TWO-LEVEL ISOMORPHIC FOLDOVERS DESIGNS

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Supplementary Material

The online Supplementary Material includes S1: the proofs of Theorems 1–3 and Corollaries 1 and 2; S2: the optimal foldover matrices for the three 6-IFDs in Table 1; S3: the initial designs of the IFDs in Tables 1–4; S4: the initial designs of the IFDs in Tables C.1 and C.2; and S5: the indicator function of design 10.48.

S1. Proof of Theorems 1–3 and Corollaries 1 and 2

Proof of Theorem 1

Proof. Consider the f-IFD defined by foldover matrix Γ, where $\Gamma = (\gamma_1^T, \ldots, \gamma_f^T)^T$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{ik})$ for $i = 1, \ldots, f$, and f is even. Without loss of generality, suppose the f rows of Γ can be paired in row order so that the product of the two rows of each pair is identical. That is, $\gamma_{q_1 j}\gamma_{q_2 j} = 1$ with $q_1 = 2q - 1$, $q_2 = 2q$ for any $q = 1, \ldots, f/2$ and $j = 1, \ldots, k$. Then the jth column of the qth pairs is equal or opposite to that of the first pair, in other words, $(\gamma_{q_1 j}^T, \gamma_{q_2 j}^T)^T = (\gamma_{1j}^T, \gamma_{2j}^T)^T$ or $(\gamma_{q_1 j}^T, \gamma_{q_2 j}^T)^T = -(\gamma_{1j}^T, \gamma_{2j}^T)^T$ for any $q = 2, \ldots, f/2$ and $j = 1, \ldots, k$. In this way, the 2-IFD defined by the qth pair is a foldover of that defined by the first pair for any $q = 2 \ldots f/2$. According to

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the definition, the f-IFD defined by foldover matrix Γ can be merged into an $f/2$-IFD, and the foldover matrix is $\hat{\Gamma} = (\hat{\gamma}_{ij})$, an $f/2 \times k$ matrix determined by

$$\hat{\gamma}_{ij} = \begin{cases} 1 & i = 1, \\ \gamma_{(2i-1)j}\gamma_{1j} & i = 2, \ldots, f/2, j = 1, \ldots, k. \end{cases}$$

□

Proof of Theorem 2

Proof. As we have discussed $\tilde{z}$ is determined by B, the $u \times f$ matrix consisting of the rows $\{1, 2, 4, \ldots, 2^{u-1}\}$ of $\tilde{z}$. And the (l, i) element of B is equal to the value of lth word of D_0 in the ith foldover. Thus the f-IFD corresponding to $\tilde{z}$ has the foldover matrix $\Gamma = (\gamma_{i,j})_{f \times k}$ with

$$\begin{cases} \prod_{v \in w_l} \gamma_{iv} = b_{li} & i = 1, \ldots, f, l = 1, \ldots, u, \\ \gamma_{iv} = 1 & i = 1, \ldots, f, v \in F_0, \end{cases}$$

where $\{w_1, \ldots, w_u\}$ are the basic words of D_0, b_{li} represents the (l, i) element of B for $i = 1, \ldots, f$, $l = 1, \ldots, u$, and F_0 consists of the first $k - u$ factors of D_0 that can not be generated by its basic words. □

Proof of Theorem 3

Proof. Theorem 3 (i) and Theorem 3 (ii) can be easily established by the definitions of PFD and IFD. We now consider Theorem 3 (iii). If D_0 is a g-PFD, then it can be written as $(P_1^T, \ldots, P_g^T)^T$, where P_i is the ith flat. Any f-IFD based on D_0 consists of f foldovers of D_0, while the sign of each basic column (i.e. the factor in the set F_0 in

Theorem 2) remained unchanged in the f foldovers. The f-IFD based on D_0 consists of gf flats in the same family and thus is a (gf)-PFD. In particular, any 2-IFD based on D_0 is a $(2g)$-PFD. It can be written as $(P_1^T, P_{1'}^T, \ldots, P_g^T, P_{g'}^T)^T$ up to row permutations, where $P_{i'}$ is a foldover of P_i. It is clear that $(P_i, P_{i'}^T)^T$ is a regular design for $i = 1, \ldots, g$, while the g flats $(P_1^T, P_{1'}^T)^T, \ldots, (P_g^T, P_{g'}^T)^T$ are in the same family. Thus, the $(2g)$-PFD based on D_0 can be reduced into a g-PFD. □

Proof of Corollary 1

Proof. From Theorem 1, the f-IFD defined by the foldover matrix Γ can be merged into an $f/2$-IFD if the rows of Γ can be paired so that the product of the two rows of each pair is identical. According to the relationship between the foldover matrix and B matrix of an f-IFD, as presented in Theorem 2, an f-IFD based on B with even f, can be merged into an $f/2$-IFD if the columns of B can be paired so that the product of the two columns of each pair is identical. □

Proof of Corollary 2

Proof. As discussed in Section 1, since any initial design can be expressed as a g-PFD, every f-IFD is essentially a (gf)-PFD for some $g \leq n$. In this way, two f-IFDs based on different foldovers, say $\tilde{z}_1 = \{h_0, z_1\}$ and $\tilde{z}_2 = \{h_0, z_2\}$, are equivalent if and only if the corresponding (gf)-PFDs are equivalent. Here each $\tilde{z}_i$ is a set of f columns of length 2^u, where $u = k - log_2(n/g)$ (see Lemma 3). Note that for $i = 1, 2$, the f-CFD based on $\tilde{z}_i = \{h_0, z_i\}$ corresponds to the (gf)-PFD with flats $\tilde{s}_i = \{h_0, s_i\}$. Here $\tilde{s}_i$ is a set of gf columns of length 2^u. The columns of $\tilde{s}_i$ can be partitioned into f groups of g columns each such that the g columns $h_{p_1}, \ldots, h_{p_g}$ in any group can be obtained from the g columns $h_{q_1}, \ldots, h_{q_g}$ in any other group by multiplying a common column simultaneously, i.e. $\{h_{p_1}, \ldots, h_{p_g}\} = \{h_{q_t}h_{q_1}, \ldots, h_{q_t}h_{q_g}\}$ for some $t \in \{1, \ldots, g\}$. It is worth noting that $\tilde{s}_i$ consists of g columns from g distinct group of $\tilde{z}_i$. Then following the definition of equivalance, $\tilde{s}_1$ and $\tilde{s}_2$ belong to the same group if and only if $\tilde{z}_1$ and $\tilde{z}_2$ belong to the same group. According to the Theorem 1 of Wang and Mee (2021), two (gf)-PFDs based on $\tilde{s}_1 = \{h_0, s_1\}$ and $\tilde{s}_2 = \{h_0, s_2\}$ are equivalent (gf)-PFDs (for every given single flat) if and only if $\tilde{s}_1$ and $\tilde{s}_2$ belong to the same group. Thus, two f-IFDs based on different foldovers, $\tilde{z}_1 = \{h_0, z_1\}$ and $\tilde{z}_2 = \{h_0, z_2\}$, are equivalent if and only if $\tilde{z}_1$ and $\tilde{z}_2$ belong to the same group. □

S2: Optimal foldover matrices for the three 6-IFDs in Table 1

$$\Gamma_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\Gamma_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

S3: The initial designs of the IFDs in Tables 1–4.

Here we list the initial 16×10 designs of the IFDs in Tables 1, 2 and 3 (i.e. designs 10.44, 10,69 and 10.48), and the initial 32×15 design (denoted as H_{32}) of the 4-IFD in Table 4.

$$\text{design } 10.44: \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \end{bmatrix}$$

$$\text{design } 10.69 : \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1
\end{bmatrix}$$

$$\text{design } 10.68 : \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}$$

$$H_{32} = \begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1
\end{bmatrix}$$

S4: The initial designs of the IFDs in Tables C.1 and C.2

Here we list the the unique 12×11 strength-two design (Plackett-Burman design) labeled H_{12}, and five nonisomorphic 16×15 strength-two designs labeled $H_{16.I}, H_{16.II}$, $H_{16.III}, H_{16.IV}, H_{16.V}$.

$$H_{12} = \begin{bmatrix} 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

$$H_{16.I} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}$$

$$H_{16.II} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1
\end{bmatrix}$$

$$
H_{16.III} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}
$$

$$H_{16.IV} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}$$

$$
H_{16.V} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1
\end{bmatrix}
$$

S5: The indicator function of design 10.48

We provide the indicator function $F(x) = \sum_{V \in \mathcal{P}} b_V X_V(x)$ of design 10.48 in Sun (1993) by listing all $V \in \mathcal{P}$ with coefficient $b_V > 0$, as shown in Table S.1.

Table S.1: The indicator function of design 10.48 in Sun (1993).

V				b_V
1	5	7		8
1	5	8		8
1	5	9		8
1	5	10		-8
1	6	7		8
1	6	8		8
1	6	9		-8
1	6	10		8
2	5	7		8
2	5	8		8
2	5	9		-8
2	5	10		8
2	6	7		8
2	6	8		8
2	6	9		8
2	6	10		-8
3	5	9		8
3	5	10		8
3	6	9		8
3	6	10		8
3	7	9		8
3	7	10		-8
3	8	9		-8
3	8	10		8
4	5	9		8
4	5	10		8
4	6	9		8
4	6	10		8
4	7	9		-8
4	7	10		8
4	8	9		8
4	8	10		-8
1	2	3	4	16
1	2	5	6	16
1	2	7	8	16
1	2	9	10	16
1	3	5	7	8
1	3	5	8	-8
1	3	6	7	-8
1	3	6	8	8
1	3	7	9	8
1	3	7	10	8
1	3	8	9	8
1	3	8	10	8
1	4	5	7	-8
1	4	5	8	8
1	4	6	7	8
1	4	6	8	-8
1	4	7	9	8
1	4	7	10	8
1	4	8	9	8
1	4	8	10	8
2	3	5	7	-8
2	3	5	8	8
2	3	6	7	8
2	3	6	8	-8
2	3	7	9	8
2	3	7	10	8
2	3	8	9	8
2	3	8	10	8
2	4	5	7	8
2	4	5	8	-8
2	4	6	7	-8
2	4	6	8	8
2	4	7	9	8
2	4	7	10	8
2	4	8	9	8
2	4	8	10	8
3	4	5	6	16

V					b_V
3	4	7	8		16
3	4	9	10		16
5	6	7	8		16
5	6	9	10		16
7	8	9	10		16
1	2	3	5	9	8
1	2	3	5	10	8
1	2	3	6	9	8
1	2	3	6	10	8
1	2	3	7	9	-8
1	2	3	7	10	8
1	2	3	8	9	8
1	2	3	8	10	-8
1	2	4	5	9	8
1	2	4	5	10	8
1	2	4	6	9	8
1	2	4	6	10	8
1	2	4	7	9	8
1	2	4	7	10	-8
1	2	4	8	9	-8
1	2	4	8	10	8
1	3	4	5	7	8
1	3	4	5	8	8
1	3	4	5	9	-8
1	3	4	5	10	8
1	3	4	6	7	8
1	3	4	6	8	8
1	3	4	6	9	8
1	3	4	6	10	-8
1	5	7	8	9	-8
1	5	7	8	10	8
1	5	7	9	10	8
1	5	8	9	10	8
1	6	7	8	9	8
1	6	7	8	10	-8
1	6	7	9	10	8
1	6	8	9	10	8
2	3	4	5	7	8
2	3	4	5	8	8
2	3	4	5	9	8
2	3	4	5	10	-8
2	3	4	6	7	8
2	3	4	6	8	8
2	3	4	6	9	-8
2	3	4	6	10	8
2	5	7	8	9	8
2	5	7	8	10	-8
2	5	7	9	10	8
2	5	8	9	10	8
2	6	7	8	9	-8
2	6	7	8	10	8
2	6	7	9	10	8
2	6	8	9	10	8
3	5	6	7	9	-8
3	5	6	7	10	8
3	5	6	8	9	8
3	5	6	8	10	-8
3	5	7	8	9	8
3	5	7	8	10	8
3	6	7	8	9	8
3	6	7	8	10	8
4	5	6	7	9	8
4	5	6	7	10	-8
4	5	6	8	9	-8
4	5	6	8	10	8
4	5	7	8	9	8
4	5	7	8	10	8
4	6	7	8	9	8
4	6	7	8	10	8

V								b_V
1	3	5	6	7	9			8
1	3	5	6	7	10			8
1	3	5	6	8	9			8
1	3	5	6	8	10			8
1	3	5	7	9	10			-8
1	3	5	8	9	10			8
1	3	6	7	9	10			8
1	3	6	8	9	10			-8
1	4	5	6	7	9			8
1	4	5	6	7	10			8
1	4	5	6	8	9			8
1	4	5	6	8	10			8
1	4	5	7	9	10			8
1	4	5	8	9	10			-8
1	4	6	7	9	10			-8
1	4	6	8	9	10			8
2	3	5	6	7	9			8
2	3	5	6	7	10			8
2	3	5	6	8	9			8
2	3	5	6	8	10			8
2	3	5	7	9	10			8
2	3	5	8	9	10			-8
2	3	6	7	9	10			-8
2	3	6	8	9	10			8
2	4	5	6	7	9			8
2	4	5	6	7	10			8
2	4	5	6	8	9			8
2	4	5	6	8	10			8
2	4	5	7	9	10			-8
2	4	5	8	9	10			8
2	4	6	7	9	10			8
2	4	6	8	9	10			-8
1	2	3	5	6	7	9		8
1	2	3	5	6	7	10		-8
1	2	3	5	6	8	9		-8
1	2	3	5	6	8	10		8
1	2	3	5	7	8	9		8
1	2	3	5	7	8	10		8
1	2	3	6	7	8	9		8
1	2	3	6	7	8	10		8
1	2	4	5	6	7	9		-8
1	2	4	5	6	7	10		8
1	2	4	5	6	8	9		8
1	2	4	5	6	8	10		-8
1	2	4	5	7	8	9		8
1	2	4	5	7	8	10		8
1	2	4	6	7	8	9		8
1	2	4	6	7	8	10		8
1	3	4	5	7	8	9		8
1	3	4	5	7	8	10		-8
1	3	4	5	7	9	10		8
1	3	4	5	8	9	10		8
1	3	4	6	7	8	9		-8
1	3	4	6	7	8	10		8
1	3	4	6	7	9	10		8
1	3	4	6	8	9	10		8
2	3	4	5	7	8	9		-8
2	3	4	5	7	8	10		8
2	3	4	5	7	9	10		8
2	3	4	5	8	9	10		8
2	3	4	6	7	8	9		8
2	3	4	6	7	8	10		-8
2	3	4	6	7	9	10		8
2	3	4	6	8	9	10		8
1	2	3	4	5	6	7	8	16
1	2	3	4	5	6	9	10	16
1	2	3	4	7	8	9	10	16
1	2	5	6	7	8	9	10	16
3	4	5	6	7	8	9	10	16