Statistical Inference for Local Granger Causality

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The supplement details the proofs in the paper. The structure of the Supplement is outlined as follows:

- Section S1 reviews the basic properties of multivariate locally stationary processes by observing two lemmas: Lemmas S1 and S2.
- Section S2 summarizes the asymptotic distribution of the functionals of the pre-periodogram matrix from the multivariate locally stationary process.
- Section S3 provides proofs of main results in Section 3 in the paper.

S1 Multivariate locally stationary processes

In Section S1, we review the basic properties of multivariate locally stationary processes. Let C be a generic constant. For the function l in (1.1), we have the following inequality

$$\sum_{j=-\infty}^{\infty} \frac{1}{l(j)l(j+s)} \leq \frac{C}{l(s)}, \tag{S1.1}$$

which is repetitively used in the proof.

Let $\boldsymbol{X}_{t,T}^{(d)}$ be the dth element of the vector $\boldsymbol{X}_{t,T}$. From (2.2), $\boldsymbol{X}_{t,T}^{(d)}$ has the expression $\boldsymbol{X}_{t,T}^{(d)} = \sum_{j=-\infty}^{\infty} \sum_{m=1}^{p} A_{t,T}(j)_{dm} \boldsymbol{\epsilon}_{t-j}^{(m)}$, where $A_{t,T}(j)_{dm}$ denotes the (d,m)-element of the coefficient matrix $A_{t,T}(j)$. In view of this expression, we obtain

$$\begin{aligned}
&\operatorname{Cov}(\boldsymbol{X}_{t,T}^{(a)}, \boldsymbol{X}_{t+s,T}^{(b)}) \\
&= \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m,n=1}^{p} A_{t,T}(j)_{am} A_{t+s,T}(l)_{bn} \operatorname{Cov}\Big(\boldsymbol{\epsilon}_{t-j}^{(m)}, \boldsymbol{\epsilon}_{t+s-l}^{(n)}\Big) \\
&= \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} A_{t,T}(j)_{am} \mathcal{K}_{mn} A_{t+s,T}(j+s)_{bn} \qquad \text{(S1.2)} \\
&= \sum_{j=-\infty}^{\infty} \Big(A_{t,T}(j) \mathcal{K} A_{t+s,T}(j)^{\top}\Big)_{ab}. \qquad \text{(S1.3)}
\end{aligned}$$

On the other hand, the autovariance function $\boldsymbol{\gamma}(u,s)$ at u is

$$\boldsymbol{\gamma}(u,s) = \int_{-\pi}^{\pi} \boldsymbol{f}(u,\lambda) \exp(\mathrm{i}\lambda s)\, \mathrm{d}\lambda = \sum_{j=-\infty}^{\infty} A(u,j) \mathcal{K} A(u, j+s)^{\top}. \qquad \text{(S1.4)}$$

Especially, the (a, b)-element of the matrix $\boldsymbol{\gamma}$ is bounded by

$$\begin{aligned}
|\boldsymbol{\gamma}(u,s)_{ab}| &\leq \sum_{j=-\infty}^{\infty} \sup_u \|A(u,j)\|_\infty \|\mathcal{K}\|_\infty \sup_u \|A(u,j+s)^{\top}\|_\infty \\
&\leq C \sum_{j=-\infty}^{\infty} \frac{1}{l(j) l(j+s)} \leq \frac{C}{l(s)}, \qquad \text{(S1.5)}
\end{aligned}$$

where the second inequality follows from Assumption 1 (i) and the third inequality follows from (S1.1).

Remark S1. From (S1.5), we can see that for any fixed $u \in [0,1]$ and any

(a,b)-element of the autovariance matrix, $|\boldsymbol{\gamma}(u,0)_{ab}|$ is bounded, i.e.,

$$|\boldsymbol{\gamma}(u,0)_{ab}| \leq C.$$

Thus, the time-varying spectral densities $f(u,\lambda)_{jk}$ $(1 \leq j,k \leq p)$ are square-integrable for any fixed $u \in [0,1]$.

We first evaluate the difference between (S1.3) and (S1.4) on discrete points $u_k = k/T$ in the following.

Lemma S1. *Under Assumption 1, we have*

$$\sum_{k=1}^{T} \Big| \operatorname{Cov}\Big(\boldsymbol{X}^{(a)}_{[k+1/2-s/2],T}, \boldsymbol{X}^{(b)}_{[k+1/2+s/2],T}\Big) - \boldsymbol{\gamma}(u_k,s)_{ab} \Big| \leq C\Big(1+\frac{1}{l(s)}\Big). \quad \text{(S1.6)}$$

Proof. To evaluate (S1.6), we use the expressions (S1.2) and (S1.4). Note that

$$\begin{aligned}
&\sum_{k=1}^{T} \Big| \operatorname{Cov}\Big(\boldsymbol{X}^{(a)}_{[k+1/2-s/2],T}, \boldsymbol{X}^{(b)}_{[k+1/2+s/2],T}\Big) - \boldsymbol{\gamma}(u_k,s)_{ab} \Big| \\
\leq &\sum_{k=1}^{T} \Big| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} \Big(A_{[k+1/2-s/2],T}(j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} - A(u_k,j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} \Big) \Big| \\
&+ \sum_{k=1}^{T} \Big| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} \Big(A(u_k,j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} - A(u_k,j)_{am} \mathcal{K}_{mn} A(u_k,j+s)_{bn} \Big) \Big|.
\end{aligned} \quad \text{(S1.7)}$$

Considering the first term in the right hand side, we have

$$\begin{aligned}
&\sum_{k=1}^{T} \Big| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} \Big(A_{[k+1/2-s/2],T}(j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} - A(u_k,j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} \Big) \Big| \\
= &\sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \Big| \sum_{m,n=1}^{p} \Big(A_{[k+1/2-s/2],T}(j) - A(u_k,j) \Big)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} \Big| \\
\leq &\sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} \Big| \Big(A_{[k+1/2-s/2],T}(j) - A(u_k,j) \Big)_{am} \Big| |\mathcal{K}_{mn}| \, \Big| A_{[k+1/2+s/2],T}(j+s)_{bn} \Big|
\end{aligned}$$

$$\leq \sum_{j=-\infty}^{\infty} \frac{C_{\mathcal{K}} C_A^2}{l(j+s)} + \frac{C_{\mathcal{K}} C_A^2}{l(j) l(j+s)}$$

$$\leq C C_{\mathcal{K}} C_A^2, \tag{S1.8}$$

where the first inequality follows from $|\mathcal{K}_{mn}| \leq C_{\mathcal{K}}$, $\Big|A_{k+1/2+s/2,T}(j+s)_{bn}\Big| \leq C_A/l(j+s)$ from (2.3), and

$$\sum_{k=1}^{T} \sum_{m,n=1}^{p} \Big|\Big(A_{[k+1/2-s/2],T}(j) - A(u_k, j)\Big)_{am}\Big|$$

$$\leq \sum_{k=1}^{T} \sum_{m,n=1}^{p} \Big|\Big(A_{[k+1/2-s/2],T}(j) - A([k+1/2-s/2]/T, j)\Big)_{am}\Big| + \Big|\Big(A([k+1/2-s/2]/T, j) - A(u_k, j)\Big)_{am}\Big|$$

$$\leq C_A + \frac{C_A}{l(j)},$$

where the second inequality follows from (ii) and (iii) in Assumption 1.

Also, it holds that

$$\sum_{k=1}^{T} \Big| \sum_{j=-\infty}^{\infty} \sum_{m,n=1}^{p} \Big(A(u_k, j)_{am} \mathcal{K}_{mn} A_{[k+1/2+s/2],T}(j+s)_{bn} - A(u_k, j)_{am} \mathcal{K}_{mn} A(u_k, j+s)_{bn}\Big)\Big|$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \Big| \sum_{m,n=1}^{p} A(u_k, j)_{am} \mathcal{K}_{mn} \Big(A_{[k+1/2+s/2],T}(j+s) - A(u_k, j+s)\Big)_{bn}\Big|$$

$$\leq \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} \Big|A(u_k, j)_{am}\Big| |\mathcal{K}_{mn}| \Big|\Big(A_{[k+1/2+s/2],T}(j+s) - A([k+1/2+s/2]/T, j+s)\Big)_{bn}\Big|$$

$$+ \sum_{j=-\infty}^{\infty} \sum_{k=1}^{T} \sum_{m,n=1}^{p} \Big|A(u_k, j)_{am}\Big| |\mathcal{K}_{mn}| \Big|\Big(A([k+1/2+s/2]/T, j+s) - A(u_k, j+s)\Big)_{bn}\Big|$$

$$\leq \sum_{j=-\infty}^{\infty} \frac{C_{\mathcal{K}} C_A^2}{l(j)} + \sum_{j=-\infty}^{\infty} \frac{C_{\mathcal{K}} C_A}{l(j)} \sum_{k=1}^{T} \|A([k+1/2+s/2]/T, j+s) - A(u_k, j+s)\|_{\infty}$$

$$\leq C C_{\mathcal{K}} C_A^2 + \sum_{j=-\infty}^{\infty} \frac{C_{\mathcal{K}} C_A^2}{l(j) l(j+s)}$$

$$\leq C C_{\mathcal{K}} C_A^2 \Big(1 + \frac{1}{l(s)}\Big), \tag{S1.9}$$

where the last inequality follows from (S1.1). Combining (S1.7), (S1.8) and (S1.9), we obtain the desired result. □

Generally, higher-order cumulants of the locally stationary process $\{\boldsymbol{X}_{t,T}\}$ can be approximated by those of the stationary process $\{\boldsymbol{X}^*(u)\}$ for $u = t/T$ under Assumption 1 in a similar manner as the autocovariance. To discuss higher-order cumulants, we introduce the notation $\boldsymbol{X}(u; s)$, which means the observation $\boldsymbol{X}(s)$, $s \in \mathbb{Z}$, of the stationary process $\boldsymbol{X}^*(u)$.

Let $\gamma_{a_1,\dots,a_q}(u;\, t_1, \dots, t_{q-1})$ be the joint cumulant function of order q, i.e.,

$$
\begin{aligned}
&\gamma_{a_1,\dots,a_q}(u;\, t_1, \dots, t_{q-1}) \\
&:= \operatorname{cum}\{\boldsymbol{X}^{(a_1)}(u; t+t_1), \boldsymbol{X}^{(a_2)}(u; t+t_2), \cdots, \boldsymbol{X}^{(a_2)}(u; t+t_{q-1}), \boldsymbol{X}^{(a_q)}(u; t)\}.
\end{aligned}
$$

To discriminate this notation from the autocovariance function, we do not use γ in boldface, although it is an extension of the autocovariance function to higher-orders.

Lemma S2. *Under Assumption 1, we have*

$$
\begin{aligned}
\sum_{k=1}^{T} \Big| \operatorname{cum}(\boldsymbol{X}^{(a_1)}_{k+t_1,T}, \boldsymbol{X}^{(a_2)}_{k+t_2,T}, \cdots \boldsymbol{X}^{(a_{q-1})}_{k+t_{q-1},T}, \boldsymbol{X}^{(a_q)}_{k,T}) - \gamma_{a_1,\dots,a_q}(u_k;\, t_1, \dots, t_{q-1}) \Big| \\
\leq C \sum_{m=-\infty}^{\infty} \left(\sum_{i=1}^{q} \frac{l(m+t_i)}{\prod_{j=1}^{q} l(m+t_j)} + \frac{1}{\prod_{j=1}^{q} l(m+t_j)} \right),
\end{aligned}
$$

where C is a generic constant and $t_q = 0$.

Remark S2. Lemma S1 is a special case of Lemma S2 when $q = 2$.

Proof. Under Assumption 1, there exists a constant $\tilde{C}^{(r)}_{\epsilon}$ such that all cumulants of order r are all bounded by $\tilde{C}^{(r)}_{\epsilon}$, since all cumulants can be written in terms of polynomials of moments. Now, it holds that

$$
\begin{aligned}
&\left|\mathrm{cum}(\boldsymbol{X}^{(a_1)}_{k+t_1,T}, \boldsymbol{X}^{(a_2)}_{k+t_2,T}, \cdots, \boldsymbol{X}^{(a_{q-1})}_{k+t_{q-1},T}, \boldsymbol{X}^{(a_q)}_{k,T}) - \gamma_{a_1,\dots,a_q}(u_k; t_1, \dots, t_{q-1})\right| \\
\leq\ & \Big| \sum_{j_1,\dots,j_q=-\infty}^{\infty} \sum_{m_1,\dots,m_q=1}^{p} \Big(A_{k+t_1,T}(j_1)_{a_1m_1} \cdots A_{k,T}(j_q)_{a_qm_q} - A(u_k, j_1)_{a_1m_1} \cdots A(u_k, j_q)_{a_qm_q}\Big) \\
&\qquad \times \mathrm{cum}\Big(\boldsymbol{\epsilon}^{(m_1)}_{k+t_1-j_1}, \boldsymbol{\epsilon}^{(m_2)}_{k+t_2-j_2}, \cdots, \boldsymbol{\epsilon}^{(m_q)}_{k-j_q}\Big)\Big| \\
\leq\ & \tilde{C}^{(q)}_{\epsilon} \sum_{j_q=-\infty}^{\infty} \sum_{m_1,\dots,m_q=1}^{p} \Big|A_{k+t_1,T}(j_q+t_1)_{a_1m_1} \cdots A_{k,T}(j_q)_{a_qm_q} - A(u_k, j_q+t_1)_{a_1m_1} \cdots A(u_k, j_q)_{a_qm_q}\Big| \\
\leq\ & \tilde{C}^{(q)}_{\epsilon} \sum_{j_q=-\infty}^{\infty} \|A_{k+t_1,T}(j_q+t_1)_{a_1m_1} \cdots A_{k,T}(j_q)_{a_qm_q} - A(u_k, j_q+t_1)_{a_1m_1} A_{k+t_2,T}(j_q+t_2)_{a_qm_q} \cdots A_{k,T}(j_q)_{a_qm_q}\|_\infty \\
& + \|A(u_k, j_q+t_1)_{a_1m_1} A_{k+t_2,T}(j_q+t_2)_{a_2m_2} \cdots A_{k,T}(j_q)_{a_qm_q} - A(u_k, j_q+t_1)_{a_1m_1} A(u_k, j_q+t_2)_{a_2m_2} \cdots A_{k,T}(j_q)_{a_qm_q}\|_\infty \\
& + \|A(u_k, j_q+t_1)_{a_1m_1} \cdots A_{k,T}(j_q)_{a_qm_q} - A(u_k, j_q+t_1)_{a_1m_1} \cdots A(u_k, j_q)_{a_qm_q}\|_\infty
\end{aligned}
$$

Note that, for $1 \leq i \leq q-1$, we have

$$
\begin{aligned}
& \sum_{k=1}^{T} \|A_{k+t_i,T}(j_q+t_i) - A(u_k, j_q+t_i)\|_\infty \\
\leq\ & \sum_{k=1}^{T} \Big(\|A_{k+t_i,T}(j_q+t_i) - A(u_{k+t_i}, j_q+t_i)\|_\infty + \|A(u_{k+t_i}, j_q+t_i) - A(u_k, j_q+t_i)\|_\infty\Big) \\
\leq\ & C_A + \frac{C_A}{l(j_q+t_i)}.
\end{aligned}
$$

Thus, it holds

$$
\begin{aligned}
& \sum_{k=1}^{T} \left|\mathrm{cum}(\boldsymbol{X}^{(a_1)}_{k+t_1,T}, \boldsymbol{X}^{(a_2)}_{k+t_2,T}, \cdots \boldsymbol{X}^{(a_{q-1})}_{k+t_{K-1},T}, \boldsymbol{X}^{(a_q)}_{k,T}) - \gamma_{a_1,\dots,a_1}(u_k; t_1, \dots, t_{q-1})\right| \\
& \leq \tilde{C}^{(q)}_{\epsilon} C_A^q \sum_{j_q=-\infty}^{\infty} \left(\sum_{i=1}^{q-1} \frac{l(j_q+t_i)}{l(j_q)\prod_{j=1}^{q-1} l(j_q+t_j)} + \frac{1}{l(j_q)\prod_{j=1}^{q-1} l(j_q+t_j)} + \frac{1}{\prod_{j=1}^{q-1} l(j_q+t_j)}\right)
\end{aligned}
$$

We obtain the conclusion if we replace j_q with m and set $t_q = 0$. $\square$

S2 Asymptotic distribution

In Section S2, we consider the asymptotic distribution of the empirical spectral process for multivariate locally stationary processes.

We first impose the following assumptions on the matrix-valued functions $\boldsymbol{\phi}$, which is to be considered later. Let $V_2(\cdot)$ be the total variation of bivariate functions, i.e.,

$$V_2(f) = \sup\Big\{\sum_{k,l=1}^{m,n} |f(u_k, \lambda_l) - f(u_{k-1}, \lambda_l) - f(u_k, \lambda_{l-1}) + f(u_{k-1}, \lambda_{l-1})|; \\ 0 \le u_0 < \cdots < u_m \le 1,\, 0 \le \lambda_0 < \cdots < \lambda_n \le \pi;\; m, n \in \mathbb{N}\Big\}.$$

Let Ψ be a class of square-integrable functions, where the $\mathcal{L}_2$-norm on $\psi \in \Psi$ is defined as

$$\|\psi\|_{\mathcal{L}_2}^2 = \int_0^1 \int_{-\pi}^{\pi} |\psi(u, \lambda)|^2 \,\mathrm{d}\lambda \,\mathrm{d}u < \infty.$$

The class Ψ is considered for the elements of the matrix $\boldsymbol{\phi}$.

For any class $\Phi := \{\boldsymbol{\phi} \in \mathbb{R}^{p\times p}; \phi_{ij} \in \Psi \text{ for } i, j = 1, \dots, p\}$, let $\tau_{\infty,\mathrm{TV}}$, $\tau_{\mathrm{TV},\infty}$, $\tau_{\mathrm{TV},\mathrm{TV}}$ and $\tau_{\infty,\infty}$ be

$$\tau_{\infty,\mathrm{TV}} := \tau_{\infty,\mathrm{TV}}(\Phi) = \sup_{\phi\in\Phi} \max_{1\le i,j\le p} \sup_{u\in[0,1]} V\big(\phi_{ij}(u,\cdot)\big), \quad \tau_{\mathrm{TV},\infty} := \tau_{\mathrm{TV},\infty}(\Phi) = \sup_{\phi\in\Phi} \max_{1\le i,j\le p} \sup_{\lambda\in[0,\pi]} V\big(\phi_{ij}(\cdot,\lambda)\big),$$

$$\tau_{\mathrm{TV},\mathrm{TV}} := \tau_{\mathrm{TV},\mathrm{TV}}(\Phi) = \sup_{\phi\in\Phi} \max_{1\le i,j\le p} V_2\big(\phi_{ij}\big), \qquad \tau_{\infty,\infty} := \tau_{\infty,\infty}(\Phi) = \sup_{\phi\in\Phi} \max_{1\le i,j\le p} \sup_{\substack{u\in[0,1]\\ \lambda\in[0,\pi]}} |\phi_{ij}|.$$

Assumption S1. *Let Φ be a class of $p \times p$ matrix-valued continuous functions $\boldsymbol{\phi}(u,\cdot)$ on $[-\pi,\pi]$ such that for any $\boldsymbol{\phi} \in \Phi$, it holds that* (i) $\boldsymbol{\phi}(u,\cdot) =$

$\boldsymbol{\phi}^*(u,\cdot)$ *for any fixed* $u \in [0,1]$*; and* (ii) $\tau_{\infty,\mathrm{TV}}$*,* $\tau_{\mathrm{TV},\infty}$*,* $\tau_{\mathrm{TV},\mathrm{TV}}$ *and* $\tau_{\infty,\infty}$ *are all finite.*

For any function $\psi \in \Psi$, let ψ_T be

$$\psi_T(u,\lambda) = \frac{1}{b_T}\psi\Big(\frac{u}{b_T},\lambda\Big), \tag{S2.10}$$

where $b := b_T \to 0$ as $T \to \infty$. Let Ψ_T denotes the function class constituted by ψ_T, i.e.,

$$\Psi_T = \{\psi_T(u,\lambda) = \frac{1}{b_T}\psi\Big(\frac{u}{b_T},\lambda\Big);\ \psi \in \mathcal{L}_2\}. \tag{S2.11}$$

Assumption S2. *For any* $\psi \in \Psi$*, let* $\psi(\cdot,\lambda)$ *be a positive, symmetric function of bounded variation such that* $\psi(\cdot,\lambda)$ *has a compact support on* $[-1,1]$.

Let $\mathscr{A}_T(u)$ and $\bar{\mathscr{A}_T}(u)$ be

$$\mathscr{A}_T(u)_{ab} := \mathscr{A}_T(u;\psi)_{ab} = \frac{1}{T}\sum_{k=1}^{T}\int_{-\pi}^{\pi}\psi_T(u-u_k,\lambda)\boldsymbol{I}_T(u_k,\lambda)_{ab}\,\mathrm{d}\lambda, \tag{S2.12}$$

$$\bar{\mathscr{A}_T}(u)_{ab} := \bar{\mathscr{A}_T}(u;\psi)_{ab} = \frac{1}{T}\sum_{k=1}^{T}\int_{-\pi}^{\pi}\psi_T(u-u_k,\lambda)\boldsymbol{f}(u_k,\lambda)_{ab}\,\mathrm{d}\lambda. \tag{S2.13}$$

The empirical spectral process $\xi_T(u)_{ab}$ is

$$\xi_T(u)_{ab} := \xi_T(u;\psi)_{ab} = \sqrt{Tb_T}\big(\mathscr{A}_T(u;\psi) - \bar{\mathscr{A}_T}(u;\psi)\big)_{ab}. \tag{S2.14}$$

We use the first expression in (S2.12), (S2.13) and (S2.14) when there is no confusion with ψ.

S2.1 Preliminary Computations

Let $\hat{\psi}$ be $\hat{\psi}(u,k) = \int_{-\pi}^{\pi} \psi(u,\lambda)\exp(-\mathrm{i}k\lambda)\,\mathrm{d}\lambda$.

Lemma S3. *Let β_T be a sequence of positive numbers such that $\beta_T \to 0$ as $T \to \infty$. Suppose*

$$\limsup_{T\to\infty} \ \beta_T \sum_{s=-T}^{T} \sup_u |\hat{\psi}(u,-s)| < \infty. \tag{S2.15}$$

Then, it holds that

$$|E\mathscr{A}_T(u)_{ab} - \bar{\mathscr{A}}_T(u)_{ab}| = O(T^{-1}b_T^{-1}\beta_T^{-1}).$$

Remark S3. The condition

$$\sum_{s=-\infty}^{\infty} \sup_u |\hat{\psi}(u,-s)| < \infty \tag{S2.16}$$

satisfies (S2.15). However, if $\psi(u,\cdot)$ is only a function of bounded variation, then ψ may not satisfy the condition (S2.16). Under (S2.15), we see that

$$\sum_{s=-T}^{T} \sup_u |\hat{\psi}(u,-s)| = O(\beta_T^{-1}),$$

which we use in the following evaluations.

Proof. From (3.16), we have

$$\boldsymbol{I}_T(u,\lambda)_{ab} = \frac{1}{2\pi} \sum_{\ell:1\le[uT+1/2\pm\ell/2]\le T} \boldsymbol{X}^{(a)}_{[uT+1/2+\ell/2],T}\boldsymbol{X}^{(b)}_{[uT+1/2-\ell/2],T} \exp(-\mathrm{i}\lambda\ell). \tag{S2.17}$$

In expression (S2.17), ℓ depends on u, but it can be naturally extended to

$$\boldsymbol{I}_T(u,\lambda)_{ab} = \frac{1}{2\pi}\sum_{\ell=1-T}^{T-1} \boldsymbol{X}^{(a)}_{[uT+1/2+\ell/2],T}\boldsymbol{X}^{(b)}_{[uT+1/2-\ell/2],T}\exp(-\,\mathrm{i}\lambda\ell), \quad \text{(S2.18)}$$

if we let $\boldsymbol{X}_{m,T} \equiv 0$ for any $m \leq 0$ or $m \geq T+1$.

We shall use this expression (S2.18) in the following proof. By Parseval's identity, it holds that

$$\begin{aligned}
& |E\mathscr{A}_T(u)_{ab} - \bar{\mathscr{A}}_T(u)_{ab}| \\
\leq\; & \Big|\frac{1}{T}\sum_{k=1}^{T}\int_{-\pi}^{\pi}\psi_T(u-u_k,\lambda)\big(E\boldsymbol{I}_T(u_k,\lambda)_{ab} - \boldsymbol{f}(u_k,\lambda)_{ab}\big)\,\mathrm{d}\lambda\Big| \\
=\; & \Big|\frac{1}{2\pi T}\sum_{k=1}^{T}\Big(\sum_{s=1-T}^{T-1}\hat{\psi}_T(u-u_k,-s)\Big(E\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T}\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T} - \boldsymbol{\gamma}(u_k,-s)_{ab}\Big) \\
& \qquad + \sum_{|s|\geq T}\hat{\psi}_T(u-u_k,-s)\boldsymbol{\gamma}(u_k,-s)_{ab}\Big)\Big| \\
\leq\; & \frac{1}{2\pi T}\Big|\sum_{k=1}^{T}\sum_{s=1-T}^{T-1}\hat{\psi}_T(u-u_k,-s)\Big(\operatorname{Cov}(\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T},\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T}) - \boldsymbol{\gamma}(u_k,-s)_{ab}\Big)\Big| \\
& \qquad + \frac{1}{2\pi T}\sum_{k=1}^{T}\Big|\sum_{|s|\geq T}\hat{\psi}_T(u-u_k,-s)\boldsymbol{\gamma}(u_k,-s)_{ab}\Big| \\
:=\; & B_1 + B_2, \quad \text{(say)}.
\end{aligned}$$

By Lemma S1, it holds that

$$\begin{aligned}
B_1 \;&\leq\; \frac{1}{2\pi b_T}\sum_{s=1-T}^{T-1}\sup_u|\hat{\psi}(u,-s)|\Big|\frac{1}{T}\sum_{k=1}^{T}\operatorname{Cov}(\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T},\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T}) - \boldsymbol{\gamma}(u_k,-s)_{ab}\Big| \\
&\leq\; \frac{C}{2\pi b_T T}\sum_{s=1-T}^{T-1}\sup_u|\hat{\psi}(u,-s)|\Big(1+\frac{1}{l(s)}\Big) \\
&=\; O(T^{-1}b_T^{-1}\beta_T^{-1}).
\end{aligned}$$

Further, noting (S1.5), we have

$$\begin{aligned} B_2 &\leq \frac{1}{2\pi b_T} \sup_u \sum_{|s| \geq T} |\hat{\psi}(u,-s)||\boldsymbol{\gamma}(u,-s)_{ab}| \\ &\leq \frac{C}{b_T \beta_T} \sum_{|s| \geq T} \frac{C}{l(s)} \\ &= O(T^{-1} b_T^{-1} \beta_T^{-1}), \end{aligned}$$

since $\{l(j)^{-1}\}_{j \in \mathbb{N}}$ is a convergent series, and $|l(s)| \geq T$ for $|s| \geq T$. Therefore, we obtain the assertion. □

Next, we evaluate the higher-order cumulants of $\xi_T(u)$. We first clarify the bias between those of the time-varying process $\{\boldsymbol{X}_{t,T}\}$ and those of the approximate stationary process $\{\boldsymbol{X}(u,t)\}$, and then evaluate the higher order cumulants of the stationary process.

Lemma S4. *Let β_T be a sequence of positive numbers such that $\beta_T \to 0$ as $T \to \infty$. Suppose $\psi^{(1)}(\cdot,\lambda)$, …, $\psi^{(q)}(\cdot,\lambda)$ are all functions of bounded variation and satisfy Assumption S2 and*

$$\limsup_{T \to \infty} \beta_T \sum_{s=-T}^{T} \sup_u |\hat{\psi}^{(i)}(u,-s)| < \infty, \qquad \text{for } i = 1, \dots, q. \tag{S2.19}$$

If $b_T \to 0$, $Tb_T \to \infty$ and $T^{-q/2}\beta_T^{-1} \to 0$ as $T \to \infty$, then it holds that

$$\mathrm{cum}\big(\xi_T(u^{(1)};\psi^{(1)})_{a_1 b_1}, \cdots, \xi_T(u^{(q)};\psi^{(q)})_{a_q b_q}\big) = O(T^{1-q/2} b_T^{1-q/2}).$$

Especially, when $q = 2$, we have

$$
\begin{aligned}
&\lim_{T\to\infty} \mathrm{Cov}\big(\xi_T(u^{(1)};\psi^{(1)})_{a_1b_1}, \xi_T(u^{(2)};\psi^{(2)})_{a_2b_2}\big) = \\
&2\pi\delta(u^{(1)},u^{(2)})\Bigg(\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda)\overline{\psi^{(2)}(v,\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(1)},\lambda)_{a_1a_2}\overline{\boldsymbol{f}(u^{(1)},\lambda)}_{b_1b_2}\,\mathrm{d}\lambda \\
&\qquad + \int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda)\overline{\psi^{(2)}(v,-\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(1)},\lambda)_{a_1b_2}\overline{\boldsymbol{f}(u^{(1)},\lambda)}_{b_1a_2}\,\mathrm{d}\lambda \\
&+ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda_1)\overline{\psi^{(2)}(v,-\lambda_2)}\,\mathrm{d}v\Big)\tilde{\gamma}_{a_1a_2b_1b_2}(u^{(1)};\lambda_1,\lambda_2,-\lambda_2)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\Bigg),
\end{aligned}
\tag{S2.20}
$$

where $\tilde{\gamma}$ is the fourth-order spectral density of the process.

Remark S4. The sequence β_T is used to alleviate the divergence of the harmonic series. There exists a sequence β_T such that $\beta_T^{-1} = O(\ln T)$ (See Remark S5 below for details). For this sequence, the condition $T^{-q/2}\beta_T^{-1} \to 0$ always holds true for $q \geq 2$.

Proof. Using the expression (S2.18), we have $\mathscr{A}_T(u)_{ab}$ as

$$
\mathscr{A}_T(u)_{ab} = \frac{1}{2\pi T}\sum_{k=1}^{T}\sum_{s=1-T}^{T-1}\hat{\psi}_T(u-u_k,-s)\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T}\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T},
\tag{S2.21}
$$

which is a linear combination of $\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T}\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T}$. We apply Lemma S2 to compute the higher order cumulants. Actually, it holds that

$$
\begin{aligned}
&\mathrm{cum}\big(\xi_T(u^{(1)};\psi^{(1)})_{a_1b_1},\cdots,\xi_T(u^{(q)};\psi^{(q)})_{a_qb_q}\big) \\
&= \mathrm{cum}\Big(\frac{1}{T}\sum_{\kappa_1}\boldsymbol{X}^{(a_1)}_{\kappa_1,T}\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T}, \frac{1}{T}\sum_{\kappa_2}\boldsymbol{X}^{(a_2)}_{\kappa_2,T}\boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T},\dots,\frac{1}{T}\sum_{\kappa_q}\boldsymbol{X}^{(a_q)}_{\kappa_q,T}\boldsymbol{X}^{(b_q)}_{\kappa_q-s_q,T}\Big)
\end{aligned}
$$

$$= \frac{1}{T^q} \sum_{\kappa_1,\ldots,\kappa_q} \mathrm{cum}\Big(\boldsymbol{X}^{(a_1)}_{\kappa_1,T}\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T}, \ldots, \boldsymbol{X}^{(a_q)}_{\kappa_q,T}\boldsymbol{X}^{(b_q)}_{\kappa_q-s_q,T}\Big),$$

where for brevity, we let

$$\kappa_1 := [k_1{+}1/2{+}s_1/2], \quad \kappa_2 := [k_2{+}1/2{+}s_2/2], \quad \ldots, \quad \kappa_q := [k_q{+}1/2{+}s_q/2].$$

To compute higher order cumulants, we have to consider all indecomposable partitions of the following table (See Brillinger (1981), Theorem 2.3.2):

$$\begin{array}{cc} \boldsymbol{X}^{(a_1)}_{\kappa_1,T} & \boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T} \\ \boldsymbol{X}^{(a_2)}_{\kappa_2,T} & \boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T} \\ \vdots & \vdots \\ \boldsymbol{X}^{(a_q)}_{\kappa_q,T} & \boldsymbol{X}^{(b_q)}_{\kappa_q-s_q,T} \end{array}.$$

In view of Lemma S2 with some tedious computation, all indecomposable partitions can be approximated by those cumulants of the stationary process with a bias of lower order for a fixed $q \geq 2$.

We give a representative example of a partition below. The other partitions can be evaluated in the same manner. Without loss of generality, let q be odd. Suppose we evaluate the following cumulant:

$$\frac{1}{T^q} \sum_{\kappa_1,\ldots,\kappa_q} \mathrm{cum}\Big(\boldsymbol{X}^{(a_1)}_{\kappa_1,T}, \boldsymbol{X}^{(a_2)}_{\kappa_2,T}\Big) \mathrm{cum}\Big(\boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T}, \boldsymbol{X}^{(b_3)}_{\kappa_3-s_3,T}\Big) \cdots \mathrm{cum}\Big(\boldsymbol{X}^{(a_q)}_{\kappa_q,T}, \boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T}\Big).$$

If we replace variables $\kappa_2, \ldots, \kappa_q$ with $\tau_2 := \kappa_2 - \kappa_1, \ldots, \tau_q := \kappa_q - \kappa_1$, then we have

$$\frac{1}{T^q}\sum_{\kappa_1,\tau_2,\dots,\tau_q}\mathrm{cum}\Big(\boldsymbol{X}^{(a_1)}_{\kappa_1,T},\boldsymbol{X}^{(a_2)}_{\kappa_1+\tau_2,T}\Big)$$
$$\mathrm{cum}\Big(\boldsymbol{X}^{(b_2)}_{\kappa_1+\tau_2-s_2,T},\boldsymbol{X}^{(b_3)}_{\kappa_1+\tau_3-s_3,T}\Big)\cdots\mathrm{cum}\Big(\boldsymbol{X}^{(a_q)}_{\kappa_1+\tau_q,T},\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T}\Big). \quad \text{(S2.22)}$$

Applying Lemma S2, (S2.22) can be approximated by

$$\frac{1}{T^q}\sum_{\kappa_1=1}^{T}\sum_{\tau_2,\cdots,\tau_q}\boldsymbol{\gamma}(u_{\kappa_1},\tau_2)_{a_1a_2}\boldsymbol{\gamma}(u_{\kappa_1},\tau_3-s_3-\tau_2+s_2)_{b_2b_3}\cdots\boldsymbol{\gamma}(u_{\kappa_1},-s_1-\tau_q)_{a_qb_1}. \quad \text{(S2.23)}$$

More precisely, the absolute bias between (S2.22) and (S2.23) is bounded by

$$T^{-q}\sum_{i=1}^{q}C_i\Big(1+\frac{1}{l(s_i)}\Big).$$

Returning back to the expression (S2.21), we see that the full expression of the absolute bias is bounded by

$$\frac{1}{2\pi b_T^q}\sum_{s_1,\cdots,s_q=1-T}^{T-1}\prod_{i=1}^{q}\sup_u|\hat{\psi}^{(i)}(u,-s_i)|$$
$$\times\Big|\frac{1}{T^q}\sum_{\kappa_1,\tau_2,\dots,\tau_q}\mathrm{cum}\Big(\boldsymbol{X}^{(a_1)}_{\kappa_1,T},\boldsymbol{X}^{(a_2)}_{\kappa_1+\tau_2,T}\Big)\mathrm{cum}\Big(\boldsymbol{X}^{(b_2)}_{\kappa_1+\tau_2-s_2,T},\boldsymbol{X}^{(b_3)}_{\kappa_1+\tau_3-s_3,T}\Big)\cdots\mathrm{cum}\Big(\boldsymbol{X}^{(a_q)}_{\kappa_1+\tau_q,T},\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T}\Big)$$
$$-\frac{1}{T^q}\sum_{\kappa_1=1}^{T}\sum_{\tau_2,\cdots,\tau_q}\boldsymbol{\gamma}(u_{\kappa_1},\tau_2)_{a_1a_2}\boldsymbol{\gamma}(u_{\kappa_1},\tau_3-s_3-\tau_2+s_2)_{b_2b_3}\cdots\boldsymbol{\gamma}(u_{\kappa_1},-s_1-\tau_q)_{a_qb_1}\Big|$$
$$=O(T^{-q}b_T^{-q}\beta_T^{-q}).$$

In summary, all cumulants of order q for $\mathscr{A}_T$ can be approximated by those of the stationary process with a bias of order $O(T^{-q}b_T^{-q}\beta_T^{-q})$. Thus, the bias in those cumulants for ξ_T is $O(T^{-q/2}b_T^{-q/2}\beta_T^{-q})$. Furthermore, it holds that

$$\mathrm{cum}\big(\xi_T(u^{(1)};\psi^{(1)})_{a_1b_1},\cdots,\xi_T(u^{(q)};\psi^{(q)})_{a_qb_q}\big)=O(T^{1-q/2}b_T^{1-q/2}),\quad \text{(S2.24)}$$

since $\psi^{(1)}(\cdot,\lambda)$, ..., $\psi^{(q)}(\cdot,\lambda)$ are all functions of bounded variation. Therefore, the bias is asymptotically negligible. A representative example of (S2.24) is shown below.

Let us consider the case $q = 2$ for ξ_T. Note that q is even now. We have three terms of the type (S2.23) , i.e.,

(i) the approximation for $\mathrm{cum}(\boldsymbol{X}^{(a_1)}_{\kappa_1,T},\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T})\mathrm{cum}(\boldsymbol{X}^{(a_2)}_{\kappa_2,T},\boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T})$:

$$\frac{b_T}{T}\sum_{\kappa_1=1}^{T}\sum_{s_1,s_2,\tau_2}\hat{\psi}_T^{(1)}(u^{(1)}-u_{\kappa_1},s_1)\hat{\psi}_T^{(2)}(u^{(2)}-u_{\kappa_1},s_2)\boldsymbol{\gamma}(u_{\kappa_1},\tau_2)_{a_1a_2}\boldsymbol{\gamma}(u_{\kappa_1},\tau_2-s_2+s_1)_{b_1b_2};\quad \text{(S2.25)}$$

(ii) the approximation for $\mathrm{cum}(\boldsymbol{X}^{(a_1)}_{\kappa_1,T},\boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T})\mathrm{cum}(\boldsymbol{X}^{(b_1)}_{\kappa_1-s_1,T},\boldsymbol{X}^{(a_2)}_{\kappa_2,T})$:

$$\frac{b_T}{T}\sum_{\kappa_1=1}^{T}\sum_{s_1,s_2,\tau_2}\hat{\psi}_T^{(1)}(u^{(1)}-u_{\kappa_1},s_1)\hat{\psi}_T^{(2)}(u^{(2)}-u_{\kappa_1},s_2)\boldsymbol{\gamma}(u_{\kappa_1},\tau_2-s_2)_{a_1b_2}\boldsymbol{\gamma}(u_{\kappa_1},\tau_2+s_1)_{b_1a_2};\quad \text{(S2.26)}$$

(iii) the approximation for $\mathrm{cum}(\boldsymbol{X}^{(a_1)}_{\kappa_1,T},\boldsymbol{X}^{(b_2)}_{\kappa_2-s_2,T},\boldsymbol{X}^{(a_2)}_{\kappa_1-s_1,T},\boldsymbol{X}^{(b_1)}_{\kappa_2,T})$:

$$\frac{b_T}{T}\sum_{\kappa_1=1}^{T}\sum_{s_1,s_2,\tau_2}\hat{\psi}_T^{(1)}(u^{(1)}-u_{\kappa_1},s_1)\hat{\psi}_T^{(2)}(u^{(2)}-u_{\kappa_1},s_2)\gamma_{a_1a_2b_1b_2}(u_{\kappa_1};-s_1,\tau_2,\tau_2-s_2).\quad \text{(S2.27)}$$

We first explain the term (S2.25). By repeated application of the Parseval equality (see, e.g., the proof of Lemma 2.2 in Hosoya and Taniguchi

(1982) for details) and by Lemma P5.1 in Brillinger (1981), the term (S2.25) is equivalent to

$$2\pi b_T \int_0^1 \int_{-\pi}^{\pi} \psi_T^{(1)}(u^{(1)}-u,\lambda)\overline{\psi_T^{(2)}(u^{(2)}-u,\lambda)}\boldsymbol{f}(u,\lambda)_{a_1a_2}\overline{\boldsymbol{f}(u,\lambda)}_{b_1b_2}\,\mathrm{d}\lambda\,\mathrm{d}u+O(T^{-1}b_T^{-1}).$$

Under Assumption S2, if $u^{(1)} \neq u^{(2)}$, we have

$$\int_0^1 \int_{-\pi}^{\pi} \psi_T^{(1)}(u^{(1)}-u,\lambda)\overline{\psi_T^{(2)}(u^{(2)}-u,\lambda)}\boldsymbol{f}(u,\lambda)_{a_1b_2}\overline{\boldsymbol{f}(u,\lambda)}_{b_1a_2}\,\mathrm{d}\lambda\,\mathrm{d}u = o(b_T^{-1}),$$

since the supports of $\psi^{(1)}$ and $\psi^{(2)}$ are compact. Thus, the term (S2.25) converges to

$$2\pi\delta(u^{(1)},u^{(2)})\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda)\overline{\psi^{(2)}(v,\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(1)},\lambda)_{a_1a_2}\overline{\boldsymbol{f}(u^{(1)},\lambda)}_{b_1b_2}\,\mathrm{d}\lambda, \tag{S2.28}$$

where δ is a delta function such that $\delta(a,b)=1$ if $a=b$, and 0 otherwise. Similarly, the term (S2.26) converges to

$$2\pi\delta(u^{(1)},u^{(2)})\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda)\overline{\psi^{(2)}(v,-\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(1)},\lambda)_{a_1b_2}\overline{\boldsymbol{f}(u^{(1)},\lambda)}_{b_1a_2}\,\mathrm{d}\lambda. \tag{S2.29}$$

The term (S2.27) converges to

$$2\pi\delta(u^{(1)},u^{(2)})\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(1)}(v,\lambda_1)\overline{\psi^{(2)}(v,-\lambda_2)}\,\mathrm{d}v\Big)\tilde{\gamma}_{a_1a_2b_1b_2}(u^{(1)};\lambda_1,\lambda_2,-\lambda_2)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2, \tag{S2.30}$$

by repeated application of the Parseval equality. Combining all terms (S2.28), (S2.29) and (S2.30), we obtain the results of Lemma S4. □

S2.2 Asymptotic Normality

Here, we show the asymptotic normality of the empirical spectral process $\xi_T(\psi)$ in (S2.14). To this goal, we adopt the idea in Dahlhaus and Polonik (2009) to use the Gaussian kernel as the mollifier with the property of being rapidly decreasing. Let G be the Gaussian kernel, that is,

$$G(x) := \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{1}{2}x^2\Big),$$

and G_b the mollifier

$$G_\beta(x) = \frac{1}{\beta} G\Big(\frac{x}{\beta}\Big),$$

with $\beta := \beta_T \to 0$ as $T \to \infty$. From the convolution theorem, the Fourier coefficients $\hat{\psi}^{*T}$ of $\psi^{*T} := \psi * G_\beta$ are

$$\hat{\psi}^{*T}(u,k) = \hat{\psi}(u,k)\hat{G}_\beta(k), \quad k \in \mathbb{Z}. \tag{S2.31}$$

Remark S5. The remarkable feature of this manipulation is that

$$\sum_{k\in\mathbb{Z}} \sup_{u\in[0,1]} |\hat{\psi}^{*T}(u,k)| \leq \sum_{k\in\mathbb{Z}} \sup_{u\in[0,1]} |\hat{\psi}(u,k)|,$$

since for any fixed $k \in \mathbb{Z}$,

$$|\hat{G}_\beta(k)| = \Big|\exp\Big(\frac{-\beta^2 k^2}{2}\Big)\Big| \leq 1.$$

In addition, the following result holds.

$$\sum_{k\in\mathbb{Z}} \sup_{u\in[0,1]} |\hat{\psi}^{*T}(u,k)| = O\Big(\ln\big(\beta_T^{-1}\big)\Big). \tag{S2.32}$$

If we take β_T as $\beta_T = T^{-k}$ for any $k \geq 1$, then we have

$$\sum_{k\in\mathbb{Z}} \sup_{u\in[0,1]} |\hat{\psi}^{*T}(u,k)| = O(\ln T).$$

Proof of Remark S5. For any $1 \leq i, j \leq p$, let $\psi := \phi_{ij} \in \Psi$ as in Assumption S1. Note that $\psi(u, \cdot)$ is a continuous function of bounded variation.

(i) Let $k \neq 0$. From Jordan decomposition theorem, there exists a signed measure g_ψ such that

$$\hat{\psi}(u,k) = \int_{-\pi}^{\pi} \frac{\exp(-\mathrm{i}k\lambda) - 1}{-\mathrm{i}k} g_\psi(u, \mathrm{d}\lambda),$$

which leads to

$$\sup_{u\in[0,1]} |\hat{\psi}(u,k)| \leq \frac{C}{|k|} \sup_{u\in[0,1]} V\big(\psi(u,\cdot)\big) \leq \frac{C\tau_{\infty,\mathrm{TV}}}{|k|}. \tag{S2.33}$$

(ii) Let $k = 0$.

$$\sup_{u\in[0,1]} |\hat{\psi}(u,0)| \leq 2\pi \sup_{u\in[0,1]} \sup_{\lambda\in[-\pi,\pi]} \psi(u,\lambda) \leq 2\pi\tau_{\infty,\infty}. \tag{S2.34}$$

Combing (S2.33) and (S2.34) with the relation (S2.31), we obtain

$$\sup_{u\in[0,1]} |\hat{\psi}^{*T}(u,k)| \leq C\Big(1 + \sum_{k=1}^{\infty} \frac{1}{|k|} \exp\Big(\frac{-\beta^2 k^2}{2}\Big)\Big) = O\Big(\ln\big(\beta^{-1}\big)\Big).$$

Thus, the equation (S2.32) is shown. □

Next result shows that the asymptotic normality of $\xi_T(u_k)_{ab}$ for $k \geq 1$.

Theorem S1. *Suppose Assumptions 1 and S1 hold. Let $b_T \to 0$ and $Tb_T \to \infty$, as $T \to \infty$. For any q, and $u^{(1)}, \ldots, u^{(q)} \in [0,1]$, it holds that*

$$\big(\xi_T(u^{(1)};\psi^{(1)})_{a_1b_1}, \cdots, \xi_T(u^{(q)};\psi^{(q)})_{a_qb_q}\big)^\top \xrightarrow{d} \mathcal{N}\big(\mathbf{0}, (V_{jk})_{j,k=1,\ldots q}\big), \qquad \text{as } T \to \infty,$$

where V_{jk} is

$$\begin{aligned}V_{jk} = 2\pi\delta(u^{(j)}, u^{(k)})\Bigg(&\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(j)}(v,\lambda)\overline{\psi^{(k)}(v,\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(j)},\lambda)_{a_ja_k}\overline{\boldsymbol{f}(u^{(j)},\lambda)}_{b_jb_k}\,\mathrm{d}\lambda \\ &+\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}\psi^{(j)}(v,\lambda)\overline{\psi^{(k)}(v,-\lambda)}\,\mathrm{d}v\Big)\boldsymbol{f}(u^{(j)},\lambda)_{a_jb_k}\overline{\boldsymbol{f}(u^{(j)},\lambda)}_{b_ja_k}\,\mathrm{d}\lambda \\ +\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty}&\psi^{(j)}(v,\lambda_1)\overline{\psi^{(k)}(v,-\lambda_2)}\,\mathrm{d}v\Big)\tilde{\gamma}_{a_ja_kb_jb_k}(u^{(j)};\lambda_1,\lambda_2,-\lambda_2)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\Bigg),\end{aligned}$$

where $\tilde{\gamma}$ is the fourth-order spectral density of the process.

Proof. First we show that $\operatorname{var}\Big(\xi_T(u;\psi)_{ab} - \xi_T(u;\psi^{*T})_{ab}\Big) \to 0$, which, in turn, shows that

$$\xi_T(u;\psi)_{ab} - \xi_T(u;\psi^{*T})_{ab} \to_P 0. \tag{S2.35}$$

As in Remark S5, let $\beta_T = T^{-k}$ for any $k \geq 1$. Following this choice, we have $O(\beta_T/b_T) = o(1)$. Note that

$$\begin{aligned}&\operatorname{var}\Big(\xi_T(u;\psi)_{ab} - \xi_T(u;\psi^{*T})_{ab}\Big) \\ =&Tb_T\operatorname{var}\Big(\frac{1}{2\pi T}\sum_{k=1}^{T}\sum_{s=1-T}^{T-1}\big\{\hat{\psi}_T(u-u_k,-s) - \hat{\psi}_T^{*T}(u-u_k,-s)\big\}\boldsymbol{X}^{(a)}_{[k+1/2+s/2],T}\boldsymbol{X}^{(b)}_{[k+1/2-s/2],T}\Big) \\ \leq& b_T^{-1}\Big(\sup_u\sum_{s=-\infty}^{\infty}|\hat{\psi}(u,-s) - \hat{\psi}^{*T}(u,-s)|\Big)^2 \\ \leq& Cb_T^{-1}\sum_{s=-\infty}^{\infty}\frac{|\exp(-s^2\beta_T^2/2) - 1|^2}{s^2},\end{aligned}$$

where the last inequality follows from (S2.31). Since $|\exp(-s^2\beta_T^2/2)-1| \le \min(1, s^2\beta_T^2/2)$, the order of the last term is $O(\beta_T/b_T) = o(1)$. Thus, (S2.35) is shown.

Now, we only have to consider the finite distributions of $\xi_T(u;\psi^{*T})$. However, from Remark S5, we find that the condition (S2.19) is satisfied and thus the covariance matrix of $\xi_T(u;\psi^{*T})$ can be expressed in the form of (S2.20). Therefore, the proof is completed. □

Finally, remembering the matrix $\boldsymbol{\phi}$ satisfies Assumption S1, we define $\mathscr{A}_T^{\circ}(u)$ and $\bar{\mathscr{A}}_T^{\circ}(u)$ as

$$\mathscr{A}_T^{\circ}(u) := \frac{1}{T}\sum_{k=1}^{T}\int_{-\pi}^{\pi} \boldsymbol{\phi}_T(u-u_k,\lambda)\boldsymbol{I}_T(u_k,\lambda)\,\mathrm{d}\lambda,$$
$$\bar{\mathscr{A}}_T^{\circ}(u) := \frac{1}{T}\sum_{k=1}^{T}\int_{-\pi}^{\pi} \boldsymbol{\phi}_T(u-u_k,\lambda)\boldsymbol{f}(u_k,\lambda)\,\mathrm{d}\lambda,$$

and let $\zeta_T(u)$ be

$$\zeta_T(u) = \sqrt{Tb_T}\,\operatorname{Tr}\big(\mathscr{A}_T^{\circ}(u) - \bar{\mathscr{A}}_T^{\circ}(u)\big). \tag{S2.36}$$

Corollary S1. *Suppose Assumptions 1, S1 and S2 hold. If* $b_T = o(1)$ *and* $b_T^{-1} = o\big(T(\ln T)^{-6}\big)$, *then it holds that*

$$\big(\zeta_T(u^{(1)}), \cdots, \zeta_T(u^{(q)})\big)^{\top} \xrightarrow{d} \mathcal{N}\big(\mathbf{0}, (\tilde{V}_{jk})_{j,k=1,\dots q}\big), \qquad \textit{as } T \to \infty,$$

where $\tilde{V}_{jk}$ is given by

$$\tilde{V}_{jk} = 4\pi\delta(u^{(j)}, u^{(k)})\Bigg(\int_{-\pi}^{\pi} \mathrm{Tr}\Big(\int_{-\infty}^{\infty} \boldsymbol{f}(u^{(j)}, \lambda)\boldsymbol{\phi}(v, \lambda)\boldsymbol{f}(u^{(j)}, \lambda)\boldsymbol{\phi}(v, \lambda)\, \mathrm{d}v\Big)\, \mathrm{d}\lambda$$
$$+\frac{1}{2}\sum_{r,t,u,v=1}^{p} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Big(\int_{-\infty}^{\infty} \boldsymbol{\phi}_{rt}(v, \lambda_1)\boldsymbol{\phi}_{uv}(v, \lambda_2)\tilde{\gamma}_{rtuv}(u^{(j)}; -\lambda_1, \lambda_2, -\lambda_2)\, \mathrm{d}v\Big)\, \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2\Bigg), \tag{S2.37}$$

where $\tilde{\gamma}$ is the fourth-order spectral density of the process.

Proof. From the definition of $\zeta_T(u)$ in (S2.36), we see that $\zeta(u)$ is a linear combination of the processes $\xi_T(u)$ in (S2.14). With a similar computation to the latter part in Lemma A.3.3. in Hosoya and Taniguchi (1982), we obtain (S2.37). □

S3 Proofs

In Section S3, we provide proofs of results in Section 3 in the paper.

S3.1 Proof of Theorem 1

Proof. From equations (3.17) and (3.19), we have

$$\begin{aligned}\mathcal{L}_T(\boldsymbol{\theta}, u) - \mathcal{L}(\boldsymbol{\theta}, u) =& \Big\{\frac{1}{T}\sum_{k=1}^{T}\frac{1}{b_T}K\Big(\frac{u-u_k}{b_T}\Big) - 1\Big\}\int_{-\pi}^{\pi} \ln\det \boldsymbol{f}_{\boldsymbol{\theta}}(\lambda)\, \mathrm{d}\lambda + \mathrm{Tr}\Big(\boldsymbol{f}(u, \lambda)\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)\Big)\, \mathrm{d}\lambda \\ &+ \frac{1}{T}\sum_{k=1}^{T}\frac{1}{b_T}K\Big(\frac{u-u_k}{b_T}\Big)\int_{-\pi}^{\pi} \mathrm{Tr}\Big\{\Big(\boldsymbol{f}(u_k, \lambda) - \boldsymbol{f}(u, \lambda)\Big)\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)\Big\}\, \mathrm{d}\lambda\end{aligned}$$

$$+\frac{1}{T}\sum_{k=1}^{T}\frac{1}{b_T}K\Big(\frac{u-u_k}{b_T}\Big)\int_{-\pi}^{\pi}\mathrm{Tr}\Big\{\Big(\boldsymbol{I}_T(u_k,\lambda)-\boldsymbol{f}(u_k,\lambda)\Big)\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)\Big\}\,\mathrm{d}\lambda$$

$$=L_1+L_2+L_3,\quad \text{(say)}.$$

Since K is a function of bounded variation, applying Lemma P5.1 in Brillinger (1981), it holds that

$$\frac{1}{T}\sum_{k=1}^{T}\frac{1}{b_T}K\Big(\frac{u-u_k}{b_T}\Big)-1=\int_0^1\frac{1}{b_T}K\Big(\frac{u-v}{b_T}\Big)\,\mathrm{d}v-1+O(T^{-1})$$
$$=\int_{\frac{u-1}{b_T}}^{\frac{u}{b_T}}K(x)\,\mathrm{d}x-1+O(T^{-1}),$$

which implies that $L_1=O(T^{-1})$, since the kernel K has a compact support. Under Assumption 2 (i), $\boldsymbol{f}$ is of bounded variation, and again, applying Lemma P5.1 in Brillinger (1981), we have

$$\frac{1}{T}\sum_{k=1}^{T}\frac{1}{b_T}K\Big(\frac{u-u_k}{b_T}\Big)\Big(\boldsymbol{f}(u_k,\lambda)-\boldsymbol{f}(u,\lambda)\Big)$$
$$=\int_0^1\frac{1}{b_T}K\Big(\frac{u-v}{b_T}\Big)\big(\boldsymbol{f}(v,\lambda)-\boldsymbol{f}(u,\lambda)\big)\,\mathrm{d}v+O(T^{-1})$$
$$=\int_{\frac{u-1}{b_T}}^{\frac{u}{b_T}}K(x)\big(\boldsymbol{f}(u-b_Tx,\lambda)-\boldsymbol{f}(u,\lambda)\big)\,\mathrm{d}x+O(T^{-1}),$$
$$=\int_{\frac{u-1}{b_T}}^{\frac{u}{b_T}}K(x)\bigg(-b_Tx\frac{\partial}{\partial u}\boldsymbol{f}(u,\lambda)+(b_Tx)^2\frac{\partial^2}{\partial u^2}\boldsymbol{f}(u,\lambda)+O(b_T^3)\bigg)\,\mathrm{d}x+O(T^{-1}).$$

Since K has a compact support and it is symmetric, we have $\int_{-\infty}^{\infty}xK(x)\,\mathrm{d}x=0$, which implies that

$$L_2=O(b_T^2)+O(T^{-1}),$$

as $T \to \infty$. In summary, we have $\sqrt{Tb_T}L_1 \to 0$, and $\sqrt{Tb_T}L_2 \to 0$, since $b_T = o(T^{-1/5})$. Finally, we apply Corollary S1 to L_3 to show

$$\sqrt{Tb_T}\big(\mathcal{L}_T(\boldsymbol{\theta}, u) - \mathcal{L}(\boldsymbol{\theta}, u)\big) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}^{\mathcal{L}}(u)). \tag{S3.38}$$

In fact, we only have to check Assumptions S1 and S2 for

$$\boldsymbol{\phi}(u, \lambda) = K(u)\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda), \tag{S3.39}$$

or equivalently, $\psi(u, \lambda) = K(u)f_{\boldsymbol{\theta}}^{ij}(\lambda)$ for $i, j = 1, \dots, p$, which is expressed in the Einstein notation. From the definition (2.4) of the time-varying spectral density matrix, $\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)$ is obviously Hermitian. Additionally, Assumption S1 (ii) is satisfied if both K and $\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)$ are bounded functions of bounded variation, which follows Assumptions 2 (ii) and 3 (iii). Applying Corollary S1 to (S3.39), we obtain (S3.38). □

S3.2 Proof of Theorem 2

Proof. Note that we have

$$\sqrt{Tb_T}\big(\mathcal{L}_T(\boldsymbol{\theta}, u) - \mathcal{L}(\boldsymbol{\theta}, u)\big) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}^{\mathcal{L}}(u)).$$

The consequence (3.22) follows, if the following conditions are guaranteed for the theorem, i.e.,

(i) both $\mathcal{L}_T(\boldsymbol{\theta}, u)$ and $\mathcal{L}(\boldsymbol{\theta}, u)$ are convex in $\boldsymbol{\theta}$ for each u and continuous in u for each $\boldsymbol{\theta}$;

(ii) $\boldsymbol{\theta}_0(u)$ is the unique minimizer of $\mathcal{L}(\boldsymbol{\theta}, u)$ for each $u \in [0,1]$

According to (i), the convexity of $\mathcal{L}_T(\boldsymbol{\theta}, u)$ and $\mathcal{L}(\boldsymbol{\theta}, u)$ in $\boldsymbol{\theta}$ follows from Assumption 3 (v-b). Especially, note that $\mathcal{L}_T(\boldsymbol{\theta}, u)$ is a linear combination of $\int_{-\pi}^{\pi} \ln\det \boldsymbol{f}_{\boldsymbol{\theta}}(\lambda) + \mathrm{Tr}\Big(\boldsymbol{I}_T(u_k, \lambda)\boldsymbol{f}_{\boldsymbol{\theta}}^{-1}(\lambda)\Big) d\lambda$ with nonnegative coefficients, which implies that $\mathcal{L}_T(\boldsymbol{\theta}, u)$ is convex. The continuity of $\mathcal{L}_T(\boldsymbol{\theta}, u)$ and $\mathcal{L}(\boldsymbol{\theta}, u)$ in u follows from Assumption 2, i.e., the continuity of K and $\boldsymbol{f}(\cdot, \lambda)$. According to (ii), it is assumed in Assumption 3 (v-a). Since $\boldsymbol{\theta}_0(u)$ is the unique minimizer and $\boldsymbol{f}_{\boldsymbol{\theta}}(\lambda)$ is twice continuously differentiable with respect to $\boldsymbol{\theta}$, again, (3.23) follows from Corollary S1. □

S3.3 Proof of Theorem 3

Proof. For simplicity, denote

$$\boldsymbol{f}^{Z}_{\boldsymbol{\theta}(u)}(\lambda)_{11} := \boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11} - 2\pi \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{12}\left(\tilde{\Sigma}_{\boldsymbol{\theta}(u),22}\right)^{-1} \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{21}.$$

Accordingly, $\mathrm{GC}^{(2\to 1)}(u;\boldsymbol{\theta})$ in (3.24) is simply

$$\mathrm{GC}^{2\to 1}(u;\boldsymbol{\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln \frac{|\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11}|}{|\boldsymbol{f}^{Z}_{\boldsymbol{\theta}(u)}(\lambda)_{11}|}\, \mathrm{d}\lambda.$$

Note that the domain of the integration is bounded, and under Assumption 4, $\ln|\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11}|$ is integrable in λ for $u \in [0,1]$, which implies that $\ln|\boldsymbol{f}^{Z}_{\boldsymbol{\theta}(u)}(\lambda)_{11}|$ is also integrable.

Now if we show $\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11}$ and $\boldsymbol{f}^{Z}_{\boldsymbol{\theta}(u)}(\lambda)_{11}$ are continuously differentiable with respect to $\boldsymbol{\theta}$, then applying the delta-method to (3.22) leads to the conclusion. We summarize the parametric expressions used in the causality measure. Suppose that $\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)$ admits the decomposition

$$\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda) = \begin{bmatrix} \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{11} & \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{12} \\ \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{21} & \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{22} \end{bmatrix}.$$

With an abuse of notation, under Assumption 4, $\boldsymbol{f}_{\boldsymbol{\theta}(u)}$, defined on the unit disk $\mathcal{D}$ in the complex plane, can be factorized as

$$\boldsymbol{f}_{\boldsymbol{\theta}(u)}(z) = \frac{1}{2\pi}\boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(z)\boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(z)^*, \quad z \in \mathcal{D}. \tag{S3.40}$$

Especially, as shown in Rozanov (1967), it holds that

$$\Sigma_{\boldsymbol{\theta}(u)} = \boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(0)\boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(0)^*. \tag{S3.41}$$

From Lemmas 2.2 and 2.3 in Hosoya (1991), we have

$$\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{11} = \boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11}, \qquad \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{12} = \boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{21}^*, \tag{S3.42}$$

$$\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{21} = \begin{bmatrix} -\Sigma_{\boldsymbol{\theta}(u),21}\Sigma_{\boldsymbol{\theta}(u),11}^{-1} & I_M \end{bmatrix} \boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(0)\boldsymbol{\Lambda}_{\boldsymbol{\theta}(u)}(e^{\mathrm{i}\lambda})^{-1} \begin{pmatrix} \boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11} \\ \boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{21} \end{pmatrix}, \tag{S3.43}$$

and

$$\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{22} = \frac{1}{2\pi}\tilde{\Sigma}_{\boldsymbol{\theta}(u),22} := \frac{1}{2\pi}\left\{\Sigma_{\boldsymbol{\theta}(u),22} - \Sigma_{\boldsymbol{\theta}(u),21}\Sigma_{\boldsymbol{\theta}(u),11}^{-1}\Sigma_{\boldsymbol{\theta}(u),12}\right\}. \tag{S3.44}$$

The continuous differentiability of $\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)_{11}$ with respect to $\boldsymbol{\theta}$ directly follows from that of $\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)$ under Assumption 3 (iv). Note that $\tilde{\Sigma}_{\boldsymbol{\theta}(u),22}$

is a continuous function of $\Sigma_{\boldsymbol{\theta}(u)}$ from (S3.44). Using the expression for $\Sigma_{\boldsymbol{\theta}(u)}$ in (S3.41) and the relation in (S3.40), the continuous differentiability of $\Sigma_{\boldsymbol{\theta}(u)}$ with respect to $\boldsymbol{\theta}$ follows from that of $\boldsymbol{f}_{\boldsymbol{\theta}(u)}(\lambda)$. In addition, this implies the continuous differentiability of $\boldsymbol{g}_{\boldsymbol{\theta}(u)}(\lambda)_{21}$ from (S3.43), which in turn implies the continuous differentiability of $\boldsymbol{f}^Z_{\boldsymbol{\theta}(u)}(\lambda)_{11}$. This completes the proof of Theorem 3. $\square$

S3.4 Proof of Theorem 4

Proof. By Theorem 6.8 of Magnus and Neudecker (2007), we have

$$\begin{aligned}\mathrm{GC}^{(2\to1)}(u;\hat{\boldsymbol{\theta}}_T) - \mathrm{GC}^{(2\to1)}(u;\boldsymbol{\theta}_0) &= \nabla\mathrm{GC}^{(2\to1)}(u;\boldsymbol{\theta}_0)^\top\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big)\\ &+ \frac{1}{2}\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big)^\top\mathcal{H}(u)\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big) + o_P\Big(\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big)^2\Big).\end{aligned}$$

Since $\nabla\mathrm{GC}^{(2\to1)}(u;\boldsymbol{\theta}_0) = \mathbf{0}$ from (3.26), we have

$$\begin{aligned}&Tb_T\big(\mathrm{GC}^{(2\to1)}(u;\hat{\boldsymbol{\theta}}_T) - \mathrm{GC}^{(2\to1)}(u;\boldsymbol{\theta}_0)\big)\\ &\quad= \frac{1}{2}\sqrt{Tb_T}\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big)^\top\mathcal{H}(u)\sqrt{Tb_T}\big(\hat{\boldsymbol{\theta}}_T(u) - \boldsymbol{\theta}_0(u)\big) + o_P(1).\end{aligned}$$

We arrived at the conclusion (3.29) by the continuous mapping theorem. $\square$

Finally, Theorems 5 and 6 are readily obtained by Theorem 4.

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