Spatial-temporal Model with Heterogeneous Random Effects

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Supplementary Material

This supplementary material includes a residual-based bootstrapping procedure for the QMLE together with its asymptotic validity, and provides additional simulation results on the finite-sample performance of the WCQE and WQAE. It also includes the detailed forms of the matrices $\Sigma_N$, $\Sigma_N^*$, $\Omega_{NT}^{-1}(\beta^*)$ in Section 3 and the vector $Y_{-1}$, as well as technical details for Theorems 1-6 and Propositions 1-2. Moreover, Lemmas 1-10 are introduced to establish Theorems 1-6.

Throughout the appendix, $\otimes$ denotes the Kronecker product of two matrices, $| \cdot |$ denotes the absolute value of a scalar/vector or the determinant of a matrix, $\| \cdot \|$ denotes the Euclidean norm of a vector, and $tr(\cdot)$ denotes the trace of a matrix. Denote $N$ and $T$ as the total numbers of spatial units and time periods, respectively. Define $I(\cdot)$ as the indicator function. For a positive integer $m$, denote $I_m$ as an $m \times m$ identity matrix, $0_m$ as an $m \times 1$ vector of zeros and $e_m$ as an $m \times 1$ vector of ones, and let $J_m \equiv e_m e_m'$. For an $n \times m$ matrix $A$, denote $A = [a_{ij}]$, where $a_{ij}$ is the $i$th row and $j$th column element of $A$. The operator $E$ denotes the expectation with respect to the probability measure, $\mathbb{E}_n$ denotes the expectation with respect to the empirical measure, and $G_n = \sqrt{n}(\mathbb{E}_n - E)$. In addition, $\ell^\infty(T)$ denotes the space of all uniformly bounded functions on $T$. Moreover, $\rightarrow_p$ denotes convergence in probability, $\rightarrow_d$ denotes convergence in distribution, and $\Rightarrow$ denotes weak convergence.
S1 Bootstrapping Procedure

To circumvent difficulties of calculating the covariance matrix of $\hat{\zeta}$ in practice, we propose a residual-based bootstrapping procedure to approximate matrix $\Gamma$ in (3.9); see also Su and Yang (2015). Denote the score function $\partial \ln L_{NT}(\zeta)/\partial \zeta$ by $s(Y_{-1}, V_{NT}, \zeta)$. Let $\bar{A}_N(\beta^*) = A_N(\beta^*) + I_N$ and $\bar{A}_0N = \bar{A}_N(\beta^*_0)$. The bootstrapping procedure for estimating $\Gamma$ is summarized as follows:

1. Based on the QMLE $\hat{\zeta}$, compute the residuals $\hat{v}_t = y_t - \hat{\alpha}y_{t-1} - \hat{\lambda}W_Ny_t - Z_t\hat{\gamma} + X\hat{\psi}$ and the transformed residuals $\hat{r}_t = \bar{A}_N^{-1/2}(\beta^*)\hat{v}_t$ for $t = 1, \ldots, T$. For each $t$, center $\hat{r}_t$ by its mean. (Notice that for $r_t = \bar{A}_{0N}^{-1/2}v_t$ with $v_t = y_t - \alpha_0y_{t-1} - \lambda_0W_Ny_t - Z_t\gamma_0 - X\psi_0$, $E(r_t) = 0$ and $\text{cov}(r_t) = \sigma_0^2 I_N$.)

2. Make $N$ random draws from the rows of $(\hat{r}_1, \ldots, \hat{r}_T)$ to give $T$ matched bootstrap samples, $(\hat{r}^b_1, \ldots, \hat{r}^b_T)$, of the transformed residuals.

3. Conditional on $y_0, X, Z_1, \ldots, Z_T$, and the QMLE $\hat{\zeta}$, generate the bootstrap data according to

$$
\hat{y}_1^b = \left(I_N - \hat{\lambda}W_N\right)^{-1}\left(\hat{\alpha}y_0 + Z_1\hat{\gamma} + X\hat{\psi} + \hat{v}_1^b\right),
$$

$$
\hat{y}_t^b = \left(I_N - \hat{\lambda}W_N\right)^{-1}\left(\hat{\alpha}\hat{y}_{t-1}^b + Z_t\hat{\gamma} + X\hat{\psi} + \hat{r}_t^b\right),
\quad t = 2, \ldots, T,
$$

where $\hat{v}_t^b = \bar{A}_N^{1/2}(\beta^*)r_t^b$ for $t = 1, \ldots, T$. Then the bootstrapped values of $V_{NT}, Y$ and $Y_{-1}$ are given by $\hat{V}^b_{NT} = \text{vec}(\hat{v}^b_1, \ldots, \hat{v}^b_T)$, $\hat{Y}^b = \text{vec}(\hat{y}^b_1, \ldots, \hat{y}^b_T)$ and $\hat{Y}^b_{-1} = \text{vec}(y_0, \ldots, \hat{y}^b_{T-1})$, respectively, where the operator $\text{vec}(\cdot)$ vectorizes a matrix to a vector by columns.

4. Compute the bootstrapped score function by $s^b = s\left(\hat{Y}^b_{-1}, \hat{V}^b_{NT}, \hat{\zeta}\right)$.
(5) Repeat steps (2)–(4) $B$ times, and the bootstrap estimator of $\Gamma$ is defined as

$$\hat{\Gamma}_B = \frac{1}{NT} \left[ \frac{1}{B} \sum_{b=1}^{B} (s^b - \bar{s}_B)(s^b - \bar{s}_B)' \right],$$

where $\bar{s}_B = \frac{1}{B} \sum_{b=1}^{B} s^b$.

**Proposition 2.** Suppose the conditions of Theorem 2 hold, then as $N \to \infty$,

$$\frac{1}{NT} E^* \left[ \left( s^b - E^* (s^b) \right) \left( s^b - E^* (s^b) \right)' \right] = o_p(1),$$

where the expectation operator $E^*$ corresponds to the bootstrap probability space.

Proposition 2 implies $\hat{\Gamma}_B - \Gamma = o_p(1)$ as $N \to \infty$ and $B \to \infty$, which establishes the validity of the proposed residual-based bootstrapping procedure.

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**S2 Additional simulation results**

In this section, we provide additional results for Tables 2 and 3 in simulation study.

From Table 2 in the manuscript, we can see that the ESDs of WCQE are obviously smaller than the corresponding ASDs at $\tau = 0.5$ due to the same sign restriction in optimization. Here, we provide the results of WCQE at $\tau = 0.5$ for larger $N$ in Table S.1. It can be found that the ESDs will move closer to their corresponding ASDs for larger $N$.

Table S.2 reports the results of WQAE $\hat{\beta}_1(\hat{\pi}_{opt,K})$ for settings $(N, T) = (1500, 300)$ and $(3000, 600)$. Comparing with the results of small samples in Table 3 of the manuscript, we can see that the ESDs of $\hat{\beta}_1(\hat{\pi}_{opt,K})$ will move closer to their corresponding ASDs for larger $N$. 
S3 Structures of Matrices and Vectors

S3.1 $\Sigma_N$ and $\Sigma_N^*$

This subsection includes detailed structure of matrices $\Sigma_N$ and $\Sigma_N^*$ in Section 3. For simplicity, hereinafter the first derivative of $\ln L_N^T(\zeta_0)$ is denoted as $d_\zeta = (d'_\phi, d_{\sigma^2}, d_\lambda, d'_\beta, \ldots)' \triangleq \left( \frac{\partial \ln L_N^T(\zeta_0)}{\partial \phi'}, \frac{\partial \ln L_N^T(\zeta_0)}{\partial \sigma^2}, \frac{\partial \ln L_N^T(\zeta_0)}{\partial \lambda}, \frac{\partial \ln L_N^T(\zeta_0)}{\partial \beta'}, \ldots \right)'$, and its second derivatives are denoted as $\ddot{d}_\phi \triangleq \frac{\partial^2 \ln L_N^T(\zeta_0)}{\partial \phi'^2}$, $\ddot{d}_{\sigma^2} \triangleq \frac{\partial^2 \ln L_N^T(\zeta_0)}{\partial \sigma^2 \partial \phi'}$, $\ddot{d}_{\sigma^2 \sigma^2} \triangleq \frac{\partial^2 \ln L_N^T(\zeta_0)}{\partial \sigma^2 \partial \sigma^2}$, and so on.

Then $\Sigma_N$ and $\Sigma_N^*$ can be rewritten as follows,

$$\Sigma_N = -E \left( \frac{1}{NT} \frac{\partial^2 \ln L_N^T(\zeta_0)}{\partial \zeta \partial \zeta'} \right) = -\frac{1}{NT} \left( \begin{array}{c} \ddot{d}_\phi' \ \ * \ \ * \\ \ddot{d}_{\sigma^2} \ \ * \ \ * \\ \ddot{d}_\lambda \ \ * \ \ * \\ \ddot{d}_\beta' \ \ * \ \ * \end{array} \right),$$

$$\Sigma_N^* = E \left( \frac{1}{\sqrt{NT}} \frac{\partial \ln L_N^T(\zeta_0)}{\partial \zeta} \frac{1}{\sqrt{NT}} \frac{\partial \ln L_N^T(\zeta_0)}{\partial \zeta'} \right) - \Sigma_N = \frac{1}{NT} \left( \begin{array}{c} \ddot{d}_\phi \ \ * \ \ * \\ \ddot{d}_{\sigma^2} \ \ * \ \ * \\ \ddot{d}_\lambda \ \ * \ \ * \\ \ddot{d}_\beta \ \ * \ \ * \end{array} \right) - \Sigma_N,$$

where

$$\dddot{d}_\zeta = \left( \begin{array}{c} \ddot{d}_\phi \\ \ddot{d}_{\sigma^2} \\ \ddot{d}_\lambda \\ \ddot{d}_\beta' \end{array} \right), \quad \dddot{d}_\phi' = -\frac{1}{\sigma_{e_0}^2} \tilde{Z}' \Omega_{0NT}^{-1} \tilde{Z}, \quad \dddot{d}_{\sigma^2} = -\frac{1}{\sigma_{e_0}^2} V' \Omega_{0NT}^{-1} \tilde{Z}.$$
\[
\dd_{\sigma_2\sigma_1} = -\frac{1}{\sigma_{e0}^2} V'_{NT} \Omega_{0NT}^{-1} V_{NT} + \frac{NT}{2\sigma_{e0}^2}, \quad \dd_{\lambda\sigma'} = -\frac{1}{\sigma_{e0}^2} Y'(I_T \otimes W_N)' \Omega_{0NT}^{-1} \dd_{Z},
\]

\[
\dd_{\lambda\sigma} = -\frac{1}{\sigma_{e0}^2} Y'(I_T \otimes W_N)' \Omega_{0NT}^{-1} V_{NT},
\]

\[
\dd_{\lambda\lambda} = -\text{tr} \left( \left( S_{0NT}^{-1} (I_T \otimes W_N) \right)^2 \right) - \frac{1}{\sigma_{e0}^2} Y'(I_T \otimes W_N)' \Omega_{0NT}^{-1} (I_T \otimes W_N) Y,
\]

\[
\dd_{\beta_{*,\sigma'}} = \left( \begin{array}{c}
\dd_{\beta_{*,\sigma'}} \\
\vdots \\
\dd_{\beta_{*,\sigma'}}
\end{array} \right), \quad \dd_{\beta_{*,\sigma}} = \left( \begin{array}{c}
\dd_{\beta_{*,\sigma}} \\
\vdots \\
\dd_{\beta_{*,\sigma}}
\end{array} \right), \quad \dd_{\beta_{*,\lambda}} = \left( \begin{array}{c}
\dd_{\beta_{*,\lambda}} \\
\vdots \\
\dd_{\beta_{*,\lambda}}
\end{array} \right),
\]

\[
\dd_{\beta_{*,\beta'}} = \left( \begin{array}{c}
\dd_{\beta_{*,\beta'}} \\
\vdots \\
\dd_{\beta_{*,\beta'}}
\end{array} \right),
\]

with

\[
\dd_{\beta_k} = -\frac{1}{2} \text{tr} \left( \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_k^*} \right) + \frac{1}{2\sigma_{e0}^2} V'_{NT} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_k^*} \Omega_{0NT}^{-1} V_{NT},
\]

\[
\dd_{\beta_{*,\sigma'}} = -\frac{1}{\sigma_{e0}^2} V'_{NT} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_{*,\sigma'}} \Omega_{0NT}^{-1} \dd_{Z},
\]

\[
\dd_{\beta_{*,\sigma}} = -\frac{1}{\sigma_{e0}^4} V'_{NT} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_{*,\sigma}} \Omega_{0NT}^{-1} V_{NT},
\]

\[
\dd_{\beta_{*,\lambda}} = -\frac{1}{\sigma_{e0}^2} Y'(I_T \otimes W_N)' \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_{*,\lambda}} \Omega_{0NT}^{-1} V_{NT},
\]

\[
\dd_{\beta_{*,\beta'}} = -\frac{1}{2} \text{tr} \left( -\Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_{*,\beta'}} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_k^*} \right) + \frac{1}{2\sigma_{e0}^2} \left[ -2V'_{NT} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_{*,\beta'}} \Omega_{0NT}^{-1} \frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_k^*} \Omega_{0NT}^{-1} V_{NT} \right],
\]

and

\[
\frac{\partial \Omega_{0NT}(\beta_0^*)}{\partial \beta_k^*} = J_T \otimes \text{diag} \{ 2|x_{k1}|x'_{a1}\beta^*, \ldots, 2|x_{kn}|x'_{aN}\beta^* \},
\]
\[
\frac{\partial^2 \Omega_{NT}(\beta_0^*)}{\partial \beta_i^* \partial \beta_k^*} = J_T \otimes \text{diag} \left\{ 2|x_{t1}||x_{k1}|, \ldots, 2|x_{tN}||x_{kN}| \right\}.
\]

**S3.2 \( \Omega_{NT}^{-1}(\beta^*) \)**

Recall that \( \Omega_{NT}(\beta^*) = J_T \otimes A_N(\beta^*) + I_{NT} \) with \( A_N(\beta^*) = \text{diag} \{ (x_{a1}^i \beta^i)^2, \ldots, (x_{aN}^i \beta^i)^2 \} \).

By Lemma 2.1 of Magnus (1982), we have that

\[
\Omega_{NT}^{-1}(\beta^*) = \frac{1}{T} J_T \otimes (I_N + T A_N(\beta^*))^{-1} + \left(I_T - \frac{1}{T} J_T\right) \otimes I_N. \quad (S3.1)
\]

**S3.3 \( Y_{-1} \)**

By model (2.1), for \( t = 1, \ldots, T \), we have

\[
y_t = \alpha_0 (B_{0N}^{-1})^t y_0 + \left( \alpha_0^{t-1} (B_{0N}^{-1})^t Z_1 + \alpha_0^{t-2} (B_{0N}^{-1})^{t-1} Z_2 + \cdots + B_{0N}^{-1} Z_t \right) \gamma_0
\]

\[
+ \left( \alpha_0^{t-1} (B_{0N}^{-1})^t + \alpha_0^{t-2} (B_{0N}^{-1})^{t-1} + \cdots + B_{0N}^{-1} \right) \theta
\]

\[
+ \left( \alpha_0^{t-1} (B_{0N}^{-1})^t \varepsilon_1 + \alpha_0^{t-2} (B_{0N}^{-1})^{t-1} \varepsilon_2 + \cdots + B_{0N}^{-1} \varepsilon_t \right),
\]

and then we have

\[
Y_{-1} = (y_0', \ldots, y_{T-1}')' = M_1 y_0 + M_2 Z \gamma_0 + M_3 \theta + M_2 \varepsilon, \quad (S3.3)
\]

where

\[
M_1 = \begin{pmatrix}
I_N \\
\alpha_0 B_{0N}^{-1} \\
\vdots \\
\alpha_0^{T-2} (B_{0N}^{-1})^{T-2} \\
\alpha_0^{T-1} (B_{0N}^{-1})^{T-1}
\end{pmatrix}_{NT \times N}, \quad M_3 = \begin{pmatrix}
0 \\
B_{0N}^{-1} \\
\vdots \\
\sum_{t=1}^{T-2} \alpha_0^{t-1} (B_{0N}^{-1})^t \\
\sum_{t=1}^{T-1} \alpha_0^{t-1} (B_{0N}^{-1})^t
\end{pmatrix}_{NT \times N}.
\]
S4. Lemmas

This section includes some lemmas to show Theorems 1–6, and their proofs are relegated at the end of this section. Particularly, Lemmas 1–7 are general results that are useful for all theorems. Throughout this section, \( n \) and \( t \) are positive integers and \( nt \) denotes \( n \times t \).

**Lemma 1.** Suppose that elements \( h_{n,ij}(x) \) of the sequence of \( n \times n \) matrices \( \{H_n(x)\} \), are uniformly bounded with respect to all \( i, j, n, \) and \( x \) in \( X \); and \( \{G_n(x)\} \) is a sequence of conformable \( n \times n \) matrices.

(i) If \( \{G_n(x)\} \) are uniformly bounded in column sums, uniformly on \( X \), then elements \( c_{n,ij}(x) \) of \( \{H_n(x)G_n(x)\} \) are uniformly bounded with respect to all \( i, j \) and \( n \) on \( X \).

(ii) If \( \{G_n(x)\} \) are uniformly bounded in row sums, uniformly on \( X \), then elements \( c_{n,ij}(x) \) of \( \{G_n(x)H_n(x)\} \) are uniformly bounded with respect to all \( i, j \) and \( n \) on \( X \).

For both cases (i) and (ii), it holds that \( \text{tr}[H_n(x)G_n(x)] = \text{tr}[G_n(x)H_n(x)] = O(n) \), uniformly on \( X \).

**Lemma 2.** For \( x \in X \), let \( \{H_n(x)\} \) and \( \{G_n(x)\} \) be sequences of conformable \( n \times n \) matrices.

(i) If \( \{H_n(x)\} \) and \( \{G_n(x)\} \) are both uniformly bounded in row sums, uniformly on \( X \), then it holds that \( \{H_n(x)G_n(x)\} \) are also uniformly bounded in row sums, uniformly on \( X \).
(ii) If \( \{H_n(x)\} \) and \( \{G_n(x)\} \) are both uniformly bounded in column sums, uniformly on \( X \), then it holds that \( \{H_n(x)G_n(x)\} \) are also uniformly bounded in column sums, uniformly on \( X \).

(iii) If \( \{H_n(x)\} \) and \( \{G_n(x)\} \) are both uniformly bounded in both row and column sums, uniformly on \( X \), then it holds that \( \{H_n(x)G_n(x)\} \) are also uniformly bounded in both row and column sums, uniformly on \( X \).

**Lemma 3.** Suppose that elements of the sequences of vectors \( p_n = (p_{n,1}, \ldots, p_{n,n})' \) and \( q_n = (q_{n,1}, \ldots, q_{n,n})' \) are uniformly bounded. For \( x \in X \), \( \{H_n(x)\} \) is a sequence of conformable \( n \times n \) matrices.

(i) If \( \{H_n(x)\} \) are uniformly bounded in either row or column sums, uniformly in \( x \) on \( X \), then it holds that \( |q_n' H_n(x)p_n| = O(n) \) uniformly on \( X \).

(ii) If \( \{H_n(x)\} \) are uniformly bounded in row sums, uniformly on \( X \), and \( \{G_n\} \) are uniformly bounded in row sums, then it holds that \( |g_{i,n} H_n(x)p_n| = O(1) \) uniformly in all \( i \), uniformly on \( X \), where \( g_{i,n} \) is the \( i \)th row of \( G_n \).

**Lemma 4.** Suppose that elements of random vectors \( u_n = (u_{n,1}, \ldots, u_{n,n})' \) and \( \bar{u}_{nt} = (\bar{u}_{nt,1}, \ldots, \bar{u}_{nt,nt})' \) are i.i.d. with zero mean and finite variance \( \sigma^2 \), and elements of the sequence of vector \( p_n = (p_{n,1}, \ldots, p_{n,n})' \) are uniformly bounded for all \( n \). For \( x \in X \), \( \{H_n(x)\} \) and \( \{\overline{H}_{nt,n}(x)\} \) are sequences of conformable \( n \times n \) and \( nt \times n \) matrices, respectively.

(i) If \( \{H_n(x)\} \) are uniformly bounded in row sums, uniformly in \( x \) on \( X \), then it holds that \( \frac{1}{n} u_n' H_n(x)p_n = O_p(1) \) uniformly on \( X \).

(ii) If \( \{\overline{H}_{nt,n}(x)\} \) are uniformly bounded in row sums, uniformly on \( X \), then it holds that \( \frac{1}{nt} \bar{u}_{nt}' \overline{H}_{nt,n}(x)p_n = O_p(1) \) uniformly on \( X \).
(iii) If \( \{ H_n(x) \} \) are uniformly bounded in both row and column sums, uniformly on \( X \),
\[ \left\{ \frac{\partial}{\partial x} H_n(x) \right\} \]
are uniformly bounded in row sums, uniformly on \( X \), and \( X \) is a compact set,
then it holds that \( E(u'_n H_n(x)p_n) = 0 \), \( \text{var}(u'_n H_n(x)p_n) = O(n) \), and \( \left| \frac{1}{n} u'_n H_n(x)p_n \right| = O_p(1) \) uniformly on \( X \).

(iv) If \( \{ \Pi_{nt,n}(x) \} \) and \( \{ \frac{\partial}{\partial x} \Pi_{nt,n}(x) \} \) are both uniformly bounded in row sums, uniformly on \( X \), the column sums of \( \{ \Pi_{nt,n}(x) \} \) are \( O(t) \) uniformly on \( X \), and \( X \) is a compact set,
then it holds that \( E(\bar{\Pi}_{nt,n}(x)p_n) = 0 \), \( \text{var}(\bar{\Pi}_{nt,n}(x)p_n) = O(n) \), and \( \left| \frac{1}{nt} \bar{\Pi}_{nt,n}(x)p_n \right| = O_p(1) \) uniformly on \( X \).

If elements of \( u_n \) and \( \bar{u}_{nt} \) are not i.i.d., but \( E(u_{n,i}) \) and \( E(\bar{u}_{nt,i}) \) are uniformly bounded, (i) and (ii) still hold. If elements of \( u_n \) and \( \bar{u}_{nt} \) are not i.i.d., but the covariance matrices of \( u_n \) and \( \bar{u}_{nt} \) are uniformly bounded in row (column) sums, (iii) and (iv) still hold.

Lemma 5. Suppose that elements of random vector \( u_n = (u_{n,1}, \ldots, u_{n,n})' \) are i.i.d. with finite variance \( \sigma^2 \). For \( x \in X \), \( \{ H_n(x) \} \) is a sequence of conformable \( n \times n \) matrices.

(i) If elements of \( \{ H_n(x) \} \) are uniformly bounded, uniformly in \( x \) on \( X \), then it holds that
\[ |u'_n H_n(x)u_n| = O_p(n) \], \( E(u'_n H_n(x)u_n) = O(n) \), and \( \left| \frac{1}{n} u'_n H_n(x)u_n - \frac{1}{n} E(u'_n H_n(x)u_n) \right| = O_p(1) \) uniformly on \( X \).

(ii) If \( E(u^4_{n,i}) < \infty \), \( \{ H_n(x) \} \) are uniformly bounded in either row or column sums, uniformly on \( X \), elements of \( \{ \frac{\partial}{\partial x} H_n(x) \} \) are uniformly bounded, uniformly on \( X \), and \( X \) is a compact set,
then it holds that \( E(u'_n H_n(x)u_n) = O(n) \), \( \text{var}(u'_n H_n(x)u_n) = O(n) \), \( |u'_n H_n(x)u_n| = O_p(n) \), and \( \left| \frac{1}{n} u'_n H_n(x)u_n - \frac{1}{n} E(u'_n H_n(x)u_n) \right| = o_p(1) \) uniformly on \( X \).

If elements of \( u_n \) are not i.i.d., but the covariance matrix of \( |u_n| \) is uniformly bounded in
row (column) sums, (i) still holds, furthermore, if for any \( 1 \leq i, j \leq N \), we also have
\[
\sum_{s=1}^{N} \sum_{t=1}^{N} \left[ E(u_i u_j u_s u_t) - E(u_i u_j) E(u_s u_t) \right] = O(1),
\]
(S4.4)
(ii) still holds, where equation (S4.4) is to ensure that \( \text{var}(u'_n H_n(x) u_n) = O(n) \).

**Lemma 6.** Suppose that elements of random vectors \( \bar{u}_{nt} = (\bar{u}_{nt,1}, \ldots, \bar{u}_{nt,n})' \), \( u_n = (u_{n,1}, \ldots, u_{n,n})' \)
and \( v_n = (v_{n,1}, \ldots, v_{n,n})' \) are i.i.d. with finite variance, respectively, and both \( \bar{u}_{nt,i} \) and \( u_{n,i} \)
are independent with \( v_{n,j} \). For \( x \in X \), \( \{H_n(x)\} \) and \( \{\bar{H}_{nt,n}(x)\} \)
are sequences of conformable \( n \times n \) and \( nt \times n \) matrices, respectively.

(i) If elements of \( \{H_n(x)\} \) are uniformly bounded, uniformly in \( x \) on \( X \), then it holds that
\[
\left| \frac{1}{n} u'_n H_n(x) v_n \right| = O_p(1) \text{ uniformly on } X.
\]

(ii) If elements of \( \{\bar{H}_{nt,n}(x)\} \) are uniformly bounded, uniformly in \( x \) on \( X \), then it holds that
\[
\left| \frac{1}{nt} u'_n \bar{H}_{nt,n}(x) v_n \right| = O_p(1) \text{ uniformly on } X.
\]

(iii) If \( E(u^4_{n,i}) < \infty \) and \( E(v^4_{n,i}) < \infty \), \( \{H_n(x)\} \) are uniformly bounded in either row or column sums, uniformly on \( X \), elements of \( \{\frac{\partial}{\partial x} H_n(x)\} \) are uniformly bounded, uniformly on \( X \), and \( X \) is a compact set, then it holds that \( E(u'_n H_n(x) v_n) = 0 \), \( \text{var}(u'_n H_n(x) v_n) = O(n) \), and \( \left| \frac{1}{n} u'_n H_n(x) v_n \right| = o_p(1) \text{ uniformly on } X \).

(iv) If \( E(\bar{u}^4_{nt,i}) < \infty \) and \( E(v^4_{nt,i}) < \infty \), elements of \( \{\bar{H}_{nt,n}(x)\} \) and \( \{\frac{\partial}{\partial x} \bar{H}_{nt,n}(x)\} \) are uniformly bounded, uniformly on \( X \), \( X \) is a compact set, and either (a) \( \{\bar{H}_{nt,n}(x)\} \)
are uniformly bounded in row sums, uniformly on \( X \); or (b) \( \{\frac{1}{nt} \bar{H}_{nt,n}(x)\} \) are uniformly bounded in column sums, uniformly on \( X \), then it holds that \( E(\bar{u}'_n \bar{H}_{nt,n}(x) v_n) = 0 \), \( \text{var}(\bar{u}'_n \bar{H}_{nt,n}(x) v_n) = O(nt) \), and \( \left| \frac{1}{nt} \bar{u}'_n \bar{H}_{nt,n}(x) v_n \right| = o_p(1) \text{ uniformly on } X \).

If elements in \( \bar{u}_{nt} \), \( u_n \) and \( v_n \) are not i.i.d., (i) and (ii) still hold. Furthermore, denote \( \Sigma_\bar{u} \), \( \Sigma_u \) and \( \Sigma_v \) as covariance matrices of \( \bar{u}_{nt} \), \( u_n \) and \( v_n \), respectively. If (c) \( \Sigma_u \) is uniformly
bounded in row (column) sums when \( \{H_n(x)\} \) are uniformly bounded in row sums; or (d) \( \Sigma_v \) is uniformly bounded in row (column) sums when \( \{H_n(x)\} \) are uniformly bounded in column sums, (iii) still holds. If (e) \( \Sigma_\pi \) is uniformly bounded in row (column) sums when (a) holds; or (f) \( \Sigma_v \) is uniformly bounded in row (column) sums when (b) holds, (iv) still holds.

**Lemma 7.** Suppose that \( \{H_{1,n}\} \) and \( \{H_{2,nt}\} \) are sequences of \( n \times n \) and \( nt \times nt \) symmetric matrices, respectively, \( \{G_{1,nt,n}\}, \{G_{2,nt}\} \) and \( \{K_{nt,n}\} \) are \( nt \times n \), \( nt \times nt \) and \( nt \times n \) matrices, respectively, all of the sequences of aforementioned matrices are uniformly bounded in both row and column sums. Let \( \{c_{1,n}\} \) and \( \{c_{2,nt}\} \) be sequences of constant vectors with uniformly bounded elements. Denote \( u_n = (u_1, \ldots, u_n)' \), \( \varepsilon_{nt} = (\varepsilon_1, \ldots, \varepsilon_{nt})' \) and \( b_{nt} = (b_1, \ldots, b_{nt})' \), for \( 1 \leq i \leq n, 1 \leq j \leq nt \), elements \( u_i \)'s, \( \varepsilon_j \)'s, \( b_j \)'s are mutually independent with zero mean, moreover, \( \sup_{1 \leq i \leq n} \mathbb{E}|u_i|^{4+\epsilon} < \infty \), \( \sup_{1 \leq j \leq nt} \mathbb{E}|\varepsilon_j|^{4+\epsilon} < \infty \) and \( \sup_{1 \leq j \leq nt} \mathbb{E}|b_j|^{4+\epsilon} < \infty \) hold for some \( \epsilon > 0 \). Let \( \mu_Q = \mathbb{E}(Q_{n(t+1)}) \) and \( \sigma^2_Q = \text{var}(Q_{n(t+1)}) \), where

\[
Q_{n(t+1)} = c'_{1,n}u_n + b'_{nt}G_{1,nt,n}u_n + u_n'G_{1,n}u_n +
+c'_{2,nt}\varepsilon_{nt} + b'_{nt}G_{2,nt}\varepsilon_{nt}
+ \varepsilon_{nt}'H_{2,nt}\varepsilon_{nt}
+ \varepsilon_{nt}'K_{nt,n}u_n.
\]  
(S4.5)

Suppose that \( t \) is a nondecreasing function of \( n \), then as \( n \to \infty \), we have

\[
\frac{Q_{n(t+1)} - \mu_Q}{\sigma_Q} \to_d N(0, 1). \tag{S4.6}
\]

Notice that \( u_n'H_{1,n}u_n = u_n'H_{1,n} + H_{1,n}'u_n \) and \( \varepsilon_{nt}'H_{2,nt}\varepsilon_{nt} = \varepsilon_{nt}'H_{2,nt} + H_{2,nt}'\varepsilon_{nt} \), hence the symmetry assumption on \( \{H_{1,n}\} \) and \( \{H_{2,nt}\} \) can be relaxed. Moreover, if elements in \( b_{nt} \) are not mutually independent, but the covariance matrix of \( b_{nt} \) is uniformly bounded in row (column) sums, then (S4.6) still holds.

Under Assumptions 3.1, 3.4–3.6, Lemmas 8 and 9 hold for our proposed model, and these two lemmas are mainly used in the proof of Theorems 1, 2.
Lemma 8. $E \left( \tilde{Z}' \Omega^{-1} \Omega' \Omega^{-1}_{0NT} V_{NT} \right) = 0$.

Lemma 9. Denote $\Delta(\beta^*)$ as the parameter space of $\beta^*$. For $\Omega_{NT}(\beta^*)$, the following results hold:

(i) $\{T^{-1} \Omega_{NT}(\beta^*)\}$ are uniformly bounded in both row and column sums, uniformly in $\Delta(\beta^*)$, and thus $\{T^{-1} \Omega_{0NT}\}$ are uniformly bounded in both row and column sums.

(ii) $\{T \Omega^{-1}_{NT}(\beta^*)\}$ are uniformly bounded in both row and column sums, uniformly in $\Delta(\beta^*)$, and thus $\{T \Omega^{-1}_{0NT}\}$ are uniformly bounded in both row and column sums.

(iii) $\{\frac{T}{\partial \beta_i} \Omega_{NT}(\beta^*)\}$ are uniformly bounded in both row and column sums, uniformly in $\Delta(\beta^*)$, for all $\ell = 0, 1, \ldots, p$.

Under Assumptions 3.3–3.9, Lemma 10 holds for our proposed model, and this lemma is mainly used in the proof of Theorem 5.

Lemma 10. Denote $\Psi_\tau(u) = \tau - I(u < 0)$ and $D_0 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{x_{ai}x'_{ai} \beta}{(x'_{ai} \beta_0)^2}$.

(i) $\mathbb{G}_N \left( \frac{x_{ai}}{x'_{ai} \beta_0} \Psi_\tau(\eta_i - Q_0(\cdot)) \right) \rightsquigarrow \mathbb{G} \left( \cdot \right)$ in $\ell^\infty(T)$, where $\mathbb{G}$ is a Gaussian process with zero mean and covariance function $E [\mathbb{G}(\tau) \mathbb{G}'(\tau')] = (\min\{\tau, \tau'\} - \tau \tau')D_0$, and $D_0$ is assumed to be well-defined.

(ii) If $\sup_{\tau \in T} \|\varphi(\tau) - \varphi_0(\tau)\| = o_p(1)$ and $\max_{1 \leq i \leq N} |\tilde{r}_i| = o_p(1)$, then it holds that

$$\sup_{\tau \in T} \left\| \mathbb{G}_N \left( \frac{x_{ai}}{x'_{ai} \beta_0} \Psi_\tau(\vartheta_i - x'_{ai} \varphi(\tau)) + \tilde{r}_i \right) - \mathbb{G}_N \left( \frac{x_{ai}}{x'_{ai} \beta_0} \Psi_\tau(\vartheta_i - x'_{ai} \varphi_0(\tau)) \right) \right\| = o_p(1).$$

Proof of Lemma 10. This lemma is an extension of Lemma A.8 of Lee (2004).

We first show (i). There exist constants $h_0$ and $a_0$ such that $\sup_x |h_{n,ij}(x)| \leq h_0$ and $\sup_x \sum_{k=1}^{n} |g_{n,kj}(x)| \leq a_0$ for all $i, j$ and $n$, where $G_n(x) = [g_{n,ij}(x)]$. It follows that $\sup_x |c_{n,ij}(x)| = \sup_x \sum_{k=1}^{n} h_{n,ik}g_{n,kj}(x) \leq h_0a_0$ for all $i, j$ and $n$, that is elements of $\{H_n(x)G_n(x)\}$ are uniformly bounded, uniformly on $X$. 
Similarly, we can show (ii) holds, and \( tr [H_n(x)G_n(x)] = tr [G_n(x)H_n(x)] = O(n) \) uniformly on \( X \) holds.

**Proof of Lemma 4.** This lemma is an extension of Lemma B.1 of Su and Yang (2015).

We first show that (i) holds. Let \( H_n(x) = [h_{n,ij}(x)] \) and \( G_n(x) = [g_{n,ij}(x)] \), there exist constants \( a_1 \) and \( b_1 \), such that \( \sup_{x} \sum_{j=1}^{n} |h_{n,ij}(x)| \leq a_1 \) and \( \sup_{x} \sum_{j=1}^{n} |g_{n,ij}(x)| \leq b_1 \) for all \( i \) and \( n \). Then, let \( H_n(x)G_n(x) = [c_{n,ij}(x)] \), for all \( i \) and \( n \) it holds that
\[
\sup_{x} \sum_{j=1}^{n} |c_{n,ij}(x)| = \sup_{x} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} h_{n,ik}(x)g_{n,kj}(x) \right| \leq \sup_{x} \sum_{k=1}^{n} |h_{n,ik}(x)| \sum_{j=1}^{n} |g_{n,kj}(x)| \leq a_1 b_1.
\]
As a result, \( \{H_n(x)G_n(x)\} \) are uniformly bounded in row sums, uniformly on \( X \).

Similarly, we can prove (ii), and then (iii) follows with (i) and (ii).

**Proof of Lemma 5.** This lemma is an extension of Lemma A.6 of Lee (2004).

There exists constants \( c_1 \) and \( c_2 \) such that \( |p_{n,i}| \leq c_1 \) and \( |q_{n,i}| \leq c_2 \) for all \( i \) and \( n \).

We first show (i). Let \( H_n(x) = [h_{n,ij}(x)] \). There exists a constant \( a_1 \) such that \( \sup_{x} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{n,ij}(x)| \leq a_1 \). It then follows that
\[
\sup_{x} |q'_{n}H_n(x)p_n| = \sup_{x} \left| \sum_{i=1}^{n} \sum_{j=1}^{n} q_{n,i}h_{n,ij}(x)p_{n,j} \right| \leq na_1 c_1 c_2.
\]
Hence, we obtain that \( |q'_{n}H_n(x)p_n| = O(n) \) uniformly on \( X \).

Next we show (ii). Let \( G_n = [g_{n,ij}] \). There exist constants \( b_1 \) and \( b_2 \) such that \( \sup_{x} \sum_{j=1}^{n} |h_{n,ij}(x)| \leq b_1 \) and \( \sum_{j=1}^{n} |g_{n,ij}| \leq b_2 \) for all \( i \) and \( n \). Then \( \sum_{j=1}^{n} |h_{n,ij}(x)p_{n,j}| \leq b_1 c_1 \) for all \( i \) and \( n \), so it follows that
\[
\sup_{x} |g_{i,n}H_n(x)p_n| = \sup_{x} \left| \sum_{k=1}^{n} \sum_{j=1}^{n} g_{n,ik}h_{n,kj}(x)p_{n,j} \right| \leq \sum_{k=1}^{n} |g_{n,ik}| \sup_{x} \sum_{j=1}^{n} |h_{n,kj}(x)p_{n,j}| \leq b_2 b_1 c_1,
\]
for all \( i \). Thus \( |g_{i,n}H_n(x)p_n| = O(1) \) holds uniformly in all \( i \), uniformly on \( X \).

**Proof of Lemma 6.** There exist constants \( c_1 \) and \( c_2 \) such that \( |p_{n,i}| \leq c_1 \) and \( |p_{nt,i}| \leq c_2 \) for all \( i \), \( n \) and \( t \). Let \( H_n(x) = [h_{n,ij}(x)] \) and \( \mathcal{H}_{n,nt}(x) = [\mathcal{H}_{n,nt,ij}(x)] \).
To show (i), it suffices to show that \( \sup_n E \left( \sup_x \left| \frac{1}{n} u'_n H_n(x) p_n \right| \right) < \infty \) by page 339 of Davidson (1994). There exists a constant \( a_1 \) such that \( \sup_x \sum_{j=1}^{n} |h_{n,ij}(x)| \leq a_1 \) for all \( i \) and \( n \). Then it holds that

\[
\sup_n E \left( \sup_x \left| \frac{1}{n} u'_n H_n(x) p_n \right| \right) = \sup_n E \left( \sup_x \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{n,i} h_{n,ij}(x) p_{n,j} \right| \right) \\
\leq \sup_n E \left( \frac{1}{n} \sum_{i=1}^{n} |u_{n,i}| \sup_x \sum_{j=1}^{n} |h_{n,ij}(x)||p_{n,j}| \right) \\
\leq \sup_n E \left( \frac{a_1 c_1}{n} \sum_{i=1}^{n} |u_{n,i}| \right) = a_1 c_1 E |u_{n,j}| < \infty.
\]

Therefore \( \frac{1}{n} u'_n H_n(x) p_n = O_p(1) \) uniformly on \( X \). Similarly, we can show (ii) holds.

Next we show (iii). It is obvious that \( E( u'_n H_n(x) p_n ) = 0 \). Notice that \( \{ H'_n(x) H_n(x) \} \) are uniformly bounded in both row and column sums, uniformly on \( X \) by Lemma 2.9(iii), then by Lemma 3.1(i), we have that

\[
\text{var}( u'_n H_n(x) p_n ) = E( ( u'_n H_n(x) p_n )^2 ) = \sigma'^2 \text{p}' H'_n(x) H_n(x) \text{p}_n = O(n),
\]

which holds uniformly on \( X \). It then follows that \( \frac{1}{n} u'_n H_n(x) p_n = o_p(1) \) for all \( x \in X \) by the Chebyshev’s inequality, which verifies the pointwise convergence. Furthermore, for \( x_1, x_2 \in X \), by the mean value theorem, it holds that

\[
\left| \frac{1}{n} u'_n H_n(x_1) p_n - \frac{1}{n} u'_n H_n(x_2) p_n \right| = \left| \frac{1}{n} u'_n \frac{\partial}{\partial x} H_n(\bar{x}) p_n \right| |x_1 - x_2|
\]

\[
\leq \sup_x \left| \frac{1}{n} u'_n \frac{\partial}{\partial x} H_n(x) p_n \right| |x_1 - x_2|,
\]

where \( \bar{x} \) lies between \( x_1 \) and \( x_2 \). This together with \( \sup_x \left| \frac{1}{n} u'_n \frac{\partial}{\partial x} H_n(x) p_n \right| = O_p(1) \) by (i), implies that \( \frac{1}{n} u'_n H_n(x) p_n \) is stochastically equicontinuous by Theorem 21.10 of Davidson (1994). The result that \( \frac{1}{n} u'_n H_n(x) p_n = o_p(1) \) holds uniformly on \( X \), by Lemma 2.8 of Newey and McFadden (1994).
Finally, we show that (iv) holds. Notice that the sequence \( \left\{ \frac{1}{n} \mathcal{T}_{nt,n}(x) \mathcal{T}_{nt,n}(x) \right\} \) are uniformly bounded in row sums, uniformly on \( X \) by Lemma 3(i), it then follows that \( \text{var} \left( \frac{1}{n} \mathcal{T}_{nt,n}(x) p_n \right) = O(n) \) holds uniformly on \( X \) by Lemma 3(i) and \( \sup_x \left| \frac{1}{n} \mathcal{T}_{nt,n}(x) \mathcal{T}_{nt,n}(x) p_n \right| = O_p(1) \) by (ii). As a result, we can prove (iv) with the similar argument as that of (iii).

Proof of Lemma 3. We first show (i). Let \( H_n(x) = [h_{n,ij}(x)] \). There exists a constant \( c_1 \) such that \( \sup_x |h_{n,ij}(x)| \leq c_1 \) for all \( i, j \) and \( n \). On one hand, it can be shown that

\[
\sup_n E \left( \sup_x \left| \frac{1}{n} u_n' H_n(x) u_n \right| \right) = \sup_n E \left( \sup_x \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{n,i} h_{n,ij}(x) u_{n,j} \right| \right) 
\leq \sup_n \frac{c_1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E|u_{n,i}||u_{n,j}|
= \sup_n \frac{c_1}{n} \sum_{i=1}^{n} E u_{n,i}^2 = c_1 \sigma^2 < \infty,
\]

so we have \( \sup_x \left| \frac{1}{n} u_n' H_n(x) u_n \right| = O_p(1) \) by the argument of page 339 in Davidson (1994), that is \( |u_n' H_n(x) u_n| = O_p(n) \) uniformly on \( X \). On the other hand, we have

\[
\sup_x \left| E \left( \frac{1}{n} u_n' H_n(x) u_n \right) \right| = \sup_x \left| E \left( \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{n,i} h_{n,ij}(x) u_{n,j} \right) \right| \leq c_1 \sigma^2 < \infty,
\]

which implies \( E (u_n' H_n(x) u_n) = O(n) \) uniformly on \( X \). As a result, we have that \( \frac{1}{n} u_n' H_n(x) u_n - \frac{1}{n} E (u_n' H_n(x) u_n) = o_p(1) \), uniformly on \( X \).

We next show (ii) holds. \( \frac{1}{n} u_n' H_n(x) u_n - \frac{1}{n} E (u_n' H_n(x) u_n) = o_p(1) \) holds for all \( x \in X \) by Lemma A.12 of Lee (2004). For \( x_1, x_2 \in X \), by the mean value theorem, it holds that

\[
\left| \frac{1}{n} u_n' H_n(x_1) u_n - \frac{1}{n} E (u_n' H_n(x_1) u_n) \right| - \left| \frac{1}{n} u_n' H_n(x_2) u_n - \frac{1}{n} E (u_n' H_n(x_2) u_n) \right|
\leq \sup_x \left| \frac{1}{n} u_n' \frac{\partial}{\partial x} H_n(x) u_n - \frac{1}{n} E \left( u_n' \frac{\partial}{\partial x} H_n(x) u_n \right) \right| |x_1 - x_2|,
\]

where \( \bar{x} \) lies between \( x_1 \) and \( x_2 \), and \( \sup_x \left| \frac{1}{n} u_n' \frac{\partial}{\partial x} H_n(x) u_n - \frac{1}{n} E (u_n' \frac{\partial}{\partial x} H_n(x) u_n) \right| = O_p(1) \) by (i), then \( \frac{1}{n} u_n' H_n(x) u_n - \frac{1}{n} E (u_n' H_n(x) u_n) \) is stochastically equicontinuous by Theorem
21.10 of Davidson (1994). This together with \( \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{u}_n - \frac{1}{n} E (\mathbf{u}'_n H_n(x) \mathbf{u}_n) \right| = o_p(1) \) for all \( x \in X \), implies that \( \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{u}_n - \frac{1}{n} E (\mathbf{u}'_n H_n(x) \mathbf{u}_n) \right| = o_p(1) \) holds uniformly on \( X \) by Lemma 2.8 of Newey and McFadden (1994).

Proof of Lemma 6. (i) and (ii) follows from that \( \sup_{x} E \left( \sup_{n} \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{v}_n \right| \right) = 0 \) and that \( \sup_{x} E \left( \sup_{n} \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{v}_n \right| \right) = 0 \).

We next show (iii) holds. Let \( \sigma^2_u = \text{var}(u_i) \) and \( \sigma^2_v = \text{var}(v_i) \). \( E (\mathbf{u}'_n H_n(x) \mathbf{v}_n) = 0 \), and it can be shown that

\[
\text{var} (\mathbf{u}'_n H_n(x) \mathbf{v}_n) = \sigma^2_u \sigma^2_v \text{tr} [H_n(x) H'_n(x)] = O(n),
\]

by Lemma 6. Then \( \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{v}_n \right| = o_p(1) \) holds for all \( x \in X \) by Chebyshev’s inequality. This together with the stochastical equicontinuity of \( \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{v}_n \) by similar argument of the proof of Lemma 6 (ii), implies that \( \left| \frac{1}{n} \mathbf{u}'_n H_n(x) \mathbf{v}_n \right| = o_p(1) \) holds uniformly on \( X \) by Lemma 2.8 of Newey and McFadden (1994).

The proof of (iv) is analogous to that of (iii). 

Proof of Lemma 7. Let \( \mathbf{v}_{n(t+1)} = (\mathbf{u}'_t, \mathbf{e}'_{nt})' = \{v_i, 1 \leq i \leq n(t+1)\} \), \( \mathbf{c}_{nt} = (\mathbf{c}'_{1,nt}, \mathbf{c}'_{2,nt})' = \{c_i, 1 \leq i \leq n(t+1)\} \), \( \sigma^2_{e_i} = \text{var}(v_i) \), and

\[
G_{nt,n(t+1)} = (G_{1,nt,n}, G_{2,nt}) = [g_{ij}, 1 \leq i \leq nt, 1 \leq j \leq n(t+1)],
\]

\[
H_{nt,n(t+1)} = \begin{pmatrix}
H_{1,nt} & \frac{1}{2} K'_{nt,n} \\
\frac{1}{2} K_{nt,n} & H_{2,nt}
\end{pmatrix} = [h_{ij}, 1 \leq i \leq n(t+1), 1 \leq j \leq n(t+1)].
\]

Note that \( E(\mathbf{v}_{n(t+1)}) = \mathbf{0} \) and \( v_i \)'s are mutually independent, so it holds that

\[
Q_{n(t+1)} - \mu_Q = \mathbf{c}'_{n(t+1)} \mathbf{v}_{n(t+1)} + \mathbf{b}_{nt} G_{nt,n(t+1)} \mathbf{v}_{n(t+1)} + [\mathbf{v}'_{n(t+1)} H_{n(t+1)} \mathbf{v}_{n(t+1)} - E(\mathbf{v}'_{n(t+1)} H_{n(t+1)} \mathbf{v}_{n(t+1)})].
\]
Let \( X \) since \( f \in F \) form a martingale difference array.\(^a\)
\[^a\] Kelejian and s
\[*\]
Y
2 + b)
a)
i,n
T o prove that \( \sigma \)
W e first show a). T ake \( 1 \leq i \leq n(t + 1) \). For each \( i \), \( F_{i-1,n(t+1)} \subset F_{i,n(t+1)} \), \( Y_{i,n(t+1)}^* \) is known given \( F_{i,n(t+1)} \) and \( E \left( Y_{i,n(t+1)}^* | F_{i-1,n(t+1)} \right) = 0 \), so \( \{Y_{i,n(t+1)}^*, F_{i,n(t+1)}, 1 \leq i \leq n(t + 1), n, t \geq 1\} \) forms a martingale difference array. Moreover, since \( Y_{i,n(t+1)} \)'s are martingale differences, it holds that

\[
\sigma_Q^2 = E \left( \sum_{i=1}^{n(t+1)} Y_{i,n(t+1)}^* \right)^2 = \sum_{i=1}^{n(t+1)} E \left( Y_{i,n(t+1)}^{*2} \right).
\]

Let \( X_{i,n(t+1)}^* = \sigma_Q^{-1} Y_{i,n(t+1)}^* \). We have that \( \{X_{i,n(t+1)}^*, F_{i,n(t+1)}, 1 \leq i \leq n(t + 1), n, t \geq 1\} \) also form a martingale difference array.

To prove that \( \frac{Q_{n(t+1)} - \mu_Q}{\sigma_Q} = \sum_{i=1}^{n(t+1)} X_{i,n(t+1)}^* \rightarrow_d N(0, 1) \), by Theorem A.1 of Kelejian and Prucha (2001), it suffices to show

a) \[ \sum_{i=1}^{n(t+1)} E \left[ E \left( \left| X_{i,n(t+1)}^* \right|^{2+\delta} | F_{i-1,n(t+1)} \right) \right] \rightarrow 0 \text{ for some } \delta > 0; \]

b) \[ \sum_{i=1}^{n(t+1)} E \left( X_{i,n(t+1)}^* \right) \rightarrow 1. \]

We first show a). T ake \( 0 < \delta \leq \epsilon/2 \), for any arbitrary \( 1 \leq i, j \leq n(t + 1), 1 \leq \ell \leq nt \) and \( s \leq 2 + \delta \), there exist constants \( k_e \) and \( k_p \) such that

\[
\sigma_{v_i}^2 \leq k_e, \ E |v_i|^s \leq k_e, \ E |v_i|^s E |v_j|^s \leq k_e, \ E |v_i|^s E |b_\ell|^s \leq k_e, \ E \left| v_i^2 - \sigma_{v_i}^2 \right|^s \leq k_e, \text{ and}
\]

\[
\sum_{i=1}^{n(t+1)} |h_{ij}| \leq k_p, \sum_{j=1}^{n(t+1)} |h_{ij}| \leq k_p, \sum_{i=1}^{nt} |g_{ij}| \leq k_p, \sum_{j=1}^{n(t+1)} |g_{ij}| \leq k_p, \text{ and } |c_i| < k_p.
\]
Let $q = 2 + \delta$ and $\frac{1}{q} + \frac{1}{p} = 1$. By Minkowski inequality and Hölder inequality, we obtain that

$$\left| Y_{i,n(t+1)}^* \right|^q = \left| h_{ii} \left( v_i^2 - \sigma_{v_i}^2 \right) + 2v_i \sum_{j=1}^{i-1} h_{ij} v_j + c_i v_i + v_i \sum_{j=1}^{nt} g_{ji} b_j \right|^q$$

$$\leq 2^q \left| h_{ii} \left( v_i^2 - \sigma_{v_i}^2 \right) + 2v_i \sum_{j=1}^{i-1} h_{ij} v_j \right|^q + 2^q \left| c_i v_i \right|^q + 2^q \left| v_i \sum_{j=1}^{nt} g_{ji} b_j \right|^q$$

$$\leq 2^q \left( \left| h_{ii} \right|^\frac{1}{p} \left| h_{ii} \right|^\frac{1}{q} \left| v_i^2 - \sigma_{v_i}^2 \right| \left| v_i \right|^q + \sum_{j=1}^{i-1} \left| h_{ij} \right|^\frac{1}{p} \left| h_{ij} \right|^\frac{1}{q} \left| 2v_i v_j \right| \right)^q + 2^q \left| c_i \right|^q \left| v_i \right|^q$$

$$+ 2^q \left| v_i \right|^q \left( \sum_{j=1}^{nt} \left| g_{ji} \right|^\frac{1}{p} \left| g_{ji} \right|^\frac{1}{q} \left| b_j \right| \right)^q$$

$$\leq 2^q \left( \sum_{j=1}^i \left| h_{ij} \right| \right)^\frac{q}{p} \left( \left| h_{ii} \right|^p \left| v_i^2 - \sigma_{v_i}^2 \right|^q \left| v_i \right|^q + \sum_{j=1}^{i-1} \left| h_{ij} \right|^p \left| v_i \right|^q \left| 2v_i v_j \right| \right)^\frac{q}{p}$$

$$+ 2^q \left| c_i \right|^q \left| v_i \right|^q + 2^q \left| v_i \right|^q \left( \sum_{j=1}^{nt} \left| g_{ji} \right| \right)^\frac{q}{p} \left( \sum_{j=1}^{nt} \left| g_{ji} \right| \left| b_j \right| \right)^q$$

Then it follows that

$$\sum_{i=1}^{n(t+1)} E \left[ E \left( \left| Y_{i,n(t+1)}^* \right|^{2+\delta} \left| \mathcal{F}_{i-1,n(t+1)} \right| \right) \right]$$

$$\leq 2^q \sum_{i=1}^{n(t+1)} \left( \sum_{j=1}^i \left| h_{ij} \right| \right)^\frac{q}{p} \left( \left| h_{ii} \right| E \left| v_i^2 - \sigma_{v_i}^2 \right|^q + 2^q \sum_{j=1}^{i-1} \left| h_{ij} \right| E \left| v_i \right|^q E \left| v_j \right|^q \right)$$

$$+ 2^q \sum_{i=1}^{n(t+1)} \left| c_i \right|^q E \left| v_i \right|^q + 2^q \sum_{i=1}^{n(t+1)} E \left| v_i \right|^q \left( \sum_{j=1}^{nt} \left| g_{ji} \right| \right)^\frac{q}{p} \left( \sum_{j=1}^{nt} \left| g_{ji} \right| \left| b_j \right| \right)^q$$

$$\leq 2^q \left( n(t+1) k_e h_p^{\frac{q}{p}+1} + n(t+1) k_e k_p^q + n(t+1) k_e k_p^{\frac{q}{p}+1} \right) = cn(t+1),$$

(S4.7)

where $c$ is a constant. Moreover, by algebra calculation, we obtain that

$$\sigma_Q^2 = \sum_{i=1}^{n(t+1)} \left[ h_{ii}^2 (\mu_{vi}^{(4)} - \sigma_{vi}^4) + 4 \sum_{j=1}^{i-1} h_{ij}^2 \sigma_{vi}^2 \sigma_{v_j}^2 + c_i^2 \sigma_{v_i}^2 + 2c_i h_{ii} \sigma_{v_i}^2 + \sum_{j=1}^{nt} g_{ji}^2 \sigma_{b_j}^2 \sigma_{v_i}^2 \right]$$
\[ = O(n(t+1)), \quad \text{(S4.8)} \]

where \( \mu^{(3)}_{v_i} = E(v^3_i), \mu^{(4)}_{v_i} = E(v^4_i) \) and \( \sigma^2_{b_j} = \text{var}(b_j) \). Then by (S4.7) and (S4.8), it holds that

\[
\sum_{i=1}^{n(t+1)} E \left[ \left| X_{i,n(t+1)}^* \right|^{2+\delta} \mid \mathcal{F}_{i-1,n(t+1)} \right] \leq \left( \frac{n(t+1)}{\sigma^2_Q} \right)^{1+\frac{2}{\delta}} \frac{c}{(n(t+1))^{\frac{2}{\delta}}} \to 0,
\]

as \( n \to \infty \). Hence a) holds.

It remains to show that b) holds. It can be verified that

\[
\sum_{i=1}^{n(t+1)} E \left( X_{i,n(t+1)}^* \mid \mathcal{F}_{i-1,n(t+1)} \right) - 1
\]

\[
= \frac{1}{\sigma^2_Q} \sum_{i=1}^{n(t+1)} E \left( Y_{i,n(t+1)}^* \mid \mathcal{F}_{i-1,n(t+1)} \right) - \frac{1}{\sigma^2_Q} \sum_{i=1}^{n(t+1)} E \left( Y_{i,n(t+1)}^* \right)
\]

\[
= \frac{1}{\sigma^2_Q} \sum_{i=1}^{n(t+1)} \left[ E \left( Y_{i,n(t+1)}^* \mid \mathcal{F}_{i-1,n(t+1)} \right) - E \left( Y_{i,n(t+1)}^* \right) \right]
\]

\[
n(t+1) - 1 = \sum_{i=1}^{n(t+1)} \left( \frac{i-1}{n(t+1)} \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} h_{ij}h_{ik}\sigma^2_{v_i}v_jv_k + 4\sum_{j=1}^{i-1} h_{ij}^2\sigma^2_{v_i}(v^2_j - \sigma^2_{v_j}) + 2\sum_{j=1}^{i-1} g_{ji}\sigma^2_{v_j}b_j + \sum_{j=1}^{i-1} g^2_{ji}\sigma^2_{v_j}(b^2_j - \sigma^2_{b_j}) + 4\sum_{j=1}^{i-1} \sum_{k=1}^{j} h_{ij}g_{jk}\sigma^2_{v_i}v_jv_k + 4\sum_{j=1}^{i-1} h_{iii}^2\sigma^2_{v_i}(v^2_j - \sigma^2_{v_j}) \right).
\]

Note that \( \frac{n(t+1)}{\sigma^2_Q} = O(1) \). It suffices to show that the rest of the last equality is \( o_p(1) \). We next show that b1) \( s_1 \triangleq \frac{1}{n(t+1)} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} h_{ij}h_{ik}\sigma^2_{v_i}v_jv_k = o_p(1) \) and b2) \( s_2 \triangleq \frac{1}{n(t+1)} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{i-1} h_{ij}^2\sigma^2_{v_i}(v^2_j - \sigma^2_{v_j}) = o_p(1) \), and the results on other terms can be established similarly.

For b1), obviously \( E(s_1) = 0 \). Moreover, we have

\[
E \left( s^2_1 \right) \leq \frac{k^4_e}{(n(t+1))^2} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{i-1} \sum_{k=1}^{j-1} \sum_{r=j+1}^{j-1} |h_{ij}||h_{ik}||h_{rj}||h_{rk}|
\]

where \( \mu^{(3)}_e = E(v^3_e), \mu^{(4)}_e = E(v^4_e) \) and \( \sigma^2_e = \text{var}(e_j) \). Then by (S4.7) and (S4.8), it holds that
\[
\begin{align*}
&= \frac{k_e^4}{(n(t+1))^2} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{n(t+1)} |h_{ij}| \sum_{k=1}^{n(t+1)} |h_{ik}| \sum_{r=1}^{n(t+1)} |h_{rj}| |h_{rk}| \\
&\leq \frac{k_e^4}{(n(t+1))^2} (n(t+1)k_p^4) = \frac{k_e^4k_p^4}{n(t+1)} \to 0,
\end{align*}
\]

as \( n \to \infty \). By Chebyshev inequality, we have \( s_1 = o_p(1) \).

For (b2), it holds that
\[
s_2 = \frac{1}{n(t+1)} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{n(t+1)} h_{ij}^2 \sigma_{v_i}^2 (v_j^2 - \sigma_{v_j}^2)
= \frac{1}{n(t+1)} \sum_{j=1}^{n(t+1)} \sum_{i=1}^{n(t+1)-1} h_{ij}^2 \sigma_{v_i}^2 (v_j^2 - \sigma_{v_j}^2)
= \frac{1}{n(t+1)} \sum_{i=1}^{n(t+1)-1} \sum_{j=1}^{n(t+1)} h_{ij}^2 \sigma_{v_i}^2 (v_j^2 - \sigma_{v_j}^2)
\triangleq \sum_{i=1}^{n(t+1)-1} \chi_{i,n(t+1)} (v_i^2 - \sigma_{v_i}^2),
\]

where \( \chi_{i,n(t+1)} = \frac{1}{n(t+1)} \sum_{j=1}^{n(t+1)} h_{ij}^2 \sigma_{v_i}^2 \). Note that \( \chi_{i,n(t+1)} (v_i^2 - \sigma_{v_i}^2) \)'s are mutually independent with zero mean. To verify \( s_2 \to_p 0 \) which is implied by \( s_2 \to 0 \) in \( L_1 \) space, by Theorem 19.7 of Davidson (1994), it suffices to show that \( \{ |v_i^2 - \sigma_{v_i}^2| \} \) is uniformly integrable, \( \limsup_{n \to \infty} \sum_{i=1}^{n(t+1)-1} \chi_{i,n(t+1)} < \infty \) and \( \lim_{n \to \infty} \sum_{i=1}^{n(t+1)-1} \chi_{i,n(t+1)}^2 = 0 \). Recall that \( E |v_i^2 - \sigma_{v_i}^2|^{1+\delta} \leq K_e \).

Thus \( \{ |v_i^2 - \sigma_{v_i}^2| \} \) is uniformly integrable. Furthermore,
\[
\limsup_{n \to \infty} \sum_{i=1}^{n(t+1)-1} \chi_{i,n(t+1)} \leq \limsup_{n \to \infty} \frac{1}{n(t+1)} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{n(t+1)} h_{ij}^2 \sigma_{v_i}^2 \leq k_e k_p^2 < \infty, \quad \text{and}
\]
\[
\lim_{n \to \infty} \sum_{i=1}^{n(t+1)-1} \chi_{i,n(t+1)}^2 \leq \lim_{n \to \infty} \frac{1}{(n(t+1))^2} \sum_{i=1}^{n(t+1)} \sum_{j=1}^{n(t+1)} \sum_{k=1}^{n(t+1)} h_{ij}^2 h_{ik}^2 \sigma_{v_i}^2 \sigma_{v_k}^2 \leq \lim_{n \to \infty} \frac{k_e^2 k_p^4}{n(t+1)} = 0.
\]

Then \( s_2 \to 0 \) in \( L_1 \) space, and hence \( s_2 \to_p 0 \). We complete this lemma by a) and b).

**Proof of Lemma 8.** Recall that \( \bar{Z} = (Y_{-1}, Z, (\uptimes T \otimes I_N) X) \). Note that \( E (Z' \Omega_{0NT}^{-1} V_{NT}) = 0 \) and \( E (X' (\uptimes T \otimes I_N)' \Omega_{0NT}^{-1} V_{NT}) = 0 \), so it suffices to to show \( E (Y_{-1}' \Omega_{0NT}^{-1} V_{NT}) = 0 \) to verify that \( E (\bar{Z}' \Omega_{0NT}^{-1} V_{NT}) = 0 \). By (S3.1) and (S3.3), it can be shown that
\[
E (Y_{-1}' \Omega_{0NT}^{-1} V_{NT}) = tr \left[ E (V_{NT} Y_{-1}' \Omega_{0NT}^{-1}) \right] = \sigma_{00}^2 tr \left[ \Omega_{0NT}^{-1} ((\uptimes T \otimes I_N) A_{0N} M_3' + M_2') \right],
\]
where \( A_{0N} = A_N(\beta_0^*) \), and

\[
tr \left[ \Omega_{0NT}^{-1} \left( (\nu_T \otimes I_N) A_{0N} M_3' + M_2' \right) \right] \\
= tr \left[ \left( \frac{1}{T} J_T \otimes (I_N + TA_{0N})^{-1} + \left( I_T - \frac{1}{T} J_T \right) \otimes I_N \right) \left( (\nu_T \otimes I_N) A_{0N} M_3' + M_2' \right) \right] \\
= tr \left[ \left( \frac{1}{T} J_T \otimes (I_N + TA_{0N})^{-1} + \left( I_T - \frac{1}{T} J_T \right) \otimes I_N \right) (\nu_T \otimes I_N) A_{0N} M_3' \right] \\
+ tr \left[ \left( \frac{1}{T} J_T \otimes (I_N + TA_{0N})^{-1} + \left( I_T - \frac{1}{T} J_T \right) \otimes I_N \right) M_2' \right] \\
= tr \left[ (\nu_T \otimes ((I_N + TA_{0N})^{-1} A_{0N})) M_3' \right] + tr \left[ \left( \frac{J_T}{T} \otimes ((I_N + TA_{0N})^{-1} - I_N) \right) M_2' \right] = 0.
\]

Then the proof of this lemma is accomplished.

**Proof of Lemma**. We first show (i). Recall that \( \Omega_{NT}(\beta^*) = J_T \otimes A_N(\beta^*) + I_{NT} \) with \( A_N(\beta^*) = \text{diag} \{ (x_{a1}^* \beta^*)^2, \ldots, (x_{aN}^* \beta^*)^2 \} \triangleq \text{diag} \{ a_1(\beta^*), \ldots, a_N(\beta^*) \} \). For \( T^{-1} \Omega_{NT}(\beta^*) \), it suffices to show that the row sums corresponding to the first \( N \) rows are uniformly bounded, uniformly on \( \Delta(\beta^*) \). Since \( \Delta(\beta^*) \) is a compact set and \( x_i \) is bounded by Assumption 3.6(i), there exists a positive constant \( a_0 \) such that \( \sup_{\beta^*} a_i(\beta^*) \leq a_0 < \infty \) for \( 1 \leq i \leq N \). The \( i \)th row sum is \( a_i(\beta^*) + T^{-1} \), then it follows that

\[
0 < \sup_{\beta^*} \left( a_i(\beta^*) + T^{-1} \right) \leq a_0 + 1 < \infty.
\]

Then, \( \{ T^{-1} \Omega_{NT}(\beta^*) \} \) are uniformly bounded in both row and column sums, uniformly in \( \Delta(\beta^*) \).

Next we show (ii). Recall that \( \Omega_{NT}^{-1}(\beta^*) = \frac{1}{T} J_T \otimes (I_N + TA_N(\beta^*))^{-1} + \left( I_T - \frac{1}{T} J_T \right) \otimes I_N \), then it follows that

\[
T \Omega_{NT}^{-1}(\beta^*) = J_T \otimes \text{diag} \left\{ \frac{1}{1 + Ta_1(\beta^*)} - 1, \ldots, \frac{1}{1 + Ta_N(\beta^*)} - 1 \right\} + TI_{NT}.
\]

For \( T \Omega_{NT}^{-1}(\beta^*) \), we only need to show that the row sums corresponding to the first \( N \) rows are uniformly bounded, uniformly on \( \Delta(\beta^*) \). For \( 1 \leq i \leq N \), note that \( \inf_{\beta^*} a_i(\beta^*) =
\[ \inf(x_i^\prime \beta^*)^2 \geq \inf \beta_0^2 = \inf \beta_0^2 \sigma_0^2 > 0. \]

Then the ith row sum is
\[ T \left( \frac{1}{1+T \sigma_i(\beta^*)} \right) + T = \left( 1 - \frac{1}{1+T \sigma_i(\beta^*)} \right) \frac{1}{a_i(\beta^*)}, \]
so it holds that
\[ 0 < \sup_{\beta^*} \left( 1 - \frac{1}{1+T \sigma_i(\beta^*)} \right) \frac{1}{a_i(\beta^*)} \leq \sup_{\beta^*} \frac{1}{a_i(\beta^*)} = \frac{1}{\inf a_i(\beta^*)} < \infty. \]

Hence, (ii) follows.

The proof of (iii) is analogous to that of (i). The proof of this lemma is complete. \( \square \)

**Proof of Lemma 10.** This lemma directly follows from Lemma A.1 of Canay (2011) and Lemma B.2 of Chernozhukov and Hansen (2006), so its proof is omitted here. \( \square \)

## S5 Technical proofs

### S5.1 Proof of Theorem 1

To establish the consistency of \( \hat{\delta} \) and hence that of \( \hat{\phi} \) and \( \hat{\sigma}_2^2 \), by Theorem 3.4 of White (1996), it suffices to show that:

(A1) the identification uniqueness condition holds, that is
\[ \lim \sup_{N \to \infty} \max_{\delta \in N^c_\epsilon(\delta_0)} \frac{1}{NT} \left[ \ln \ell^*_N(\delta) - \ln \ell^*_N(\delta_0) \right] < 0 \]
for any \( \epsilon > 0 \), where \( N^c_\epsilon(\delta_0) \) is the complement of an open neighborhood of \( \delta_0 \) on \( \Delta \) of radius \( \epsilon \);

(A2) \( \frac{1}{NT} \left[ \ln \ell_N(\delta) - \ln \ell^*_N(\delta_0) \right] \to_p 0 \), uniformly in \( \delta \) on \( \Delta \);

where \( \ln \ell_N(\delta) = \max_{\phi, \sigma_2^2} \ln L_N(\zeta) \) and \( \ln \ell^*_N(\delta) = \max_{\phi, \sigma_2^2} E[\ln L_N(\zeta)] \) can be rewritten for our model in the following forms

\[ \ln \ell_N(\delta) = -\frac{NT}{2} (\ln (2\pi) + 1) - \frac{NT}{2} \ln \hat{\sigma}_2^2(\delta) - \frac{1}{2} \ln |\Omega_N(\beta^*)| + \ln |S_N(\lambda)|, \]
\[ \ln \ell^*_N(\delta) = -\frac{NT}{2} (\ln (2\pi) + 1) - \frac{NT}{2} \ln \hat{\sigma}_2^2(\delta) - \frac{1}{2} \ln |\Omega_N(\beta^*)| + \ln |S_N(\lambda)|. \]
First, we show (A1) by contradiction, which needs \( \frac{1}{NT} \ln \ell^*_NT(\delta) - \ln \ell^*_NT(\delta_0) \leq 0 \) for all \( \delta \) on \( \Delta \), and \( \frac{1}{NT} \ln \ell^*_NT(\delta) \) is uniformly equicontinuous on \( \Delta \).

(Show that \( \frac{1}{NT} [\ln \ell^*_NT(\delta) - \ln \ell^*_NT(\delta_0)] \leq 0 \) for all \( \delta \) on \( \Delta \)) By Jensen’s inequality, it can be verified that

\[
0 = \ln 1 = \ln E \frac{\mathcal{L}_NT(\zeta)}{\mathcal{L}_NT(\zeta_0)} \geq E \ln \frac{\mathcal{L}_NT(\zeta)}{\mathcal{L}_NT(\zeta_0)} = E \ln \mathcal{L}_NT(\zeta) - E \ln \mathcal{L}_NT(\zeta_0),
\]

so it follows that

\[
\ln \ell^*_NT(\delta) = \max_{\phi, \sigma^2} E \ln \mathcal{L}_NT(\zeta) \leq \max_{\phi, \sigma^2} E \ln \mathcal{L}_NT(\zeta_0) = \ln \ell^*_NT(\delta_0),
\]

and thus \( \frac{1}{NT} [\ln \ell^*_NT(\delta) - \ln \ell^*_NT(\delta_0)] \leq 0 \) holds for all \( \delta \) on \( \Delta \).

(Show that \( \frac{1}{NT} \ln \ell^*_NT(\delta) \) is uniformly equicontinuous on \( \Delta \)) It suffices to prove that \( \frac{1}{NT} \ln |\Omega_{NT}(\beta^*)|, \frac{1}{NT} \ln |S_{NT}(\lambda)| \) and \( \ln \bar{\sigma}^2_T(\delta) \) are all uniformly equicontinuous on \( \Delta \). Recall that \( \Delta \) is the parameter space of \( \delta = (\lambda, \beta^*)' \), and denote \( \Delta(\beta^*) \) and \( \Delta(\lambda) \) as the parameter space of \( \beta^* \) and \( \lambda \), respectively.

For \( \frac{1}{NT} \ln |\Omega_{NT}(\beta^*)| \), let \( \beta^*_1 = (\beta^*_{10}, \ldots, \beta^*_{1p})' \) and \( \beta^*_2 = (\beta^*_{20}, \ldots, \beta^*_{2p})' \) be in \( \Delta(\beta^*) \), by the mean value theorem, it holds that

\[
\left| \frac{1}{NT} \ln |\Omega_{NT}(\beta^*_1)| - \frac{1}{NT} \ln |\Omega_{NT}(\beta^*_2)| \right| = \left| \sum_{\ell=0}^p \frac{1}{NT} \frac{\partial}{\partial \beta^*_\ell} \ln |\Omega_{NT}(\beta^*)| \left( \beta^*_\ell - \beta^*_{2\ell} \right) \right| \\
\leq \sum_{\ell=0}^p \left| \frac{1}{NT} \frac{\partial}{\partial \beta^*_\ell} \ln |\Omega_{NT}(\beta^*)| \right| \left| \beta^*_\ell - \beta^*_{2\ell} \right|,
\]

where \( \beta^* \) lies between \( \beta^*_1 \) and \( \beta^*_2 \). Moreover, by Lemmas 4 and 5, we have that

\[
\frac{\partial}{\partial \beta^*_\ell} \ln |\Omega_{NT}(\beta^*)| = tr \left[ T\Omega^{-1}_{NT}(\beta^*) \frac{\partial}{\partial \beta^*_\ell} T^{-1} \Omega_{NT}(\beta^*) \right] = O(NT)
\]

holds uniformly in \( \beta^* \in \Delta(\beta^*) \). Therefore, we can obtain that \( \frac{1}{NT} \ln |\Omega_{NT}(\beta^*)| \) is uniformly equicontinuous on \( \Delta \) by its definition. Similarly, we can show the uniform equicontinuity of \( \frac{1}{NT} \ln |S_{NT}(\lambda)| \).
For \( \ln \tilde{\sigma}_\varepsilon^2(\delta) \), we show the uniform equicontinuity of \( \ln \tilde{\sigma}_\varepsilon^2(\delta) \) by showing that 1) \( \tilde{\sigma}_\varepsilon^2(\delta) \) is uniformly equicontinuous on \( \Delta \), and 2) \( \tilde{\sigma}_\varepsilon^2(\delta) \) is uniformly bounded away from zero on \( \Delta \). We first consider 1). Notice that \( Y = S_{0NT}^{-1} \left( \bar{Z}\phi_0 + V_{NT} \right) \), and (3.8) implies that

\[
\tilde{\sigma}_\varepsilon^2(\delta) = \frac{1}{NT} E \left( S_{NT}(\lambda)Y - \bar{Z}\phi(\delta) \right)' \Omega_{NT}^{-1}(\delta^*) \left( S_{NT}(\lambda)Y - \bar{Z}\phi(\delta) \right) = \sum_{i=1}^3 \tilde{\sigma}_{\varepsilon i}^2(\delta),
\]

where

\[
\tilde{\sigma}_{\varepsilon 1}^2(\delta) = \frac{1}{NT} E \left( V_{NT}' \left( S_{0NT}^{-1} \right)' S_{NT}(\lambda) \Omega_{NT}^{-1}(\delta^*) S_{NT}(\lambda) S_{0NT}^{-1} V_{NT} \right),
\]

\[
\tilde{\sigma}_{\varepsilon 2}^2(\delta) = \frac{2}{NT} E \left[ \left( S_{NT}(\lambda)S_{0NT}^{-1} \bar{Z}\phi_0 - \bar{Z}\phi(\delta) \right)' \Omega_{NT}^{-1}(\delta^*) \left( S_{NT}(\lambda)S_{0NT}^{-1} V_{NT} \right) \right],
\]

and

\[
\tilde{\sigma}_{\varepsilon 3}^2(\delta) = \frac{1}{NT} E \left[ \left( S_{NT}(\lambda)S_{0NT}^{-1} \bar{Z}\phi_0 - \bar{Z}\phi(\delta) \right)' \Omega_{NT}^{-1}(\delta^*) \left( S_{NT}(\lambda)S_{0NT}^{-1} \bar{Z}\phi_0 - \bar{Z}\phi(\delta) \right) \right].
\]

Then the uniform equicontinuity of \( \tilde{\sigma}_\varepsilon^2(\delta) \) follows from that of \( \tilde{\sigma}_{\varepsilon i}^2(\delta) \), \( \tilde{\sigma}_{\varepsilon 2}^2(\delta) \) and \( \tilde{\sigma}_{\varepsilon 3}^2(\delta) \). Similar to the proof of that of \( \frac{1}{NT} \ln \ell_{NT}^*(\delta) \), we can verify the uniform equicontinuity of \( \tilde{\sigma}_{\varepsilon i}^2(\delta) \) for \( i = 1, 2, 3 \). Hence, 1) holds. Next we prove 2) by contradiction. We have shown that \( \frac{1}{NT} [\ln \ell_{NT}^*(\delta) - \ln \ell_{NT}^*(\delta_0)] \leq 0 \) for all \( \delta \) on \( \Delta \), and

\[
\frac{1}{NT} [\ln \ell_{NT}^*(\delta) - \ln \ell_{NT}^*(\delta_0)] = \frac{1}{2} \ln \tilde{\sigma}_\varepsilon^2(\delta) + \frac{1}{2} \ln \sigma_{\varepsilon 0}^2 - \frac{1}{2NT} [\ln |\Omega_{NT}(\beta^*)| - \ln |\Omega_{0NT}|] + \frac{1}{NT} [\ln |S_{NT}(\lambda)| - \ln |S_{0NT}|].
\]

Then by the mean value theorem, it follows that

\[
-\frac{1}{2} \ln \tilde{\sigma}_\varepsilon^2(\delta) \leq - \frac{1}{2} \ln \sigma_{\varepsilon 0}^2 + \frac{1}{2NT} [\ln |\Omega_{NT}(\beta^*)| - \ln |\Omega_{0NT}|] - \frac{1}{NT} [\ln |S_{NT}(\lambda)| - \ln |S_{0NT}|]
\]

\[
= - \frac{1}{2} \ln \sigma_{\varepsilon 0}^2 + \frac{1}{2NT} \sum_{\ell=0}^p \frac{\partial}{\partial \beta_\ell^*} \ln |\Omega_{NT}(\beta^*)| (\beta_\ell^* - \beta_0^*) - \frac{1}{NT} \frac{\partial}{\partial \lambda} \ln |S_{NT}(\lambda)| (\lambda - \lambda_0),
\]

where \( \delta = (\bar{\lambda}, \bar{\beta}^*) \) lies between \( \delta = (\lambda, \beta^*) \) and \( \delta_0 = (\lambda_0, \beta_0^*) \). This together with compactness of \( \Delta \), \( \frac{\partial}{\partial \lambda} \ln |\Omega_{NT}(\beta^*)| = O(NT) \) and \( \frac{\partial}{\partial x} \ln |S_{NT}(\lambda)| = O(NT) \) which are shown previously, implies that \( -\frac{1}{2} \ln \tilde{\sigma}_\varepsilon^2(\delta) \) is bounded above. Suppose that 2) does not hold, then
there exists a sequence \( \{\delta_N\} \) in \( \Delta \) such that \( \lim_{N \to \infty} \tilde{\sigma}_\epsilon^2(\delta_N) = 0 \), which is in contradiction with \( -\frac{1}{2} \ln \tilde{\sigma}_\epsilon^2(\delta) \) is bounded above. Hence 2) holds by contradiction. As a result, the uniform equicontinuity of \( \ln \tilde{\sigma}_\epsilon^2(\delta) \) follows from 1) and 2).

(Show (A1) by contradiction) Suppose that the identification uniqueness condition does not hold, then there exists a constant \( \epsilon_1 > 0 \) and a sequence \( \{\delta_N\} \) in \( N_{\epsilon_1}^c(\delta_0) \) such that \( \limsup_{N \to \infty} \frac{1}{N_T} [\ln \ell_{NT}(\delta_N) - \ln \ell_{NT}(\delta_0)] = 0 \). Furthermore, because of the compactness of \( N_{\epsilon_1}^c(\delta_0) \), there exists a convergent subsequence \( \{\delta_{N_k}\} \) of \( \{\delta_N\} \) such that \( \lim_{N_k \to \infty} \delta_{N_k} = \delta_1 \in N_{\epsilon_1}^c(\delta_0) \), obviously \( \delta_1 \neq \delta_0 \). It follows that \( \lim_{N_k \to \infty} \frac{1}{N_{k,T}} [\ln \ell_{N_{k,T}}(\delta_{N_k}) - \ln \ell_{N_{k,T}}(\delta_0)] = 0 \), and then by the uniform equicontinuity of \( \frac{1}{N_T} \ln \ell_{NT}^*(\delta) \), we have that

\[
\lim_{N_k \to \infty} \frac{1}{N_{k,T}} [\ln \ell_{N_{k,T}}^*(\delta_1) - \ln \ell_{N_{k,T}}^*(\delta_0)] = 0. \tag{S5.9}
\]

However, by Assumption \( \text{[5.7]} \), it holds that, for any \( \delta \neq \delta_0 \),

\[
\lim_{N \to \infty} \frac{1}{N_T} [\ln \ell_{NT}^*(\delta) - \ln \ell_{NT}^*(\delta_0)] = \lim_{N \to \infty} \frac{1}{2N_T} [\ln |\tilde{\sigma}_\epsilon^2(\delta)\Omega_{NT}^{-1}(\beta^*)S_{NT}(\lambda)| - \ln |\sigma_\epsilon^2\Omega_{0NT}^{-1}S_{0NT}|] \neq 0,
\]

which is in contradiction with \( \text{[S5.9]} \). Hence, the identification uniqueness condition holds.

It remains to show (A2). Since \( \frac{1}{N_T} [\ln \ell_{NT}(\delta) - \ln \ell_{NT}(\delta_0)] = -\frac{1}{2} [\ln \tilde{\sigma}_\epsilon^2(\delta) - \ln \tilde{\sigma}_\epsilon^2(\delta)] \), it suffices to show that \( \tilde{\sigma}_\epsilon^2(\delta) - \tilde{\sigma}_\epsilon^2(\delta) = o_p(1) \) uniformly in \( \delta \in \Delta \), and \( \tilde{\sigma}_\epsilon^2(\delta) \) is uniformly bounded away from zero on \( \Delta \). The latter has been verified in (A1), hence we are left to show \( \tilde{\sigma}_\epsilon^2(\delta) - \tilde{\sigma}_\epsilon^2(\delta) = o_p(1) \) uniformly in \( \delta \in \Delta \). Denote

\[
\Omega(\delta) = \Omega_{NT}^{-1}(\beta^*)S_{NT}(\lambda)S_{0NT}^{-1}, \quad \tilde{\Omega}(\delta) = (S_{0NT}^{-1})' S_{NT}(\lambda)\Omega_{NT}^{-1}(\beta^*)S_{NT}(\lambda)S_{0NT}^{-1},
\]

\[
K_0(\delta) = \frac{1}{N_T} \tilde{Z}'\Omega_{NT}^{-1}(\beta^*)\tilde{Z}, \quad K_{10}(\delta) = \frac{1}{N_T} \tilde{Z}'\tilde{\Omega}(\delta)\tilde{Z}, \quad K_{11}(\delta) = \frac{1}{N_T} \tilde{Z}'\tilde{\Omega}(\delta)V_{NT}, \quad K_{20}(\delta) = \frac{1}{N_T} \tilde{Z}'\tilde{\Omega}(\delta)\tilde{Z}, \quad K_{21}(\delta) = \frac{1}{N_T} \tilde{Z}'\tilde{\Omega}(\delta)V_{NT}, \quad K_{22}(\delta) = \frac{1}{N_T} V_{NT}'\tilde{\Omega}(\delta)V_{NT}.
\]
Then by (3.3), (3.7) and \( Y = S_{0\text{NT}}^{-1} \left( \tilde{Z} \phi_0 + V_{\text{NT}} \right) \), it follows that

\[
\tilde{V}_{\text{NT}}(\delta) = S_{\text{NT}}(\lambda) Y - \tilde{\phi}(\delta)
\]

\[
= S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} \left( \tilde{Z} \phi_0 + V_{\text{NT}} \right) + \left[ S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} \tilde{Z} \phi_0 - \tilde{Z} K_{0\text{NT}}^{-1}(\delta) K_{10}(\delta) \phi_0 \right] + \left[ S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} V_{\text{NT}} - \tilde{Z} K_{0\text{NT}}^{-1}(\delta) K_{11}(\delta) \right]
\]

\[
\triangleq Q_1(\delta) + Q_2(\delta),
\]

\[
\tilde{V}_{\text{NT}}(\delta) = S_{\text{NT}}(\lambda) Y - \tilde{\phi}(\delta)
\]

\[
= S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} \left( \tilde{Z} \phi_0 + V_{\text{NT}} \right) - \tilde{Z} \left[ E \left( \tilde{Z} \Omega_{\text{NT}}^{-1}(\beta^*) \tilde{Z} \right) \right]^{-1} E \left[ \tilde{Z} \Omega_{\text{NT}}^{-1}(\beta^*) S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} \left( \tilde{Z} \phi_0 + V_{\text{NT}} \right) \right]
\]

\[
= \left\{ S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} \tilde{Z} \phi_0 - \tilde{Z} \left[ E \left( K_0(\delta) \right) \right]^{-1} E \left( K_{10}(\delta) \phi_0 \right) \right\} + \left\{ S_{\text{NT}}(\lambda) S_{0\text{NT}}^{-1} V_{\text{NT}} - \tilde{Z} \left[ E \left( K_0(\delta) \right) \right]^{-1} E \left( K_{11}(\delta) \right) \right\}
\]

\[
\triangleq P_1(\delta) + P_2(\delta).
\]

These together with (3.4) and (3.8), imply that

\[
\hat{\sigma}^2(\delta) - \tilde{\sigma}^2(\delta) = \frac{1}{N_T} \tilde{V}_{\text{NT}}(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) \tilde{V}_{\text{NT}}(\delta) - \frac{1}{N_T} E \left( \tilde{V}_{\text{NT}}(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) \tilde{V}_{\text{NT}}(\delta) \right)
\]

\[
= \frac{1}{N_T} \left[ Q_1(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) Q_1(\delta) - E \left( P_1(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) P_1(\delta) \right) \right] + \frac{1}{N_T} \left[ Q_2(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) Q_2(\delta) - E \left( P_2(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) P_2(\delta) \right) \right] + \frac{2}{N_T} \left[ Q_1(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) Q_2(\delta) - E \left( P_1(\delta) \Omega_{\text{NT}}^{-1}(\beta^*) P_2(\delta) \right) \right]
\]

\[
\triangleq \Pi_1(\delta) + \Pi_2(\delta) + 2\Pi_3(\delta).
\]

Therefore, to show \( \hat{\sigma}^2(\delta) - \tilde{\sigma}^2(\delta) = o_p(1) \) uniformly in \( \delta \in \Delta \), it suffices to show that \( \Pi_1(\delta) \),
\( \Pi_2(\delta) \) and \( \Pi_3(\delta) \) are all \( o_p(1) \) uniformly in \( \delta \in \Delta \). Next we rewrite \( \Pi_i(\delta) \) for \( i = 1, 2, 3 \) below

\[
\begin{align*}
\Pi_1(\delta) &= \frac{1}{NT} \left[ Q_1'(\delta) \Omega_{NT}^{-1}(\beta^*) Q_1(\delta) - E \left( P_1'(\delta) \Omega_{NT}^{-1}(\beta^*) P_1(\delta) \right) \right] \\
&= \phi_0' \left\{ [K_{20}(\delta) - E(K_{20}(\delta))] \\
&- [K_{10}'(\delta) K_0^{-1}(\delta) K_{10}(\delta) - E(K_{10}(\delta))]' [E(K_0(\delta))]^{-1} E(K_{10}(\delta)) \right\} \phi_0 \\
&= \phi_0' \left\{ [K_{20}(\delta) - E(K_{20}(\delta))] \\
&- [K_{10}(\delta) - E(K_{10}(\delta))]' K_0^{-1}(\delta) K_{10}(\delta) \\
&+ [E(K_{10}(\delta))]' K_0^{-1}(\delta) [K_0(\delta) - E(K_0(\delta))] [E(K_0(\delta))]^{-1} K_{10}(\delta) \\
&- [E(K_{10}(\delta))]' [E(K_0(\delta))]^{-1} [K_{10}(\delta) - E(K_{10}(\delta))] \right\} \phi_0 \\
&\triangleq \phi_0' \{ \Pi_{11}(\delta) - \Pi_{12}(\delta) + \Pi_{13}(\delta) - \Pi_{14}(\delta) \} \phi_0.
\end{align*}
\]

\[
\Pi_2(\delta) = \frac{1}{NT} \left[ Q_2'(\delta) \Omega_{NT}^{-1}(\beta^*) Q_2(\delta) - E \left( P_2'(\delta) \Omega_{NT}^{-1}(\beta^*) P_2(\delta) \right) \right] \\
= [K_{22}(\delta) - E(K_{22}(\delta))] \\
- [K_{11}(\delta) - E(K_{11}(\delta))]' K_0^{-1}(\delta) K_{11}(\delta) \\
+ [E(K_{11}(\delta))]' K_0^{-1}(\delta) [K_0(\delta) - E(K_0(\delta))] [E(K_0(\delta))]^{-1} K_{11}(\delta) \\
- [E(K_{11}(\delta))]' [E(K_0(\delta))]^{-1} [K_{11}(\delta) - E(K_{11}(\delta))] \\
\triangleq \Pi_{21}(\delta) - \Pi_{22}(\delta) + \Pi_{23}(\delta) - \Pi_{24}(\delta),
\]

\[
\Pi_3(\delta) = \frac{1}{NT} \left[ Q_3'(\delta) \Omega_{NT}^{-1}(\beta^*) Q_3(\delta) - E \left( P_3'(\delta) \Omega_{NT}^{-1}(\beta^*) P_3(\delta) \right) \right] \\
= \phi_0' \left\{ [K_{21}(\delta) - E(K_{21}(\delta))] \\
- [K_{10}(\delta) - E(K_{10}(\delta))]' K_0^{-1}(\delta) K_{11}(\delta) \\
+ [E(K_{10}(\delta))]' K_0^{-1}(\delta) [K_0(\delta) - E(K_0(\delta))] [E(K_0(\delta))]^{-1} K_{11}(\delta) \\
- [E(K_{10}(\delta))]' [E(K_0(\delta))]^{-1} [K_{11}(\delta) - E(K_{11}(\delta))] \right\} \\
\triangleq \phi_0' \{ \Pi_{31}(\delta) - \Pi_{32}(\delta) + \Pi_{33}(\delta) - \Pi_{34}(\delta) \}.
\]
(Show that \( \Pi_1(\delta) = o_p(1) \) uniformly in \( \delta \in \Delta \)) It suffices to show that all elements of \( \Pi_{11}(\delta), \Pi_{12}(\delta), \Pi_{13}(\delta) \) and \( \Pi_{14}(\delta) \) are \( o_p(1) \) uniformly in \( \delta \in \Delta \), respectively. We establish the proof by verifying the following parts i)–iv).

i) We first show that \( K_{20}(\delta) - E(K_{20}(\delta)) \), \( K_{10}(\delta) - E(K_{10}(\delta)) \) and \( K_0(\delta) - E(K_0(\delta)) \) are all \( o_p(1) \) uniformly on \( \Delta \). For \( K_{20}(\delta) - E(K_{20}(\delta)) \), we have that

\[
K_{20}(\delta) - E(K_{20}(\delta)) = \begin{pmatrix} e_{11} & * & * \\ e_{21} & e_{22} & * \\ e_{31} & e_{32} & e_{33} \end{pmatrix},
\]

where

\[
e_{11} = \frac{1}{NT} \left[ Y'_{-1}\tilde{\Omega}(\delta)Y_{-1} - E\left(Y'_{-1}\tilde{\Omega}(\delta)Y_{-1}\right) \right],
\]

\[
e_{21} = \frac{1}{NT} \left[ Z'\tilde{\Omega}(\delta)Y_{-1} - E\left(Z'\tilde{\Omega}(\delta)Y_{-1}\right) \right],
\]

\[
e_{22} = \frac{1}{NT} \left[ Z'\tilde{\Omega}(\delta)Z - E\left(Z'\tilde{\Omega}(\delta)Z\right) \right],
\]

\[
e_{31} = \frac{1}{NT} \left[ X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)Y_{-1} - E\left(X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)Y_{-1}\right) \right],
\]

\[
e_{32} = \frac{1}{NT} \left[ X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)Z - E\left(X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)Z\right) \right],
\]

\[
e_{33} = \frac{1}{NT} \left[ X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)(\iota_{NT} \otimes I_N)X - E\left(X'(\iota_{NT} \otimes I_N)'\tilde{\Omega}(\delta)(\iota_{NT} \otimes I_N)X\right) \right].
\]

Here we only show the element \( e_{11} = o_p(1) \) uniformly on \( \Delta \), and the others can be verified similarly. By \( \{S3.3\} \), we have \( Y_{-1} = M_1y_0 + M_2Z\gamma_0 + M_3(X\psi_0 + \vartheta) + M_2\varepsilon \), then it follows that \( e_{11} \triangleq 2(S_1 + S_2) \), where

\[
S_1 = \frac{1}{NT}\gamma_0'Z'M_2'\tilde{\Omega}(\delta)M_1y_0 + \frac{1}{NT}\varepsilon'M_2'\tilde{\Omega}(\delta)M_1y_0 + \frac{1}{NT}\vartheta'M_2'\tilde{\Omega}(\delta)M_1y_0
\]

\[
+ \frac{1}{NT}\gamma_0'Z'M_2'\tilde{\Omega}(\delta)M_3X\psi_0 + \frac{1}{NT}\varepsilon'M_2'\tilde{\Omega}(\delta)M_3X\psi_0
\]

\[
+ \frac{1}{NT}\vartheta'M_2'\tilde{\Omega}(\delta)M_3X\psi_0,
\]

\[
S_2 = \frac{1}{2NT}\left[ \gamma_0'Z'M_2'\tilde{\Omega}(\delta)M_2Z\gamma_0 - E\left(\gamma_0'Z'M_2'\tilde{\Omega}(\delta)M_2Z\gamma_0\right) \right]
\]
\[ + \frac{1}{2NT} \left[ \varepsilon' M_2' \tilde{\Omega}(\delta) M_2 \varepsilon - E \left( \varepsilon' M_2' \tilde{\Omega}(\delta) M_2 \varepsilon \right) \right] \\
+ \frac{1}{2NT} \left[ \vartheta' M_3' \tilde{\Omega}(\delta) M_3 \vartheta - E \left( \vartheta' M_3' \tilde{\Omega}(\delta) M_3 \vartheta \right) \right] \\
+ \frac{1}{NT} \gamma_0' Z' M_2' \tilde{\Omega}(\delta) M_2 \varepsilon + \frac{1}{NT} \gamma' Z' M_2' \tilde{\Omega}(\delta) M_3 \vartheta + \frac{1}{NT} \varepsilon' M_2' \tilde{\Omega}(\delta) M_3 \vartheta. \]

Note that under Assumption 3.5, \( M_1, M_2 \) and \( \frac{1}{T} M_3 \) are all uniformly bounded in both row and column sums, and \( T \tilde{\Omega}(\delta) = \left( S_{0NT}^{-1} \right)' S_{0NT}^{-1}(\lambda) T \Omega_{NT}^{-1}(\beta^*) S_{NT}(\lambda) S_{0NT}^{-1} \) and \( T \frac{\partial}{\partial \delta} \tilde{\Omega}(\delta) \) are both uniformly bounded in both row and column sums, uniformly on \( \Delta \) by Lemmas 2 and 3. Then we have that \( M_2' \tilde{\Omega}(\delta) M_1, M_2' \tilde{\Omega}(\delta) M_2, M_3' \tilde{\Omega}(\delta) M_1 = \frac{1}{T} M_3' T \tilde{\Omega}(\delta) M_1, M_2' \tilde{\Omega}(\delta) M_3 = M_2' T \tilde{\Omega}(\delta) \frac{1}{T} M_3 \) and \( \frac{1}{T} M_3' \tilde{\Omega}(\delta) M_3 = \frac{1}{T} M_3' T \tilde{\Omega}(\delta) \frac{1}{T} M_3 \) as well as their derivatives with respect to \( \delta \) are uniformly bounded in both row and column sums uniformly on \( \Delta \) by Lemma 2(iii). Moreover, elements of previous matrices are all uniformly bounded on \( \Delta \) by Lemma 1. It follows that \( S_1 = o_p(1) \) holds uniformly on \( \Delta \) by Lemma 1, and \( S_2 = o_p(1) \) holds uniformly on \( \Delta \) by Lemma 4 and Lemma 5. As a result, \( K_{20}(\delta) - E(K_{20}(\delta)) = o_p(1) \) holds uniformly on \( \Delta \). Similarly, we can show that \( K_{10}(\delta) - E(K_{10}(\delta)) = o_p(1) \) and \( K_0(\delta) - E(K_0(\delta)) = o_p(1) \) hold uniformly on \( \Delta \).

ii) Next we prove that \( K_0^{-1}(\delta) = O_p(1) \). All elements of \( K_0^{-1}(\delta) \) are \( O_p(1) \) holds uniformly on \( \Delta \) if and only if \( \|K_0^{-1}(\delta)\|^2_F = O_p(1) \) uniformly on \( \Delta \). Denote \( d_0 = 2 + p + q \), by Bernstein (2009), we have

\[
\sup_{\beta^*} \|K_0^{-1}(\delta)\|^2_F = \sup_{\beta^*} \text{tr} \left[ \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)^{-1} \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)^{-1} \right] \
\leq d_0 \sup_{\beta^*} \lambda_{\max} \left[ \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)^{-1} \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)^{-1} \right] \
= d_0 \sup_{\beta^*} \lambda_{\max}^2 \left[ \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)^{-1} \right] \
= d_0 \sup_{\beta^*} \lambda_{\min}^{-2} \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right) 
\]
\[= d_0 \sup_{\beta^*} \left[ \lambda_{\min}^2 \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1} (\beta^*) \tilde{Z} \right) \right]^{-1},\]

where \( \| \cdot \|_F \) denotes the Frobenius norm of a matrix, \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) are the smallest and largest eigenvalues of a matrix, respectively. It follows \( \sup_N E \sup_{\beta^*} \| K_0^{-1}(\delta) \|_F^2 \leq d_0 \sup_N E \sup_{\beta^*} \left[ \lambda_{\min}^2 \left( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1} (\beta^*) \tilde{Z} \right) \right]^{-1} < \infty \) by Assumption 3.6(iv), then \( \| K_0^{-1}(\delta) \|_F^2 = \mathcal{O}_p(1) \) holds uniformly on \( \Delta \) by page 339 in Davidson (1994), and equivalently, all elements of \( K_0^{-1}(\delta) \) are \( \mathcal{O}_p(1) \) uniformly on \( \Delta \).

iii) Below we show that all elements of \( E(K_{10}(\delta)) \) and \( [E(K_0(\delta))]^{-1} \) are uniformly bounded on \( \Delta \). We only need to show that each element of \( E(K_{10}(\delta)) \) and \( E(K_0(\delta)) \) is uniformly bounded on \( \Delta \). As for (S5.10), we next show the element \( \frac{1}{NT} E \left( Y'_{-1} \Omega_{NT}^{-1}(\beta^*) Y_{-1} \right) \) in \( E(K_0(\delta)) \) is \( \mathcal{O}(1) \) uniformly on \( \Delta \). Note that elements of \( M_1, M_2 \) and \( M_3 \) are uniformly bounded, and \( M_1, M_2 \) and \( \frac{1}{t} M_3 \) are all uniformly bounded in both row and column sums. Then for \( M_2' \Omega_{NT}^{-1}(\beta^*) M_2 \) and \( M_3' \Omega_{NT}^{-1}(\beta^*) M_3 = (M_3')' T \Omega_{NT}^{-1}(\beta^*) \frac{1}{t} M_3 \), their elements are all uniformly bounded on \( \Delta \) by Lemma 1, so by Lemma 3(i), we obtain that

\[ \frac{1}{NT} E \left( Y'_{-1} \Omega_{NT}^{-1}(\beta^*) Y_{-1} \right) = \frac{1}{NT} E \left( \gamma_0' Z'M_2 \Omega_{NT}^{-1}(\beta^*) M_2 Z \gamma_0 \right) = \mathcal{O}(1) \]

holds uniformly on \( \Delta \).

iv) Analogous to the proofs of i) and iii), we can show that \( K_{11}(\delta) - E(K_{11}(\delta)) = o_p(1) \) and \( E(K_{11}(\delta)) = \mathcal{O}(1) \) hold uniformly on \( \Delta \). It follows that \( K_{11}(\delta) = o_p(1) \) holds uniformly on \( \Delta \). As a result, with i)–iv), we can obtain that \( \Pi_1(\delta) = o_p(1) \) holds uniformly in \( \delta \in \Delta \).

With the similar argument, we can show that \( \Pi_2(\delta) \) and \( \Pi_3(\delta) \) are also \( o_p(1) \) uniformly in \( \delta \in \Delta \). This completes the proof of (A2).

The consistency of \( \hat{\delta} \) and hence that of \( \hat{\phi} \) and \( \hat{\sigma}_\varepsilon^2 \) follow from (A1) and (A2). We accomplish the proof of this theorem.
S5.2 Proof of Theorem 2

By Taylor’s expansion, we have

\[
0 = \frac{1}{\sqrt{NT}} \frac{\partial \ln L_{NT}(\tilde{\zeta})}{\partial \zeta} = \frac{1}{\sqrt{NT}} \frac{\partial \ln L_{NT}(\zeta_0)}{\partial \zeta} + \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\tilde{\zeta})}{\partial \zeta \partial \zeta'} \sqrt{NT}(\tilde{\zeta} - \zeta_0),
\]

where elements of \( \tilde{\zeta} = (\tilde{\phi}', \tilde{\sigma}^2, \tilde{\lambda}, \tilde{\beta}'')' \) lie in the segment joining the corresponding elements of \( \hat{\zeta} \) and \( \zeta_0 \). It follows that

\[
\sqrt{NT}(\tilde{\zeta} - \zeta_0) = -\left( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\tilde{\zeta})}{\partial \zeta \partial \zeta'} \right)^{-1} \frac{1}{\sqrt{NT}} \frac{\partial \ln L_{NT}(\zeta_0)}{\partial \zeta}.
\]

Then, to verify Theorem 2, it suffices to show that

**(B1)** \( \frac{1}{\sqrt{NT}} \frac{\partial \ln L_{NT}(\zeta_0)}{\partial \zeta} \rightarrow_\text{d} N(0, \Sigma + \Sigma^*) \);

**(B2)** \( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\tilde{\zeta})}{\partial \zeta \partial \zeta'} - \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \zeta \partial \zeta} = o_p(1) \);

**(B3)** \( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \zeta \partial \zeta'} - E \left( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \zeta \partial \zeta'} \right) = o_p(1) \).

It can be verified that all elements of \( \frac{1}{\sqrt{NT}} \frac{\partial \ln L_{NT}(\zeta_0)}{\partial \zeta} \) can be rewritten in form of (S4.3).

For example, \( \dot{\sigma}^2 \) can be decomposed as follows

\[
\dot{\sigma}^2 = \left( \frac{NT}{2\sigma^2_{\epsilon_0}} \right)^{-1} \left[ \phi'(\mu_T \otimes I_N)\Omega^{-1}_{\epsilon_0}(\mu_T \otimes I_N)\phi + 2\phi'(\mu_T \otimes I_N)\Omega^{-1}_{\epsilon_0}\epsilon + \epsilon'\Omega^{-1}_{\epsilon_0}\epsilon \right].
\]

Note that \( \Omega^{-1}_{\epsilon_0} \) and \( \Omega^{-1}_{\epsilon_0}(\mu_T \otimes I_N) = T\Omega^{-1}_{\epsilon_0}(\mu_T \otimes I_N) \) are both uniformly bounded in both row and column sums, and \( \frac{1}{\sqrt{NT}}(\mu_T \otimes I_N)\Omega^{-1}_{\epsilon_0}(\mu_T \otimes I_N) = \sqrt{\frac{T}{N}}(\mu_T \otimes I_N)\Omega^{-1}_{\epsilon_0}(\mu_T \otimes I_N) \) is also uniformly bounded in both row and column sums. Then (B1) holds by Lemma 7.

Moreover, with the similar argument of the proof of Theorem 1, we can show that all elements of \( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \zeta \partial \zeta'} - E \left( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \zeta \partial \zeta'} \right) = o_p(1) \) and thus (B3) holds. Then we are left to prove the result of (B2).

(Show (B2)) Recall that \( \zeta = (\phi', \sigma^2, \lambda, \beta'')' \). We first show that

\[
\frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\tilde{\zeta})}{\partial \phi \partial \phi} - \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\zeta_0)}{\partial \phi \partial \phi} = o_p(1).
\]

Note that \( \frac{1}{NT} \frac{\partial^2 \ln L_{NT}(\tilde{\zeta})}{\partial \phi \partial \phi} = -\frac{1}{\sigma^2} \frac{1}{NT} \tilde{Z}'\Omega^{-1}(\beta')\tilde{Z} \), by the mean value theorem, it can
be verified that

\[
\frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\tilde{\zeta})}{\partial \phi \partial \phi'} - \frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\zeta_0)}{\partial \phi \partial \phi'}
\]

\[= - \frac{1}{\sigma^2} \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} + \frac{1}{\sigma^2} \frac{1}{NT} \tilde{Z}' \Omega_{0NT}^{-1} \tilde{Z}
\]

\[= \left( \frac{1}{\sigma^2} - \frac{1}{\sigma^2_0} \right) \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} + \frac{1}{\sigma^2} \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \left( \Omega_{NT}(\beta^*) - \Omega_{0NT} \right) \Omega_{0NT}^{-1} \tilde{Z}
\]

\[= \left( \frac{1}{\sigma^2} - \frac{1}{\sigma^2_0} \right) \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z}
\]

where \( \beta^* \) lies between \( \beta^* \) and \( \beta_0^* \). By Theorem 3, \( \sigma^2 \to_p \sigma^2_0 \) and \( \beta^* \to_p \beta_0^* \). Moreover, by Lemmas 3 and 6, it can be verified that \( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \frac{\partial}{\partial \beta_t} \Omega_{NT}(\beta^*) \Omega_{0NT}^{-1} \tilde{Z} = O_p(1) \) and \( \frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} = O_p(1) \). Hence, we have \( \frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\tilde{\zeta})}{\partial \phi \partial \phi'} - \frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\zeta_0)}{\partial \phi \partial \phi'} = o_p(1) \). Furthermore, we can show that other terms of \( \frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\tilde{\zeta})}{\partial \zeta \partial \zeta'} - \frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\zeta_0)}{\partial \zeta \partial \zeta'} \) converges to zero in probability similarly, and then (B2) holds.

The proof of this theorem is then accomplished.

### S5.3 Proof of Theorem 3

Recall that \( \hat{\theta} = \frac{1}{T} \sum_{t=1}^T \left[ B_N(\lambda) y_t - \alpha y_{t-1} - Z_t \tilde{\gamma} - X \psi \right] \), so it holds that

\[
\hat{\theta} - \theta = \frac{1}{T} \sum_{t=1}^T \left[ -(\lambda - \lambda_0) W_N y_t - (\alpha - \alpha_0) y_{t-1} - Z_t (\tilde{\gamma} - \gamma_0) - X (\tilde{\psi} - \psi_0) + \epsilon_t \right].
\]  

(S5.11)

We first show Theorem 3(i), that is \( \hat{\theta}_i - \psi_i = o_p(1) \) for \( 1 \leq i \leq N \). By (S5.11), we have

\[
\hat{\theta}_i - \psi_i = -(\lambda - \lambda_0) \frac{1}{T} \sum_{t=1}^T w_i N y_t - (\alpha - \alpha_0) \frac{1}{T} \sum_{t=1}^T y_{t-1} - \frac{1}{T} \sum_{t=1}^T z_{it}' (\tilde{\gamma} - \gamma_0)
\]
where \( w_{i,N} \) is the \( i \)th row of \( W_N \). Note that \( \hat{\zeta} - \zeta_0 = o_p(1) \) by Theorem 3.1, \( \frac{1}{T} \sum_{t=1}^{T} z_{it} = o_p(1) \), \( \frac{1}{T} \sum_{t=1}^{T} \varepsilon_{it} = o_p(1) \) and \( x_i = O(1) \) by Assumption 3.6. To prove that \( \hat{\vartheta}_i - \vartheta_i = o_p(1) \), it remains to show that \( \frac{1}{T} \sum_{t=1}^{T} w_{i,N} y_t = O_p(1) \) and \( \frac{1}{T} \sum_{t=1}^{T} y_{i,t-1} = O_p(1) \). Since the row sums of \( W_N \) are uniformly bounded, we only need to show \( \frac{1}{T} \sum_{t=1}^{T} y_{it} = O_p(1) \). By (S3.2), we have

\[
y_t = \alpha_0'(B^{-1}_{0N})' y_0 + \sum_{j=1}^{T} \left[ \alpha_0^{-1}(B^{-1}_{0N})^j z_{t-j+1} \right] \gamma_0 + \sum_{j=1}^{T} \left[ \alpha_0^{-1}(B^{-1}_{0N})^j \right] \theta + \sum_{j=1}^{T} \left[ \alpha_0^{-1}(B^{-1}_{0N})^j \varepsilon_{t-j+1} \right].
\]

By Assumption 3.3, we have \( E \left( \sum_{t=1}^{T} y_{it} \right) = O(T) \) and \( \text{var} \left( \sum_{t=1}^{T} y_{it} \right) = O(T) \), then it follows that \( \frac{1}{T} \sum_{t=1}^{T} (y_{it} - E y_{it}) = o_p(1) \) by Chebyshev inequality and thus \( \frac{1}{T} \sum_{t=1}^{T} y_{it} = O_p(1) \). Therefore, \( \frac{1}{T} \sum_{t=1}^{T} w_{i,N} y_t = O_p(1) \) and \( \frac{1}{T} \sum_{t=1}^{T} y_{i,t-1} = O_p(1) \) holds. This completes the proof of Theorem 3(i).

Next we consider Theorem 3(ii). By (S5.12), it follows that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\hat{\vartheta}_i - \vartheta_i) = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ - (\hat{\lambda} - \lambda_0) w_{i,N} y_t - (\hat{\alpha} - \alpha_0) y_{i,t-1} - z'_{it} (\hat{\gamma} - \gamma_0) - x'_i (\hat{\psi} - \psi_0) + \varepsilon_{it} \right]
\]

\[
= - \sqrt{NT} (\hat{\lambda} - \lambda_0) \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} w_{i,N} y_t - \sqrt{NT} (\hat{\alpha} - \alpha_0) \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t-1}
\]

\[
- \frac{1}{NT^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} z'_{it} \sqrt{NT} (\hat{\gamma} - \gamma_0) - \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} x'_i \sqrt{NT} (\hat{\psi} - \psi_0) + \frac{1}{N^{1/2}T} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it}.
\]

Notice that \( \sqrt{NT} (\hat{\zeta} - \zeta_0) = O_p(1) \) by Theorem 3, \( \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} z_{it} = o_p(1) \), \( \frac{1}{NT^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} x_{it} = o_p(1) \) and \( \frac{1}{N^{1/2}T} \sum_{i=1}^{N} \sum_{t=1}^{T} \varepsilon_{it} = o_p(1) \) by Assumption 3.6. Moreover, by Assumption 3.5, it can be verified that

\[
E \left( \sum_{i=1}^{N} \sum_{t=1}^{T} y_{it} \right) = O(NT) \quad \text{and} \quad \text{var} \left( \sum_{i=1}^{N} \sum_{t=1}^{T} y_{it} \right) = O(N^2T),
\]
then it follows that \( \frac{1}{N^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t} = o_p(1) \) by Chebyshev inequality. Since the row sums of 
\( W_N \) are uniformly bounded, we obtain that \( \frac{1}{N^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} w_{i,N} y_{i} = o_p(1) \) and \( \frac{1}{N^{3/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} y_{i,t-1} = o_p(1) \) holds. Thus \( \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\hat{\vartheta}_i - \vartheta_i) = o_p(1) \), that is Theorem 3(ii) holds.

The proof of Theorem 3 is accomplished.

**S5.4 Proof of Theorem 4**

Define

\[
Q_N(\tau, \varphi) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) \quad \text{and} \quad Q(\tau, \varphi) = \frac{1}{N} \sum_{i=1}^{N} E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi).
\]

We first show that \( Q_N(\tau, \varphi) - Q(\tau, \varphi) = o_p(1) \) holds uniformly on \( T \times \Phi \). To achieve this, it suffices to show that, as \( N \to \infty \),

\[
\text{(C1) } \sup_{(\tau, \varphi)} \left| \frac{1}{N} \sum_{i=1}^{N} E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - \frac{1}{N} \sum_{i=1}^{N} E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \right| \to 0;
\]

\[
\text{(C2) } \sup_{(\tau, \varphi)} \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \right| = o_p(1);
\]

\[
\text{(C3) } \sup_{(\tau, \varphi)} \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \right| = o_p(1).
\]

(Show (C1)) By Theorem 3, we obtain that \( \hat{\vartheta}_i - \vartheta_i = o_p(1) \) and hence \( \hat{\vartheta}_i \to_d \vartheta_i \). Then by Lemma 2.2 of Van der Vaart (2000), it holds that

\[
\left| E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \right| \to 0
\]
as \( N \to \infty \), because \( \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) \) is a bounded (which is due to the compactness of \( \Phi \) and \( \mathcal{B} \)) and continuous function. Furthermore, since the set \( T \times \Phi \) is compact and

\[
E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \text{ is continuous on } T \times \Phi,
\]

we have that

\[
\sup_{(\tau, \varphi)} \left| E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\hat{\vartheta}_i - x'_{ai} \varphi) - E \frac{1}{x'_{ai} \beta_c} \rho_\tau(\vartheta_i - x'_{ai} \varphi) \right| \to 0.
\]
It follows that (C1) holds.

(Show (C2)) Let $H = \{ (\tau, \varphi) \mapsto \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta - x'_a \varphi) \}$. Since $T \times \Phi$ is compact and

$\frac{1}{x'_a \beta_0} \rho_\tau(\vartheta - x'_a \varphi)$ is Lipschitz over $T \times \Phi$, $H$ is Donsker by Theorem 2.10.6 of [Van Der Vaart and Wellner (1996)]. Following Chernozhukov and Hansen (2006) and Canay (2011), donsker-ness implies a uniform law of large number, that is

$$\sup_{(\tau, \varphi)} \left| \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) - E \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) \right] \right| = o_p(1),$$

which gives

$$\sup_{(\tau, \varphi)} \left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) - \frac{1}{N} \sum_{i=1}^{N} E \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) \right| = o_p(1),$$

then (C2) holds; see also [Van der Vaart (2000)].

(Show (C3)) Note that $\tilde{\beta}_c - \beta_0 = o_p(1)$, so we have that

$$\left| \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_a \beta_c} \rho_\tau(\vartheta_i - x'_a \varphi) - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) \right| = o_p(1).$$

Furthermore, $T \times \Phi$ is compact and

$\frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi) - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{x'_a \beta_0} \rho_\tau(\vartheta_i - x'_a \varphi)$ is stochastically equicontinuous on $T \times \Phi$ (Chernozhukov and Hansen (2006)), so (C3) holds.

By (C1)–(C3), we obtain that $Q_N(\tau, \varphi) - Q(\tau, \varphi) = o_p(1)$ holds uniformly on $T \times \Phi$. This together with $\varphi_0(\tau) = \arg\min_{\varphi \in \Phi} Q(\tau, \varphi)$, implies that $\sup_{\tau \in T} \| \hat{\varphi}(\tau) - \varphi_0(\tau) \| = o_p(1)$ by Lemma B.1 of Chernozhukov and Hansen (2006). As a result, we complete the proof of Theorem 4.
S5.5 Proof of Theorem \(5\)

Following \cite{Canay2011} and \cite{ChernozhukovHansen2006}, by the computational properties of standard quantile regression, it holds that

\[
\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{x_{ai}' \beta_c} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai} = O_p \left( \frac{1}{\sqrt{N}} \right) \text{ in } \ell^\infty(\mathcal{T}). \tag{S5.13}
\]

Denote

\[
F_1 \triangleq G_N \left( \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai} \right),
\]

\[
F_2 \triangleq \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai},
\]

where \( \hat{\tau}_i \equiv \hat{\varphi}(\cdot) - \varphi(\cdot) = o_p(1) \) for \(1 \leq i \leq N\) by Theorem \(3(i)\). By \(5(\text{S5.13})\) and \( \beta_c - \beta_0 = o_p(1) \), we obtain that

\[
F_1 + F_2 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai}
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{x_{ai}' \beta_c} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai} + o_p(1)
\]

\[
= o_p(1) \text{ in } \ell^\infty(\mathcal{T}). \tag{S5.14}
\]

(For \(F_1\)) Note that \( \sup_{\tau \in \mathcal{T}} \| \hat{\varphi}(\tau) - \varphi_0(\tau) \| = o_p(1) \) by Theorem \(4\), so by Lemma \(10(\text{ii})\),

\[
F_1 = G_N \left( \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai} \right)
\]

\[
= G_N \left( \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \varphi(\cdot) \right) x_{ai} \right) + o_p(1)
\]

\[
= G_N \left( \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - Q_{\eta}(\cdot) \right) x_{ai} \right) + o_p(1) \text{ in } \ell^\infty(\mathcal{T}). \tag{S5.15}
\]

(For \(F_2\)) Following \cite{Canay2011} and \cite{ChernozhukovHansen2006}, we have

\[
F_2 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} E \frac{1}{x_{ai}' \beta_0} \Psi \left( \theta_i - x_{ai}' \hat{\varphi}(\cdot) + \hat{\tau}_i \right) x_{ai}
\]
\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ \frac{1}{x'_{ai} \beta_0} \Psi_{i1} (\vartheta_{i} - x'_{ai} \varphi_0 (\cdot) + 0) x_{ai} \right. \\
\left. + J_{i1}(\cdot, \varphi_0 (\cdot), \vartheta_{i}) (\varphi (\cdot) - \varphi_0 (\cdot)) + J_{2i}(\cdot, \varphi_0 (\cdot), \vartheta_{i}) (\vartheta_{i} - 0) \right],
\]

where \((\varphi^*(\tau), r_1^*, \ldots, r_N^*)\) lies between \((\varphi_0 (\tau), 0, \ldots, 0)\) and \((\varphi (\tau), \vartheta_1, \ldots, \vartheta_N)\), and

\[
J_{i1}(\tau, \varphi, \vartheta) = \frac{\partial}{\partial \vartheta_i} E \frac{1}{x'_{ai} \beta_0} \Psi_{i1} (\vartheta_{i} - x'_{ai} \varphi + r_i) x_{ai},
\]

\[
J_{2i}(\tau, \varphi, \vartheta) = \frac{\partial}{\partial r_i} E \frac{1}{x'_{ai} \beta_0} \Psi_{i1} (\vartheta_{i} - x'_{ai} \varphi + r_i) x_{ai}.
\]

By Theorems 3(i) and 4, \(\varphi^*(\cdot) - \varphi_0 (\cdot) = o_p(1)\) in \(\ell^\infty (\mathcal{T})\), and \(r_i^* = o_p(1)\) for \(1 \leq i \leq N\).

Then by Assumption 3.9, we have that

\[
F_2 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ \frac{1}{x'_{ai} \beta_0} \Psi_{i1} (\vartheta_{i} - x'_{ai} \varphi_0 (\cdot) + 0) x_{ai} \right. \\
\left. + J_{i1}(\cdot, \varphi_0 (\cdot), 0) (\varphi (\cdot) - \varphi_0 (\cdot)) + J_{2i}(\cdot, \varphi_0 (\cdot), 0) \vartheta_{i} \right] + o_p(1)
\]

\[
= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ \frac{1}{x'_{ai} \beta_0} \Psi_{i1} (\eta_i - Q_\eta (\cdot)) x_{ai} \right. \\
\left. + J_{i1}(\cdot, \varphi_0 (\cdot), 0) (\varphi (\cdot) - \varphi_0 (\cdot)) + J_{2i}(\cdot, \varphi_0 (\cdot), 0) \vartheta_{i} \right] + o_p(1),
\]

in \(\ell^\infty (\mathcal{T})\). Furthermore, we have

\[
J_{1i}(\tau, \varphi_0 (\tau), 0) = - \frac{x'_{ai} x_{ai}}{(x'_{ai} \beta_0)^2} f_\eta (Q_\eta (\tau)) \quad \text{and} \quad J_{2i}(\tau, \varphi_0 (\tau), 0) = \frac{x_{ai}}{(x'_{ai} \beta_0)^2} f_\eta (Q_\eta (\tau)).
\]

These together with \(E \Psi_{11} (\eta_i - Q_\eta (\cdot)) = 0\) and \(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_{ai}}{(x'_{ai} \beta_0)^2} (\hat{\vartheta}_{i} - \vartheta_{i}) = o_p (1)\) by Theorem 3(ii), imply that

\[
F_2 = -D_{1N} (\cdot) \sqrt{N} (\varphi (\cdot) - \varphi_0 (\cdot)) + \Xi (\cdot) \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{x_{ai}}{(x'_{ai} \beta_0)^2} (\hat{\vartheta}_{i} - \vartheta_{i}) + o_p (1)
\]

\[
= -D_{1N} (\cdot) \sqrt{N} (\varphi (\cdot) - \varphi_0 (\cdot)) + o_p (1) \quad \text{in} \quad \ell^\infty (\mathcal{T}),
\]

(S5.16)
By (S5.14)–(S5.16), it follows that
\[
\mathbb{G}_N \left( \frac{x_{ai}}{x_{ai}' \beta_0} \Psi_i (\eta_i - Q_\eta(\cdot)) \right) - D_{1N}(\cdot) \sqrt{N}(\varphi(\cdot) - \varphi_0(\cdot)) = o_p(1) \quad \text{in} \quad \ell^\infty(\mathcal{T}). \tag{S5.17}
\]

This together with Lemma 10 implies that
\[
\sqrt{N}(\varphi(\cdot) - \varphi_0(\cdot)) = D_{1N}^{-1}(\cdot) \mathbb{G}_N \left( \frac{x_{ai}}{x_{ai}' \beta_0} \Psi_i (\eta_i - Q_\eta(\cdot)) \right) + o_p(1) \rightsquigarrow \mathbb{G}(\cdot) \quad \text{in} \quad \ell^\infty(\mathcal{T}),
\]
where \(\mathbb{G}(\cdot)\) is a Gaussian process with zero mean and covariance kernel \(\Xi^{-1}(\tau)S(\tau, \tau')\Xi^{-1}(\tau')D_0^{-1}\).

This completes the proof of Theorem 5.

**S5.6 Proof of Theorem 6**

Recall that \(\hat{\beta}(\pi_K) = \sum_{k=1}^{K} \pi_k \varphi(\tau_k)\) and \(\sum_{k=1}^{K} \pi_k Q_\eta(\tau_k) = 1\), then by Theorem 3, we have
\[
\sqrt{N} \left( \hat{\beta}(\pi_K) - \beta_0 \right) = \sqrt{N} \left( \sum_{k=1}^{K} \pi_k \varphi(\tau_k) - \sum_{k=1}^{K} \pi_k Q_\eta(\tau_k)\beta_0 \right) \\
= \sqrt{N} \left( \sum_{k=1}^{K} \pi_k \varphi(\tau_k) - \sum_{k=1}^{K} \pi_k \varphi_0(\tau_k) \right) \\
= \sum_{k=1}^{K} \pi_k \sqrt{N}(\varphi(\tau_k) - \varphi_0(\tau_k)) \rightarrow_d N(0, \mathcal{Y}(\pi_K)),
\]
where the covariance matrix is in form of
\[
\mathcal{Y}(\pi_K) = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} \pi_{k_1} \pi_{k_2} \Xi^{-1}(\tau_{k_1})S(\tau_{k_1}, \tau_{k_2})\Xi^{-1}(\tau_{k_2})D_0^{-1} = \pi' KH\pi K D_0^{-1}.
\]

As for Theorem 5 of Zhao and Xiao (2014), the optimal weight follows from Lagrange multiplier method, that is
\[
\pi_{opt,K} = \arg\min_{\pi, s.t. \pi'q=1} \mathcal{Y}(\pi_K) = \frac{H^{-1}q}{q'H^{-1}q}.
\]
It follows that the asymptotic covariance of the optimal WQAE \(\hat{\beta}(\pi_{opt,K})\) is \(\mathcal{Y}(\pi_{opt,K}) = \frac{H^{-1}q}{q'H^{-1}q} \rightarrow_d N(0, \mathcal{Y}(\pi_{opt,K}))\). Hence, the proof of this theorem is accomplished.
S5.7 Proof of Proposition

Let \{\xi_t\} be a vector sequence of random variables with

\[
\xi_t = \sum_{m=0}^{\infty} \alpha^m (B_N^{-1})^m (\theta + Z_{t-m} \gamma + \varepsilon_{t-m}),
\]

(S5.18)

where \(B_N = B_N(\lambda)\). Recall that \{\varepsilon_{it}\} are i.i.d. across \(i\) and \(t\), we have \(E(\varepsilon_t) = 0\), \(\text{cov}(\varepsilon_t) = \sigma_\varepsilon^2 I_N\) and \(\text{cov}(\varepsilon_t, \varepsilon_{t-h}) = 0\) for \(h \neq 0\). Moreover, since \{\z_{it}\} is covariance stationary by assumption, we have \(E(\z_{it}) = \mu_z\) for a constant vector \(\mu_z\) and \(\text{cov}(\z_{it}, \z_{it+h})\) is independent of \(t\). Without loss of generality, we assume \(\mu_z = 0\). It follows that \(E(Z_t \gamma) = 0\) and \(\text{cov}(Z_t \gamma, Z_{t-h} \gamma) = E(Z_t \gamma \gamma' Z_{t-h})\) is independent of \(t\). Since \(|\alpha B_N^{-1}| < 1\) by assumption, \(\theta = \theta + X \psi\) and \(E(\theta) = 0\), it holds that

\[
E(B_N \xi_t) = E\left[ \sum_{m=0}^{\infty} (\alpha B_N^{-1})^m (\theta + X \psi + Z_{t-m} \gamma + \varepsilon_{t-m}) \right]
\]

\[
= \sum_{m=0}^{\infty} (\alpha B_N^{-1})^m X \psi = (I_N - \alpha B_N^{-1})^{-1} X \psi,
\]

and by independence of \(\theta\), \{\varepsilon_t\} and \{\z_{it}\},

\[
\text{cov}(B_N \xi_t, B_N \xi_{t-h}) = E[B_N \xi_t - E(B_N \xi_t)] [B_N \xi_{t-h} - E(B_N \xi_{t-h})]
\]

\[
= E\left\{ \sum_{m=0}^{\infty} (\alpha B_N^{-1})^m (\theta + Z_{t-m} \gamma + \varepsilon_{t-m}) \left[ \sum_{k=0}^{\infty} (\alpha B_N^{-1})^k (\theta + Z_{t-h-k} \gamma + \varepsilon_{t-h-k}) \right] \right\}
\]

\[
= \sum_{m,k=0}^{\infty} (\alpha B_N^{-1})^m E(\theta \theta') \left[ (\alpha B_N^{-1})^k \right] + \sum_{m,k=0}^{\infty} (\alpha B_N^{-1})^m E(\varepsilon_{t-m} \varepsilon'_{t-h-k}) \left[ (\alpha B_N^{-1})^k \right]
\]

\[
+ \sum_{m,k=0}^{\infty} (\alpha B_N^{-1})^m E(Z_{t-m} \gamma \gamma' Z'_{t-h-k}) \left[ (\alpha B_N^{-1})^k \right],
\]

where \(E(B_N \xi_t)\) is a vector of constants and \(\text{cov}(B_N \xi_t, B_N \xi_{t-h})\) is independent of \(t\). Thus \{\xi_t\} is covariance stationary. Note that models (2.1) and (2.2) can be rewritten as

\[
B_N y_t = \theta + \alpha y_{t-1} + Z_t \gamma + \varepsilon_t.
\]

(S5.19)

It can be shown that \{\xi_t\} satisfies the recursive equation

\[
B_N \xi_t = \theta + \alpha \xi_{t-1} + Z_t \gamma + \varepsilon_t.
\]
Hence we prove the existence of a covariance stationary solution to models (2.1) and (2.2) by setting $y_t = \xi_t$.

Next suppose that $\{y_t\}$ is a covariance stationary solution to models (2.1) and (2.2). Then, for any integer $h > 0$, by successively substituting $y_{t-1}$’s in (S5.19) $h$ times, we have

$$B_N y_t = \sum_{m=0}^{h} (\alpha B_N^{-1})^m (\theta + Z_{t-m} \gamma + \varepsilon_{t-m}) + R_{t,h},$$

where $R_{t,h} = (\alpha B_N^{-1})^h \alpha y_{t-h-1} \rightarrow 0$ almost surely as $h \rightarrow \infty$, since $|\alpha B_N^{-1}| < 1$ by assumption. Thus $B_N y_t = B_N \xi_t$ and then $y_t = \xi_t$ almost surely. This completes the proof of Proposition 5.8.

### S5.8 Proof of Proposition 5.8

Recall that $Y = S_0^{-1}(\tilde{Z}\phi_0 + V_{NT})$ and $\tilde{Z} = (Y_{-1}, Z, (\nu_T \otimes I_N)X)$. Since $M_3 = M_2(\nu_T \otimes I_N)$, $Y_{-1}$ can be rewritten as $M_1 y_0 + M_2 Z \gamma_0 + M_3 X \psi_0 + M_2 V_{NT}$ by (S3.3). Then elements of the score vector $s(Y_{-1}, V_{NT}, \zeta_0) = \partial \ln L_{nt}(\zeta_0)/\partial \zeta$ can be expressed in terms of $\zeta_0$ and $V_{NT}$ as follows:

$$s(Y_{-1}, V_{NT}, \zeta_0) = (\dot{d}_\phi, \dot{d}_\sigma^2, \dot{d}_\lambda, \dot{d}_\beta^r)'$$

where

$$\begin{align*}
\dot{d}_\phi &= (\sigma_{\phi_0}^2)^{-1} \left[ (M_1 y_0 + M_2 Z \gamma_0 + M_3 X \psi_0)' \Omega_{0NT}^{-1} V_{NT} + V_{NT}' M_2' \Omega_{0NT}^{-1} V_{NT} \right], \\
\dot{d}_\sigma^2 &= (\sigma_{\sigma_0}^2)^{-1} Z' \Omega_{0NT}^{-1} V_{NT}, \\
\dot{d}_\psi &= (\sigma_{\psi_0}^2)^{-1} X'(\nu_T \otimes I_N)' \Omega_{0NT}^{-1} V_{NT}, \\
\dot{d}_\sigma^4 &= (2 \sigma_{\sigma_0}^4)^{-1} V_{NT}' \Omega_{0NT}^{-1} V_{NT} - (2 \sigma_{\sigma_0}^2)^{-1} NT, \\
\dot{d}_\lambda &= (\sigma_{\lambda_0}^2)^{-1} \left[ \{\alpha_0 M_1 y_0 + (\alpha_0 M_2 + I_N)Z \gamma_0 + (\alpha_0 M_3 + \nu_T \otimes I_N)X \psi_0 \} ight]' \\
&\quad \left\{ (S_0^{-1})'(I_T \otimes W_N)' \Omega_{0NT}^{-1} V_{NT} + V_{NT}' M_2 + I_N)'(S_0^{-1})'(I_T \otimes W_N)' \Omega_{0NT}^{-1} V_{NT} \right\} \\
&\quad - tr \left[ S_{0NT}' (I_T \otimes W_N) \right],
\end{align*}$$

\[ \]
\[ \hat{\beta}_t^* = (2\sigma_s^2)^{-1}V'_{NT}\Omega_{0,NT}^{-1} \frac{\partial \Omega_{NT}(\beta_0^*)}{\partial \beta^*_t} \Omega_{0,NT}^{-1} V_{NT} - \frac{1}{2} \text{tr} \left[ \Omega_{0,NT}^{-1} \frac{\partial \Omega_{NT}(\beta_0^*)}{\partial \beta^*_t} \right], \ell = 0, 1, \ldots, p. \]

Let \( r_{NT} = (r'_1, \ldots, r'_T)' \) with \( r_t = \bar{A}_{0N}^{-1/2}v_t \), then \( r_{NT} = (I_T \otimes \bar{A}_{0N}^{-1/2})V_{NT} \). It follows that elements of \( s(Y_{-1}, V_{NT}, \zeta_0) \) can be rewritten in terms of \( \zeta_0 \) and \( r_{NT} \), which contains three types of terms: constant \( C \), linear \( g'r_{NT} \) and quadratic \( r'_{NT}Qr_{NT} \). Hence the result in Proposition 5 follows if the following conditions hold:

(a) \[ \frac{1}{NT} \left[ \text{var}^*(g'\tilde{r}^b_{NT}) - \text{var}(g'r_{NT}) \right] = o_p(1); \]

(b) \[ \frac{1}{NT} \left[ \text{var}^*(r'^{b'}_{NT}Q\tilde{r}^b_{NT}) - \text{var}(r'_{NT}Qr_{NT}) \right] = o_p(1); \]

(c) \[ \frac{1}{NT} \left[ \text{cov}^*(g'\tilde{r}^b_{NT}, r'^{b'}_{NT}Q\tilde{r}^b_{NT}) - \text{cov}(g'r_{NT}, r'_{NT}Qr_{NT}) \right] = o_p(1), \]

where the operators \text{var}^* and \text{cov}^* denote variance and covariance corresponding to the bootstrap probability space, \( \tilde{g} \) and \( \tilde{Q} \) are estimates of \( g \) and \( Q \) by plugging in QMLE \( \hat{\zeta} \), respectively, and \( r^b_{NT} = (r'^b_1, \ldots, r'^b_T)' \). Note that matrices \( S_0^{-1}, I_T \otimes W_N, I_T \otimes \bar{A}_{0N}^{-1/2}, M_1, M_2, T\Omega_{0,NT}^{-1} \) and \( \frac{1}{T} \frac{\partial}{\partial \beta^*_t} \Omega_{NT}(\beta_0^*) \) are uniformly bounded in both row and column sums, and \( \nu_T \otimes I_N \) and \( M_3 \) are uniformly bounded in row sums. By Lemmas 1 and 2 together with the compactness of the parameter space, it can be shown that \( g_a = Tg = O_p(1) \), and \( Q_a = TQ \) is uniformly bounded in both row and column sums. Therefore, to show (a)–(c), it suffices to show that

(i) \[ \frac{1}{NT^3} \left[ \text{var}^*(\tilde{g}'_a\tilde{r}^b_{NT}) - \text{var}(g'_a r_{NT}) \right] = o_p(1); \]

(ii) \[ \frac{1}{NT^3} \left[ \text{var}^*(r'^{b'}_{NT}Q_a \tilde{r}^b_{NT}) - \text{var}(r'_{NT}Q_ar_{NT}) \right] = o_p(1); \]

(iii) \[ \frac{1}{NT^3} \left[ \text{cov}^*(\tilde{g}'_a\tilde{r}^b_{NT}, r'^{b'}_{NT}Q_a \tilde{r}^b_{NT}) - \text{cov}(g'_a r_{NT}, r'_{NT}Q_ar_{NT}) \right] = o_p(1), \]

where \( \tilde{g}_a = T\tilde{g} \) and \( \tilde{Q}_a = T\tilde{Q} \). It can be verified that (i)–(iii) hold by the proof of Proposition 5.1 in Su and Yang (2015). As a result, the proof of this proposition is accomplished.
References


Table S.1: Biases, ESDs and ASDs of the WCQE $\hat{\varphi}(\tau)$ at $\tau = 0.5$, when the innovations $\{\eta_i\}$ follow the standard normal or uniform distribution.

| $\tau = 0.5$ | $\varphi_0$ | $N(0, 1)$ |  |  |  |  |  |  |  |  |  |  |  |
|--------------|-------------|------------|--------------|------------|--------------|------------|--------------|------------|--------------|------------|--------------|------------|--------------|------------|
| $N$ | $T$ | Bias | ESD | ASD | Bias | ESD | ASD |
| 20 | 20 | 0.0164 | 0.2520 | 0.8681 | 0.0037 | 0.2654 | 1.0016 |
| 100 | 50 | 0.0038 | 0.1092 | 0.3529 | 0.0035 | 0.1334 | 0.4541 |
| 300 | 200 | 0.0045 | 0.0589 | 0.1851 | 0.0041 | 0.0750 | 0.2427 |
| 1500 | 300 | 0.0003 | 0.0272 | 0.0792 | 0.0010 | 0.0321 | 0.1068 |
| 3000 | 600 | 0.0001 | 0.0196 | 0.0557 | 0.0003 | 0.0243 | 0.0756 |
| $\varphi_1$ | 20 | 20 | 0.0143 | 0.3411 | 1.6275 | 0.0253 | 0.4375 | 1.8757 |
| 100 | 50 | 0.0104 | 0.1718 | 0.6852 | 0.0028 | 0.2384 | 0.8802 |
| 300 | 200 | 0.0052 | 0.1125 | 0.3844 | 0.0015 | 0.1558 | 0.5042 |
| 1500 | 300 | 0.0030 | 0.0505 | 0.1680 | 0.0014 | 0.0751 | 0.2268 |
| 3000 | 600 | 0.0004 | 0.0353 | 0.1172 | 0.0021 | 0.0501 | 0.1593 |

Table S.2: Biases, ESDs and ASDs of the WQAE $\hat{\beta}(\hat{\pi}_{opt, K})$, when the innovations $\{\eta_i\}$ follow the standard normal or uniform distribution.

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