CONSISTENT COMMUNITY DETECTION IN MULTI-LAYER NETWORKS WITH PERSONALIZED EDGE DIFFERENTIAL PRIVACY

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Supplementary Material

This Supplementary Material contains all necessary lemmas and technical proofs of the main paper.

S1 Technical proofs

Proof of Lemma 1. For any $(i,j,l) \in [n] \times [n] \times [L]$, we have

$$\sup_{1\le i<j\le n, l\in[L]} \sup_{\widetilde{x}\in\{0,1\}} \sup_{x,x'\in\{0,1\}} \frac{P(\mathcal{M}(\boldsymbol{\mathcal{A}}_{i,j,l}) = \widetilde{x} | \boldsymbol{\mathcal{A}}_{i,j,l} = x)}{P(\mathcal{M}(\boldsymbol{\mathcal{A}}_{i,j,l}) = \widetilde{x} | \boldsymbol{\mathcal{A}}_{i,j,l} = x')}$$
$$= \max\{1, \frac{\theta}{1-\theta}, \frac{1-\theta}{\theta}\} = \exp(\epsilon),$$

where the last inequality follows from the assumption that $\theta = \big(1 + \exp(-\epsilon)\big)^{-1}$. ∎

Proof of Lemma 2. As in the proof of Lemma 1, $\theta_{i,j} = \big(1+\exp(-\epsilon_{i,j})\big)^{-1}$. The decomposition (4) immediately yields that $\epsilon_{i,j} = \log\frac{1+f_j f_j}{1-f_i f_j}$ for $1 \leq i < j \leq n$. ■

Proof of Lemma 3. Combining the fact that $\mathbb{E}(\boldsymbol{A}_{i,j}^{(l)}) = d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)}$ and the definition of flipping mechanism, we have

$$
\begin{aligned}
\mathbb{E}(\mathcal{M}_{\theta_{i,j}}(\boldsymbol{A}_{i,j}^{(l)})) &= \mathbb{E}(\mathcal{M}_{\theta_{i,j}}(\boldsymbol{A}_{i,j}^{(l)})|\boldsymbol{A}_{i,j}^{(l)}=1)d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)} \\
&\quad + \mathbb{E}(\mathcal{M}_{\theta_{i,j}}(\boldsymbol{A}_{i,j}^{(l)})|\boldsymbol{A}_{i,j}^{(l)}=0)(1-d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)}) \\
&= \theta_{i,j} d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)} + (1-\theta_{i,j})(1-d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)}) \\
&= (2\theta_{i,j}-1)d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)} + (1-\theta_{i,j}). \\
&= f_i f_j d_i d_j \boldsymbol{B}_{c_i,c_j}^{(l)} + \frac{1}{2}(1-f_i f_j).
\end{aligned}
$$

This completes the proof. ■

Proof of Lemma 4. We first define $\widetilde{\boldsymbol{U}}$ such that $\widetilde{\boldsymbol{U}}_{i,:} = \boldsymbol{U}_{i,:}/\|\boldsymbol{U}_{i,:}\|_2$. By the definition of $\boldsymbol{U}$, we have

$$
\boldsymbol{U}_{i,:} = f_i d_i/\sqrt{\gamma_{c_i^*}}\boldsymbol{O}_{c_i^*,:}.
$$

By the fact that $\|\boldsymbol{O}_{c_i^*,:}\| = 1$, we further have $\widetilde{\boldsymbol{U}} = \boldsymbol{O}_{c_i^*,:}$. Clearly, $\widetilde{\boldsymbol{U}}$ has only K distinct rows, and each corresponding to one community. It immediately follows that $\widetilde{\boldsymbol{U}}_{i,:} = \widetilde{\boldsymbol{U}}_{j,:}$ if $c_i^* = c_j^*$. Further, as $\boldsymbol{O}$ is an orthogonal matrix, $\widetilde{\boldsymbol{U}}_{i,:}$ and $\widetilde{\boldsymbol{U}}_{j,:}$ are perpendicular to each other if $c_i^* \neq c_j^*$ and $\|\widetilde{\boldsymbol{U}}_{i,:}\| = 1$. We thus conclude that $\|\widetilde{\boldsymbol{U}}_{i,:} - \widetilde{\boldsymbol{U}}_{j,:}\| = \sqrt{2}$ if $c_i^* \neq c_j^*$, for $i,j \in [n]$. This completes the proof. ■

Lemma 1. *Under the conditions of Lemma 4, there exists an orthogonal matrix* $\boldsymbol{O}^{(1)} \in \mathbb{R}^{K\times K}$*, such that the left singular vectors corresponding to the non-zero singular values of* $\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})$ *is* $\boldsymbol{U}\boldsymbol{O}^{(1)}$*.*

Proof of Lemma 1**.** Since the columns of $\boldsymbol{V}$ and $\boldsymbol{U}$ are all orthonormal vectors, $\boldsymbol{V}\otimes\boldsymbol{U}$ is also a column orthogonal matrix. Note that the Tucker decomposition of $\widetilde{\boldsymbol{\mathcal{P}}}$ implies that

$$\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}}) = \boldsymbol{U}\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})(\boldsymbol{V}\otimes\boldsymbol{U})^T.$$

Let $\boldsymbol{O}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}\boldsymbol{V}^T_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}$ be the singular value decomposition of $\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})$, it then follows that

$$\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U}) = (\boldsymbol{U}\boldsymbol{O}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})})\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}\boldsymbol{V}^T_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}.$$

Therefore, $(\boldsymbol{U}\boldsymbol{O}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})})\boldsymbol{\Sigma}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}\boldsymbol{V}^T_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}$ is the singular value decomposition of $\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})$. The desired result immediately follows by taking $\boldsymbol{O}^{(1)} = \boldsymbol{O}_{\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})}$. ■

Lemma 2. *Under Assumptions A, it holds true that*

$$\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big) \asymp ns_n\overline{\psi}\sqrt{L},\ \textit{for}\ k\in[K],$$

where $\overline{\psi} = \frac{1}{n}\sum_{i=1}^n f_i^2 d_i^2$.

Proof of Lemma 2. The Tucker decomposition of $\widetilde{\boldsymbol{\mathcal{P}}}$ implies that

$$\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)=\sigma_k\big(\boldsymbol{U}\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})(\boldsymbol{V}\otimes\boldsymbol{U})^T(\boldsymbol{V}\otimes\boldsymbol{U})\big)=\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})\big),$$

where the last equality follows from the fact that $\boldsymbol{U}$ has orthonormal columns and hence does not affect the singular values. It then follows from the Tucker decomposition of $\boldsymbol{\mathcal{B}}\times_1\boldsymbol{\Gamma}\times_2\boldsymbol{\Gamma}$ that

$$\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})\big)=\sigma_k\big(\boldsymbol{O}\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{C}})(\boldsymbol{V}\otimes\boldsymbol{O})^T\big)=\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{B}}\times_1\boldsymbol{\Gamma}\times_2\boldsymbol{\Gamma})\big).$$

Let $\boldsymbol{F}^{(l)}=\boldsymbol{\Gamma}\boldsymbol{\mathcal{B}}_{:,:,l}\boldsymbol{\Gamma}$, for $l\in[L]$, and then we have $\boldsymbol{\mathcal{M}}_1(\boldsymbol{\mathcal{B}}\times_1\boldsymbol{\Gamma}\times_2\boldsymbol{\Gamma})=([\boldsymbol{F}^{(1)},...,\boldsymbol{F}^{(L)}])$. With this, it follows that

$$\sigma_k\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)=\sigma_k([\boldsymbol{F}^{(1)},...,\boldsymbol{F}^{(L)}])=\Big(\lambda_k\Big(\sum_{l=1}^{L}\boldsymbol{F}^{(l)}(\boldsymbol{F}^{(l)})^T\Big)\Big)^{1/2},$$

where $\lambda_k(\cdot)$ denotes the k-th largest eigenvalue of a symmetric matrix. By Assumptions A, we have

$$\lambda_k\big(\boldsymbol{F}^{(l)}(\boldsymbol{F}^{(l)})^T\big)\asymp(ns_n\overline{\psi})^2,\ \text{for } l\in[L].$$

Further, it follows from Weyl's inequality that

$$\lambda_k\Big(\sum_{l=1}^{L}\boldsymbol{F}^{(l)}(\boldsymbol{F}^{(l)})^T\Big)\asymp n^2s_n^2\overline{\psi}^2L,\ \text{for } k\in[K].$$

The desired result then follows immediately. ■

Lemma 3. *Denote* $\delta_n=\|\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{A}}})(\boldsymbol{V}\otimes\boldsymbol{U})-\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\|$. *Then there exists an orthogonal matrix* $\boldsymbol{O}^{(2)}$ *such that*

$$\|\widehat{\boldsymbol{U}}-\boldsymbol{U}\boldsymbol{O}^{(2)}\|_F\leq\frac{2\sqrt{2}\Big(2\sigma_1\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)+\delta_n\Big)\delta_n}{\sigma_K^2\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)}.$$

Proof of Lemma 3. By Lemma 1, we know that there exists an orthogonal matrix $\boldsymbol{O}^{(1)}$ such that, $\boldsymbol{U}\boldsymbol{O}^{(1)}$ are the left singular vectors corresponding to the non-zero singular values of $\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V} \otimes \boldsymbol{U})$. Applying similar argument in Lemma 1 to $\widetilde{\boldsymbol{\mathcal{A}}}$, there exists an orthogonal matrix $\boldsymbol{O}^{(3)} \in \mathbb{R}^{K\times K}$ such that $\widehat{\boldsymbol{U}}\boldsymbol{O}^{(3)}$ are the left singular vectors corresponding to the first K leading singular values of $\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{A}}})(\boldsymbol{V} \otimes \boldsymbol{U})$. By Theorem 3 in [2], there exists an orthogonal matrix $\boldsymbol{O}^{(4)}$ such that

$$\begin{aligned}\|\widehat{\boldsymbol{U}} - \boldsymbol{U}\boldsymbol{O}^{(1)}\boldsymbol{O}^{(4)}(\boldsymbol{O}^{(3)})^T\|_F &= \|\widehat{\boldsymbol{U}}\boldsymbol{O}^{(3)} - \boldsymbol{U}\boldsymbol{O}^{(1)}\boldsymbol{O}^{(4)}\|_F \\ &\leq \frac{2\sqrt{2}\Big(2\sigma_1\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big) + \delta_n\Big)\delta_n}{\sigma_K^2\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)}.\end{aligned}$$

The desired result then follows immediately by taking $\boldsymbol{O}^{(2)} = \boldsymbol{O}^{(1)}\boldsymbol{O}^{(4)}(\boldsymbol{O}^{(3)})^T$.

■

Lemma 4. *Let δ_n be defined in Lemma 3. Under Assumptions A to D, it holds true that*

$$\delta_n = O_p\Big(\sqrt{n\varphi_n \log n}\Big),$$

where $\varphi_n = 1 - \min_{i\in[n]} f_i + 4s_n$.

Proof of Lemma 4. Note that

$$\Big\|\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{A}}} - \widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\Big\| = \Big\|\sum_{l=1}^{L}\sum_{i\leq j}(\widetilde{\boldsymbol{A}}_{i,j}^{(l)} - \widetilde{\boldsymbol{P}}_{i,j}^{(l)})\boldsymbol{E}^{(i,j)}\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\Big\|, \quad \text{(S1.1)}$$

where $\boldsymbol{E}^{(i,j)}$ is the square matrix with 1 in the (i,j)-th and (j,i)-th entries and 0 otherwise. Clearly, $(\widetilde{\boldsymbol{A}}^{(l)}_{i,j} - \widetilde{\boldsymbol{P}}^{(l)}_{i,j})\boldsymbol{E}^{(i,j)}(\boldsymbol{V}^T_{l,:} \otimes \boldsymbol{U})$ are independent zero-mean random matrices, for $i, j \in [n]$, $l \in [L]$.

Next, we proceed to verify the required conditions of the matrix Bernstein inequality (Theorem 1.6 in [1]) in order to provide an probabilistic upper bound for (S1.1). For each $i, j \in [n]$ and $l \in [L]$, we have

$$\Big\|(\widetilde{\boldsymbol{A}}^{(l)}_{i,j} - \widetilde{\boldsymbol{P}}^{(l)}_{i,j})\boldsymbol{E}^{(i,j)}\boldsymbol{V}^T_{l,:} \otimes \boldsymbol{U}\Big\| \leq \begin{cases} \Big\|\big(\boldsymbol{V}_{l,:} \otimes \boldsymbol{U}_{j,:}, \boldsymbol{V}_{l,:} \otimes \boldsymbol{U}_{i,:}\big)\Big\|, & \text{if } i < j, \\ \Big\|\big(\boldsymbol{V}_{l,:} \otimes \boldsymbol{U}_{i,:}\big)\Big\|, & \text{if } i = j. \end{cases} \tag{S1.2}$$

The right-hand side of $(S1.2)$ can be further upper bounded by $\sqrt{2}|\boldsymbol{V}_{l,:}||$ max $\{||\boldsymbol{U}_{i,:}||, ||\boldsymbol{U}_{j,:}||\}$, where the coefficient $\sqrt{2}$ is due to the case that $i < j$ and $c^*_i = c^*_j$. Note that

$$\begin{aligned} \sigma_{L_0}\Big(\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\Big) &= \sigma_{L_0}\Big(\boldsymbol{V}\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\Big) \\ &= \sigma_{L_0}\Big(\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{B}\big)(\boldsymbol{\Gamma} \otimes \boldsymbol{\Gamma})\Big) = \Omega(n\overline{\psi}\sqrt{L}s_n), \end{aligned}$$

where the last equality follows from Assumption D. Therefore, for any $l \in [L]$, we have

$$\begin{aligned} \|\boldsymbol{V}_{l,:}\|\sigma_{L_0}\Big(\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\Big) &\leq \|(\boldsymbol{V}_{l,:})^T\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\| \\ &= \|\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O} \times_3 \boldsymbol{V}\big)_{:,:,l}\|_F \\ &= \|\big(\boldsymbol{B} \times_1 \boldsymbol{\Gamma} \times_2 \boldsymbol{\Gamma}\big)_{:,:,l}\|_F = O(n\overline{\psi}s_n). \end{aligned}$$

Combining the above two bounds implies that

$$\|\boldsymbol{V}_{l,:}\| = \frac{\|\boldsymbol{V}_{l,:}\|\sigma_{L_0}\Big(\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\Big)}{\sigma_{L_0}\Big(\boldsymbol{\mathcal{M}}_3\big(\boldsymbol{\mathcal{C}} \times_1 \boldsymbol{O} \times_2 \boldsymbol{O}\big)\Big)} = O(\frac{1}{\sqrt{L}}).$$

Thus, it follows from Assumption A and B that

$$\Big\|(\widetilde{\boldsymbol{A}}_{i,j}^{(l)} - \widetilde{\boldsymbol{P}}_{i,j}^{(l)})\boldsymbol{E}^{(i,j)}\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\Big\| \leq \sqrt{\frac{2C_1}{L\min\{n_{c^*i}, n_{c_j^*}\}}} \leq C_2\sqrt{\frac{1}{nL}},$$

for some absolute constant C_2. We next proceed to bound the second-order central moment. Denote

$$\mathrm{Var}_1 = \sum_{l=1}^{L}\sum_{i\leq j}\mathbb{E}\big(\widetilde{\boldsymbol{A}}_{i,j}^{(l)} - \widetilde{\boldsymbol{P}}_{i,j}^{(l)}\big)^2\boldsymbol{E}^{(i,j)}\big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\big)\big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\big)^T\boldsymbol{E}^{(i,j)},$$
$$\mathrm{Var}_2 = \sum_{l=1}^{L}\sum_{i\leq j}\mathbb{E}\big(\widetilde{\boldsymbol{A}}_{i,j}^{(l)} - \widetilde{\boldsymbol{P}}_{i,j}^{(l)}\big)^2\Big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\Big)^T\boldsymbol{E}^{(i,j)}\boldsymbol{E}^{(i,j)}\big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\big).$$

In what follows, we proceed to bound $\|\mathrm{Var}_1\|$ and $\|\mathrm{Var}_2\|$, separately, and then obtain a upper bound for $\max\{\|\mathrm{Var}_1\|, \|\mathrm{Var}_2\|\}$.

Let $\boldsymbol{Q} = \boldsymbol{F}\boldsymbol{D}$ and then we can verify that $\big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\big)\big(\boldsymbol{V}_{l,:}^T \otimes \boldsymbol{U}\big)^T =$

$\|\boldsymbol{V}_{l,:}\|^2\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{\Gamma}^{-2}\boldsymbol{Z}^T\boldsymbol{Q}$. Thus,

$$\begin{aligned}
\mathrm{Var}_1 &= \sum_{l=1}^{L}\|\boldsymbol{V}_{l,:}\|^2\sum_{i\leq j}\mathbb{E}(\widetilde{\boldsymbol{A}}_{i,j}^{(l)}-\widetilde{\boldsymbol{P}}_{i,j}^{(l)})^2\boldsymbol{E}^{(i,j)}\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{\Gamma}^{-2}\boldsymbol{Z}^T\boldsymbol{Q}\boldsymbol{E}^{(i,j)} \\
&= \sum_{l=1}^{L}\|\boldsymbol{V}_{l,:}\|^2\sum_{i\leq j}(f_if_j\boldsymbol{\mathcal{P}}_{i,j,l}+\frac{1-f_if_j}{2})(\frac{1+f_if_j}{2}-f_if_j\boldsymbol{\mathcal{P}}_{i,j,l}) \\
&\quad\times\boldsymbol{E}^{(i,j)}\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{\Gamma}^{-2}\boldsymbol{Z}^T\boldsymbol{Q}\boldsymbol{E}^{(i,j)} \\
&= \sum_{l=1}^{L}\|\boldsymbol{V}_{l,:}\|^2\sum_{i\leq j}\big(\frac{1}{4}-\frac{f_i^2f_j^2}{4}(1-2\boldsymbol{\mathcal{P}}_{i,j,l})^2\big)\boldsymbol{E}^{(i,j)}\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{\Gamma}^{-2}\boldsymbol{Z}^T\boldsymbol{Q}\boldsymbol{E}^{(i,j)} \\
&\preceq \sum_{l=1}^{L}\frac{\varphi_n\|\boldsymbol{V}_{l,:}\|^2}{4}\sum_{i\leq j}\boldsymbol{E}^{(i,j)}\boldsymbol{Q}\boldsymbol{Z}\boldsymbol{\Gamma}^{-2}\boldsymbol{Z}^T\boldsymbol{Q}\boldsymbol{E}^{(i,j)}=\sum_{l=1}^{L}\frac{\varphi_n\|\boldsymbol{V}_{l,:}\|^2}{4}(K\boldsymbol{I}_n+\boldsymbol{G}_{\boldsymbol{c}}),
\end{aligned}$$

where $\varphi_n = 1-\min_{i\in[n]}f+4s_n$, $\boldsymbol{I}_n$ is the n-dimensional identity matrix, and $\boldsymbol{G}_{\boldsymbol{c}}$ is a $n\times n$ matrix such that $(\boldsymbol{G}_{\boldsymbol{c}})_{ii}=0$ for $i\in[n]$, $(\boldsymbol{G}_{\boldsymbol{c}})_{ij}=(f_if_jd_id_j)\gamma_{c_i^*}^{-1}$ if $c_i^*=c_j^*$ and 0 otherwise for $i\neq j$. Herein, the partial order $\preceq$ between two matrix $\boldsymbol{M}^{(1)}$ and $\boldsymbol{M}^{(2)}$ is defined as $\boldsymbol{M}^{(1)}\preceq\boldsymbol{M}^{(2)}$ if and only if $\boldsymbol{M}^{(2)}-\boldsymbol{M}^{(1)}$ is positive semi-definite. This leads to

$$\|\mathrm{Var}_1\|\leq\sum_{l=1}^{L}\frac{\varphi_n\|\boldsymbol{V}_{l,:}\|^2}{4}\Big(K+\max_{i\in[n]}\sqrt{\frac{f_i^2d_i^2n_{c_i^*}}{\gamma_{c_i^*}}}\Big)\leq\sum_{l=1}^{L}\frac{\varphi_n\|\boldsymbol{V}_{l,:}\|^2}{4}(K+\sqrt{C_1}),$$

where the first inequality follows from the triangle inequality, Gershgorin circle theorem and Cauchy-Schwarz inequality, and the second inequality follows from Assumption B.

Next, we turn to bound the spectral norm of Var_2. Note that

$$\begin{aligned}
\mathrm{Var}_2 &= \sum_{l=1}^{L}\sum_{i\le j}\mathbb{E}(\widetilde{\boldsymbol{A}}^{(l)}_{i,j}-\widetilde{\boldsymbol{P}}^{(l)}_{i,j})^2(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})^T\boldsymbol{I}^{(i,j)}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})\\
&=\sum_{l=1}^{L}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U}\Big)^T\big(\sum_{i\le j}\mathbb{E}(\widetilde{\boldsymbol{A}}^{(l)}_{i,j}-\widetilde{\boldsymbol{P}}^{(l)}_{i,j})^2\boldsymbol{I}^{(i,j)}\big)(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})\\
&=\sum_{l=1}^{L}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U}\Big)^T\big(\sum_{i\le j}\big(\frac{1}{4}-\frac{f_i^2f_j^2}{4}(1-2\boldsymbol{\mathcal{P}}_{i,j,l})^2\big)\boldsymbol{I}^{(i,j)}\big)(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})\\
&\preceq\sum_{l=1}^{L}\frac{n\varphi_n}{4}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})^T(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U}),
\end{aligned}$$

where $\boldsymbol{I}^{(i,j)}$ is the diagonal matrix with the (i,i)-th and (j,j)-th entries being 1 and all other entries being zero. Hence,

$$\|\mathrm{Var}_2\|\le\Big\|\sum_{l=1}^{L}\frac{n\varphi_n}{4}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})^T(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})\Big\|\le\sum_{l=1}^{L}\frac{n\varphi_n\|\boldsymbol{V}_{l,:}\|^2}{4}.$$

Since $n\gg(K+C_1)$, the variance condition can be met by the fact that $\sigma^2=\max\{\|\mathrm{Var}_1\|,\|\mathrm{Var}_2\|\}\le\sum_{l=1}^{L}n\varphi_n\|\boldsymbol{V}_{l,:}\|^2/4$. With this, Theorem 1.6 in [1] yields that, for any $t>0$,

$$\begin{aligned}
&\mathbb{P}\Big(\Big\|\sum_{l=1}^{L}\sum_{i\le j}(\widetilde{\boldsymbol{A}}^{(l)}_{i,j}-\widetilde{\boldsymbol{P}}^{(l)}_{i,j})\boldsymbol{E}^{(i,j)}(\boldsymbol{V}_{l,:}^T\otimes\boldsymbol{U})\Big\|\ge t\Big)\\
\le&(n+KL_0)\exp\Big\{-\frac{6t^2}{3\sum_{l=1}^{L}n\varphi_n\|\boldsymbol{V}_{l,:}\|^2+\frac{4C_2}{\sqrt{Ln}}t}\Big\}.
\end{aligned}$$

By the upper bound of $\|\boldsymbol{V}_{l,:}\|$, there exists an absolute constant C_3, such that for any $t>0$,

$$\mathbb{P}\Big(\Big\|\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{A}}}-\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\Big\|\ge t\Big)\le\exp\Big\{2\log n-\frac{C_3t^2}{n\varphi_n+\frac{t}{\sqrt{nL}}}\Big\}.$$

Taking $t = \sqrt{\frac{8}{C_3}\varphi_n n \log n}$, then with probability at least $1 - n^{-2}$, we have

$$\left\| \boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{A}}} - \widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V} \otimes \boldsymbol{U}) \right\| < \sqrt{\frac{8}{C_3}\varphi_n n \log n}. \tag{S1.3}$$

This completes the proof. ■

Proof of Theorem 1. Let $\widehat{\boldsymbol{U}} \in \mathbb{R}^{n\times K}$ be any matrix having orthonormal columns such that the column space of $\widehat{\boldsymbol{U}}$ is the same as the one that spanned by the first K leading left singular vectors of $\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{A}})$. Then the $(1+\tau)$-optimal K-medians algorithm is applied to estimate assignment matrix and spectral embedding centers, which finds a pair of solution $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$ such that

$$\|\widehat{\boldsymbol{Z}}\widehat{\boldsymbol{W}} - \widehat{\widehat{\boldsymbol{U}}}\|_{2,1} \leq (1+\tau) \min_{\boldsymbol{Z}\in\boldsymbol{\Delta}, \boldsymbol{W}\in\mathbb{R}^{K\times K}} \|\boldsymbol{Z}\boldsymbol{W} - \widehat{\widehat{\boldsymbol{U}}}\|_{2,1}.$$

Define $\mathcal{S}_k = \{i : c_i^* = k, \|\widehat{\boldsymbol{Z}}_{i,:}\widehat{\boldsymbol{W}} - \widetilde{\boldsymbol{U}}_{i,:}\boldsymbol{O}^{(2)}\| \geq \sqrt{2}/2\}$, where $\boldsymbol{O}^{(2)}$ is defined as in Lemma 3. It is easy to show that

$$\begin{aligned} \|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{W}}\|_{2,1} &\leq \|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\widehat{\boldsymbol{U}}}\|_{2,1} + \|\widehat{\widehat{\boldsymbol{U}}} - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{W}}\|_{2,1} \\ &\leq (2+\tau)\|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\widehat{\boldsymbol{U}}}\|_{2,1}, \end{aligned}$$

where the last inequality follows from the $(1+\tau)$-optimality of $(\widehat{\boldsymbol{Z}}, \widehat{\boldsymbol{W}})$.

$$\|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\widehat{\boldsymbol{U}}}\|_{2,1} = \sum_{i=1}^{n} \left\| \frac{\boldsymbol{U}_{i,:}}{\|\boldsymbol{U}_{i,:}\|}\boldsymbol{O}^{(2)} - \frac{\widehat{\boldsymbol{U}}_{i,:}}{\|\widehat{\boldsymbol{U}}_{i,:}\|} \right\| \leq 2\sum_{i=1}^{n} \frac{\|\boldsymbol{U}_{i,:}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}_{i,:}\|}{\|\boldsymbol{U}_{i,:}\boldsymbol{O}^{(2)}\|},$$

where the inequality follows from the fact that $\left\|\frac{v_1}{\|v_1\|} - \frac{v_2}{\|v_2\|}\right\| \leq 2\frac{\|v_1 - v_2\|}{\|v_1\|}$ for any two vectors with same dimension. By the Cauchy-Swartz inequality, we further have

$$
\begin{aligned}
\|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\widetilde{\boldsymbol{U}}}\|_{2,1} &\leq 2\sqrt{\sum_{i=1}^{n}\|\boldsymbol{U}_{i,:}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}_{i,:}\|^2 \sum_{j=1}^{n}\frac{1}{\|\boldsymbol{U}_{j,:}\boldsymbol{O}^{(2)}\|^2}} \\
&= 2\sqrt{\sum_{i=1}^{n}\|\boldsymbol{U}_{i,:}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}_{i,:}\|^2}\sqrt{\sum_{j=1}^{n}\frac{\gamma_{c_j^*}}{f_j^2 d_j^2}} \\
&= 2\|\boldsymbol{U}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}\|_F\sqrt{\sum_{j=1}^{n}\frac{\gamma_{c_j^*}}{f_j^2 d_j^2}} = 2\|\boldsymbol{U}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}\|_F\sqrt{\sum_{k=1}^{K} n_k^2 v_k},
\end{aligned}
$$

where $v_k = n_k^{-2}\sum_{c_i^*=k}\gamma_k(f_i d_i)^{-2}$. Here it can be verified that v_k takes value in $[1, \infty)$ and higher privacy requirement of nodes within community k leads to larger value of v_k, which then leads to a slower convergence rate.

Next, we proceed to establish the connection between the Hamming error of $\hat{\boldsymbol{c}}$ and $\|\boldsymbol{U}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}\|_F^2$. It is straightforward to verified that

$$
\begin{aligned}
Err(\hat{\boldsymbol{c}}, \boldsymbol{c}^*) &\leq \frac{1}{n}\sum_{k=1}^{K}|S_k| \\
&\leq \frac{\sqrt{2}}{n}\|\widetilde{\boldsymbol{U}}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{Z}}\widehat{\boldsymbol{W}}\|_{2,1} \\
&\leq \frac{\sqrt{2}(4+2\tau)}{n}\|\boldsymbol{U}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}\|_F\sqrt{\sum_{k=1}^{K} n_k^2 v_k}.
\end{aligned}
$$

Combined with Lemma 3 and 4, it follows that

$$
\begin{aligned}
Err(\hat{\boldsymbol{c}}, \boldsymbol{c}^*) &\leq \frac{\sqrt{2}(4+2\tau)}{n} \|\boldsymbol{U}\boldsymbol{O}^{(2)} - \widehat{\boldsymbol{U}}\|_F \sqrt{\sum_{k=1}^K n_k^2 v_k} \\
&\leq \frac{\sqrt{\sum_{k=1}^K n_k^2 v_k}(16+8\tau)}{n} \frac{\Big(2\sigma_1\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big) + \delta_n\Big)\delta_n}{\sigma_K^2\big(\boldsymbol{\mathcal{M}}_1(\widetilde{\boldsymbol{\mathcal{P}}})(\boldsymbol{V}\otimes\boldsymbol{U})\big)} \\
&= O_p\Big(\sqrt{\sum_{k=1}^K v_k} \big(\frac{\sqrt{\varphi_n \log n}}{\sqrt{nL} s_n \overline{\psi}} + \frac{\varphi_n \log n}{n s_n^2 \overline{\psi}^2 L}\big)\Big) \\
&= O_p\Big(\sqrt{\sum_{k=1}^K v_k} \frac{\sqrt{\varphi_n \log n}}{\sqrt{nL} s_n \overline{\psi}}\big)\Big).
\end{aligned}
$$

where the last equality follows from the lower of s_n in Assumption 3, yielding that $\frac{\sqrt{\varphi_n \log n}}{\sqrt{nL} s_n \overline{\psi}}$ vanishes. This completes the proof. ■

Proof of Corollary 1. (1) Suppose $f_i f_j \asymp \alpha_n^2$, for $i,j \in [n]$, under the condition of Corollary 1, we have $v_k = \frac{\gamma_k}{n_k^2} \sum_{c_i^*=k} \frac{1}{f_i^2 d_i^2} \asymp \frac{n_k \alpha_n^2}{n_k^2} \cdot \frac{n_k}{\alpha_n^2} \asymp 1$, $\varphi_n = 1 - \alpha + 4s_n = O(1)$, and $\overline{\psi} = \frac{1}{n}\sum_{i=1}^n (f_i d_i)^2 \asymp \alpha_n^2$. By the result of theorem 1, we have

$$
Err(\hat{\boldsymbol{c}}, \boldsymbol{c}^*) = O_P\big(\sqrt{\frac{\log n}{nL s_n^2 \alpha_n^4}}\big) = o_p(1).
$$

(2) Under the condition of Corollary 1, we have

$$
\begin{aligned}
v_k = \frac{\gamma_k}{n_k^2} \sum_{c_i^*=k} \frac{1}{f_i^2 d_i^2} &\asymp \frac{\big(\alpha_n^2 \beta_n n_k + (1-\beta_n) n_k\big)\big(\beta_n n_k/\alpha_n^2 + (1-\beta_n) n_k\big)}{n_k^2} \\
&\asymp (1-\beta_n)(\beta_n/\alpha_n^2 + 1 - \beta_n).
\end{aligned}
$$

Therefore, $\sqrt{\sum_k v_k} \asymp \sqrt{(1-\beta_n)(\beta_n/\alpha_n^2 + 1 - \beta_n)}$, and $\varphi_n = 1 - \alpha_n + 4s_n =$

$O(1)$. In addition,

$$\overline{\psi} = \frac{1}{n}\sum_{i=1}^{n}(f_i d_i)^2 = \frac{1}{n}\sum_{k=1}^{K}\left(\alpha_n^2\beta_n n_k + (1-\beta_n)n_k\right) = \Omega(1-\beta_n).$$

Note that the condition $\frac{\beta_n}{\alpha_n^2(1-\beta_n)} \ll \frac{nLs_n^2}{\log n}$ and $\frac{\log n}{nLs_n^2} = o(1)$ imply that $(1-\beta_n)\left(\frac{\beta_n}{\alpha_n^2} + 1 - \beta_n\right) \ll \frac{nLs_n^2\overline{\psi}^2}{\log n}$. By the result of Theorem 1, we have

$$Err(\hat{\boldsymbol{c}}, \boldsymbol{c}^*) = O_p\left(\sqrt{(1-\beta_n)\left(\frac{\beta_n}{\alpha_n^2} + 1 - \beta_n\right)}\sqrt{\frac{\varphi_n \log n}{nLs_n^2\overline{\psi}^2}}\right) = o_p(1).$$

■

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