[Z_{n,i}^*(e_n)\right]\right) - W_n(e_n)\\
& -\frac{e_n}{2}\sum_{i=1}^{n}\left(\mathbb{E}\left[Z_{n,i}^*(e_n) - Z_{n,i}^*(1)\right]\right)\\
& -\frac{e_n - 1}{2}\sum_{i=1}^{n}\mathbb{E}\left[Z_{n,i}^*(1)\right]\\
& -\frac{e_n}{2}W_n(e_n)\\
=& J_{21} + J_{22} + J_{23} + J_{24}.
\end{aligned}$$

By $e_n = o_{\mathbf{P}}(1)$ and Lemma 3, it follows that $J_{21} = o_{\mathbf{P}}(1)$, $J_{22} = o_{\mathbf{P}}(1)$, $J_{23} = o_{\mathbf{P}}(1)$, $J_{24} = -\frac{1}{2}W_n(1) + o_{\mathbf{P}}(1)$, and thus $J_2 = -\frac{1}{2}W_n(1) + o_{\mathbf{P}}(1)$.

Next, we deal with J_3. Write

$$
\begin{aligned}
J_3 =& S_n \int_0^\alpha \frac{1}{n}\sum_{i=1}^n \left(\frac{\mathbb{E}I\left(V_1<y\right)-I\left(V_i<e_n y\right)}{\alpha^2}\right) dy \times \frac{\mathbb{E}X_1 I\left(V_1<\alpha\right)}{\alpha Q_1(1-\alpha)} \\
=& \frac{M_n}{k}\int_0^1 \sum_{i=1}^n \left(I\left(V_i<e_n\alpha v\right)-\mathbb{E}I\left(V_1<\alpha v\right)\right) dy \times \int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \\
=& (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \int_0^1 \sqrt{n\alpha}\left(\frac{1}{n}\sum_{i=1}^n \frac{I\left(V_i<M_n\alpha y\right)}{\alpha}-e_n y\right)-\widetilde{W}_n\left(e_n y\right) dy \times \int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \\
&-(\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \int_0^1 \sqrt{n\alpha}\left(\frac{1}{n}\sum_{i=1}^n \frac{I\left(V_i<\alpha y\right)}{\alpha}-y\right)-\widetilde{W}_n(y) dy \times \int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \\
&+(\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \int_0^1 \sqrt{n\alpha}\left(\frac{1}{n}\sum_{i=1}^n \frac{I\left(V_i<\alpha y\right)-\mathbb{E}I\left(V_1<\alpha y\right)}{\alpha}\right)-\widetilde{W}_n(y) dy \times \int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \\
&+(\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left[\int_0^1 \left(\sqrt{n\alpha}\left(e_n-1\right)y+\widetilde{W}_n\left(e_n y\right)-\widetilde{W}_n(y)\right) dy+\frac{1}{2}\tilde{W}_n(1)\right] \times \int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \\
&+M_n \int_0^\infty \left(\tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right)-\tau\left(x,1\right)\right) dx^{-\gamma_1} \times \frac{1}{\sqrt{k}}\left(-\frac{1}{2}\widetilde{W}_n(1)+\int_0^1 \widetilde{W}_n(y)dy\right) \\
&+(\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left(-\frac{1}{2}\widetilde{W}_n(1)+\int_0^1 \widetilde{W}_n(y)dy\right) \times \int_0^\infty \tau\left(x,1\right) dx^{-\gamma_1} \\
=&: J_{31}-J_{32}+J_{33}+J_{34}+J_{35}+J_{36}.
\end{aligned}
$$

Lemma 4 and $\int_0^\infty \tau_{\frac{k}{n}}\left(s_{\frac{k}{n}}(x),1\right) dx^{-\gamma_1} \leq \infty$ imply that

$$
J_{31}=o_{\mathbf{P}}(1), \quad J_{32}=o_{\mathbf{P}}(1), \quad J_{33}=o_{\mathbf{P}}(1).
$$

Recall that $\sqrt{n\alpha}\left(e_n-1\right)=O_{\mathbf{P}}(1)$ and that $(\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \to 0$, so $J_{34}=o_{\mathbf{P}}(1)$. By (ii) and (iv) in Lemma 1, $J_{35}=o_{\mathbf{P}}(1)$ holds. $J_{36}=o_{\mathbf{P}}(1)$ also holds because $-\frac{1}{2}\widetilde{W}_n(1)+\int_0^1 \widetilde{W}_n(y)dy=O_{\mathbf{P}}(1)$ and $\int_0^\infty \tau\left(x,1\right) dx^{-\gamma_1} \leq \infty$. Therefore, $J_3=o_{\mathbf{P}}(1)$.

Similarly, for J_4 we have

$$\begin{aligned}
J_4 =& - S_n \left(\frac{1}{n\alpha} \sum_{i=1}^{n} I\left(V_i < e_n \alpha\right) - 1 \right) \times \frac{\mathbb{E} X_1 V_1 I\left(V_1 < \alpha\right)}{\alpha^2 Q_1(1-\alpha)} \\
=& - (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left(\frac{\sqrt{n\alpha}}{n\alpha} \sum_{i=1}^{n} I\left(V_i < e_n \alpha\right) - 1 \right) \times \int_0^1 \int_0^\infty \tau_{\frac{k}{n}} \left(s_{\frac{k}{n}}(x), y \right) dx^{-\gamma_1} dy \\
=& - (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left[\sqrt{n\alpha} \left(\frac{1}{n\alpha} \sum_{i=1}^{n} I\left(V_i < e_n \alpha\right) - e_n \right) - \widetilde{W}_n\left(e_n\right) \right] \times \int_0^1 \int_0^\infty \tau_{\frac{k}{n}} \left(s_{\frac{k}{n}}(x), y \right) dx^{-\gamma_1} dy \\
& - (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left(\sqrt{n\alpha}\left(e_n - 1\right) + \widetilde{W}_n(1) \right) \times \int_0^1 \int_0^\infty \tau_{\frac{k}{n}} \left(s_{\frac{k}{n}}(x), y \right) dx^{-\gamma_1} dy \\
& - (\frac{n}{k})^{-\frac{1}{2\eta}+\frac{1}{2}} \left(\widetilde{W}_n\left(e_n\right) - \widetilde{W}_n(1) \right) \times \int_0^1 \int_0^\infty \tau_{\frac{k}{n}} \left(s_{\frac{k}{n}}(x), y \right) dx^{-\gamma_1} dy \\
=& o_{\mathbf{P}}(1).
\end{aligned}$$

By taking J_1, J_2, J_3 and J_4 together, we have

$$\begin{aligned}
\sqrt{k}\frac{\phi_0}{4}\left(\frac{n}{k}\right)^{-\frac{1}{2\eta}+\frac{1}{2}} \left(\frac{\hat{\theta}_{k/n}}{\mathrm{TG}_{k/n}(X;Y)} - 1 \right) =& (\frac{n}{k})^{\frac{1}{2\eta}-\frac{1}{2}} \sqrt{\frac{\alpha}{n}} \sum_{i=1}^{n} \frac{h_1\left(X_i, \hat{V}_i\right)}{2\alpha^3 Q_1(1-\alpha)} \\
=& - \int_0^1 W_n(y) dy - \frac{1}{2} W_n(1) + o_{\mathbf{P}}(1),
\end{aligned}$$

and the proof is complete. □

S3 Simulation

In this section, we present an additional simulation study for the real value case in the following Model 3: Model 3. Let Z_1, Z_2, and Z_3 be independent Pareto random variables with parameters a_1, a_2, and a_1, respectively. Define

$$(X, Y) = (B-2)\left(Z_1, Z_3\right) - (B+1)\left(Z_2, Z_2\right),$$

Table S1: Parameters of Model 3 and the approximated true values of the tail Gini functional.

	(a_1, a_2)	γ_1	η	$-1/\eta + 1 + \gamma_1$	$p = 0.01$	$p = 0.001$
Model 3(a)	(0.35, 0.3)	0.35	6/7	0.183	2.3294	4.1145
Model 3(b)	(0.4, 0.35)	0.4	0.875	0.251	3.7148	7.2546
Model 3(c)	(0.6,0.5)	0.6	5/6	0.1	11.5657	31.6759
Model 3(d)	(0.5, 0.4)	0.5	0.8	0.3	4.3407	8.2514

where B is a Bernoulli(1/2) random variable independent of Z_i's. Here, both X and Y can take negative values. For this model, we have $\gamma_1 = a_1$, $\rho_1 = 1 - a_1/a_2, \eta = a_2/a_1$, and $\tau(x, y) = 2^{a_1/a_2-1}(x \wedge y)^{a_1/a_2}$. Here we can take $\xi = 2a_1/a_2 - 1, \delta = (2 - a_2^4)/a_2 - 1/a_1 - 2, \beta_1 = a_1(2 - a_2^3)/a_2 - 1, \beta_2 = (2a_1 - a_2^2)/(a_2(1 - a_1)) - 1/(1 - a_1)$, and we note that Model 3 satisfies Assumptions 1 to 8. We consider four settings of (a_1, a_2), see Table S1. Similar to Models 1 and 2, we calculate the true value $\mathrm{TG}_p(X; Y)$ at extreme level $p = 0.01$ and 0.001 in Table S1.

Table S2 shows means and standard errors of the ratio $\hat{\theta}_p/\,\mathrm{TG}_p(X; Y)$ with sample sizes $n = 1500$ and 5000 and number of replication $m = 2000$, comparing our proposed method (denoted by AIE) and the method in Hou and Wang (2021) (denoted by HW). We set $\alpha = 0.09, \alpha_1 = \alpha_2 = 0.05$, following the main text. The boxplots of $\log\left(\hat{\theta}_p/\mathrm{TG}_p(X; Y)\right)$ are presented

Table S2: Means of the ratios of the proposed estimators for the tail Gini functional and the true values for $n = 1500, 5000$ and $p = 0.01, 0.001$ are reported with corresponding standard deviation given in the brackets.

		AIE		HW	
		$n = 1500$	$n = 5000$	$n = 1500$	$n = 5000$
Model 3(a)	$p = 0.01$	1.0520(1.5527)	1.1248(0.8205)	1.6293(1.6278)	1.6090(0.8035)
	$p = 0.001$	1.4935(3.9785)	1.4903(1.8521)	2.9316(4.4592)	2.7631(1.7631)
Model 3(b)	$p = 0.01$	1.0358(1.4254)	1.1281(0.8222)	1.5805(1.7725)	1.5327(0.7631)
	$p = 0.001$	1.8027(5.3756)	1.5776(1.7469)	3.0919(7.0181)	2.6237(1.6641)
Model 3(c)	$p = 0.01$	1.1102(1.7850)	1.1025(0.8593)	1.6622(2.2281)	1.6511(0.9825)
	$p = 0.001$	1.6663(3.9812)	1.3056(1.4575)	3.3475(5.4001)	2.8473(2.2396)
Model 3(d)	$p = 0.01$	1.0846(1.5885)	1.1086(0.4914)	1.7924(1.7869)	1.8311(0.9753)
	$p = 0.001$	1.8434(5.7193)	1.4566(1.9148)	4.4383(8.5167)	3.7037(2.7988)

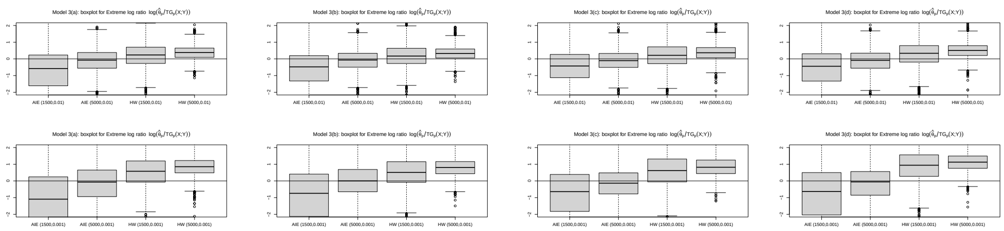

Figure S1: Boxplots of log ratios with (n, p).

in Figure S1. We can see that the proposed method has with sample size $n = 5000$ performs the best among all scenarios. The QQ plots in Figure S2 indicate no big difference from a normal distribution when comparing the sample quantiles of log-ratios with the quantiles of the theoretical limit distribution.

S4 Application

In this section, we assess the signs of $\hat{\gamma}_1$ and $\hat{\eta}$ for the 14 stocks that fail to reject asymptotic independence. We plot $\hat{\gamma}$ and $\hat{\eta}$ against different values of α_1 and α_2 in Figures S3 and S4, respectively. From Figures S3 and S4, we can see that $\hat{\gamma}_1 > 0$ and $\hat{\eta} \in (0.5, 1)$ for each pair of losses across the ranges of α_1 and α_2.

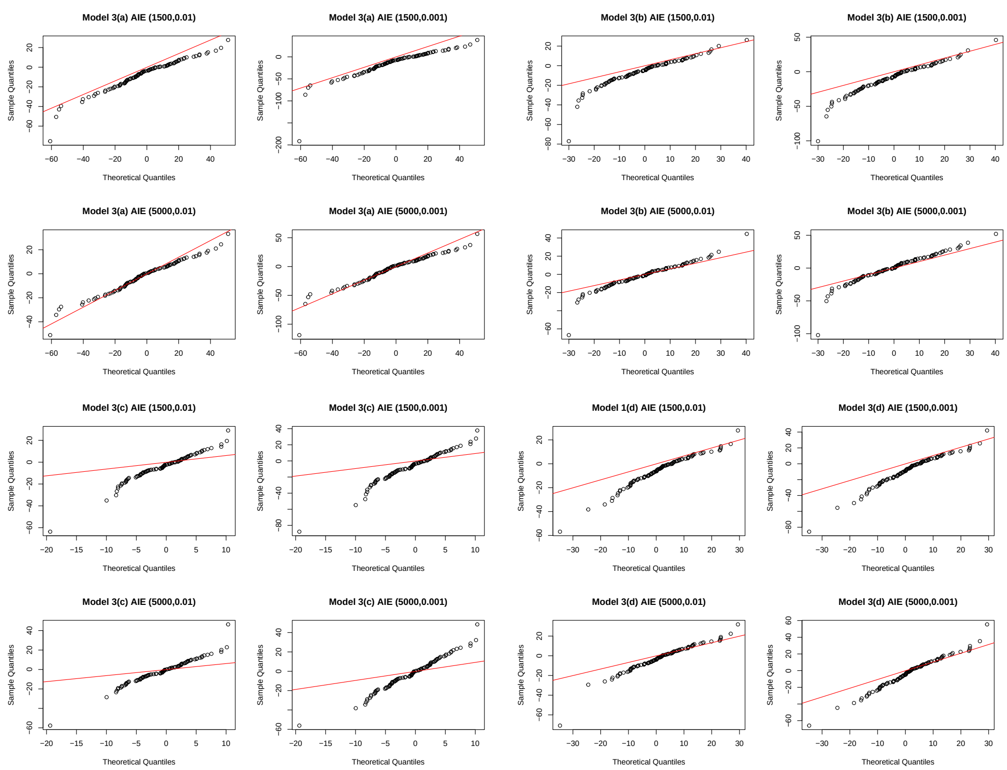

Figure S2: QQ plots of log ratios for AIE estimators for (n, p)

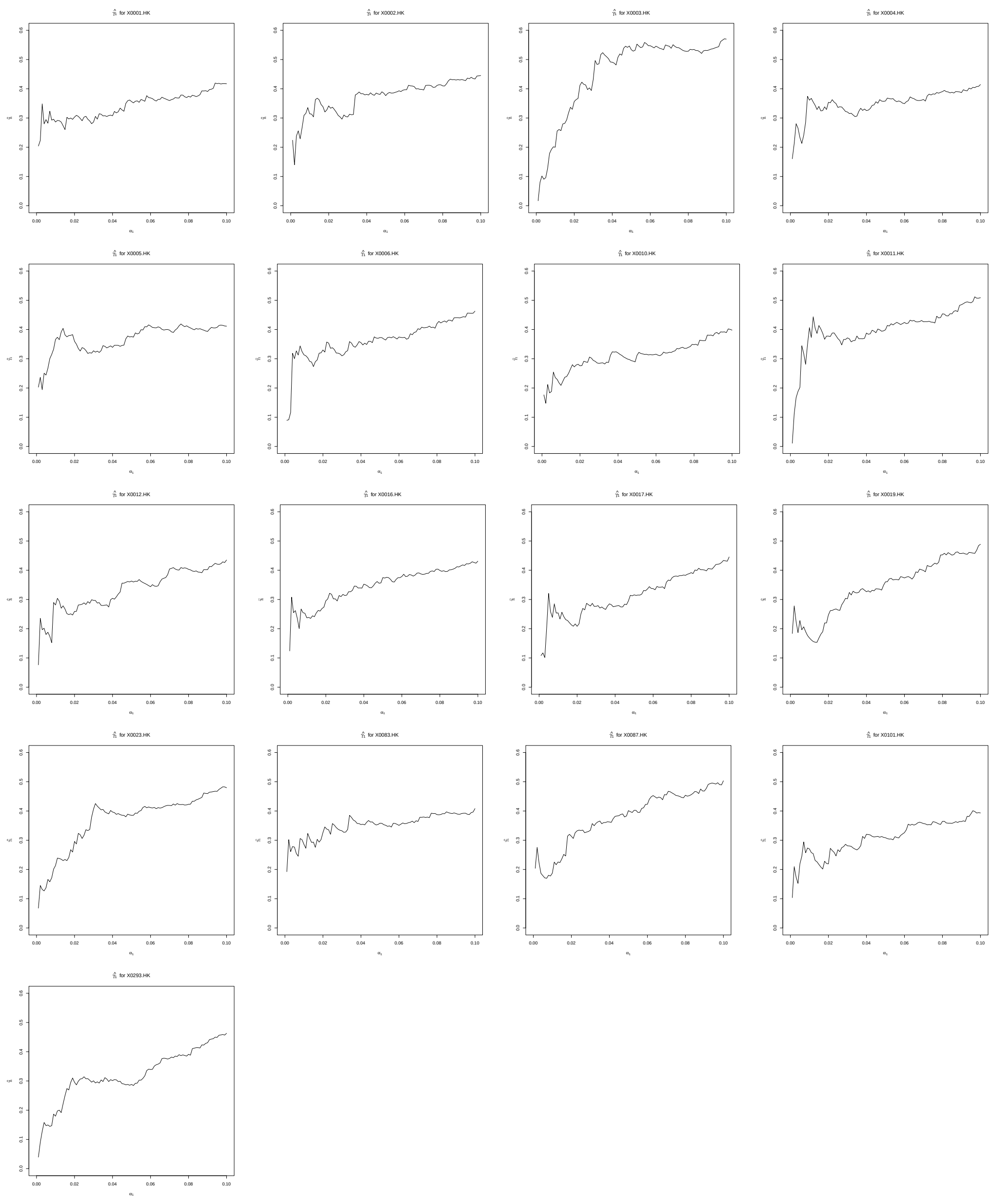

Figure S3: The Hill estimates of γ_1.

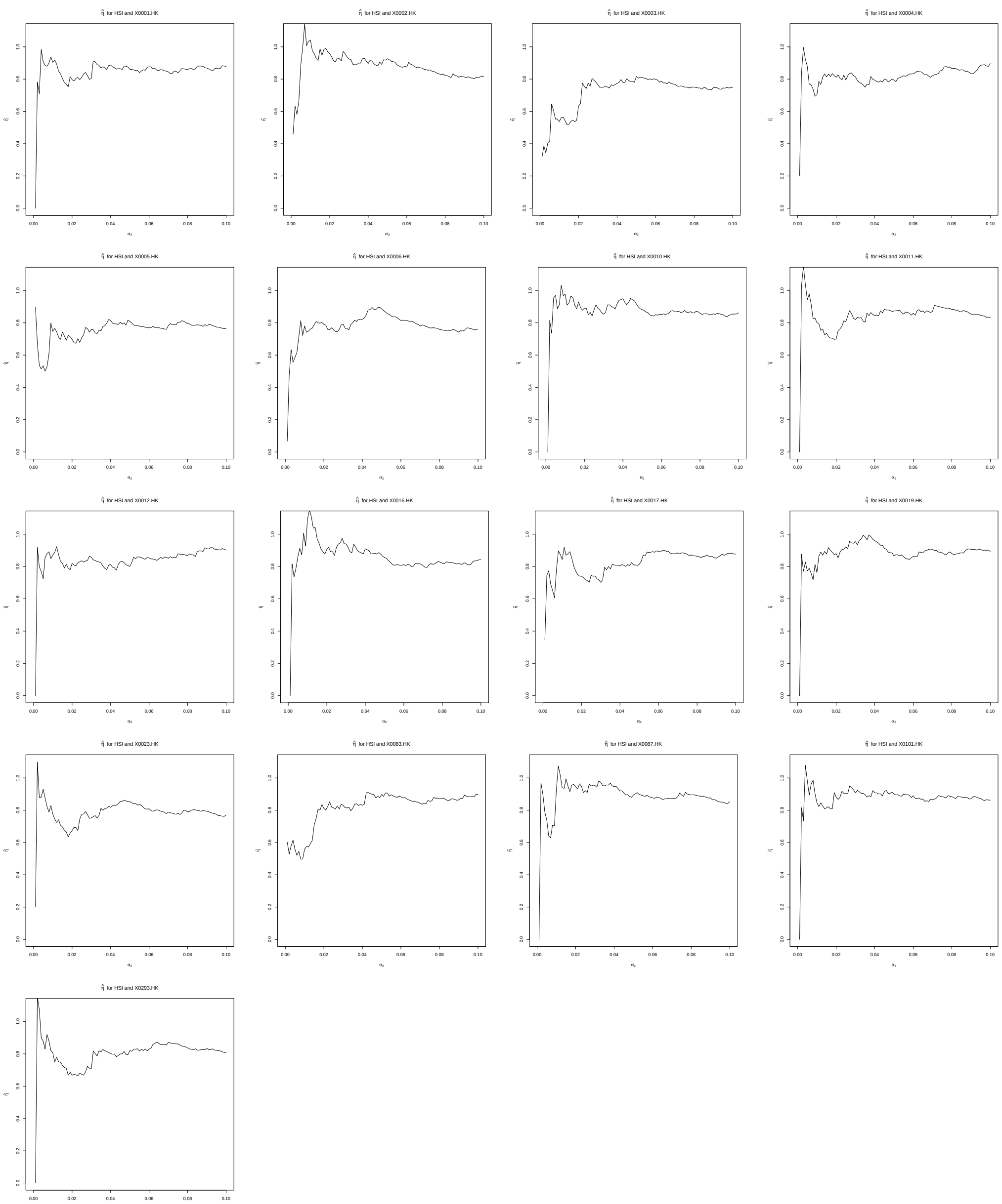

Figure S4: The estimates of η.

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