Projection-based diagnostic test for generalized functional regression models

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Supplementary Material

In this document, we present notation (in Section S.1), the proposed PD test for the functional error (in Section S.2), the proofs for the main theorems (in Section S.3), several corollaries and their proofs (in Section S.4), and the proofs of several lemmas (in Section S.5) that are used to prove the main results. We also present various auxiliary numerical results.

S.1 Notation

Let $q = Kd+1$. Denote $\varpi = (\varpi_i)_{i=1}^n := \left(\hat{\ell}(m(Y_i, \eta_i))\right)_{i=1}^n$, $V = \text{Diag}\{\varpi_1^2, \ldots, \varpi_n^2\}$, $D = \{D_i\}_{i=1}^n$ and $\tilde{D} = \left\{\left(\hat{\ell}(m(Y_i, \eta_i))\right)^{1/2}D_i\right\}_{i=1}^n$ are $n \times q$-dimensional matrices, $\Gamma = \lim_{n \to \infty} \frac{D^\top V D}{n} := \{\Gamma_{k,l}\}_{1 \leq k, l \leq q}$, $\tilde{\Gamma} = \lim_{n \to \infty} \left(\frac{\tilde{D}^\top \tilde{D}}{n}\right) = (\tilde{\Gamma}_{k,l})_{1 \leq k, l \leq q}$, $\Xi = \tilde{\Gamma}^{-1} = (\zeta_{j,k})_{1 \leq j, k \leq q}$, $\sum_{k=1}^q \zeta_{j,k}^{(1/2)} \zeta_{k,l}^{(1/2)} = \zeta_{j,l}$.

For the local alternatives, let $Y_{i,n} = g_l(a, \langle X_{i,1}, \beta_1 \rangle, \ldots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i, n^{-1/2} F(X_i))$, $Y_i = g_l(a, \langle X_{i,1}, \beta_1 \rangle, \ldots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i, 0) := g(a, \langle X_{i,1}, \beta_1 \rangle, \ldots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i)$,
S.2. PD test for the function-valued error

Under model \( (2.5) \), let \( q_f = K^2d + K \). Denote \( \epsilon_i^f = \epsilon_i(\cdot) \), and \( \varpi_f = \left( \hat{\ell}(m(Y_i(t), \eta_i^f(t))) \right)_{i=1}^{n_f} \), and \( D_f = \{ D_i^f \}_{i=1}^{n_f} \) is a \( n \times q_f \)-dimensional matrix. \( \langle D_f, \varpi_f \rangle = \left\{ \sum_{i=1}^{n_f} (D_i^f)^\top \varpi_i^f \right\}_{k=1}^{q_f} \) is a \( q_f \times q_f \)-dimensional matrix. \( \Gamma_f := \{ \Gamma_{k,l}^f \}_{1 \leq k,l \leq q_f} := \{ \lim_{n \to \infty} \frac{\sum_{i=1}^{n_f} (D_i^f)^\top \varpi_i^f (D_i^f)^\top} {n} \}_{1 \leq k,l \leq q_f} \) is a \( q_f \times q_f \)-dimensional matrix.

Let \( \hat{\Gamma}_n^f = \sum_{i=1}^{n_f} \int \hat{\ell}(m(Y_i(t), \eta_i^f(t))) D_i^f(t) (D_i^f(t))^\top dt \) be a \( q_f \times q_f \)-dimensional matrix, \( \hat{\Gamma}_f \hat{\Gamma}_f = \lim_{n \to \infty} \left( \hat{\Gamma}_n^f \right) = (\hat{\Gamma}_{k,l}^f)_{0 \leq k,l \leq q_f}, \Xi_f = \hat{\Gamma}_n^{-1} = (\hat{\zeta}_{j,k}^f)_{0 \leq j,k \leq q_f}, \sum_{k=1}^{q_f} (\hat{\zeta}_{j,k}^f)^{1/2} (\hat{\zeta}_{k,l}^f)^{1/2} = \hat{\zeta}_{j,l}^f \). Let \( \nu_f = \mathbb{E}(\nu_n^f) \) and \( \nu_n^f = n^{-1} \Xi_n \sum_{i=1}^{n} \int \hat{\ell}(m(Y_i(t), \eta_i^f(t))) m_g(Y_i(t), \eta_i^f(t)) \hat{g}_t(\eta_i^f(t), \epsilon_i(t), 0) \mathcal{F}(X_i) D_i^f(t) dt. \)

For the local alternatives, let \( Y_{i,n}(t) = g_t \left( a(t), \langle X_{i,1}, \beta_1 \rangle, \cdots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i(t), n^{-1/2} \mathcal{F}(X_i) \right) \),
Let \( \varepsilon^f(X_i; \beta)(\cdot) = m(Y_i(\cdot), a(\cdot), \langle X_{i,1}, \beta_1(\cdot) \rangle, \cdots, \langle X_{i,d}, \beta_d(\cdot) \rangle) \), and
\[
\varepsilon^f(X_i; \beta)(\cdot) = m(Y_{i,n}(\cdot), (\hat{a}_f)\tau \hat{\phi}(\cdot), (\hat{\xi}_i \otimes \hat{\phi}(\cdot))^\tau \hat{b}_f, \cdots, (\hat{\xi}_id \otimes \hat{\phi}(\cdot))^\tau \hat{b}_fd),
\]
where \( \otimes \) indicates the Kronecker product.

To avoid any subjective selection of \( \alpha \) and \( \gamma \), which may cause the test to be inconsistent (Escanciano 2006), we consider integrating out all possible \( \alpha \)'s and \( \gamma \)'s. Particularly, we consider the following PD test statistic:
\[
T_{n,F}^f(\beta) = \int_{\mathcal{G}p} \int_{\mathcal{S}pd} \int_{\mathcal{R}^2} \left( M_{n,F}(\beta; \alpha, \gamma, u, v) \right)^2 F_{n,\langle \varepsilon^f, \gamma \rangle}(du) F_{n,\langle \varepsilon^f, \alpha \rangle}(dv) d\alpha d\gamma, T_{n,F}^f
\]
To estimate \( \beta \), define the loss functions \( \ell(\varepsilon^f(X; \beta)) \) as \( \int \ell(\varepsilon^f(X; \beta)(t))dt \), where
\[
\varepsilon^f(X; \beta)(t) := m\left(Y(t), a(t), \langle X_{1,1}, \beta_1(t) \rangle, \cdots, \langle X_{d,1}, \beta_d(t) \rangle \right),
\]
is the residual between \( Y(t) \) and the predictors \( X \) based on model (2.5).
For model (2.5), assume \( \beta_j(s, t) = \sum_{k,l=1}^{j} b_{j,kl}^f \phi_k(s) \phi_l(t), j = 1, \cdots, d \) and
which is equal to the projection of $X$. We assume $0$. Thus, we obtain $\hat{\beta}$.

Next we will show the details on calculating the test statistics. Let

$$ a(t) = \sum_{j=1}^{K} a_{j}^T \phi_j(t). $$

Denote $b_{j}^T = \{b_{j,k,l}^T\}_{k,l}^{K}$, for $j = 1, \cdots, d$, $a_f = \{a_{j}^T\}_{j=1}^{K}$, and $\theta_f = ((a_f)^T, (b_1^T)^T, \cdots, (b_d^T)^T)^T$. Then, the estimating equation takes the form:

$$ U_f(\theta_f) := \sum_{i=1}^{n} \int \epsilon (m(Y_i(t), \hat{\eta}_i(t))) \hat{D}_i(t) dt = 0, $$

where $\hat{\eta}_i(t) = \left( (a_f)^T \phi(t), (b_1^T)^T \left( \hat{\xi}_1 \phi(t) \right), \cdots, (b_d^T)^T \left( \hat{\xi}_d \phi(t) \right) \right)$, $\phi(t) = \hat{\phi}_j(t)$, $\hat{D}_i(t) = \left( m_0(Y_i(t), \hat{\eta}_i(t)) \phi(t), m_1(Y_i(t), \hat{\eta}_i(t)) \left( \hat{\xi}_1 \phi(t) \right), \cdots, m_d(Y_i(t), \hat{\eta}_i(t)) \left( \hat{\xi}_d \phi(t) \right) \right)^T$. $\hat{\theta}_f$ is denoted as the solution to $U_f(\theta_f) = 0$. Thus, we obtain $\hat{\beta}_j(s, t) = \sum_{k=1}^{K} \hat{b}_{j,k,l}^T \phi_k(s) \phi(t), j = 1, \cdots, d$. Further, we assume $p \leq K$. Since for $p \geq K$, the projection $\langle X_i, \gamma \rangle_m = \sum_{k=1}^{K} \xi_i, k \gamma_k$, which is equal to the projection of $X_i$ along the direction $\gamma \in \mathbb{R}^K$.

Next we will show the details on calculating the test statistics. Let

$$ B_{ijkl}(\beta) = \int_{\gamma} I(\langle \epsilon_i(X_i; \beta), \gamma \rangle_m $$

$$ \leq \langle \epsilon_i(X_i; \beta), \gamma \rangle_m I(\langle \epsilon_i(X_i; \beta), \gamma \rangle_m \leq \langle \epsilon_i(X_i; \beta), \gamma \rangle_m d\gamma. $$

After some calculations, we can obtain that

$$ T_{n,F}^{ij}(\beta) = n^{-3} \sum_{i,j,k,l} A_{ijkl} B_{ijkl}(\beta) - 2n^{-4} \sum_{i,j,k,l,s} A_{ijkl} B_{is,l}(\beta) + n^{-5} \sum_{i,j,k,l,s_1,s_2} A_{ijkl} B_{s_1s_2l}(\beta). $$

Note that $B_{ijkl}(\beta)$ involves $p$-dimensional integrals, we solve the computational problem similarly to the way of obtaining $A_{ijkl}$.

The following conditions are needed to establish asymptotic properties for model (2.5):
(C1f) \( \{ (Y_i, \{ X_{i,j} \}_{j=1}^d) \}_{i=1}^n \) is a sequence of i.i.d. function-valued random vectors with

\[
0 < \inf_{0 < t < 1} \mathbb{E}|Y_i(t)| < \sup_{0 < t < 1} \mathbb{E}|Y_i(t)| < \infty.
\]

(C2f) \( \epsilon_i(\cdot) \) is an independent mean zero process and independent of \( \{ X_{i,j}(\cdot) \}_{j=1}^d \), and \( X_{i,j}(\cdot) \in L^2[0, 1] \), for \( j = 1, \cdots, d \). The covariance function of each \( X_{i,j} \) is positive definite with a spectral decomposition \( \Sigma_j(s, t) = \sum_{k=1}^{\infty} \theta_{j,k} \phi_k(s) \phi_k(t) \), where \( \theta_{j,1} > \theta_{j,2} > \cdots \) and \( C^{-1}k^{-\alpha} \leq \theta_{j,k} \leq Ck^{-\alpha}, \theta_{j,k} - \theta_{j,k+1} \geq C^{-1}k^{-\alpha-1} \), for \( \alpha > 1, j = 1, \cdots, d, k \geq 1 \). Furthermore, the true coefficient function \( \beta_j(s, t) = \sum_{k_1, k_2=1}^{\infty} b_{j,k_1,k_2} \phi_{k_1}(t) \phi_{k_2}(s) \) with \( |b_{j,k_1,k_2}| \leq C_1 \min(k_1^{-\kappa_1}, k_2^{-\kappa_2}) \), for \( k_1, k_2 \geq 1, \kappa \geq \alpha + 3 \).

(C3f) \( \mathbb{E}(\langle \varpi_i^f, D_i^f \rangle | X_i) = 0, \sup_{1 \leq k \leq q_f} \mathbb{E}(\langle \varpi_i^f, D_i^f \rangle^2) < c_2 < \infty \), and

\[
0 < c_1 \leq \inf_{0 < t < 1, 1 \leq i \leq n} \bar{\ell}(m(Y_i(t), \eta_i^f(t))) \leq \sup_{0 < t < 1, 1 \leq i \leq n} \bar{\ell}(m(Y_i(t), \eta_i^f(t))) \leq c_u < \infty.
\]

(C4f) \( m(\cdot) \) has continuous bounded first-order derivatives, and \( \| D_f \|_\infty \leq c < \infty, \tilde{\Gamma}_f \) is positive definite, and has a bounded maximum eigenvalue.

(C5f) \( Kn^{-\frac{1}{2\kappa+\alpha-1}} \) is bounded away from zero and infinity as \( n \to \infty \). The
following equations hold:

\[
\sum_{k_1,k_2,k_3,k_4=1}^{q_f} \mathbb{E}\left( \langle D_{i,k_1}^f, \varsigma_{i}^f \rangle \langle D_{i,k_2}^f, \varsigma_{i}^f \rangle \langle D_{i,k_3}^f, \varsigma_{i}^f \rangle \langle D_{i,k_4}^f, \varsigma_{i}^f \rangle \right) = o\left( \frac{n}{q_f} \right),
\]

\[
\sum_{k_1,\ldots,k_8=1}^{q_f} \mathbb{E}\left( \langle D_{i,k_1}^f, \varsigma_{i}^f \rangle \langle D_{i,k_2}^f, \varsigma_{i}^f \rangle \langle D_{i,k_3}^f, \varsigma_{i}^f \rangle \langle D_{i,k_4}^f, \varsigma_{i}^f \rangle \right) \times \mathbb{E}\left( \langle D_{i,k_5}^f, \varsigma_{i}^f \rangle \langle D_{i,k_6}^f, \varsigma_{i}^f \rangle \langle D_{i,k_7}^f, \varsigma_{i}^f \rangle \langle D_{i,k_8}^f, \varsigma_{i}^f \rangle \right) = o\left( n^2 q_f^2 \right).
\]

Conditions (C1f)–(C4f) are general conditions easy to be satisfied in practice. Particularly, Condition (C1f) is the regularity condition on the upper bound of the expectation of the outcome. Condition (C2f) is about the independence between error and covariates, and smoothness of the covariance functions of covariates $X$ and true coefficient function $\beta_j$. Noting that condition (C2f) is stronger than Condition (C2) for the scalar response model in terms of $\kappa \geq \alpha + 3$ instead of $\kappa \geq \alpha + 2$ required in Condition (C2). This is because the dimension of parameters involved in function-valued response model is larger than that in scalar response model. That is, $q_f = K^2d + K$ in comparison to $q = Kd + K$. Conditions (C3f) and (C4f) are conditions for the link function $g$ and loss function $\ell$, which are expressed through the conditions on $m(\cdot)$ and $\ell$ since function $m(\cdot)$ is determined by the link function $g$. For example, the continuous bounded first-order derivatives required for $m(\cdot)$ imply $g(\cdot)$ has continuous bounded first-order derivatives.

In addition, the order of the second and fourth moment of estimation equa-
tions, which involves \(m(\cdot)\) and \(\ell(\cdot)\), required by condition (C5f) is also related to the link function \(g\). These conditions are easily to be satisfied by commonly used canonical link functions \(g(\cdot)\) which are monotonic with high-order smoothness. Condition (C5f) is to ensure that plug-in \(\hat{\beta}\) does not affect the convergence property and the order of local power. With the condition (C5f), the number of the parameters \(\theta\) satisfies
\[
q_f := K^2 d + K \approx \left. \frac{n^2}{\alpha + 2\alpha - 1} \right. + \frac{n^{2\alpha - 3}}{\alpha + 2\alpha - 1},
\]

it follows that
\[
\int_{0}^{1} \left( \hat{\beta}_j(s, t) - \beta_j(s, t) \right)^2 dsdt = O_p(n^{-\frac{2\alpha - 3}{\alpha + 2\alpha - 1}})
\]
and
\[
\mathbb{E} \int_{0}^{1} \left( \langle x, \hat{\beta}_j(t) \rangle - \langle x, \beta_j(t) \rangle \right)^2 dt = O(n^{-\frac{2\alpha - 3}{\alpha + 2\alpha - 1}})\]
for \(j = 1, \ldots, d\) and for any fixed function \(x(t) = \sum_{k=1}^{\infty} x_k \phi_k(t)\) with \(|x_k| \leq C k^{-\alpha/2}\).

**Theorem S.1.** Under conditions (C1f) and (C2f), and the null hypothesis (3), if \(p = o(n)\), for model (2.5) and any \(m \in \mathcal{R}\),
\[
\text{Prob}(T_{n,F}^{0,f}(\beta_0) < m) - \text{Prob}(T_{\infty,F}^{0,f} < m) \to 0,
\]
where
\[
T_{\infty,F}^{0,f} := \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} \int_{\mathbb{R}^2} \left( M_{\infty,F}^{0,f}(\alpha, \gamma, u, v) \right)^2 F(\epsilon, \gamma, m)(dv)F(x, \alpha, m)(du)d\alpha d\gamma,
\]
and \(M_{\infty,F}^{0,f}(\cdot, \cdot, \cdot, \cdot)\) is a Gaussian process with zero mean and covariance function
\[
K((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) = \left\{ \mathbb{E} \left[ I \left( \langle \epsilon_1, \gamma_1 \rangle_m \leq v_1 \right) I \left( \langle \epsilon_1, \gamma_2 \rangle_m \leq v_2 \right) \right] \right.
\]
\[
- F(\epsilon, \gamma_1, m)(v_1) F(\epsilon, \gamma_2, m)(v_2) \left\} \mathbb{E} (\Delta(X_i; \alpha_1, u_1) \Delta(X_i; \alpha_2, u_2)) \right. .
\]

Next we establish the asymptotic distribution of test statistics with
estimated parameters. Let \( \zeta^f_m(\alpha, \gamma, u) = E \left[ \Delta(X_i; \alpha, u) \langle D_i^f, \gamma \rangle_m \right] \), and

\[
\sigma^2_f(\alpha_1, \gamma_1, u_1, \alpha_2, \gamma_2, u_2) = \sum_{k_1,k_2,l_1,l_2} \zeta^f_{j_1,k_1} \zeta^f_{j_2,k_2} \Gamma^f_{k_1,k_2} \zeta^f_{l_1,l_2} \Gamma^f_{l_1,l_2}(\alpha_1, \gamma_1, u_1) \zeta^f_{l_2,F}(\alpha_2, \gamma_2, u_2),
\]

\[
\sigma^f_{c,F}(\alpha_1, \gamma_1, u_1, \alpha_2, \gamma_2, u_2) = \sum_{k,l=1}^{q_f} \zeta^f_{i,k,l,F}(\alpha_2, \gamma_2, u_2) E \left[ I(\langle \epsilon_i^f, \gamma_1 \rangle_m \leq v_1) \langle D_i^f, \epsilon_i^f \rangle \Delta(X_i; \alpha_1, u_1) \right].
\]

**Theorem S.2.** For model (2.5), if conditions (C1f)–(C5f) hold, and under null hypothesis (3), for any \( m \in \mathcal{R} \),

\[
\text{Prob}(T^f_{n,F}(\bar{\beta}) < m) - \text{Prob}(T^f_{\infty,F} < m) \to 0,
\]

where \( T^f_{\infty,F} := \int_{S^p} \int_{S^p} \int_{\mathbb{R}^2} \left( M^f_{\infty,F}(\alpha, \gamma, u, v) \right)^2 F_{(\epsilon^f, \gamma)_m}(dv) F_{X,\alpha}_m(du) d\alpha d\gamma, \)

and \( M^f_{\infty,F} \equiv M^0_{\infty,F} + M^{e,f}_{\infty,F} \), and \( M^{e,f}_{\infty,F}(\cdot, \cdot, \cdot) \) is a Gaussian process with zero mean and covariance function \( K_1((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) = \sigma^2_f(\alpha_1, \gamma_1, u_1, \alpha_2, \gamma_2, u_2) \)

\[
f_{(\epsilon^f, \gamma_1)_m}(v_1)f_{(\epsilon^f, \gamma_2)_m}(v_2), \quad \text{and cov}\left(M^0_{\infty,F}(\alpha_1, \gamma_1, u_1, v_1), M^{e,f}_{\infty,F}(\alpha_2, \gamma_2, u_2, v_2)\right) = \sigma^f_{c,F}\]

\[
(\alpha_1, \gamma_1, u_1, v_1, \alpha_2, \gamma_2, u_2)f_{(\epsilon^f, \gamma_2)_m}(v_2).
\]

Now we study the asymptotic distribution of \( T^f_{n,F} \) under a sequence of local alternatives converging to null at a parametric rate \( n^{-1/2} \). We consider the local alternative

\[
H^f_{A,n} : Y_{i,n}(t) = g(t, (X_{i,1}, \beta_1), \ldots, (X_{i,d}, \beta_d), e_i(t), n^{-1/2} F(X_i)) S, 1
\]
where \( g_l(a, \langle X_{i,1}, \beta_1 \rangle, \cdots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i(t), 0) := g(a, \langle X_{i,1}, \beta_1 \rangle, \cdots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i(t)) \).

**Theorem S.3.** For model (2.5), under conditions (C1f)–(C5f) and (C6), and local alternative (S.1). For any \( m \in \mathcal{R} \),

\[
\text{Prob}(T_{n,f}(\tilde{\beta}) < m) - \text{Prob}(T_{\infty,F}^{a,f} < m) \to 0,
\]

where \( T_{\infty,F}^{a,f} := \int_{S_{pd}} \int_{S_p} \int_{\mathcal{R}^2} \left( M_{\infty,F}^{a,f}(\alpha, \gamma, u, v) \right)^2 F_{(\epsilon', \gamma, m)}(dv) F_{(x, \alpha, m)}(du) d\alpha d\gamma, \)

and \( \tilde{\beta} \) is the estimate obtained from model (2.5) using data \( \{Y_{i,n}(t), X_i\}_{i=1}^n \),

\[
M_{\infty,F}^{a,f} \equiv M_{\infty,F}^{1,f} - D_F^{a,f}, \quad \text{and}
\]

\[
D_F^{a,f}(\alpha, \gamma, u, v) = \mathbb{E} \left\{ \Delta(X_i; \alpha, u) \left( \left\langle m_g(Y_{i}(t), \eta_{i}^{f}(t))g_l(\eta_{i}^{f}(t), \epsilon_i(t), 0) \mathcal{F}(X_i), \gamma \right\rangle_m 
- \sum_{j=1}^q \left\langle D_{i,j}^{f}, \gamma \right\rangle_m v_{i,j}^{f} \right) \right\} f_{(\epsilon', \gamma, m)}(v).
\]

Consider the following local alternative hypothetical model:

\[
H_{A,n}^f : Y_{i,n}(t) = g_l(a(t), \langle X_{i,1}, \beta_1 \rangle, \cdots, \langle X_{i,d}, \beta_d \rangle, \epsilon_i(t), n' \mathcal{F}(X_i))(S.2)
\]

**Corollary S.1.** Under the conditions of Theorem S.3 and the alternative (S.2) with \(-1/2 < \nu \leq 0\), we have \( \text{Prob}(T_{n,f}^{f} \to \infty) = 1 \) as \( n \to \infty \).
S.3 Proofs

Proof of Proposition 1. Denote $X_{i,j}^{(N)}(t) = \sum_{k=1}^{N} \langle X_{i,j}, \phi_k \rangle \phi_k(t)$. For any $p$ and $\gamma \in \mathcal{R}^p$, let

\[
F_{(U,\gamma)_m|X^{(N)}}(u) = \text{Prob} \left( \langle U_i, \gamma \rangle_m \leq u \mid X_{i,1}^{(N)}, \ldots, X_{i,d}^{(N)} \right),
\]

\[
F_{(U,\gamma)_m|(X^{(N)},\alpha)_m}(u) = \text{Prob} \left( \langle U_i, \gamma \rangle_m \leq u \mid \langle X^{(N)}, \alpha \rangle_m \right).
\]

According to Lemma 2.1 (A) in [Lavergne & Patilea (2008)] and $U, X_j \in L^2[0,1]$, we obtain that, for any $N$,

\[
F_{(U,\gamma)_m|X^{(N)}}(u) \equiv \mathbb{E} \left[ I(\langle U_i, \gamma \rangle_m \leq u) \mid X^{(N)} \right] = \mathbb{E} (I(\langle U_i, \gamma \rangle_m \leq u)) \equiv F_{(U,\gamma)_m}(u),
\]

holds if and only if

\[
F_{(U,\gamma)_m|(X^{(N)},\alpha)_m}(u) \equiv \mathbb{E} \left[ I(\langle U_i, \gamma \rangle_m \leq u) \mid \langle X^{(N)}, \alpha \rangle_m \right] = \mathbb{E} (I(\langle U_i, \gamma \rangle_m \leq u)) \equiv F_{(U,\gamma)_m}(u), \quad \forall \alpha_j \in S^p, p \geq 1.
\]

On the other hand, as $N \to \infty$, $F_{(U,\gamma)_m|X^{(N)}}(u) \to F_{(U,\gamma)_m|X}(u)$. It then follows from Cramer-Wold theorem that $F_{(U,\gamma)_m|X^{(N)}}(u) \to F_{(U,\gamma)_m|X}$ holds if and only if $F_{(U,\gamma)_m|(X^{(N)},\alpha)_m}(u) \to F_{(U,\gamma)_m|(X,\alpha)_m}(u), \forall \alpha_j \in S^p, p \geq 1$. Thus, Proposition 1 follows.

Proofs of Theorems 1 and S.1 Note that under the null hypothesis (1.3), we have $\varepsilon(X; \beta_0) \equiv \epsilon$ for a scalar response, and $\varepsilon^f(X; \beta_0)(\cdot) \equiv \epsilon^f$ for a function-valued response.
We first prove Theorem 1. It follows from Theorem 1 of Escanciano (2006), and \( \mathbb{E}[I(\epsilon_i \leq v) - F_v(v)] = 0, \mathbb{E}[I(\epsilon_i \leq v) - F_v(v)]^2 = F_v(v)(1 - F_v(v)), \) that

\[
M_{n,F}(\beta_0; \alpha, u, v) = n^{-1/2} \sum_{i=1}^{n} [I(\epsilon_i \leq v) - F_{n,\epsilon}(v)] I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
= n^{-1/2} \sum_{i=1}^{n} [I(\epsilon_i \leq v) - F_v(v)] I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
- n^{-1/2} (F_{n,\epsilon}(v) - F_v(v)) \sum_{i=1}^{n} I(\langle X_i, \alpha \rangle_m \leq u) := T_{n,1}(\alpha, u, v) - T_{n,2}(\alpha, u, v).
\]

Note that \( \mathbb{E}(T_{n,1}(\alpha, u, v))^2 = F_v(v)(1 - F_v(v)) F_{\langle X, \alpha \rangle_m}(u), \)

\[
K_{T_{n,1}}((\alpha_{1, \bullet}, u_1, v_1), (\alpha_{2, \bullet}, u_2, v_2)) := \mathbb{E}(T_{n,1}(\alpha_{1, \bullet}, u_1)T_{n,1}(\alpha_{2, \bullet}, u_2, v_2))
\]

\[
= \{ \mathbb{E}[I(\epsilon_i \leq v_1) I(\epsilon_i \leq v_2)] - F_v(v_1)F_v(v_2) \} \mathbb{E}[I(\langle X, \alpha_{1, \bullet} \rangle_m \leq u_1) I(\langle X, \alpha_{2, \bullet} \rangle_m \leq u_2)],
\]

\[
\mathbb{E}(T_{n,2}(\alpha, u, v))^2 = F_v(v)(1 - F_v(v)) F_{\langle X, \alpha \rangle_m}^2(u) \left( 1 + O(n^{-1}) \right),
\]

\[
K_{T_{n,2}}((\alpha_{1, \bullet}, u_1, v_1), (\alpha_{2, \bullet}, u_2, v_2)) = \{ \mathbb{E}[I(\epsilon_i \leq v_1) I(\epsilon_i \leq v_2)] - F_v(v_1)F_v(v_2) \}
\]

\[
\times F_{\langle X, \alpha_{1, \bullet} \rangle_m}(u_1) F_{\langle X, \alpha_{2, \bullet} \rangle_m}(u_2),
\]

\[
\mathbb{E}\{T_{n,1}(\alpha, u, v)T_{n,2}(\alpha, u, v)\} = F_v(v)(1 - F_v(v)) F_{\langle X, \alpha \rangle_m}^2(u) \left( 1 + O(n^{-1}) \right),
\]

\[
K_{T_{n,1}T_{n,2}}((\alpha_{1, \bullet}, u_1, v_1), (\alpha_{2, \bullet}, u_2, v_2)) = \{ \mathbb{E}[I(\epsilon_i \leq v_1) I(\epsilon_i \leq v_2)] - F_v(v_1)F_v(v_2) \}
\]

\[
\times F_{\langle X, \alpha_{1, \bullet} \rangle_m}(u_1) F_{\langle X, \alpha_{2, \bullet} \rangle_m}(u_2).
\]

Thus, \( M_{n,F}(\beta_0; \alpha, u, v) \) converges to \( M_{\infty,F}^0(\alpha, u, v) \). Consequently, we have

\[
M_{n,F}(\beta_0; \alpha, u, v) \overset{d}{\to} M_{\infty,F}^0(\alpha, u, v)
\]

for any fixed \( u, v, \) and \( \alpha_j \in \mathbb{S}^p := \{ \alpha \in \} \)
$R^p : \|\alpha\|_p = 1$, for $j = 1, \cdots, d$, where $M^0_{\infty,F}$ is a Gaussian process with zero mean and covariance function

$$K((\alpha_{1,\bullet}, u_1, v_1), (\alpha_{2,\bullet}, u_2, v_2)) = \{\mathbb{E}[I(\epsilon_i \leq v_1)I(\epsilon_i \leq v_2)] - F_\epsilon(v_1)F_\epsilon(v_2)\} \times \mathbb{E}[\{I(\langle X_i, \alpha_{1,\bullet}\rangle_m \leq u_1) - F_{\langle X, \alpha_{1,\bullet}\rangle_m}(u_1)\} \{I(\langle X_i, \alpha_{2,\bullet}\rangle_m \leq u_2) - F_{\langle X, \alpha_{2,\bullet}\rangle_m}(u_2)\}].$$

Furthermore, based on the derivation, we have

$$|\text{Prob}(M_{n,F}(\beta_0; \alpha, u, v) < m) - \text{Prob}(M^0_{\infty,F}(\alpha, u, v) < m)| = O(n^{-1/2}).$$

Since the dimension of $\alpha$ is $pd$, which changes with $n$. From the central limit theorem for classes of functions changing with $n$ (Van Der Vaart & Wellner 1996, Thm. 2.11.23), as long as

$$P^*F_n^2 = O(1),$$

$$P^*F_n^2\{F_n > \eta\sqrt{n}\} \to 0, \text{ for every } \eta > 0,$$

$$\sup_{\rho((\alpha_{1,\bullet}, u_1, v_1), (\alpha_{2,\bullet}, u_2, v_2)) < \delta_n} P(M_{n,F}(\beta_0; \alpha_{1,\bullet}, u_1, v_1) - M_{n,F}(\beta_0; \alpha_{2,\bullet}, u_2, v_2)) \to 0, \text{ for every } \delta_n \downarrow 0,$$

$$\int_0^{\delta_n} \sqrt{\log N_{[1]}(\epsilon\|F_n\|_{P^2,F_n,L^2(P)})}d\epsilon \to 0, \text{ for every } \delta_n \downarrow 0, \quad (S.3)$$

where $F_n$ are envelope functions, then we have the sequence $\{M_{n,F}(\beta_0; \alpha, u, v)\}$ is asymptotically tight in $\ell^\infty(S^{pd} \times R^2)$ and converges in distribution to a Gaussian process provided point-wised convergence on $\ell^\infty(S^{pd} \times R^2)$. Since $S^{pd}$ is a $pd$-unit ball, combining with the probability measure, the $\ell^\infty$ norm
indicates that

\[
\left| \text{Prob}(T_{n,F}(\beta_0) < m) - \text{Prob}(T_{\infty,F}^0 < m) \right| \\
:= \left| \text{Prob}\left( \int_{\mathbb{S}^d} \int_{R^2} M_{n,F}(\beta_0; \alpha, u, v) F_{n,\epsilon}(dv) F_{n,\langle \alpha \rangle_m}(du) d\alpha < m \right) \\
- \text{Prob}\left( \int_{\mathbb{S}^d} \int_{R^2} M_{\infty,F}^0(\alpha, u, v) F_{\epsilon}(dv) F_{\langle \alpha \rangle_m}(du) d\alpha < m \right) \right| \\
\rightarrow 0.
\]

Taking \( F_n \) as a constant function, and \( \delta_n = n^{-1/2} \), then conditions (S.3) hold from

\[
\log N(\|F_n\|_{P,2}, F_n, L_2(P)) = pd \log \left( \frac{1}{\epsilon} \right),
\]

and \( pd = o(n) \). Therefore, as long as \( pd = o(n) \), Theorem 1 follows.

Next, we prove Theorem S.1. Recall that \( \epsilon(f(X_i; \beta_0)) = \epsilon^f_i \), then we have

\[
M_{n,F}^f(\beta_0; \alpha, \gamma, u, v) = n^{-1/2} \sum_{i=1}^n \left[ I \left( \langle \epsilon^f_i, \gamma \rangle_m \leq v \right) - F_{n,\langle \epsilon^f_i, \gamma \rangle_m}(v) \right] I \left( \langle X_i, \alpha \rangle_m \leq u \right)
\]

\[
= n^{-1/2} \sum_{i=1}^n \left[ I \left( \langle \epsilon^f_i, \gamma \rangle_m \leq v \right) - F_{\langle \epsilon^f_i, \gamma \rangle_m}(v) \right] I \left( \langle X_i, \alpha \rangle_m \leq u \right)
- n^{1/2} \left( F_{n,\langle \epsilon^f_i, \gamma \rangle_m}(v) - F_{\langle \epsilon^f_i, \gamma \rangle_m}(v) \right) F_{n,\langle X, \alpha \rangle_m}(u)
\]

\[
:= T_{n,1}^f(\alpha, \gamma, u, v) - T_{n,2}^f(\alpha, \gamma, u, v),
\]
and

\[ \mathbb{E} \left( T_{n,1}^f(\alpha, \gamma, u, v) \right)^2 \rightarrow F_{(\epsilon_f, \gamma)_m}(v) \left( 1 - F_{(\epsilon_f, \gamma)_m}(v) \right) F_{X, \alpha_m}(u), \]

\[ K_{T_{n,1}}((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) \rightarrow \left\{ \mathbb{E} \left[ I \left( \langle \epsilon_{i_1}^f, \gamma_1 \rangle_m \leq v_1 \right) I \left( \langle \epsilon_{i_2}^f, \gamma_2 \rangle_m \leq v_2 \right) \right] - F_{(\epsilon_f, \gamma_1)_m}(v_1)F_{(\epsilon_f, \gamma_2)_m}(v_2) \right\} \mathbb{E} \left[ I \left( \langle X_i, \alpha_{1,\bullet} \rangle_m \leq u_1 \right) I \left( \langle X_i, \alpha_{2,\bullet} \rangle_m \leq u_2 \right) \right], \]

\[ \mathbb{E} \left( T_{n,2}^f(\alpha, \gamma, u, v) \right)^2 \rightarrow F_{(\epsilon_f, \gamma)_m}(v) \left( 1 - F_{(\epsilon_f, \gamma)_m}(v) \right) F_{X, \alpha_m}(u), \]

\[ K_{T_{n,2}^f}((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) \rightarrow \left\{ \mathbb{E} \left[ I \left( \langle \epsilon_{i_1}^f, \gamma_1 \rangle_m \leq v_1 \right) I \left( \langle \epsilon_{i_2}^f, \gamma_2 \rangle_m \leq v_2 \right) \right] - F_{(\epsilon_f, \gamma_1)_m}(v_1)F_{(\epsilon_f, \gamma_2)_m}(v_2) \right\} F_{X, \alpha_{1,\bullet}}(u_1)F_{X, \alpha_{2,\bullet}}(u_2), \]

\[ \mathbb{E} \left( T_{n,1}^f(\alpha, \gamma, u, v)T_{n,2}^f(\alpha, \gamma, u, v) \right) \rightarrow F_{(\epsilon_f, \gamma)_m}(v) \left( 1 - F_{(\epsilon_f, \gamma)_m}(v) \right) F_{X, \alpha_m}(u), \]

\[ K_{T_{n,1}^f, T_{n,2}^f}((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) \rightarrow \left\{ \mathbb{E} \left[ I \left( \langle \epsilon_{i_1}^f, \gamma_1 \rangle_m \leq v_1 \right) I \left( \langle \epsilon_{i_2}^f, \gamma_2 \rangle_m \leq v_2 \right) \right] \times \mathbb{E} \left[ I \left( \langle X_i, \alpha_{1,\bullet} \rangle_m \leq u_1 \right) - F_{X, \alpha_{1,\bullet}}(u_1) \right] \left[ I \left( \langle X_i, \alpha_{2,\bullet} \rangle_m \leq u_2 \right) - F_{X, \alpha_{2,\bullet}}(u_2) \right] \right\}. \]

Then, Theorem S.1 follows from similar arguments to those in the proof of Theorem 1. In particular, we have \( M_{n,F}^f(\beta_0; \alpha, \gamma, u, v) \Rightarrow M_{\infty,F}^0(\alpha, \gamma, u, v) \) for any fixed \( u, v, \) and \( \alpha, \gamma \in \mathbb{S}_p := \{ \alpha \in R^p : \| \alpha \|_p = 1 \}, \) for \( j = 1, \cdots, d, \)

where \( M_{\infty,F}^0 \) is a Gaussian process with zero mean and covariance function

\[ K((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) \]

\[ = \left\{ \mathbb{E} \left[ I(\langle \epsilon_{i_1}^f, \gamma_1 \rangle_m \leq v_1)I(\langle \epsilon_{i_2}^f, \gamma_2 \rangle_m \leq v_2) \right] - F_{(\epsilon_f, \gamma_1)_m}(v_1)F_{(\epsilon_f, \gamma_2)_m}(v_2) \right\} \times \mathbb{E} \left[ I(\langle X_i, \alpha_{1,\bullet} \rangle_m \leq u_1) - F_{X, \alpha_{1,\bullet}}(u_1) \right] \left[ I(\langle X_i, \alpha_{2,\bullet} \rangle_m \leq u_2) - F_{X, \alpha_{2,\bullet}}(u_2) \right] \].
Furthermore, based on the derivation, we have

$$|\text{Prob}(M_{m,F}^{f}(\beta_0; \alpha, \gamma, u, v) < m) - \text{Prob}(M_{\infty,F}^{0,f}(\alpha, \gamma, u, v) < m)| = O(n^{-1/2}).$$

Similarly to the proof of Theorem 1, as long as \( p(d+1) = o(n) \), Theorem S.1 follows.

\[\square\]

**Proofs of Theorems 2 and S.2** We first prove Theorem 2. Note that

$$M_{n,F}(\hat{\beta}; \alpha, u, v) = n^{-1/2} \sum_{i=1}^{n} \left[ I(\varepsilon(X_i; \hat{\beta}) \leq v) - \hat{F}_{n,\varepsilon}(v) \right] I(\langle X_i, \alpha \rangle_m \leq u)$$

$$= M_{n,F}(\beta_0; \alpha, u, v) + n^{-1/2} \sum_{i=1}^{n} \left[ I(\varepsilon(X_i; \hat{\beta}) \leq v) - I(\varepsilon \leq v) \right]$$

$$\times \left[ I(\langle X_i, \alpha \rangle_m \leq u) - n^{-1} \sum_{j=1}^{n} I(\langle X_i, \alpha \rangle_m \leq u) \right]:= M_{n,F}(\beta_0; \alpha, u, v) + T_{n}.\]$$

It follows from Lemma S.1 that \( \varepsilon(X_i; \hat{\beta}) = \varepsilon_i + h_i \), where \( \mathbb{E}(h_i) = 0 \), and

\[h = \max_j h_j = O_p(q^{1/2}n^{-1/2}).\] Then, we have

\[\mathbb{E} [I(v-h_i < \varepsilon_i \leq v)] = h_i f_i(v) - 0.5h_i^2 \dot{f_i}(v) + o(h_i^2),\]

\[\text{Var} [I(v-h_i < \varepsilon_i \leq v)] = h_i f_i(v) - h_i^2 \left( 0.5 \dot{f_i}(v) + f_i^2(v) \right) + o(h_i^2).\]
Substituting the above equalities into $T_{n}^e$, we have

$$T_{n}^e = -n^{-1/2} \sum_{i=1}^{n} I(v - h_i < \epsilon_i \leq v) \Delta_{n,i}(X; \alpha, u)$$

$$= -n^{-1/2} \sum_{i=1}^{n} \Delta_{n,i}(X; \alpha, u) \left( h_i f_{\epsilon}(v) - 0.5h_i^2 \dot{f}_{\epsilon}(v) + o(h_i^2) \right)$$

$$- n^{-1/2} \sum_{i=1}^{n} \Delta_{n,i}(X; \alpha, u) \{ I(v - h_i < \epsilon_i \leq v) - \mathbb{E} [ I(v - h_i < \epsilon_i \leq v) ] \}$$

$$= -n^{-1/2} \sum_{i=1}^{n} \Delta_{n,i}(X; \alpha, u) \left( h_i f_{\epsilon}(v) - 0.5h_i^2 \dot{f}_{\epsilon}(v) \right) + O_p(h^{1/2}),$$

$$(T_{n}^e)^2 = n^{-1} \sum_{i,j} \Delta_{n,i}(X; \alpha, u) \Delta_{n,j}(X; \alpha, u) \left( h_i f_{\epsilon}(v) - 0.5h_i^2 \dot{f}_{\epsilon}(v) \right)$$

$$\left( h_j f_{\epsilon}(v) - 0.5h_j^2 \dot{f}_{\epsilon}(v) \right) + O_p(h),$$

$$M_{n,p}(\beta_0; \alpha, u, v)T_{n}^e = -n^{-1} \sum_{i,j} I(\epsilon_i \leq v) - F_{n,\epsilon}(v) I(v - h_j < \epsilon_j \leq v)$$

$$\times I(\langle X_i, \alpha \rangle_m \leq u) \Delta_{n,i}(X; \alpha, u)$$

$$= -n^{-1} \sum_{i=1}^{n} \left( h_i f_{\epsilon}(v) - 0.5h_i^2 \dot{f}_{\epsilon}(v) \right) I(\langle X_i, \alpha \rangle_m \leq u) \Delta_{n,i}(X; \alpha, u)$$

$$+ n^{-2} \sum_{i,j} I(v - h_i < \epsilon_i \leq v) I(\epsilon_j \leq v) I(\langle X_i, \alpha \rangle_m \leq u) \Delta_{n,i}(X; \alpha, u)$$

$$- n^{-1} \sum_{i \neq j} I(\epsilon_i \leq v) I(v - h_j < \epsilon_j \leq v) I(\langle X_i, \alpha \rangle_m \leq u) \Delta_{n,j}(X; \alpha, u)$$

$$+ n^{-2} \sum_{i \neq j} \sum_{k=1}^{n} I(v - h_j < \epsilon_j \leq v) I(\epsilon_k \leq v) I(\langle X_i, \alpha \rangle_m \leq u) \Delta_{n,j}(X; \alpha, u),$$

where $\Delta_{n,j}(X; \alpha, u) = I(\langle X_j, \alpha \rangle_m \leq u) - F_{n,\langle X, \alpha \rangle_m}(u)$. Note that

$$n^{-1} \sum_{i,j} \Delta_{n,i}(X; \alpha, u) \Delta_{n,j}(X; \alpha, u) T_{i2} T_{j2} = o_p(1), \quad \text{and} \quad n^{-1} \sum_{i,j} I(\epsilon_i \leq v) T_{j2} = o_p(1).$$
After some calculations, we can obtain that

$$\mathbb{E}(T_n^c) = O(h^{1/2} + n^{1/2}h^2), \quad \mathbb{E}(T_n^c)^2 = \sigma_F^2(\alpha, u, \alpha, u) f_\epsilon^2(v) + O(h),$$

$$\mathbb{E}(M_n,F(\beta_0; \alpha, u,v)) = \sigma_{c,F}(\alpha, u,v, \alpha, u) f_\epsilon(v).$$

where

$$\sigma_{c,F}(\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2) = \sum_{k,l=1}^q \zeta_{l,k} \zeta_{l,F}(\alpha_2,\bullet, u_2) \mathbb{E} [I(\epsilon_i \leq v_1) \omega_i D_{i,k} \Delta(X_i; \alpha_1, u_1)],$$

$$\sigma_F^2(\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2) = \sum_{k_1,k_2,l_1,l_2=1}^q \zeta_{l_1,k_1} \zeta_{l_2,k_2} \Gamma_{k_1,k_2} S_{l_1,F}(\alpha_1,\bullet, u_1) S_{l_2,F}(\alpha_2,\bullet, u_2),$$

$$\zeta_{l,F}(\alpha, u) = \mathbb{E} [\Delta(X_j; \alpha, u) D_{j,l}], \Delta(X_i; \alpha, u) = I(\langle X_i, \alpha \rangle_m \leq u) - F(\langle X, \alpha \rangle_m (u)).$$

Consequently, we have

$$M_n,F(\hat{\beta}; \alpha, u,v) \xrightarrow{d} M_{\infty,F}^1(\alpha, u,v)$$

for any fixed $u$, $v$, and $\alpha_j \in S^p := \{ \alpha \in R^p : \|\alpha\|_p = 1 \}$, for $j = 1, \ldots, d$, where

$$M_{\infty,F}^1 \equiv M_{\infty,F}^0 + M_{\infty,F}^e,$$

$M_{\infty,F}^e$ is a Gaussian process with zero mean and covariance function

$$K_1((\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2)) = \sigma_F^2(\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2) f_\epsilon(v_1) f_\epsilon(v_2),$$

$$\text{cov}(M_{\infty,F}^0(\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2)) = \sigma_{c,F}(\alpha_1,\bullet, u_1, \alpha_2,\bullet, u_2) f_\epsilon(v_2).$$

Furthermore, based on the derivation, we have

$$|\text{Prob}(M_n,F(\hat{\beta}; \alpha, u,v) < m) - \text{Prob}(M_{\infty,F}^1(\alpha, u,v) < m)| = O(n^{-1/2} + h^{1/2}).$$

Again, similar to the proof of Theorem 1, as long as $p^2 d^2 q = o(n)$, Theorem 2 follows.
Now we prove Theorem S.2. Note that

\[
M_{n,F}^f(\beta; \alpha, \gamma, u, v) = n^{-1/2} \sum_{i=1}^{n} \left[ I \left( \langle \epsilon^f_i(X_i; \hat{\beta}), \gamma \rangle_m \leq v \right) - \hat{F}_{n,(\epsilon^f_i, \gamma)_m}(v) \right] I \left( \langle X_i, \alpha \rangle_m \leq u \right)
\]

\[
= M_{n,F}^f(\beta_0; \alpha, \gamma, u, v) - n^{-1/2} \sum_{i=1}^{n} I \left( v - \langle h^f_i, \gamma \rangle_m < \langle \epsilon^f_i, \gamma \rangle_m \leq v \right) \Delta_{n,i}(X; \alpha, u)
\]

\[
:= M_{n,F}^f(\beta_0; \alpha, \gamma, u, v) - T_n^{e,f}.
\]

Let \( f_{(\epsilon^f_i, \gamma)_m}(v) = E \left[ I \left( \langle \epsilon^f_i, \gamma \rangle_m = v \right) \right] \). Let \( \langle h^f, \gamma \rangle_m = sup_j \langle h^f_j, \gamma \rangle_m \). Similarly, using Lemma S.2, we obtain

\[
E(T_n^{e,f}) = O((h^f, \gamma)_m^{1/2}), \quad E(T_n^{e,f})^2 = \sigma_{f}^{2f}(\alpha, \gamma, u, \alpha, \gamma, u) f_{(\epsilon^f_i, \gamma)_m}(v) + (h^f, \gamma)_m),
\]

\[
E \left( M_{n,F}^f(\beta_0; \alpha, \gamma, u, v)T_n^{e,f} \right) = \sigma_{c,F}^{f}(\alpha, \gamma, u, v, \alpha, \gamma, u) f_{(\epsilon^f_i, \gamma)_m}(v),
\]

where

\[
\sigma_{f}^{2f}(\alpha_1, \gamma_1, u_1, \alpha_2, \gamma_2, u_2) = \sum_{k_1, k_2, l_1, l_2} \zeta_{j_1, k_1}^{f} \zeta_{j_2, k_2}^{f} \Gamma_{k_1, k_2}^{f} \sigma_{i,F}^{f}(\alpha_1, \gamma_1, u_1) \sigma_{i,F}^{f}(\alpha_2, \gamma_2, u_2),
\]

\[
\sigma_{c,F}^{f}(\alpha_1, \gamma_1, u_1, \alpha_2, \gamma_2, u_2) = \sum_{k_1, k_2, l_1, l_2} \zeta_{l_1, l_2}^{f} \sigma_{i,F}^{f}(\alpha_2, \gamma_2, u_2) \times
\]

\[
E \left[ I(\langle \epsilon^f_i, \gamma_1 \rangle_m \leq v_1) (D_i^{f,k}, \omega_i^{f}) \Delta(X_i; \alpha_1, u_1) \right],
\]

\[
\sigma_{i,F}^{f}(\alpha, \gamma, u) = E \left[ \Delta(X_i; \alpha, u) (D_i^{f,k}, \gamma)_m \right].
\]

Then, Theorem S.2 follows. In particular, we have \( M_{n,F}^f(\beta; \alpha, \gamma, u, v) \overset{d}{\to} M_{\infty,F}^1(\alpha, \gamma, u, v) \) for any fixed \( u, v \), and \( \alpha, \gamma \in \mathbb{S}^p := \{ \alpha \in R^p : \| \alpha \|_p = 1 \} \), \( j = 1, \cdots, d \), where \( M_{\infty,F}^1 \equiv M_{\infty,F}^0 + M_{\infty,F}^e \), \( M_{\infty,F}^e \) is a Gaussian process.
with zero mean and covariance function

\[ K_1 ((\alpha_1, \gamma_1, u_1, v_1), (\alpha_2, \gamma_2, u_2, v_2)) = \sigma_f^2 F_1(\alpha_1, \gamma_1, u_1) f_{\gamma_1}(v_1) f_{\gamma_2}(v_2); \]

\[ \text{cov} \left( M_{\infty,F}(\alpha_1, \gamma_1, u_1, v_1), M_{\infty,F}(\alpha_2, \gamma_2, u_2, v_2) \right) = \sigma_f^2 F_2(\alpha_1, \gamma_1, u_1) f_{\gamma_2}(v_2). \]

Furthermore, based on the derivation, we have

\[ |Prob(M_n,F(\hat{\beta}; \alpha, \gamma, u, v) < m) - Prob(M_{\infty,F}(\alpha, \gamma, u, v) < m)| = O(n^{-1/2} + h_f, \gamma_{1/2}). \]

Similarly to the proof of Theorem S.1, as long as \( p^2(d + 1)^2 q_f = o(n) \), Theorem S.2 follows.

\[ \Box \]

Proofs of Theorems 3 and S.3: We first prove Theorem 3. Note that

\[ \tilde{F}_{n,v}(v) = n^{-1} \sum_{i=1}^{n} I \left( \varepsilon(X_i; \hat{\beta}) + \hat{h}_i - h_i \leq v \right). \]

Under the local alternative (3.6), and Lemma S.3, we have that

\[ M_{n,F}(\hat{\beta}; \alpha, u, v) = n^{-1/2} \sum_{i=1}^{n} \left[ I(\varepsilon(X_i; \hat{\beta}) \leq v) - \tilde{F}_{n,v}(v) \right] I(\langle X_i, \alpha \rangle_m \leq u) \]

\[ = M_{n,F}(\hat{\beta}; \alpha, u, v) - n^{-1/2} \sum_{i=1}^{n} \left[ I(v - \hat{h}_i < \varepsilon_i \leq v - h_i) \right] \Delta_{n,i}(X; \alpha, u) \]

\[ = M_{n,F}(\hat{\beta}; \alpha, u, v) - n^{-1/2} \sum_{i=1}^{n} \Delta_{n,i}(X; \alpha, u) \left[ (\hat{h}_i - h_i) f_{\varepsilon}(v) - 0.5(\hat{h}_i - h_i)^2 \hat{f}_{\varepsilon}(v) \right] \]

\[ = M_{n,F}(\hat{\beta}; \alpha, u, v) - n^{-1} \sum_{i=1}^{n} \Delta_{n,i}(X; \alpha, u) f_{\varepsilon}(v) \left( m_y(Y_i, \eta_i) \hat{g}(\eta_i, \varepsilon_i, 0) F(X_i) - \sum_{j=1}^{q} D_{i,j} v_j \right). \]

Then, Theorem 3 follows from similar arguments to those in the derivation of Theorem 2.
For Theorem S.3, it follows from Lemma S.4 that

$$\tilde{F}_{n,\epsilon,\gamma,m}(v) = n^{-1} \sum_{i=1}^{n} I\left(\langle \varepsilon^f(X_i; \hat{\beta}), \gamma \rangle_m + \langle h_i^f - h_i^\epsilon, \gamma \rangle_m \leq v \right).$$

Similar to the proof of Theorem 2, we have

$$M_{n,F}(\bar{\beta}; \alpha, \gamma, u, v) = n^{-1/2} \sum_{i=1}^{n} \left[ I\left(\langle \varepsilon^f(X_i; \hat{\beta}), \gamma \rangle_m \leq v \right) - \tilde{F}_{n,\epsilon,\gamma,m}(v) \right] I\left(\langle X_i, \alpha \rangle_m \leq u \right)$$

$$= M_{n,F}(\bar{\beta}; \alpha, \gamma, u, v) - n^{-1/2} \sum_{i=1}^{n} \left[ I\left( v - \langle h_i^f, \gamma \rangle_m < \langle \epsilon^f_i, \gamma \rangle_m \leq v - \langle h_i^f, \gamma \rangle_m \right) \right] \Delta_n,i(X; \alpha, u)$$

$$= M_{n,F}(\bar{\beta}; \alpha, \gamma, u, v) - n^{-1} \sum_{i=1}^{n} \Delta_n,i(X; \alpha, u) f_{\epsilon,\gamma,m}(v)$$

$$\times \left( m(Y_i(t), \eta_i(t)) \tilde{g}_i(\eta_i(t), \epsilon_i(t), 0) F(X_i; \gamma) - \sum_{j=1}^{q} \langle D_{i,j}^f; \gamma \rangle m v_j^f \right).$$

Then, Theorem S.3 follows. \qed

Proof of Theorem 4. Define \( \tilde{\eta}_i(\theta) = \left( \hat{a}_i, \hat{b}_i^\top \hat{\xi}_{i1}, \cdots, \hat{b}_i^\top \hat{\xi}_{id} \right) \), and \( D_i^*(\hat{\theta}) = \)

\[
(m_0(Y_{i,n}, \tilde{\eta}_i(\theta)),
\]

\[
m_1(Y_{i,n}, \tilde{\eta}_i(\theta))(\hat{\xi}_{i1})^\top, \cdots, m_d(Y_{i,n}, \tilde{\eta}_i(\theta))(\hat{\xi}_{id})^\top \hat{g}_i. \]

Note that

$$M_{n,F}^* = n^{-1/2} \sum_{i=1}^{n} \left[ I\left\{ \varepsilon^*(X_i; \hat{\beta}^*) \leq v \right\} - F_{n,F}^*(v) \right] I\left(\langle X_i, \alpha \rangle_m \leq u \right)$$

$$= M_{n,F}(g) + n^{-1/2} \sum_{i=1}^{n} \left[ I(h_i^* + \epsilon_i g_i \leq v) - I(\epsilon_i g_i \leq v) \right] \Delta_n,i(X; \alpha, u)$$

where \( h_i^* = (D_i^*(\theta))^\top(\hat{\theta}^* - \theta) + \left[ m(Y_{i,n}, \tilde{\eta}_i(\theta)) - m(Y_i, \tilde{\eta}_i(\theta)) + D_i^T(\hat{\theta} - \theta) \right] g_i \),

and \( M_{n,F}(g) = n^{-1/2} \sum_{i=1}^{n} \left[ I(\epsilon_i g_i \leq v) - F_{n,c,g}(v) \right] I\left(\langle X_i, \alpha \rangle_m \leq u \right), \) and \( F_{n,c,g}(v) = \).
$n^{-1} \sum_{i=1}^{n} I(\epsilon_i \varrho_i \leq v)$. It follows from Taylor expansion that

\[
0 \equiv n^{-1/2} \sum_{i=1}^{n} \hat{\ell} \left( m \left( Y_{i,n}^*, \hat{\eta}_i(\hat{\theta}^*) \right) \right) \hat{D}_i^*(\hat{\theta}^*)
\]

\[
= n^{-1/2} \sum_{i=1}^{n} \hat{\ell} \left( m \left( Y_{i,n}^*, \hat{\eta}_i(\hat{\theta}) \right) \right) D_i^*(\hat{\theta}) + n^{1/2} \tilde{\Gamma}^*(\hat{\theta}^* - \hat{\theta}) + o_p(1)
\]

\[
= n^{-1/2} \sum_{i=1}^{n} \hat{\ell} (m (Y_{i,n}, \eta_i) \varrho_i) D_i + n^{1/2} \tilde{\Gamma}^*(\hat{\theta}^* - \hat{\theta})
\]

\[
+ n^{-1/2} \sum_{i=1}^{n} \hat{\ell} (m (Y_{i,n}, \eta_i) \varrho_i) D_i D_i^\top (\hat{\theta} - \theta) \varrho_i + o_p(1),
\]

where $\tilde{\Gamma}^* = n^{-1} \sum_{i=1}^{n} \hat{\ell} \left( m \left( Y_{i,n}^*, \hat{\eta}_i(\hat{\theta}) \right) \right) D_i^*(\hat{\theta}) (D_i^*(\hat{\theta}))^\top$. Then from Lemma S.3, we have

\[
\hat{\theta}^* - \hat{\theta} = -n^{-1} (\tilde{\Gamma}^*)^{-1} \sum_{i=1}^{n} \left[ \hat{\ell}(\epsilon_i \varrho_i)
\right.
\]

\[
+ \left. \left( \hat{\ell} (m (Y_{i,n}, \eta_i) \varrho_i) - \hat{\ell} (m (Y_i, \eta_i) \varrho_i) \right) \right] D_i (1 + o_p(1))
\]

Substituting expression [S.4] into $h_i^*$ and using condition (C7), we obtain

\[
h_i^* = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \zeta_{i,k} D_{a,k} D_{i,j} \hat{\ell}(\epsilon_s \varrho_s).
\]

Then Theorem 4 follows using similar arguments to the proof of Theorem 2 and condition (C7). □

### S.4 Corollaries

**Corollary S.2.** Under conditions of Theorem 1 and null hypothesis (1.1), if $p = o(n)$,
(a) For model (2.4), for any $m \in \mathbb{R}$, $\text{Prob}(T_{n,M}(\beta_0) < m) - \text{Prob}(T_{\infty,M}^0 < m) \to 0$, where $T_{\infty,M}^0 := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \left( M_{\infty,M}^0(\alpha, u) \right)^2 F_{(X,\alpha),m}(du)d\alpha$, and $M_{\infty,M}^0(\cdot, \cdot)$ is a Gaussian process with zero mean and covariance function

$$K_M(\alpha_1, u_1, \alpha_2, u_2) = \mathbb{E}(\epsilon_i^2) \mathbb{E}[I((X,\alpha_{1,\cdot})_m \leq u_1) I((X,\alpha_{2,\cdot})_m \leq u_2)].$$

(b) For model (2.5), for any $m \in \mathbb{R}$, $\text{Prob}(T_{n,M}^f(\beta_0) < m) - \text{Prob}(T_{\infty,M}^0 < m) \to 0$, where $T_{\infty,M}^0 := \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \int_0^1 \left( M_{\infty,M}^{0,f}(\alpha, u; t) \right)^2 dt F_{(X,\alpha),m}(du)d\alpha$, where $M_{\infty,M}^{0,f}(\cdot, \cdot; t)$ is a Gaussian process with zero mean and covariance function

$$K_{M}^{f}(\alpha_1, u_1, \alpha_2, u_2; t) = \mathbb{E}[\epsilon(t)^2] \mathbb{E}[I((X,\alpha_{1,\cdot})_m \leq u_1) I((X,\alpha_{2,\cdot})_m \leq u_2)].$$

**Proof.** For model (2.4), it follows from Theorem 1 in Escanciano (2006), and

$$M_{n,M}^0(\alpha, u) = n^{-1/2} \sum_{i=1}^n \epsilon_i I((X,\alpha)_m \leq u),$$

that $M_{n,M}(\beta_0; \alpha, u)$ converges to $M_{\infty,M}^0(\alpha, u)$. For model (2.5), similarly, we have $M_{n,M}^{f}(\beta_0; \alpha, u, t)$ converges to $M_{\infty,M}^{0,f}(\alpha, u; t)$ from

$$M_{n,M}^{f}(\beta_0; \alpha, u, t) = n^{-1/2} \sum_{i=1}^n \epsilon_i(t) I((X,\alpha)_m \leq u).$$

Then, Corollary [S.2] holds from similar arguments as those in the proof of Theorems 1 and [S.1]
Corollary S.3. Under conditions (C1)–(C2),

(a) for model (2.4), if further conditions (C3)–(C5) hold, and under null hypothesis (1.1), for any $m \in \mathbb{R}$,

$$ \text{Prob}(T_{n,R}^n(\hat{\beta}) < m) - \text{Prob}(T_{\infty,M}^1 < m) \to 0, $$

where $T_{\infty,M}^1 := \int_{\mathcal{R}} \int_{\mathcal{S}} (M_{\infty,M}^1(\alpha, u))^2 F(x, \alpha)(du)d\alpha$, and $M_{\infty,M}^1 \equiv M_{\infty,M}^0 + M_{\infty,M}^e$, $M_{\infty,M}^e$ is a Gaussian process with zero mean and covariance function $K_{1,M}(\alpha_1, u_1, \alpha_2, u_2) = \sigma_M^2(\alpha_1, u_1, \alpha_2, u_2)$, and

$$ \text{cov}(M_{\infty,M}^0(\alpha_1, u_1), M_{\infty,M}^e(\alpha_2, u_2)) = -\sigma_{c,M}(\alpha_1, u_1, \alpha_2, u_2). $$

(b) for model (2.5), if further conditions (C3f)–(C5f) hold, and under null hypothesis (1.1), for any $m \in \mathcal{R}$,

$$ \text{Prob}(T_{n,R}^f(\hat{\beta}) < m) - \text{Prob}(T_{\infty,M}^{1,f} < m) \to 0, $$

where $T_{\infty,M}^{1,f} := \int_{\mathcal{R}} \int_{\mathcal{S}} \int_0^1 (M_{\infty,M}^{1,f}(\alpha, u; t))^2 dF(x, \alpha)(du)d\alpha$, and $M_{\infty,M}^{1,f} \equiv M_{\infty,M}^{0,f} + M_{\infty,M}^{e,f}$, $M_{\infty,M}^{e,f}$ is a Gaussian process with zero mean and covariance function $K_{1,M}^{f}(\alpha_1, u_1, \alpha_2, u_2, t) = \sigma_M^2(\alpha_1, u_1, \alpha_2, u_2, t)$, and

$$ \text{cov}(M_{\infty,M}^{0,f}(\alpha_1, u_1, t), M_{\infty,M}^{e,f}(\alpha_2, u_2, t)) = -\sigma_{c,M}^f(\alpha_1, u_1, \alpha_2, u_2, t). $$
**Proof.** We first show part (a). Note that

\[
M_{n,M}(\hat{\beta}; \alpha, u) = n^{-1/2} \sum_{i=1}^{n} \varepsilon(X_i; \hat{\beta}) I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
= M_{n,M}(\beta_0; \alpha, u) + n^{-1/2} \sum_{i=1}^{n} \left( \varepsilon(X_i; \hat{\beta}) - \varepsilon_i \right) I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
:= M_{n,M}(\beta_0; \alpha, u) + \text{T}_{n,u}.
\]

It follows from Lemma S.1 that \( \varepsilon(X_i; \hat{\beta}) = \varepsilon_i + h_i \). Then, we have

\[
\mathbb{E}(M_{n,M}(\beta_0; \alpha, u) | \text{T}_{n,u}) = o_p(1), \quad \mathbb{E}(\text{T}_{n,u}^2) = \sigma^2_M(\alpha_1, u_1, \alpha_2, u_2) + o(1), \quad \text{and}
\]

\[
\mathbb{E}(M_{n,M}(\beta_0; \alpha, u) \text{T}_{n,u}^c) = \mathbb{E} \left[ n^{-1} \sum_{i=1}^{n} \varepsilon_i h_i I(\langle X_i, \alpha \rangle_m \leq u) I(\langle X_j, \alpha \rangle_m \leq u) \right]
\]

\[= -\sigma_{c,M}(\alpha, u, \alpha, u).
\]

where \( \sigma_{c,M}(\alpha_1, u_1, \alpha_2, u_2) = \sum_{k,l=1}^{q} \zeta_{k,l,M}(\alpha_2, u_2) \mathbb{E}[\varepsilon_i \varpi_i D_{i,k} I(\langle X_i, \alpha_1 \rangle_m \leq u_1)], \)

\[
\sigma^2_M(\alpha_1, u_1, \alpha_2, u_2) = \sum_{k_1,k_2,l_1,l_2=1}^{q} \zeta_{k_1,k_2,l_1,l_2,M}(\alpha_2, u_2) \mathbb{E}[\varepsilon_i \varpi_i D_{i,k_1,k_2} I(\langle X_i, \alpha_1 \rangle_m \leq u_1) I(\langle X_j, \alpha_2 \rangle_m \leq u_2)],
\]

and \( \zeta_{l,M}(\alpha, u) = \mathbb{E}[I(\langle X_i, \alpha \rangle_m \leq u) D_{i,l}]. \)

For part (b), similarly, we have

\[
M_{n,M}(\beta; \alpha, u, t) = n^{-1/2} \sum_{i=1}^{n} \varepsilon^f(X_i; \hat{\beta}) I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
= M_{n,M}(\beta_0; \alpha, u, t) + n^{-1/2} \sum_{i=1}^{n} h_i^f I(\langle X_i, \alpha \rangle_m \leq u) := M_{n,M}(\beta_0; \alpha, u, t) + \text{T}_{n,u}^f.
\]

It follows from Lemma S.2 that \( \varepsilon^f(X_i; \hat{\beta}) = \varepsilon_i^f + h_i^f \). Then, we have
S.4. COROLLARIES

\[ \mathbb{E}(T_n^{e,u,f}) = o_p(1), \mathbb{E}(T_n^{e,u,f})^2 = \sigma^2_M(\alpha, u, \alpha, u; t) + o(1), \text{ and} \]

\[ \mathbb{E} \left( M_{n,h}^f(\beta_0, \alpha, u, t) T_n^{e,u,f} \right) = \mathbb{E} \left[ n^{-1} \sum_{i,j} \epsilon_i(t) h_j^f I(\langle X_i, \alpha \rangle_m \leq u) I(\langle X_j, \alpha \rangle_m \leq u) \right] \]

\[ = - \sum_{k,l=1}^{q_f} \zeta_{i,k}^f \zeta_{i,M}^f(\alpha, u; t) \mathbb{E} \left[ \epsilon_i(t) \langle D_{i,k}, \varpi_i^f \rangle I(\langle X_i, \alpha \rangle_m \leq u) \right], \]

where

\[ \sigma_{c,M}^f(\alpha_1, u_1, \alpha_2, u_2; t) = \sum_{k,l=1}^{q_f} \zeta_{i,k}^f \zeta_{i,M}^f(\alpha_2, u_2; t) \mathbb{E} \left[ \epsilon_i(t) \langle D_{i,k}, \varpi_i^f \rangle I(\langle X_i, \alpha_1 \rangle_m \leq u_1) \right], \]

\[ \sigma_M^2(\alpha_1, u_1, \alpha_2, u_2; t) = \sum_{k_1,k_2,l_1,l_2} \zeta_{j_1,k_1}^f \zeta_{j_2,k_2}^f \Gamma_{i_1,k_1,l_1}^f \zeta_{i_2,M}^f(\alpha_1, u_1; t) \zeta_{i_2,M}^f(\alpha_2, u_2; t), \]

\[ \zeta_{i,M}^f(\alpha, u; t) = \mathbb{E} \left[ I(\langle X_i, \alpha \rangle_m \leq u) D_{i,t}^f(t) \right]. \]

Corollary S.4. Under conditions of Theorem 3,

(a) for model (2.4), under conditions (C1)–(C6), and local alternative (3.6), for any \( m \in \mathbb{R} \),

\[ \text{Prob}(T_{n,h}(\beta) < m) - \text{Prob}(T_{\infty,M}^a < m) \to 0, \]

where \( T_{\infty,M}^a := \int_{S^{p+d}} \int_{\mathbb{R}} (M_{\infty,M}^a(\alpha, u))^2 F_{X,\alpha} m(du) d\alpha, \) and \( \beta \) is the estimate obtained from model (2.4) using data \( \{Y_{i,n}, X_i\}_{i=1}^n \), \( M_{\infty,M}^a(\alpha, u) \equiv M_{\infty,M}^1(\alpha, u) + D_{M}^2(\alpha, u), \) and

\[ D_{M}^2(\alpha, u) = \mathbb{E} \left\{ I(\langle X_i, \alpha \rangle_m \leq u) \left( m_y(Y_i, \eta_i) \hat{g}_t(\eta_i, \epsilon_i, 0) F(X_i) - \sum_{j=1}^{q} D_{i,j} v_j \right) \right\}, \]

where \( m_y, \hat{g}_t, v_j \) are defined in Section S.4.
(b) for model (2.5), under conditions (C1)–(C2), (C3f)–(C5f), (C6), and local alternative \( S.1 \), for any \( m \in \mathbb{R} \),

\[
\text{Prob}(T_{\infty,M}^f(\hat{\beta}) < m) - \text{Prob}(T_{\infty,M}^{a,f} < m) \to 0,
\]

where \( T_{\infty,M}^{a,f} := \int_{\mathbb{R}} \int_{\mathcal{R}} \int_0^1 \left( M_{\infty,M}^{a,f}(\alpha, u; t) \right)^2 dt \mathcal{F}(X, \alpha, m)(du) d\alpha \), and \( \hat{\beta} \) is the estimate obtained from model (2.5) using data \( \{Y_{i,n}(t), X_i\}_{i=1}^n \),

\[
M_{\infty,M}^{a,f}(\alpha, u; t) = M_{\infty,M}^{1,f}(\alpha, u; t) + D_{\infty,M}^{a,f}(\alpha, u; t), \quad \text{and}
\]

\[
D_{\infty,M}^{a,f}(\alpha, u; t) = E \left\{ I(\langle X_i, \alpha \rangle_m \leq u) \left( m_y(Y_i(t), \eta_i^f(t)) \hat{g}_l(\eta_i^f(t), \epsilon_i(t), 0) \mathcal{F}(X_i) \right. \\
- \left. \sum_{j=1}^q D_{i,j}(t) v_j^f \right) \right\},
\]

where \( v_j^f \) is defined in Appendix A.1.

**Proof.** For model (2.4), under the local alternative (3.6), and Lemma \( S.3 \), we have that

\[
M_{n,M}(\hat{\beta}; \alpha, u) = n^{-1/2} \sum_{i=1}^n \varepsilon(X_i; \hat{\beta}) I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
= M_{n,M}(\hat{\beta}; \alpha, u) + n^{-1} \sum_{i=1}^n \left( m_y(Y_i(t), \eta_i^f(t)) \hat{g}_l(\eta_i^f(t), \epsilon_i(t), 0) \mathcal{F}(X_i) - \sum_{j=1}^q D_{i,j}(t) v_j^f \right) I(\langle X_i, \alpha \rangle_m \leq u).
\]

For model (2.5), similarly, we have

\[
M_{n,M}^f(\hat{\beta}; \alpha, u) = n^{-1/2} \sum_{i=1}^n \varepsilon^f(X_i; \hat{\beta}) I(\langle X_i, \alpha \rangle_m \leq u)
\]

\[
= M_{n,M}^f(\hat{\beta}; \alpha, u) + n^{-1} \sum_{i=1}^n \left( m_y(Y_i(t), \eta_i^f(t)) \hat{g}_l(\eta_i^f(t), \epsilon_i(t), 0) \mathcal{F}(X_i) \\
- \sum_{j=1}^q D_{i,j}(t) v_j^f \right) I(\langle X_i, \alpha \rangle_m \leq u).
\]
Corollary S.5. Under the null hypothesis (1.1) or alternative hypothesis (3.7), if Conditions (C1)–(C7) are satisfied, the conditional distribution of $T_{n,M}^*$ converges in distribution to the null limiting distribution $T_{n,M}$ giving

\{Y_i, X_i\}_{i=1}^n.$

Proof. We consider the alternative (3.7) with $\nu \leq 0$. For the null hypothesis, we assume $F(X_i) = 0$. The result can be proved similarly and the details are omitted. Define $\hat{\eta}_i(\theta) = (\hat{a}, \hat{b}_1^\top \hat{\xi}_{i1}, \ldots, \hat{b}_d^\top \hat{\xi}_{id})$. Note that

$M_{n,M}^* = n^{-1/2} \sum_{i=1}^n \varepsilon(X_i; \hat{\beta}^*) I(\langle X_i, \alpha \rangle_m \leq u)$

$= n^{-1/2} \sum_{i=1}^n \varepsilon_i \varrho_i I(\langle X_i, \alpha \rangle_m \leq u)$

$+ n^{-1/2} \sum_{i=1}^n \left[ m\left( Y_{i,n}^*, \hat{\eta}_i(\hat{\theta}^*) \right) - m\left( Y_{i,n}, \hat{\eta}_i(\hat{\theta}) \right) \right] I(\langle X_i, \alpha \rangle_m \leq u)$

$+ n^{-1/2} \sum_{i=1}^n \left[ m\left( Y_{i,n}, \hat{\eta}_i(\hat{\theta}) \right) - \varepsilon_i \right] \varrho_i I(\langle X_i, \alpha \rangle_m \leq u) \equiv M_1^* + M_2^* + M_3^*.$

It follows from Lemma S.3 and random variable sequence $\{\varrho_i, i = 1, \ldots, n\}$ are i.i.d. with mean zero, and $\varrho_i$ is independent of $(Y_i, X_i)$ that,

$M_2^* = n^{-1/2} \sum_{i=1}^n (D_i^*(\hat{\theta}))^\top (\hat{\theta}^* - \hat{\theta}) I(\langle X_i, \alpha \rangle_m \leq u)(1 + o_p(1)),$

$M_3^* = n^{-1/2} \sum_{i=1}^n \left[ m\left( Y_{i,n}, \hat{\eta}_i(\theta) \right) - m\left( Y_i, \hat{\eta}_i(\theta) \right) + D_i^\top(\theta - \hat{\theta}) \right] V_i I(\langle X_i, \alpha \rangle_m \leq u)(1 + o_p(1)),$

where $D_i^*(\theta) = (m_0(Y_{i,n}, \hat{\eta}_i(\theta)), m_1(Y_{i,n}, \hat{\eta}_i(\theta))(\xi_{i1})^\top, \ldots, m_d(Y_{i,n}, \hat{\eta}_i(\theta))(\xi_{id})^\top)^\top \varrho_i.$
Next we show the asymptotic expression for $\theta^* - \hat{\theta}$. It follows from Taylor expansion that

$$0 \equiv n^{-1/2} \sum_{i=1}^{n} \left( m\left(Y^*_{i,n}, \hat{\eta}_i(\hat{\theta}^*)\right) - D_i^*(\hat{\theta}^*) \right)$$

$$= n^{-1/2} \sum_{i=1}^{n} \left( m\left(Y^*_{i,n}, \hat{\eta}_i(\hat{\theta})\right) \right) D_i^*(\hat{\theta}) + n^{1/2} \tilde{\Gamma}^*(\hat{\theta}^* - \hat{\theta}) + o_p(1)$$

$$= n^{-1/2} \sum_{i=1}^{n} \left( m\left(Y_{i,n}, \eta_i\right) \right) D_i + n^{1/2} \tilde{\Gamma}^*(\hat{\theta}^* - \hat{\theta})$$

$$+ n^{-1/2} \sum_{i=1}^{n} \left( m\left(Y_{i,n}, \eta_i\right) \right) D_i D_i^\top (\hat{\theta} - \theta) g_i + o_p(1),$$

where $\tilde{\Gamma}^* = n^{-1} \sum_{i=1}^{n} \left( m\left(Y^*_{i,n}, \hat{\eta}_i(\theta)\right) \right) D_i^*(\theta) (D_i^*(\theta))^\top$. Then from Lemma S.3, we have

$$\hat{\theta}^* - \hat{\theta} = -n^{-1}(\tilde{\Gamma}^*)^{-1} \sum_{i=1}^{n} \left[ \hat{\ell}(\epsilon_i g_i) + \left( \hat{\ell}(m(Y_{i,n}, \eta_i) g_i) - \hat{\ell}(m(Y_{i,n}, \eta_i) g_i) \right) \right] D_i (1 + o_p(1)).$$

Substituting above expression into $M^*_n$, under the condition that $\epsilon_i$ and $\epsilon_i g_i$ have the same distribution and $g_i$ has the variance 1 and mean 0, we have

$$M^*_{n,M} = n^{-1/2} \sum_{i=1}^{n} \epsilon_i g_i I(\langle X_i, \alpha \rangle_m \leq u)$$

$$- n^{-1/2} \sum_{i=1}^{n} \left( n^{-1} \sum_{s=1}^{n} \sum_{k,j} \zeta_{j,k} D_{s,k} D_{i,j} \hat{\ell}(\epsilon_s g_s) \right) I(\langle X_i, \alpha \rangle_m \leq u) + o_p(1).$$
S.5 Preliminary Lemmas

Let \( q = Kd + 1 \). Denote \( \varpi = \left( \hat{\ell}(m(Y_i, \eta_i)) \right)_{i=1}^{n}, V = \text{Diag}\{\varpi_1^2, \ldots, \varpi_n^2\} \), \( D = \{D_i\}_{i=1}^{n} \) and \( \tilde{D} = \{(\hat{\ell}(m(Y_i, \eta_i)))^{1/2}D_i\}_{i=1}^{n} \) are \( n \times q \)-dimensional matrices, \( \Gamma = \lim_{n \to \infty} \frac{D^\top V D}{n} := \{\Gamma_{k,l}\}_{1 \leq k, l \leq q}, \tilde{\Gamma} = \lim_{n \to \infty} \left( \frac{D^\top \tilde{D}}{n} \right) = (\tilde{\Gamma}_{k,l})_{1 \leq k, l \leq q} \), \( \Xi = \tilde{\Gamma}^{-1} = (\zeta_{j,k})_{1 \leq j, k \leq q} \), \( \sum_{k=1}^{q} \zeta_{j,k}^{(1/2)} \zeta_{k,l}^{(1/2)} = \zeta_{j,l}. \) Denote \( T_{i2} = \sum_{j=1}^{d} m_j(Y_i, \eta_i) \left( \hat{\xi}_{ij} - \xi_{ij} \right)^\top b_j \).

Lemma S.1. Under conditions (C1)–(C5), \( \varepsilon(X_i; \hat{\beta}) = \epsilon_i + h_i \), where

\[
\begin{align*}
    h_i &= -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \zeta_{j,k} D_{s,k} D_{i,j} \varpi_s + T_{i2}, \\
    \mathbb{E}(h_i) &= 0, \quad h_i = O_p(q^{1/2} n^{-1/2}).
\end{align*}
\]

Proof. Using a Taylor expansion, for a \( \tilde{\theta} \) between \( \theta \) and \( \hat{\theta} \), we have

\[
U(\theta) = U(\hat{\theta}) - J_{\hat{\theta}} (\hat{\theta} - \theta) = - \left[ \tilde{D}^\top \tilde{D} + (J_{\hat{\theta}} - J_\theta) + (J_\theta - \tilde{D}^\top \tilde{D}) \right] (\hat{\theta} - \theta),
\]

where

\[
J_\theta = \tilde{D}^\top \tilde{D} + \sum_{i=1}^{n} \varpi_i \frac{\partial D_i}{\partial \theta} + \sum_{i=1}^{n} \partial \left\{ \hat{\ell}(m(Y_i, \hat{\eta}_i)) \hat{D}_i - \varpi_i D_i \right\} / \partial \theta.
\]

Denote \( Z_n = \left( \frac{\tilde{D}^\top \tilde{D}}{n} \right)^{-1} \frac{\varpi}{\sqrt{n}}. \) Then it follows that

\[
\sqrt{n}(\hat{\theta} - \theta) = - \left[ I_p + \left( \frac{\tilde{D}^\top \tilde{D}}{n} \right)^{-1} \left( \frac{J_{\hat{\theta}} - J_\theta}{n} \right) + \left( \frac{\tilde{D}^\top \tilde{D}}{n} \right)^{-1} \left( \frac{J_\theta - \tilde{D}^\top \tilde{D}}{n} \right) \right]^{-1}
\times \left[ Z_n + \left( \frac{\tilde{D}^\top \tilde{D}}{n} \right)^{-1} \frac{U(\theta) - \tilde{D}^\top \varpi}{\sqrt{n}} \right].
\]
Note that under conditions (C3)-(C4), $\mathbb{E}\left(\left\|\frac{J_{\theta} - \tilde{D}^\top \tilde{D}}{n}\right\|^2\right) = O(q^2/n)$. Combining with condition (C3)–(C5) leads to

$$\left\|\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1}\left(\frac{J_{\theta} - \tilde{D}^\top \tilde{D}}{n}\right)\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1} \frac{\tilde{D}^\top \varpi}{\sqrt{n}}\right\|_2 = o_p(1),$$

$$\left\|\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1}\left(\frac{U(\theta) - \tilde{D}^\top \varpi}{n}\right)\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1} \frac{\varpi}{\sqrt{n}}\right\|_2 = o_p(1),$$

$$\left\|\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1} \left(\frac{U(\theta) - \tilde{D}^\top \varpi}{n}\right)\left(\frac{\tilde{D}^\top \tilde{D}}{n}\right)^{-1} \frac{\varpi}{\sqrt{n}}\right\|_2 = o_p(1).$$

Thus, $\left\|\sqrt{n}(\hat{\theta} - \theta) + Z_n\right\|_2 = o_p(1)$, which implies the asymptotically term to be $\sqrt{n}(\hat{\theta} - \theta) \sim -Z_n$. Denote $X_n = \frac{\tilde{D}^\top \varpi}{\sqrt{n}}$, that is, $X_n = \left(\sum_{i=1}^n \sum_{k=1}^q \left(\frac{D_{i,k}}{\sqrt{n}}\right)^{1/2} \frac{D_{i,k} \varpi_i}{\sqrt{n}}\right)^q$. Thus, it follows that

$$X_n^T X_n = \frac{1}{n} \sum_{j=1}^q \sum_{i_1 \neq i_2} \sum_{k_1,k_2} \varpi_{i_1} \varpi_{i_2} D_{i_1,k_1} D_{i_2,k_2} \zeta_{j,k_1}^{(1/2)} \zeta_{j,k_2}^{(1/2)} = A_n + B_n.$$

Using Condition (C5) that $q$ and $n^{1/(2\kappa + \alpha - 1)}$ are at the same order and Condition (C2) that $\kappa \geq \alpha + 2$ and $\alpha > 1$, it follows that $q^4/n \leq n^{\frac{4}{\kappa + \alpha - 1}} \leq n^{\frac{4}{\kappa + 3} - 1} < n^{4/\delta - 1} = o(1)$. Combining the regularity Conditions (C3)–(C4) and using similar arguments as those in Lemmas 7.3-7.5 in Muller & Stadtmuller (2005), we have $A_n = \mathbb{E}[\operatorname{Tr}(\Xi \Gamma)] + O_p(\sqrt{q}) = O_p(q)$, $B_n = O_p(\sqrt{q})$.

Therefore, we obtain that $n(\hat{\theta} - \theta)^\top \tilde{\Gamma}(\hat{\theta} - \theta) = O_p(q)$. 
For \( \varepsilon(X_i; \hat{\beta}) \), we have that

\[
\varepsilon(X_i; \hat{\beta}) = \varepsilon_i + \varepsilon(X_i; \hat{\beta}) - \varepsilon_i = \varepsilon_i + m(Y, \hat{\beta}_1, \cdots, \hat{\beta}_d) - m(Y, a_0, (X_{i1}, \beta_{01}), \cdots, (X_{id}, \beta_{0d}))
\]

\[
= \varepsilon_i + \sum_{j=1}^{d} m_j(Y, \eta_i) \left\{ (\hat{\xi}_{ij} - \xi_{ij})^\top b_j - \sum_{k=K+1}^{\infty} \xi_{ij,k} b_{j,k} \right\}
\]

\[
:= \varepsilon_i + T_{i1} + T_{i2} + T_{i3},
\]

where \( T_{i1} = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \xi_{ij,k} D_{s,k} D_{i,j} \varpi_s \), and \( T_{i3} = \sum_{j=1}^{d} m_j(Y, \eta_i) \sum_{k=K+1}^{\infty} \xi_{ij,k} b_{j,k} \).

Note that \( \mathbb{E}(T_{i2}) = \mathbb{E}(T_{i3}) = 0 \). Under condition (C2) that \( \alpha > 1 \) and \( \kappa \geq \alpha + 2 \) and condition (C5) that \( q \) and \( n^{\frac{1}{2\alpha - 1}} \) are at the same order as \( n \to \infty \), it follows from [Cai & Hall (2006)](2006) that \( T_{i3}^2 = O(n^{-\frac{2\alpha + 2}{\alpha + 2}}) \) (the discussion on page 2169) and \( T_{i2}^2 = O_p(n^{-1}) \) (inequality (5.20)). On the other hand, we have \( \mathbb{E}(T_{i1}) = 0 \), and

\[
\mathbb{E}(T_{i1}^2) = \mathbb{E} \left( n^{-2} \sum_{s=1}^{n} \sum_{k_1,k_2,j_1,j_2=1}^{q} \xi_{ij_1,k_1} \xi_{ij_2,k_2} D_{s,k_1} D_{s,k_2} D_{i,j_1} D_{i,j_2} \right)
\]

\[
= n^{-1} \mathbb{E} \left( \sum_{k_1,k_2,j_1,j_2=1}^{q} \xi_{ij_1,k_1} \xi_{ij_2,k_2} D_{i,j_1} D_{i,j_2} \right) = O(qn^{-1}).
\]

Thus, \( \varepsilon(X_i; \hat{\beta}) = \varepsilon_i + h_i = \varepsilon_i + O_p(q^{1/2}n^{-1/2}) \), where \( h_i = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \xi_{ij,k} D_{s,k} D_{i,j} \varpi_s + T_{i2} \), and \( \mathbb{E}(h_i) = 0 \). \( \square \)

Recall that \( q_f = K^2d + K \), \( \varpi_f = \left( \hat{\ell}(m(Y, \eta_i^f(\cdot))) \right)^n \), and \( D_f = \)
\{ \mathbf{D}_f^i \}_{i=1}^n \text{ is a } n \times q_f \text{-dimensional matrix.} \\
\langle \mathbf{D}_f, \mathbf{\omega}_f \rangle = \left( \sum_{i=1}^n \langle \mathbf{D}^i_{f,k}, \mathbf{\omega}^i_{f} \rangle \right)_{k=1}^{q_f}, \\
\Gamma_f := \{ \Gamma^f_{k,l} \}_{1 \leq k,l \leq q_f} := \{ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \langle \mathbf{D}^i_{f,k}, \mathbf{\omega}^i_{f} \rangle \langle \mathbf{D}^i_{f,l}, \mathbf{\omega}^i_{f} \rangle \}_1 \leq k,l \leq q_f. \\
\hat{\Gamma}_n^f = \sum_{i=1}^n \int \tilde{\ell}(m(Y_i(t), \eta_i^f(t))) \mathbf{D}_f^i(t)(\mathbf{D}_f^i(t))^\top dt \text{ be a } q_f \times q_f \text{-dimensional matrix,} \\
\hat{\mathbf{f}} = \lim_{n \to \infty} \left( \frac{\hat{\Gamma}_n^f}{n} \right) = (\hat{\Gamma}^f_{k,l})_{0 \leq k,l \leq q_f}, \quad \Xi_f = \hat{\Gamma}^{-1}_f = (\zeta_{j,k})_{0 \leq j,k \leq q_f}, \\
\sum_{k=1}^{q_f}(\zeta_{j,k}^{1/2}(\zeta_{k,l}^{1/2} = \zeta_{j,l}^{1/2}. \text{ Denote } T_{12} = m_0(Y_i(\cdot), \eta_i^f(\cdot))(\hat{\phi}(\cdot) - \phi(\cdot))^\top a_f + \\
\sum_{j=1}^d m_j(Y_i(\cdot), \eta_i^f(\cdot)) \left( \xi_{ij} \otimes \hat{\phi}(\cdot) - \xi_{ij} \otimes \phi(\cdot) \right)^\top b_j^f. \\
\textbf{Lemma S.2. Under conditions (C1)--(C2), (C3f)--(C5f), for any } \gamma \in \mathbb{R}^p,
\langle \varepsilon^f(X_i, \hat{\beta}), \gamma \rangle_m = \langle \epsilon_i^f, \gamma \rangle_m + \langle h_i^f, \gamma \rangle_m, \text{ where} \\
\langle h_i^f, \gamma \rangle_m = -n^{-1} \sum_{s=1}^n \sum_{k,j=1}^{q_f} \zeta_{j,k}^{1/2}(D_s^i, \mathbf{\omega}_s^i \langle \mathbf{D}_s^i, \gamma \rangle_m + \langle T_{i2}, \gamma \rangle_m, \\
\text{ and } \mathbb{E}(\langle h_i^f, \gamma \rangle_m) = 0, \langle h_i^f, \gamma \rangle_m = O_p(a_f^{1/2}n^{-1/2}). \\
\textbf{Proof. Using a Taylor expansion, for a } \theta_f \text{ between } \theta_f \text{ and } \hat{\theta}_f, \text{ we have} \\
U_f(\theta_f) = U_f(\hat{\theta}_f) - \mathbf{J}_{\theta_f}(\theta_f - \hat{\theta}_f) = -\left[ \hat{\Gamma}_n^f + \left( \mathbf{J}_{\hat{\theta}_f} - \mathbf{J}_{\theta_f} \right) + \left( \mathbf{J}_{\theta_f} - \hat{\Gamma}_n^f \right) \right] (\theta_f - \hat{\theta}_f), \\
\text{ where } \mathbf{J}_{\theta_f} = \hat{\Gamma}_n^f + \sum_{i=1}^n \int \mathcal{w}(t) \phi_f(t) dt + \sum_{i=1}^n \int \phi_f(t) \hat{\mathbf{D}}_i(t) - \mathcal{w}(t) \mathbf{D}_i(t) \phi_f(t) dt. \\
\text{ Denote } Z_n^f = \left( \frac{\hat{\Gamma}_n^f}{\sqrt{n}} \right)^{-1}(\mathbf{D}_f, \mathbf{\omega}_f). \text{ Note that} \\
\sqrt{n}(\hat{\theta}_f - \theta_f) = -\left[ 1_{q_f} + \left( \frac{\hat{\Gamma}_n^f}{n} \right)^{-1} \left( \mathbf{J}_{\hat{\theta}_f} - \mathbf{J}_{\theta_f} \right) + \left( \frac{\hat{\Gamma}_n^f}{n} \right)^{-1} \left( \mathbf{J}_{\theta_f} - \hat{\Gamma}_n^f \right) \right]^{-1} \\
\times \left\{ Z_n^f + \frac{\hat{\Gamma}_n^f}{\sqrt{n}} U_f(\theta_f) - \langle \mathbf{D}_f, \mathbf{\omega}_f \rangle \right\}. \\
\text{ S.5. PRELIMINARY LEMMAS}
Under conditions (C3f)-(C5f), we have \( \| \sqrt{n} \left( \hat{\theta}_f - \theta_f \right) + Z^f_n \|_2 = o_p(1) \).

Thus, the asymptotically term is seen to be \( \sqrt{n}(\hat{\theta}_f - \theta_f) \sim -Z^f_n \). Denote \( X^f_n = \Xi^{1/2} f \langle D^f, \varpi^f \rangle / \sqrt{n} \), that is, \( X^f_n = (\sum_{i=1}^{n} \sum_{k=1}^{q_f} (\zeta^f_{j,k})^{1/2} \langle D^f_{i,k}, \varpi^f \rangle)_{j=1}^{q_f} \).

Thus, it follows that

\[
(\mathcal{X}_n^f)^\top \mathcal{X}_n^f = \frac{1}{n} \sum_{j=1}^{q_f} \sum_{i_1,i_2} \sum_{k_1,k_2} \langle D^f_{i_1,k_1}, \varpi^f \rangle \langle D^f_{i_2,k_2}, \varpi^f \rangle (\zeta^f_{j,k_1})^{1/2} (\zeta^f_{j,k_2})^{1/2} = A^f_n + B^f_n.
\]

Using condition (C5f) that \( K \) and \( n^{1/(2\kappa+\alpha-1)} \) are at the same order and condition (C2) that \( \kappa \geq \alpha + 3 \) and \( \alpha > 1 \), it follows that \( q_f^2/n \leq n^{\frac{8}{3\alpha + 5} - 1} \leq n^{-\frac{8}{3\alpha + 5} - 1} = o(1) \). Then, using similar arguments as those in Lemmas 7.3-7.5 in Muller & Stadtmuller (2005), under condition (C5f), we have \( A^f_n = \mathbb{E}[\text{Tr} (\Xi^f \Gamma^f)] + O_p(\sqrt{q_f}) = O_p(q_f) \), and \( B^f_n = O_p(\sqrt{q_f}) \). Therefore, \( n(\hat{\theta}_f - \theta_f)^\top \Gamma^f (\hat{\theta}_f - \theta_f) = O_p(q_f) \).

For \( \langle \varepsilon^f (X^i; \hat{\beta}), \gamma \rangle_m \), we have that
\begin{align*}
\langle \varepsilon^f(X_i; \hat{\beta}), \gamma \rangle_m &= \langle \epsilon_i^f, \gamma \rangle_m + \langle \varepsilon^f(X_i; \hat{\beta}), \gamma \rangle_m - \langle \epsilon_i^f, \gamma \rangle_m \\
&= \langle \epsilon_i^f, \gamma \rangle_m + \left( \sum_{i} m(Y_i(\cdot), \hat{n}_i^f(\cdot)) - m(Y_i(\cdot), a_0(\cdot), \langle X_{i,1}, \beta_0,1(\cdot) \rangle, \cdots, \langle X_{i,d}, \beta_0, d(\cdot) \rangle, \gamma \right)_m \\
&= \langle \epsilon_i^f, \gamma \rangle_m + \left( (D_i^f)^\top (\hat{\theta}_f - \theta_f), \gamma \right)_m \\
&\quad + \left( m_0(Y_i(\cdot), \eta_i^f(\cdot)) \left( \hat{\phi}(\cdot) - \phi(\cdot) \right)^\top a_f - \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} b^f_{j,ik} \xi_{ij,k} \phi(l) \right) \\
&\quad - \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} b^f_{j,ik} \xi_{ij,k} \phi(l) \\
&= \langle \epsilon_i^f, \gamma \rangle_m + \langle T^f_{i1}, \gamma \rangle_m + \langle T^f_{i2}, \gamma \rangle_m + \langle T^f_{i3}, \gamma \rangle_m,
\end{align*}

where \( \langle T^f_{i1}, \gamma \rangle_m = -n^{-1} \sum_{s=1}^{n} \sum_{j=1}^{q_f} \zeta^f_{j,k} \langle D^f_{s,k}, \omega^f_s \rangle \langle D^f_{i,j}, \gamma \rangle_m \) and \( T^f_{i3} = m_0(Y_i(\cdot), \eta_i^f(\cdot)) \times \sum_{k=K+1}^{\infty} \phi_k(\cdot) a_k^f(\cdot) + \sum_{j=1}^{d} m_j(Y_i(\cdot), \eta_i^f(\cdot)) \left( \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} b^f_{j,ik} \xi_{ij,k} \phi(l) \right) + \sum_{k=K+1}^{\infty} \sum_{l=K+1}^{\infty} b^f_{j,ik} \xi_{ij,k} \phi(l) \). Following similarly arguments as the proof of Lemma \textbf{S.1} we obtain that \( \mathbb{E}(\langle T^f_{i2}, \gamma \rangle_m) = \mathbb{E}(\langle T^f_{i3}, \gamma \rangle_m) = 0, \langle T^f_{i3}, \gamma \rangle_m^2 = O_p(qfn^{-1}) \). On the other hand, we have

\begin{align*}
\mathbb{E}(\langle T^f_{i1}, \gamma \rangle_m) = 0, \quad \text{and} \\
\mathbb{E}(\langle T^f_{i1}, \gamma \rangle_m^2) &= n^{-2} \sum_{s=1}^{n} \sum_{k_1,k_2,j_1,j_2=1}^{q_f} \mathbb{E}\left( \zeta^f_{j_1,k_1} \zeta^f_{j_2,k_2} \langle D^f_{s,k_1}, \omega^f_s \rangle \langle D^f_{s,k_2}, \omega^f_s \rangle \langle D^f_{i,j_1}, \gamma \rangle_m \langle D^f_{i,j_2}, \gamma \rangle_m \right) \\
&= n^{-1} \mathbb{E}\left( \sum_{k_1,k_2,j_1,j_2=1}^{q_f} \zeta^f_{j_1,k_1} \Gamma^f_{k_1,k_2} \zeta^f_{j_2,k_2} \langle D^f_{i,j_1}, \gamma \rangle_m \langle D^f_{i,j_2}, \gamma \rangle_m \right) = O(qfn^{-1}).
\end{align*}
Thus, we have $\langle \varepsilon^f (X_i; \beta), \gamma \rangle = \langle \varepsilon^f, \gamma \rangle_m + \langle h^f, \gamma \rangle_m = \langle \varepsilon^f, \gamma \rangle_m + O_p(q_1^{1/2} n^{-1/2})$,
where $\langle h^f, \gamma \rangle_m = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \zeta^{f}_{j,k} (D^{f}_{s,k}, \omega^f_{s}) (D^{f}_{i,j}, \gamma)(\gamma) m + \langle T^f_{i2}, \gamma \rangle_m$, and $E(\langle h^f, \gamma \rangle_m) = 0$. \hfill $\square$

Recall that $Y_{i,n} = g_l (a, (X_{i,1}, \beta_1), \cdots , (X_{i,d}, \beta_d), \epsilon_i, n^{-1/2} \mathcal{F}(X_i))$,
$Y_i = g_l (a, (X_{i,1}, \beta_1), \cdots , (X_{i,d}, \beta_d), \epsilon_i, 0) := g (a, (X_{i,1}, \beta_1), \cdots , (X_{i,d}, \beta_d))$,
$\varepsilon (X_i; \beta) = m(Y_i, a, (X_{i,1}, \beta_1), \cdots , (X_{i,d}, \beta_d))$, and $\varepsilon (X_i; \tilde{\beta}) = m(Y_{i,n}, \tilde{a}, (\tilde{\xi}_{i1})^\top \tilde{b}_1, \cdots , (\tilde{\xi}_{id})^\top \tilde{b}_d)$.

**Lemma S.3.** Under conditions (C1)–(C6), $\varepsilon (X_i; \tilde{\beta}) = \epsilon_i + \hat{h}_i$, where

$$\hat{h}_i = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \zeta^{f}_{j,k} D_{s,k} D_{i,j} \omega_s + T_{i2} + n^{-1/2} \left( m_y (Y_i, \eta_i) \hat{g}_l (\eta_i, \epsilon_i, 0) \mathcal{F}(X_i) - \sum_{j=1}^{q} D_{i,j} v_j \right).$$

**Proof.** Note that $\tilde{\beta}$ is obtained by solving the following estimating equation,

$$U(\theta) = \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \hat{\eta}_i)) \hat{D}_i$$

$$= \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \eta_i)) D_i + \left[ \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \hat{\eta}_i)) \hat{D}_i - \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \eta_i)) D_i \right]$$

$$= \sum_{i=1}^{n} \hat{\ell} (m(Y_i, \eta_i)) D_i + \sum_{i=1}^{n} \hat{\ell} (m(Y_i, \eta_i)) m_y (Y_i, \eta_i) \hat{g}_l (\eta_i, \epsilon_i, 0) n^{-1/2} \mathcal{F}(X_i) D_i$$

$$+ \left[ \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \hat{\eta}_i)) \hat{D}_i - \sum_{i=1}^{n} \hat{\ell} (m(Y_{ni}, \eta_i)) D_i \right],$$

where $m_y (Y, \eta) = \partial m(Y, \eta) / \partial Y$, and $\hat{g}_l (\eta_i, \epsilon_i, b) = \partial g_l (\eta_i, \epsilon_i, b) / \partial b$.

Similar to Lemma [S.1] we have that

$$\sqrt{n} (\tilde{\theta} - \theta) \approx -\mathcal{Z}_n - \nu_n,$$
where \( \nu_n = n^{-1} \Xi \sum_{i=1}^n \hat{\epsilon}(m(Y_i, \eta_i))m_y(Y_i, \eta_i)\hat{g}_t(\eta_i, \epsilon_i, 0)\mathcal{F}(X_i)D_i. \)

For \( \varepsilon(X_i; \hat{\beta}) \), it is easy to obtain that

\[
\varepsilon(X_i; \hat{\beta}) = \varepsilon_i + \varepsilon(X_i; \hat{\beta}) - \varepsilon_i = \varepsilon_i + m(Y_{i,n}, \hat{a}, (\hat{\xi}_{i1})^\top \hat{b}_1, \cdots, (\hat{\xi}_{id})^\top \hat{b}_d) - m(Y_i, a, (X_{i,1}, \beta_1), \cdots, (X_{i,d}, \beta_d))
\]

\[
= \varepsilon_i + m_y(Y_i, \eta_i)\hat{g}_t(\eta_i, \epsilon_i, 0)n^{-1/2}\mathcal{F}(X_i) + D_i^\top (\hat{\theta} - \theta) + T_{i2} + T_{i3}.
\]

Thus, it follows from Lemma S.1 that

\[
\hat{h}_i = -n^{-1} \sum_{s=1}^n \sum_{k,j=1}^q \zeta_{j,k}D_{s,k}D_{i,j} \omega_s + T_{i2} + n^{-1/2} \left( m_y(Y_i, \eta_i)\hat{g}_t(\eta_i, \epsilon_i, 0)\mathcal{F}(X_i) - \sum_{j=1}^q D_{i,j}v_j \right).
\]

Recall that \( Y_{i,n}(t) = g_t(a(t), (X_{i,1}, \beta_1), \cdots, (X_{i,d}, \beta_d), \epsilon_i(t), n^{-1/2}\mathcal{F}(X_i)) \),

\( Y_i(t) = g_t(a(t), (X_{i,1}, \beta_1), \cdots, (X_{i,d}, \beta_d), \epsilon_i(t), 0) := g(a(t), (X_{i,1}, \beta_1), \cdots, (X_{i,d}, \beta_d)), \)

\( \varepsilon_i^f = m(Y_i(\cdot), a(\cdot), (X_{i,1}, \beta_1(\cdot)), \cdots, (X_{i,d}, \beta_d(\cdot))), \)

and

\[
\varepsilon^f(X_i; \hat{\beta}) = m(Y_{i,n}(\cdot), (\hat{a}_f)^\top \hat{\phi}(\cdot), (\hat{\xi}_{i1} \otimes \hat{\phi}(\cdot))^\top \hat{b}_1^f, \cdots, (\hat{\xi}_{id} \otimes \hat{\phi}(\cdot))^\top \hat{b}_d^f).
\]

**Lemma S.4.** Under conditions (C1)–(C2), (C3f)–(C5f), and (C6), \( \langle \varepsilon^f(X_i; \hat{\beta}), \gamma \rangle_m = \langle \varepsilon_i^f, \gamma \rangle_m + \langle \hat{h}_i^f, \gamma \rangle_m \), where

\[
\langle \hat{h}_i^f, \gamma \rangle_m = -n^{-1} \sum_{s=1}^n \sum_{k,j=1}^q \zeta_{j,k}^f \langle D_{s,k}^f, \omega_s^f \rangle \langle D_{i,j}^f, \gamma \rangle_m + \langle T_{i2}^f, \gamma \rangle_m
\]

\[+ n^{-1/2} \left( \langle m_y(Y_i(\cdot), \eta_i^f(\cdot))\hat{g}_t(\eta_i^f(\cdot), \epsilon_i(\cdot), 0)\mathcal{F}(X_i), \gamma \rangle_m - \sum_{j=1}^q \langle D_{i,j}^f, \gamma \rangle_m v_j^f \right). \]

**Proof.** Note that \( \hat{\beta} \) is obtained by solving the following estimating equa-
tion,

\[ U_f(\theta_f) = \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i,n}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt \]

\[ = \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i,n}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt \]

\[ + \left[ \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i,n}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt - \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i,n}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt \right] \]

\[ = \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt \]

\[ + \left[ \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt - \sum_{i=1}^{n} \int \dot{\ell} \left( m(Y_{i,n}(t), \eta_i^f(t)) \right) \mathbf{D}_i^f(t) dt \right]. \]

Similar to Lemma S.2, we have that

\[ \sqrt{n}(\hat{\theta}_f - \theta_f) \approx -Z_n^f - v_n^f, \]

where \( v_n^f = n^{-1} \Xi_f \sum_{i=1}^{n} \int \dot{\ell}(m(Y_{i}(t), \eta_i^f(t)))m_y(Y_{i}(t), \eta_i^f(t))g_i(\eta_i^f(t), \epsilon_i(t), 0)n^{-1/2} \mathcal{F}(X_i) \mathbf{D}_i^f(t) dt. \)

For \( \langle \varepsilon^f(X_i; \tilde{\beta}), \gamma \rangle_m \), it is easy to obtain that

\[ \langle \varepsilon^f(X_i; \tilde{\beta}), \gamma \rangle_m = \langle \varepsilon_i^f, \gamma \rangle_m + \langle \varepsilon_i^f(X_i, \tilde{\beta}), \gamma \rangle_m - \langle \varepsilon_i^f, \gamma \rangle_m \]

\[ = \langle \varepsilon_i^f, \gamma \rangle_m + \langle m(Y_{i,n}(\cdot), (\bar{a}_f)^\top \phi(\cdot), (\tilde{\xi}_{i1} \otimes \phi(\cdot))^\top \mathbf{b}_1^f, \ldots, (\xi_{i,d} \otimes \phi(\cdot))^\top \mathbf{b}_d^f) \]

\[ -m(Y_{i}(\cdot), a_0(\cdot), X_{i,1}, \beta_{0,1}(\cdot)), \ldots, (X_{i,d}, \beta_{0,d}(\cdot)), \gamma \rangle_m \]

\[ = \langle \varepsilon_i^f, \gamma \rangle_m + n^{-1/2} \langle m_y(Y_{i}(\cdot), \eta_i^f(\cdot))g(\eta_i^f(\cdot), \epsilon_i(\cdot), 0) \mathcal{F}(X_i), \gamma \rangle_m \]

\[ + \langle (\mathbf{D}_i^f)^\top (\hat{\theta}_f - \theta_f), \gamma \rangle_m + \langle T_{i2}, \gamma \rangle_m + \langle T_{i3}, \gamma \rangle_m. \]
Thus, it follows from Lemma S.2 that

\[
\langle \hat{h}, \gamma \rangle_m = -n^{-1} \sum_{s=1}^{n} \sum_{k,j=1}^{q} \zeta_{s,k}^{f} \langle D_{s,k}^{f}, \omega_{s}^{f} \rangle \langle D_{i,j}^{f}, \gamma \rangle_m + \langle T_{i2}^{f}, \gamma \rangle_m + n^{-1/2} \left( \langle m(y(Y(\cdot), \eta_i^{f}(-\cdot)), \epsilon_i(\cdot), 0) F(X_i), \gamma \rangle_m - \sum_{j=1}^{q} \langle D_{i,j}^{f}, \gamma \rangle_m v_j^{f} \right).
\]

\[\square\]

S.6 Additional results from numerical study

In this section, we provide more simulation results. The following part includes the non-Gaussian noise cases, Brownian Bridge and Pareto noise with sample size \( n = 40 \). We also list the computation time for different test methods for Examples 1.1 of the main paper and 1.2 of the Supplementary Material.

**Example 1.2 (FLMsR)** We consider a model where the response, \( Y_i \), is a scalar and the predictor, \( X_i(t) \), is a univariate functional predictor. It is given by a functional linear model as follows:

\[
Y_i = \int_0^1 c_1 \cdot \beta(t) X_i(t) dt + \int_0^1 c_2 \cdot \beta(t) X_i^2(t) dt + \left( \int_0^1 \{X_i(t)\} c_3 dt \right) \epsilon_i, \quad 1 \leq i \leq n
\]

where \( \{X_i(t)\}_{i=1}^{n} \) are generated independently from Brownian bridges, \( \{\epsilon_i\}_{i=1}^{n} \) follows \( N(0,0.1^2) \), \( \beta(t) = \exp(t^2)/2 \) and \( c_1 = 0.25 \). We consider \( c_2 = 0 \) and \( c_3 = 0 \) for the null hypothesis, and \( c_2 = n^{-1/2}, n^{-2/5} \) or \( c_3 = 2 \) for the alter-
Table 1: Simulation results for **Example 1.2** based on the proposed test and FMDD under models (S.5) respectively. The rows of $\beta_0$ and $\hat{\beta}$ show the results based on using the true $\beta_0(t)$ and the estimated value $\hat{\beta}(t)$, respectively.

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<th>Level=5%</th>
<th>Level=10%</th>
<th>Level=5%</th>
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<td>$c_2 = 0$, $c_3 = 0$</td>
<td>$c_2 = n^{-2/3}$, $c_3 = 0$</td>
<td>$c_2 = n^{-1/2}$, $c_3 = 0$</td>
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<td>$T_{n,F}$</td>
<td>$\beta_0$</td>
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<td>0.100</td>
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<td></td>
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<td>0.090</td>
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<td></td>
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</tr>
<tr>
<td>FMDD</td>
<td>$\beta_0$</td>
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<td>0.078</td>
<td>0.052</td>
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</tr>
<tr>
<td></td>
<td>$\hat{\beta}$</td>
<td>0.958</td>
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<td>0.296</td>
<td>0.266</td>
<td>0.212</td>
<td>0.154</td>
</tr>
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</table>

In this setting, the estimation $\hat{\beta}(t)$ is obtained by using the function `fregre.basis` in the fda.usc package.

Table 1 shows the empirical sizes and power of our proposed test compared with FMDD based on 500 repetitions. Similar conclusions with those for Table 1 in the main text can be obtained.

**Example 1.3 (FLMfR)** Consider the same linear regression settings as in the main paper. The response, $Y_i$, is a functional response and the predictor, $X_i(t)$, is a univariate functional predictor. It is given by a functional linear...
model as follows:

\[ Y_i(t) = \int_0^1 c_1 \cdot \beta(s,t) X_i(s) ds + \int_0^1 c_2 \cdot \beta(s,t) X_i^2(s) ds + \{X_i(t)\}^{c_3} \epsilon_i(t), \quad 1 \leq i \leq n \] (S.6)

where \( \{X_i(t)\}_{i=1}^n \) are generated independently from Brownian bridges, \( \beta(s,t) = \exp(s^2 + t^2)/2 \), and \( c_1 = 0.25 \). \( \{\epsilon_i(t)\}_{i=1}^n \) follows Pareto distribution with location 0.01 and shape 2.5, and Brownian bridge, respectively. Setting homogeneous scenario \((c_3 = 0)\), we consider \( c_2 = 0 \) for the null hypothesis, and \( c_2 = 0.05, n^{-1/2}, n^{-2/5}, \) and 1 for the alternatives, respectively.

The left block of Table 2 shows the results for the Pareto distribution while the right block shows the results for the Brownian bridge. It presents our proposed test’s empirical sizes and power compared with fdapss and FMDD, when sample size \( n = 40 \). We use 500 bootstrap samples for each original sample to compute the critical value. Five hundred samples are generated to approximate the percentages of rejection. The results are based on the true value of coefficient function \( \beta_0(s,t) \), we demonstrate our proposed test for comparison with the test fdapss proposed by Patilea et al. (2016) and the test FMDD proposed by Lee et al. (2020). The number of basic components \( K \) is chosen for each simulated sample so that the percentage of explained variance is larger than 95%, and \( p = K \) is taken. Since the Pareto distribution is a skewed positive distribution, we first centralize the random measurement error to satisfy the mean zero assumption for the
From Table 2, we can see that the empirical size of $T_{n,M}^f$ is zero for the Pareto distribution. The reason is that when we use wild bootstrap to calculate the distribution, the skewed Pareto distribution may be concentrated around zero. The wild bootstrap can not generate the empirical distribution evenly. When the null hypothesis does not hold, $T_{n,F}^f$ is more powerful than $T_{n,F}^f$. When the alternative part becomes more significant as $c_2$ increases, $T_{n,F}^f$ performs slightly better than FMDD in test power.

Further, to consider the impact of the heterogeneous variance, we generate data with $c_2 = 0, c_3 = 0$ for the null and $c_2 = 0, c_3 = 2$ for the alternative. The bottom right block of Table 2 presents the results of the test statistics $T_{n,F}^f$ and $T_{n,M}^f$ under heterogeneity compared with FMDD and fdapss. The results show that under model (S.6), only the distribution-based test $T_{n,F}^f$ can detect the heterogeneity, whereas both $T_{n,M}^f$, FMDD and fdapss fail to do so.

Compared to the fdapss test and FMDD test, we show the CPU time of computations for our methods in Table 3. It is conducted on a personal computer (Processor: Inter Core i7-9700, CPU 3.00GHz, RAM: 16GB). Our approach is realized by R. When the sample size is small, our $T_{n,M}^f$ test is comparable to the FMDD test. And our $T_{n,F}^f$ test is slightly slower than
S.6. ADDITIONAL RESULTS FROM NUMERICAL STUDY

fdapss as the fdapss test uses FORTRAN routines for inner loop iterations. Although when sample size increases, fdapss and FMDD tests outperform our $T_{n,F}^f$ test method significantly in terms of computing speed. And our $T_{n,F}^f$ test method needs to apply parallel methods as the sample size rises to complete the 500 simulations. Our $T_{n,F}^f$ test can detect heterogeneity, whereas both $T_{n,M}^f$ and FMDD fail to do so.

Table 2: Simulation results for Example 1.3 based on the proposed test, FMDD, and fdapss and $\epsilon \sim$ Pareto distribution with location 0.01 and shape 2.5 and Brownian Bridge, respectively. The rows of $\beta_0$ show the results based on using the true $\beta_0(s,t)$.

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<th>Level=5%</th>
<th>Level=10%</th>
<th>Level=5%</th>
<th>Level=10%</th>
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<th>Level=10%</th>
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<tbody>
<tr>
<td></td>
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<td>$n = 40$</td>
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<td>$n = 40$</td>
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<tr>
<td>$T_{n,F}^f$</td>
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<tr>
<td>$T_{n,M}^f$</td>
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<td>0.000</td>
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<td>0.158</td>
<td>0.080</td>
<td>0.880</td>
<td>0.554</td>
<td>0.122</td>
<td>0.084</td>
<td>0.134</td>
<td>0.080</td>
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<tr>
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<td>0.618</td>
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<td>0.004</td>
<td>0.115</td>
<td>0.065</td>
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<tr>
<td>FMDD</td>
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<td>0.556</td>
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<td>fdapss</td>
<td>0.808</td>
<td>0.638</td>
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<td>0.108</td>
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Table 3: Computation time for Examples 1.1 and 1.2 based on the proposed test, FMDD, and fdapss under models FLMfR and FLMsR, respectively. The rows of $\beta_0$ and $\hat{\beta}$ show the results using the true $\beta_0(s,t)$, $\beta_0(t)$, the estimated value $\hat{\beta}(s,t)$, and $\hat{\beta}(t)$ (seconds for one replicate), respectively.

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$c_2 = 0, c_3 = 0$

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References


