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Center-Outward Ranks and Signs for Testing Conditional Quantile Independence

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Abstract: This article treats the problem of constructing nonparametric asymptotically distribution-free tests for conditional quantile independence in multidimensions. Our approach combines the quantile martingale difference divergence with a notion of the recently introduced multivariate center-outward ranks and signs. We derive the asymptotic null representation of the proposed test statistics by exploiting the degenerate V-type and U-type structures of the quantile martingale difference divergence and the Glivenko-Cantelli strong consistency and distribution-freeness of the center-outward ranks and signs. This representation permits direct calculation of limiting null distributions without requiring bootstrap calibration. We further show that our center-outward versions of the quantile martingale difference divergence tests are consistent against all fixed alternatives. A local power analysis provides strong support for the center-outward approach by establishing the nontrivial power of our center-outward rank-based tests over root-n neighborhoods. Moreover, the proposed tests are computationally feasible and well-defined without any moment assumptions. We illustrate the advantages of the proposed methods via extensive simulation studies and a gene expression dataset analysis.

Key words and phrases: center-outward ranks and signs, conditional quantile, Hájek representation, Le Cam's third lemma, quantile martingale difference divergence, V-type and U-type processes

1. Introduction

Inference for regression models is one of the most important issues in statistics. Since Koenker and Bassett (1978), quantile regression has attracted increasing attention in recent years, mainly due to the robustness against outliers in the response and the ability to capture heterogeneity in the set of important predictors. More references about quantile regression estimation and interpretation can be found in the seminal book by Koenker (2005). For a scalar response variable $Y \in \mathbb{R}$ and a set of predictors $\mathbf{x} \in \mathbb{R}^q$, we are interested in exploring whether $\mathbf{x} = (X_1, \ldots, X_q)^{\mathrm{T}}$ is useful in modeling a certain aspect of the quantiles of Y, before constructing a parametric/nonparametric quantile regression model. For theoretical development, a fixed dimension is assumed to derive the asymptotic results.

To test for significant predictors at the τth $(0 < \tau < 1)$ conditional quantile of Y, a natural methodology is to carry out a hypothesis test comparing the null model of no predictors and the full model consisting of all

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q predictors. Zheng (1998) first considered a kernel-based test for a general parametric quantile regression model. He and Zhu (2003) extended the approach in Stute (1997) and proposed a test based on a weighted cumulative sum process of the residuals. See also Horowitz and Spokoiny (2002), Whang (2006), Otsu (2008), Escanciano and Velasco (2010) and Escanciano and Goh (2014) for more lack-of-fit tests based on cumulative sum processes. Conde-Amboage et al. (2015) suggested projecting the covariates $\mathbf{x}$ into a random variable first, and then applying He and Zhu (2003)'s method to form a lack-of-fit test. Xu and Chen (2020) generalized the mean restriction test of Su and Zheng (2017) to the quantile restriction case. Wang et al. (2018) and Xu and An (2024) suggested marginal testing procedures based on fitting the working marginal quantile regression models by regressing Y on X_j, $j = 1, \ldots, q$, for each j separately. Their approach is in the spirit of McKeague and Qian (2015), who proposed an adaptive resampling test for detecting significant predictors based on marginal linear mean regression. Dong et al. (2019) novelly transformed lack-of-fit tests for parametric quantile regression models into checking the equality of two conditional distributions, and then employed Baringhaus and Franz (2004)'s method to construct a reliable test. Most of these tests are non-parametric, and they can detect the departures at all directions when the sample size tends to in-

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finity. Despite their usefulness, the null limit distributions, which involved in the aforementioned methodologies, depend on the unknown density of the error distribution and, therefore, are not Asymptotically Distribution-Free (ADF).

In this article, we study the problem of testing conditional quantile independence in multidimensions and develop nonparametric testing procedures that are ADF. For this, we first introduce a variant of the martingale difference divergence (Shao and Zhang, 2014; Park et al., 2015; Zhang et al., 2018; Lee and Shao, 2018; Lee et al., 2020), the so-called quantile martingale difference divergence (Lee and Hilafu, 2022). Our proposed tests are based on applying the quantile martingale difference divergence to center-outward ranks and signs (Chernozhukov et al., 2017; Hallin, 2017; Hallin et al., 2021; Shi et al., 2022). Among the existing literature, the most closely related papers to ours are the ones by Shao and Zhang (2014) and Lee and Hilafu (2022). For clarity we discuss the differences in the following four important aspects.

- Exploiting the degenerate V-type and U-type structures of the quantile martingale difference divergence and the Glivenko-Cantelli strong consistency and distribution-freeness of the center-outward ranks and signs, we derive the asymptotic null representation of the proposed

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test statistics. This representation permits direct calculation of limiting null distributions without requiring bootstrap calibration. In contrast, we need bootstrap to estimate the critical values for the original martingale difference divergence test statistic.

- The proposed testing procedures are computationally feasible and are well-defined without any moment assumptions. They are consistent against all fixed alternatives, that is, the probability of rejecting the null, calculated under the alternative, converges to one as the sample size increases. Unlike the original martingale difference divergence test, we can use our tests on generally distributed predictors and errors including the heavy-tailed ones.

- Combined with a nontrivial use of Le Cam's third lemma in a context of non-Gaussian limits, we conduct local power analyses of the proposed quantile martingale difference divergence tests. We provide strong support for the center-outward approach by establishing the nontrivial power of our center-outward rank-based tests over root-n neighborhoods. However, previous work does not perform any power analysis for the original martingale difference divergence test.

- Unlike the original martingale difference divergence test, our center-

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outward rank-based tests allow for the incorporation of score functions, which may improve their performance. In addition, our proposed method can be used to conduct joint tests across multiple quantiles.

The remainder of the paper is organized as follows. In Section 2, we give a brief review of the quantile martingale difference divergence and the center-outward ranks and signs. Our center-outward versions of the quantile martingale difference divergence tests are proposed in Section 3, after establishing the asymptotic null representation of the test statistics, followed by results on the consistency and the local power analysis of the tests. In Section 4, we consider extensions across multiple quantiles. In Section 5, we report some simulation studies and a real data example. We conclude the article with a short discussion in Section 6. Technical proofs are relegated to the Supplementary Material.

2. Preliminaries

2.1 Center-outward ranks and signs

Let $\mathbf{x}$ have an absolutely continuous probability measure on $\mathbb{R}^q$. Let $\mathbb{S}_q$ and $\mathcal{S}_{q-1}$ denote the open unit ball and the unit sphere in $\mathbb{R}^q$, respectively. Denote by W_q the spherical uniform measure on $\mathbb{S}_q$, that is, the product of the uniform measures on $[0, 1)$ (for the distance to the origin) and on

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$\mathcal{S}_{q-1}$ (for the direction). The center-outward distribution function $\mathbf{F}_\pm$ of $\mathbf{x}$ is defined as the almost surely unique gradient of convex function mapping $\mathbb{R}^q$ to $\mathbb{S}_q$ and pushing $\mathbf{x}$ forward to W_q, that is, $\mathbf{F}_\pm(\mathbf{x}) \sim W_q$. We refer the readers to Chernozhukov et al. (2017), Hallin (2017), Hallin et al. (2021) and Shi et al. (2022) for details about the existence and almost everywhere uniqueness of a center-outward distribution function. In dimension $q = 1$, $\mathbf{F}_\pm$ reduces to $2F - 1$ ~Uniform(-1, 1), where F is the usual cumulative distribution function.

The sample counterpart $\widehat{\mathbf{F}}_\pm$ of $\mathbf{F}_\pm$ is based on an n-tuple of data points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^q$. As in Hallin (2017), let n factorize into $n = n_R n_S + n_0$, for $n_R, n_S \in \mathbb{Z}_+$ and $0 \leq n_0 < \min(n_R, n_S)$, where $\mathbb{Z}_+$ is the set of positive integer numbers, $n_R \to \infty$ and $n_S \to \infty$ as $n \to \infty$. Consider a sequence $\mathfrak{G}_n$ of grids, where each grid consists of the intersection between an n_S-tuple $(\mathbf{u}_1, \ldots, \mathbf{u}_{n_S})$ of unit vectors, and the n_R hyperspheres with radii $1/(n_R + 1), \ldots, n_R/(n_R + 1)$ centered at the origin, along with n_0 copies of the origin. It is required that the sequence $\mathfrak{G}_n$ of grids is such that the discrete distribution with probability masses $1/n$ at each gridpoint and probability mass n_0/n at the origin converges weakly to the uniform W_q over the ball $\mathbb{S}_q$. Let $\mathcal{T}$ be the collection of all bijective mappings between $\{\mathbf{x}_i\}_{i=1}^n$ and $\mathfrak{G}_n$. The empirical center-outward distribution function is defined as

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$\widehat{\mathbf{F}}_{\pm} = \arg\min_{T\in\mathcal{T}} \sum_{i=1}^{n} \|\mathbf{x}_i - T(\mathbf{x}_i)\|^2$, where $\|\cdot\|$ stands for the Euclidean norm. The center-outward rank of $\mathbf{x}_i$ is defined as $(n_R+1)\|\widehat{\mathbf{F}}_{\pm}(\mathbf{x}_i)\|$, and the center-outward sign of $\mathbf{x}_i$ is defined as $\widehat{\mathbf{F}}_{\pm}(\mathbf{x}_i)/\|\widehat{\mathbf{F}}_{\pm}(\mathbf{x}_i)\|$ if $\|\widehat{\mathbf{F}}_{\pm}(\mathbf{x}_i)\| \neq 0$, and 0 otherwise.

2.2 Quantile martingale difference divergence

Shao and Zhang (2014), Park et al. (2015), Lee and Shao (2018) and Lee et al. (2020) advocated using the martingale difference divergence (MDD) and its standardized version to measure the conditional mean independence between two variables. The MDD of Y given $\mathbf{x}$ whose square is defined by

$$\begin{aligned} & \mathrm{MDD}^2(Y \mid \mathbf{x}) \\ = \; & c_q^{-1} \int_{\mathbb{R}^q} \mid E\{Y\exp(\imath \mathbf{s}^{\mathrm{T}}\mathbf{x})\} - E(Y)E\{\exp(\imath \mathbf{s}^{\mathrm{T}}\mathbf{x})\} \mid^2 \|\mathbf{s}\|^{-1-q} d\mathbf{s}, \end{aligned}$$

where $\imath = (-1)^{1/2}$, $c_q = \pi^{(1+q)/2}/\Gamma\{(1+q)/2\}$, and $\Gamma(\cdot)$ is the complete gamma function: $\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)dt$. The definition is an analogue of the popular distance covariance (Székely et al., 2007) and has many remarkable properties. For instance, $\mathrm{MDD}^2(Y \mid \mathbf{x}) = 0$ if and only if $\mathrm{pr}\{E(Y \mid \mathbf{x}) = E(Y)\} = 1$. If $E(\mid Y \mid^2 + \|\mathbf{x}\|^2) < \infty$, Theorem 1 in Shao and Zhang (2014) stated that

$$\mathrm{MDD}^2(Y \mid \mathbf{x}) = -E\left[\{Y_1 - E(Y_1)\}\{Y_2 - E(Y_2)\}\|\mathbf{x}_1 - \mathbf{x}_2\|\right], \qquad (2.1)$$

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where $(Y_1, \mathbf{x}_1)$ and $(Y_2, \mathbf{x}_2)$ are independent copies of $(Y, \mathbf{x})$.

Given that $(Y_i, \mathbf{x}_i), i = 1, \ldots, n$ is a random sample from the population $(Y, \mathbf{x})$, Shao and Zhang (2014) proposed the V-type sample martingale difference divergence defined as

$$\widehat{\text{MDD}}_1^2(Y \mid \mathbf{x}) \;=\; -n^{-2} \sum_{i_1, i_2 = 1}^{n} A_{1, i_1 i_2} B_{1, i_1 i_2}, \tag{2.2}$$

where

$$\begin{aligned} A_{1,i_1 i_2} &= a_{1,i_1 i_2} - n^{-1} \sum_{i_3=1}^{n} a_{1,i_1 i_3} - n^{-1} \sum_{i_4=1}^{n} a_{1,i_4 i_2} + n^{-2} \sum_{i_3,i_4=1}^{n} a_{1,i_3 i_4}, \\ \text{and } B_{1,i_1 i_2} &= b_{1,i_1 i_2} - n^{-1} \sum_{i_3=1}^{n} b_{1,i_1 i_3} - n^{-1} \sum_{i_4=1}^{n} b_{1,i_4 i_2} + n^{-2} \sum_{i_3,i_4=1}^{n} b_{1,i_3 i_4}, \end{aligned}$$

with $a_{1,i_1 i_2} = Y_{i_1} Y_{i_2}$ and $b_{1,i_1 i_2} = \|\mathbf{x}_{i_1} - \mathbf{x}_{i_2}\|$. In some statistical applications, one may prefer to use U-type statistic. Adopting the idea of the double centred distance (Székely and Rizzo, 2014; Park et al., 2015; Yao et al., 2018), Zhang et al. (2018) proposed the U-type sample martingale difference divergence defined as

$$\widehat{\text{MDD}}_2^2(Y \mid \mathbf{x}) = \{n(n-3)\}^{-1} \sum_{i_1 \neq i_2}^{n} A_{2,i_1 i_2} B_{2,i_1 i_2}, \tag{2.3}$$

where

$$\begin{aligned} A_{2,i_1 i_2} &= a_{2,i_1 i_2} - (n-2)^{-1} \sum_{i_3=1}^{n} a_{2,i_1 i_3} - (n-2)^{-1} \sum_{i_4=1}^{n} a_{2,i_4 i_2} \\ &\quad + \{(n-1)(n-2)\}^{-1} \sum_{i_3,i_4=1}^{n} a_{2,i_3 i_4}, \end{aligned}$$

$$\text{and } B_{2,i_1i_2} = b_{2,i_1i_2} - (n-2)^{-1}\sum_{i_3=1}^{n} b_{2,i_1i_3} - (n-2)^{-1}\sum_{i_4=1}^{n} b_{2,i_4i_2} + \{(n-1)(n-2)\}^{-1}\sum_{i_3,i_4=1}^{n} b_{2,i_3i_4},$$

with $a_{2,i_1i_2} = | Y_{i_1} - Y_{i_2} |^2 /2$ and $b_{2,i_1i_2} = b_{1,i_1i_2}$.

Using the MDD in (2.1), Lee and Hilafu (2022) stated a formal definition of the quantile martingale difference divergence (QMDD). For a continuous random variable $Y \in \mathbb{R}$, a random vector $\mathbf{x} \in \mathbb{R}^d$ and a quantile level $\tau \in (0,1)$, the τth QMDD is defined as

$$\text{QMDD}_\tau^2(Y \mid \mathbf{x}) = -E\left([I\{Y_1 \le Q_\tau(Y)\} - \tau][I\{Y_2 \le Q_\tau(Y)\} - \tau]\|\mathbf{x}_1 - \mathbf{x}_2\|\right),$$

where $Q_\tau(Y)$ is the unconditional τth quantile of Y. As the QMDD is a special case of the MDD, it inherits a number of the latter's desirable features. In particular, Lee and Hilafu (2022, Proposition 1) showed that $\text{QMDD}_\tau^2(Y \mid \mathbf{x}) = 0$ if and only if $\text{pr}\{Y \le Q_\tau(Y) \mid \mathbf{x}\} = \text{pr}\{Y \le Q_\tau(Y)\}$ almost surely, under the assumption that $E(\|\mathbf{x}\|^2) < \infty$. The quantile independence-zero equivalence property motivates us to use it in a distribution-free testing procedure. Inspired by the sample estimators of MDD^2 in (2.2) and (2.3), we construct the QMDD estimators as below:

$$\widehat{\text{QMDD}}_{1,\tau}^2(Y \mid \mathbf{x}) = -n^{-2}\sum_{i_1,i_2=1}^{n} \widehat{A}_{1,i_1i_2}B_{1,i_1i_2}, \tag{2.4}$$

$$\text{and } \widehat{\text{QMDD}}_{2,\tau}^2(Y \mid \mathbf{x}) = \{n(n-3)\}^{-1}\sum_{i_1\neq i_2}^{n} \widehat{A}_{2,i_1i_2}B_{2,i_1i_2}. \tag{2.5}$$

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Let $\widehat{Q}_\tau(Y)$ be the τth sample quantile of Y. Write $\varepsilon_{i,\tau} = Y_i - Q_\tau(Y)$, $\widehat{\varepsilon}_{i,\tau} = Y_i - \widehat{Q}_\tau(Y)$ and $\psi_\tau(\widehat{\varepsilon}_{i,\tau}) = \tau - I(\widehat{\varepsilon}_{i,\tau} \leq 0)$, where the symbol $I(\cdot)$ stands for the indicator function. The statistics $\widehat{A}_{1,i_1i_2}$ and $\widehat{A}_{2,i_1i_2}$ are the natural analogues of A_{1,i_1i_2} and A_{2,i_1i_2} that replace a_{1,i_1i_2} and a_{2,i_1i_2} by $\psi_\tau(\widehat{\varepsilon}_{i_1,\tau})\psi_\tau(\widehat{\varepsilon}_{i_2,\tau})$ and $\mid \psi_\tau(\widehat{\varepsilon}_{i_1,\tau}) - \psi_\tau(\widehat{\varepsilon}_{i_2,\tau}) \mid^2 /2$, respectively.

3. Method proposed

3.1 Score transformed test statistics

Let $Q_\tau(Y \mid \mathbf{x})$ denote the quantile of Y conditional on $\mathbf{x}$ at the τth quantile level, and assume that $\mathbf{x}$ is absolutely continuous with respect to the Lebesgue measure. Let $J : [0, 1) \to \mathbb{R}_+$ be the score function and $\mathbb{R}_+$ denote the set of nonnegative reals. Define the population scored center-outward distribution functions as $\mathbf{G}_\pm(\mathbf{x}) = J\{\|\mathbf{F}_\pm(\mathbf{x})\|\}\{\mathbf{F}_\pm(\mathbf{x})/\|\mathbf{F}_\pm(\mathbf{x})\|\}I(\|\mathbf{F}_\pm(\mathbf{x})\| \neq 0)$. Thanks to Shi et al. (2022, Proposition 4.2), the relation $\mathrm{pr}[\mathrm{pr}\{Y \leq Q_\tau(Y) \mid \mathbf{x}\} = \tau] = 1$ is equivalent to

$$\mathrm{pr}[\mathrm{pr}\{Y \leq Q_\tau(Y) \mid \mathbf{G}_\pm(\mathbf{x})\} = \tau] = 1, \tag{3.1}$$

provided that the score function J is strictly monotone. Classical examples include the normal or van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ with $F_{(\chi^2_q)^{1/2}}$ the $(\chi^2_q)^{1/2}$ distribution function, the Wilcoxon score function

$J(x) = x$ and the sign test score function $J(x) = 1$. Following Shao and Zhang (2014, Assumption B1) and Zhang et al. (2018, Assumption 3.2), we assume throughout this paper that in a small neighborhood of $Q_\tau = Q_\tau(Y)$, the cumulative distribution function of Y is continuously differentiable, and $(Y \le Q_\tau)$ is independent of $\mathbf{x}$ under the conditional quantile independence.

Define the sample scored center-outward distribution functions as

$$\widehat{\mathbf{G}}_\pm(\mathbf{x}) = J\{\|\widehat{\mathbf{F}}_\pm(\mathbf{x})\|\}\{\widehat{\mathbf{F}}_\pm(\mathbf{x})/\|\widehat{\mathbf{F}}_\pm(\mathbf{x})\|\}I(\|\widehat{\mathbf{F}}_\pm(\mathbf{x})\| \neq 0).$$

Let $\widehat{B}_{1,i_1 i_2}$ denote the analog of $B_{1,i_1 i_2}$ except for replacing $b_{1,i_1 i_2}$ by $\|\widehat{\mathbf{G}}_\pm(\mathbf{x}_{i_1}) - \widehat{\mathbf{G}}_\pm(\mathbf{x}_{i_2})\|$. Analogously, construct $\widehat{B}_{2,i_1 i_2}$ by replacing $b_{2,i_1 i_2}$ by $\|\widehat{\mathbf{G}}_\pm(\mathbf{x}_{i_1}) - \widehat{\mathbf{G}}_\pm(\mathbf{x}_{i_2})\|$. Motivated by (3.1) and using the QMDD in (2.4) and (2.5), we consider the V-type and U-type quantile MDD statistics defined as

$$\widehat{V}_\tau = -n^{-2} \sum_{i_1, i_2 = 1}^{n} \widehat{A}_{1,i_1 i_2} \widehat{B}_{1,i_1 i_2}, \tag{3.2}$$

$$\text{and } \widehat{U}_\tau = \{n(n-3)\}^{-1} \sum_{i_1 \neq i_2}^{n} \widehat{A}_{2,i_1 i_2} \widehat{B}_{2,i_1 i_2}, \tag{3.3}$$

respectively.

3.2 Asymptotic null representation

In order to develop our multivariate asymptotic null representation, we first introduce formally the oracle counterparts to the rank-based quantile MDD statistics in (3.2) and (3.3). The oracle versions of $\widehat{V}_\tau$ and $\widehat{U}_\tau$ are

$$\widehat{V}_\tau^\natural = -n^{-2} \sum_{i_1,i_2=1}^{n} \widehat{A}_{1,i_1 i_2}^\natural \widehat{B}_{1,i_1 i_2}^\natural,$$

$$\text{and } \widehat{U}_\tau^\natural = \{n(n-3)\}^{-1} \sum_{i_1 \neq i_2}^{n} \widehat{A}_{2,i_1 i_2}^\natural \widehat{B}_{2,i_1 i_2}^\natural,$$

which are obtained by replacing the empirical $\widehat{Q}_\tau(Y)$ and $\widehat{\mathbf{G}}_\pm(\cdot)$ in (3.2) and (3.3) with their population analogs. The oracle $\widehat{V}_\tau^\natural$ and $\widehat{U}_\tau^\natural$ cannot be computed from the observations as they involve the population unconditional quantile $Q_\tau(Y)$ and the population scored center-outward distribution function $\mathbf{G}_\pm(\cdot)$. However, the limiting null distributions of $\widehat{V}_\tau^\natural$ and $\widehat{U}_\tau^\natural$, unlike those of $\widehat{V}_\tau$ and $\widehat{U}_\tau$, follow from standard theory for degenerate statistics of V-type and U-type (Serfling, 1980, Chapters 5 and 6).

For $\tau \in (0,1)$, take

$$\begin{aligned} h[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^4] &= 8^{-1}\{\mid \psi_\tau(\varepsilon_{1,\tau}) - \psi_\tau(\varepsilon_{3,\tau}) \mid^2 + \mid \psi_\tau(\varepsilon_{2,\tau}) - \psi_\tau(\varepsilon_{4,\tau}) \mid^2 \\ &\quad - \mid \psi_\tau(\varepsilon_{1,\tau}) - \psi_\tau(\varepsilon_{4,\tau}) \mid^2 - \mid \psi_\tau(\varepsilon_{2,\tau}) - \psi_\tau(\varepsilon_{3,\tau}) \mid^2\} \\ &\quad \times\{\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_3)\| + \|\mathbf{G}_\pm(\mathbf{x}_2) - \mathbf{G}_\pm(\mathbf{x}_4)\| \\ &\quad -\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_4)\| - \|\mathbf{G}_\pm(\mathbf{x}_2) - \mathbf{G}_\pm(\mathbf{x}_3)\|\}. \end{aligned}$$

We symmetrize h by

$$\begin{aligned} \widetilde{h}[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^4] &= 3^{-1}(h[\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}, \{\psi_\tau(\varepsilon_{2,\tau}), \mathbf{G}_\pm(\mathbf{x}_2)\}, \\ &\quad \{\psi_\tau(\varepsilon_{3,\tau}), \mathbf{G}_\pm(\mathbf{x}_3)\}, \{\psi_\tau(\varepsilon_{4,\tau}), \mathbf{G}_\pm(\mathbf{x}_4)\}] \end{aligned}$$

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$$+h[\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}, \{\psi_\tau(\varepsilon_{3,\tau}), \mathbf{G}_\pm(\mathbf{x}_3)\},$$
$$\{\psi_\tau(\varepsilon_{2,\tau}), \mathbf{G}_\pm(\mathbf{x}_2)\}, \{\psi_\tau(\varepsilon_{4,\tau}), \mathbf{G}_\pm(\mathbf{x}_4)\}]$$
$$+h[\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}, \{\psi_\tau(\varepsilon_{4,\tau}), \mathbf{G}_\pm(\mathbf{x}_4)\},$$
$$\{\psi_\tau(\varepsilon_{2,\tau}), \mathbf{G}_\pm(\mathbf{x}_2)\}, \{\psi_\tau(\varepsilon_{3,\tau}), \mathbf{G}_\pm(\mathbf{x}_3)\}]).$$

Let $\widetilde{h}_c[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^c] = E[\widetilde{h}[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^4] \mid \{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^c]$ be projections of $\widetilde{h}$ to lower-dimensional sample spaces, for $c = 1, \ldots, 4$. Further denote the double centred version of $\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_2)\|$ as

$$\begin{aligned} & D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} \\ = \; & \|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_2)\| - E\{\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_2)\| \mid \mathbf{G}_\pm(\mathbf{x}_1)\} \\ & - E\{\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_2)\| \mid \mathbf{G}_\pm(\mathbf{x}_2)\} + E\{\|\mathbf{G}_\pm(\mathbf{x}_1) - \mathbf{G}_\pm(\mathbf{x}_2)\|\}. \end{aligned}$$

The useful property of the double centred distance, that is, $E[D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} \mid \mathbf{x}_1] = E[D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} \mid \mathbf{x}_2] = 0$ is helpful to obtain the asymptotic representation of $\widehat{V}_\tau^\natural$ and $\widehat{U}_\tau^\natural$ under conditional quantile independence. This point can be summarized as follows.

Proposition 1. Let the score function J satisfy $\int_0^1 J^2(x)dx < \infty$ and $\mathrm{var}[D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\}] > 0$. Under the null hypothesis H_0 that $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ are independent, as $n \to \infty$, $n\ \widehat{V}_\tau^\natural$ and $n\ \widehat{U}_\tau^\natural$ converge in distribution to $\tau(1-\tau)\sum_{k=1}^{\infty}\lambda_k N_k^2$ and $\tau(1-\tau)\sum_{k=1}^{\infty}\lambda_k(N_k^2-1)$, respectively, where $N_1, N_2, \ldots$ are independent standard normal random variables and $\lambda_{k,\tau} = 6^{-1}\tau(1-\tau)\lambda_k > 0, k \geq 1$ are the associated eigenvalues corresponding to the symmetric kernel $\widetilde{h}_2$ admitting a spectral decomposition

$$\begin{aligned}\widetilde{h}_2[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^2] &= -6^{-1}\psi_\tau(\varepsilon_{1,\tau})\psi_\tau(\varepsilon_{2,\tau})D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} \\ &= \sum_{k=1}^{\infty} \lambda_{k,\tau}\phi_{k,\tau}\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}\phi_{k,\tau}\{\psi_\tau(\varepsilon_{2,\tau}), \mathbf{G}_\pm(\mathbf{x}_2)\}.\end{aligned}$$

Moreover, $\phi_{k,\tau}\{\psi_\tau(\varepsilon_\tau), \mathbf{G}_\pm(\mathbf{x})\} = \{\tau(1-\tau)\}^{-1/2}\psi_\tau(\varepsilon_\tau)\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}$, $E[\phi_{k_1}\{\mathbf{G}_\pm(\mathbf{x})\}\phi_{k_2}\{\mathbf{G}_\pm(\mathbf{x})\}] = I(k_1 = k_2)$, and $\{\lambda_k\}_{k\geq 1}$ and $\{\phi_k\}_{k\geq 1}$ are the eigenvalues and eigenfunctions, defined in relation to the symmetric kernel D admitting a spectral decomposition

$$D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} = -\sum_{k=1}^{\infty} \lambda_k\phi_k\{\mathbf{G}_\pm(\mathbf{x}_1)\}\phi_k\{\mathbf{G}_\pm(\mathbf{x}_2)\}.$$

Thanks to the distribution-freeness of $\mathbf{G}_\pm$, the distribution of $D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\}$ is exactly the same as that of $D\{J(\|\mathbf{w}_{1,q}\|)(\mathbf{w}_{1,q}/\|\mathbf{w}_{1,q}\|), J(\|\mathbf{w}_{2,q}\|)(\mathbf{w}_{2,q}/\|\mathbf{w}_{2,q}\|)\}$, where $\mathbf{w}_{1,q}$ and $\mathbf{w}_{2,q}$ are independent and distributed according to the spherical uniform measure W_q, that is, the product of the uniform measures on $[0,1)$ and on $\mathcal{S}_{q-1}$. The random variable $\psi_\tau(\varepsilon_\tau)$ has two point distribution with the probabilities $\text{pr}\{\psi_\tau(\varepsilon_\tau) = \tau\} = 1-\tau$ and $\text{pr}\{\psi_\tau(\varepsilon_\tau) = \tau - 1\} = \tau$. The conditional quantile independence of $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ implies the independence of $\psi_\tau(\varepsilon_\tau)$ and $[\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}]_{k\geq 1}$. Given any fixed dimension q, both $\tau(1-\tau)\sum_{k=1}^{\infty}\lambda_k N_k^2$ and $\tau(1-\tau)\sum_{k=1}^{\infty}\lambda_k(N_k^2-1)$

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are therefore independent of the underlying distributions.

Remark 1. Typical examples that satisfy $\int_0^1 J^2(x)dx < \infty$ include the van der Waerden, Wilcoxon and sign test score functions, where J is equal to $F^{-1}_{(\chi^2_q)^{1/2}}(x)$, x and 1, respectively. According to Theorem 7 from Székely and Rizzo (2009), $\mathrm{var}[D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\}]$ can be expressed as the squared population distance covariance between $\mathbf{G}_\pm(\mathbf{x})$ and $\mathbf{G}_\pm(\mathbf{x})$. Therefore, $\mathrm{var}[D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\}] > 0$ basically assumes that $\mathbf{G}_\pm(\mathbf{x})$ is not a constant.

Remark 2. For any prespecified significance level α, let $z_{1,1-\alpha}$ and $z_{2,1-\alpha}$ be the upper-α quantiles of $\sum_{k=1}^{\infty} \lambda_k N_k^2$ and $\sum_{k=1}^{\infty} \lambda_k (N_k^2 - 1)$, respectively. The values of λ_k's, and hence also the critical values of $z_{1,1-\alpha}$ and $z_{2,1-\alpha}$ themselves, are distribution-free and only depend on the dimension q and the score function J. The critical values may thus be calculated using numerical methods for each pair of q and J.

Remark 3. In the univariate case of $q = 1$ and when $J(x) = x$, it can be easily checked that

$$\begin{aligned} & \widetilde{h}_2[\{\psi_\tau(\varepsilon_{i,\tau}), \mathbf{G}_\pm(\mathbf{x}_i)\}_{i=1}^2] = \widetilde{h}_2[\{\psi_\tau(\varepsilon_{i,\tau}), 2F(X_i) - 1\}_{i=1}^2] \\ = \ & 4\psi_\tau(\varepsilon_{1,\tau})\psi_\tau(\varepsilon_{2,\tau})[2^{-1}F^2(X_1) + 2^{-1}F^2(X_2) - \max\{F(X_1), F(X_2)\} + 1/3]. \end{aligned}$$

We obtain from van der Vaart (1998, Example 12.13) that the kernel func-

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tion $2^{-1}F^2(X_1) + 2^{-1}F^2(X_2) - \max\{F(X_1), F(X_2)\} + 1/3$ has eigenvalues $\pi^{-2}k^{-2}$ with corresponding eigenfunctions $2^{1/2}\cos(\pi k X)$ for $k \geq 1$. Since $\psi_\tau(\varepsilon_\tau)$ and X are independent and $E\{\psi_\tau^2(\varepsilon_\tau)\} = \tau(1-\tau)$, we find that the nonzero eigenvalues of the kernel function $\widetilde{h}_2[\{\psi_\tau(\varepsilon_{i,\tau}), 2F(X_i) - 1\}_{i=1}^2]$ are $4\tau(1-\tau)\pi^{-2}k^{-2}$ for $k \geq 1$, with eigenfunctions $2^{1/2}\cos(\pi k X)\psi_\tau(\varepsilon_\tau)/\{\tau(1-\tau)\}^{1/2}$. Then as $n \to \infty$, $n\,\widehat{V}_\tau^\natural$ and $n\,\widehat{U}_\tau^\natural$ converge in distribution to $4\tau(1-\tau)\pi^{-2}\sum_{k=1}^{\infty} k^{-2}N_k^2$ and $4\tau(1-\tau)\pi^{-2}\sum_{k=1}^{\infty} k^{-2}(N_k^2 - 1)$, respectively.

Intuitively, the asymptotic null behavior of $\widehat{V}_\tau$ and $\widehat{U}_\tau$ follows from that of their Hájek asymptotic representations, which are oracle versions in Proposition 1. In the next, we show the correctness of this intuition by proving asymptotic equivalence between rank-based quantile MDD statistics and their oracle versions.

Theorem 1. Assume that the conditions in Proposition 1 hold. Then, under the conditional quantile independence of $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$, the random vector $(n\,\widehat{V}_\tau, n\,\widehat{U}_\tau)$ has the same limit distribution as the vector $(n\,\widehat{V}_\tau^\natural, n\,\widehat{U}_\tau^\natural)$ and thus each of $n\,\widehat{V}_\tau$ and $n\,\widehat{U}_\tau$ does enjoy distribution-freeness.

Remark 4. For $q = 1$, we take the nonzero eigenvalues $\lambda_{k,\tau} = 4\tau(1-\tau)\pi^{-2}k^{-2}, k \geq 1$. We are not aware of any closed form formulas for the eigenvalues when $q \geq 2$, but the asymptotic null distributions of $\widehat{V}_\tau$ and $\widehat{U}_\tau$ do not depend on the joint distribution of $(Y, \mathbf{x})$. This inspires us to randomly generate new samples from uniform distribution to approximate the asymptotic null distributions. To be precise, we generate $(Y_1^\ddagger, \mathbf{x}_1^\ddagger), \ldots, (Y_n^\ddagger, \mathbf{x}_n^\ddagger)$ independently from uniform distribution, and re-estimate $\widehat{V}_\tau$ and $\widehat{U}_\tau$ based on $\{(Y_i^\ddagger, \mathbf{x}_i^\ddagger)\}_{i=1}^n$. We repeat this procedure for B times and set the simulation-based critical values to be the upper α quantile of the estimates of $\widehat{V}_\tau$ and $\widehat{U}_\tau$ obtained from the randomly generated samples. The Dvoretzky-Kiefer-Wolfowitz inequality guarantees the asymptotic control of sizes using the simulation-based thresholds for a sufficiently large B, say, $B = 1000$.

For any significance level $\alpha \in (0,1)$, let $z_{1,1-\alpha}$ and $z_{2,1-\alpha}$ be as in Remark 2 and define the tests

$$\widehat{T}_{1,\tau,\alpha} = I\{n\,\widehat{V}_\tau > \tau(1-\tau)z_{1,1-\alpha}\}, \quad \text{and} \quad \widehat{T}_{2,\tau,\alpha} = I\{n\,\widehat{U}_\tau > \tau(1-\tau)z_{2,1-\alpha}\}.$$

By Theorem 1, in conjunction with Proposition 1 and Slutsky's theorem, the following proposition summarizes the asymptotic type I error control of the rank-based quantile MDD tests.

Corollary 1. Suppose the conditions in Theorem 1. Then, $\lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{1,\tau,\alpha} = 1) = \alpha$ and $\lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{2,\tau,\alpha} = 1) = \alpha$ under the conditional quantile independence of $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$.

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3.3 Consistency and local asymptotic power

To begin with, we discuss the consistency of the rank-based quantile MDD tests $\widehat{T}_{1,\tau,\alpha}$ and $\widehat{T}_{2,\tau,\alpha}$. For this, we need the projection of $\widetilde{h}$ to $(q+1)$-dimensional sample spaces. When $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ are not independent, it has the expression $\widetilde{h}_1\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\} - 2^{-1}\mathrm{MDD}\{\psi_\tau(\varepsilon_\tau) \mid \mathbf{G}_\pm(\mathbf{x})\} = -2^{-1}E[\psi_\tau(\varepsilon_{1,\tau})\psi_\tau(\varepsilon_{2,\tau})D\{\mathbf{G}_\pm(\mathbf{x}_1), \mathbf{G}_\pm(\mathbf{x}_2)\} \mid \psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)]$. The following theorem shows that each of $\widehat{T}_{1,\tau,\alpha}$ and $\widehat{T}_{2,\tau,\alpha}$ yields a universally consistent test for the conditional quantile independence testing problem.

Theorem 2. Let the score function J satisfy $\int_0^1 J^2(x)dx < \infty$ and $\mathrm{var}[\widetilde{h}_1\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}] > 0$. Under the fixed alternative H_1 that $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ are dependent, as $n \to \infty$, both $n^{1/2}[\widehat{V}_\tau - \mathrm{MDD}\{\psi_\tau(\varepsilon_\tau) \mid \mathbf{G}_\pm(\mathbf{x})\}]/4$ and $n^{1/2}[\widehat{U}_\tau - \mathrm{MDD}\{\psi_\tau(\varepsilon_\tau) \mid \mathbf{G}_\pm(\mathbf{x})\}]/4$ converge in distribution to a normal distribution with mean zero and variance $\mathrm{var}[\widetilde{h}_1\{\psi_\tau(\varepsilon_{1,\tau}), \mathbf{G}_\pm(\mathbf{x}_1)\}]$. In particular, under any fixed alternative, both $n\,\widehat{V}_\tau$ and $n\widehat{U}_\tau$ converge in probability to ∞ if $n \to \infty$ and thus $\lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{1,\tau,\alpha} = 1) = \lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{2,\tau,\alpha} = 1) = 1$.

Having established distribution-freeness and consistency of $\widehat{T}_{1,\tau,\alpha}$ and $\widehat{T}_{2,\tau,\alpha}$ previously, we now shift our attention to their local powers against contiguous alternatives. To quantify the notion of local alternatives, we adopt the standard smooth parametric model assumptions from the theory of local asymptotic normality (van der Vaart, 1998, Chapter 7).

For a given quantile level $\tau \in (0,1)$, consider a parametric model $\{P_\tau(\cdot \mid \theta), \theta \in \Theta\}$, where, throughout this section, Θ is assumed to be an open subset of $\mathbb{R}$. Each $P_\tau(\cdot \mid \theta)$ is absolutely continuous with respect to a σ-finite measure μ and set $p_\tau(\cdot \mid \theta) = dP_\tau(\cdot \mid \theta)/d\mu(\cdot)$. Following the paper of Shi et al. (2022), we assume the following standard regularity conditions on this parametric family.

- Conditional quantile dependence of $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$: $p_\tau(Y, \mathbf{x} \mid \theta) = [(1-\tau)I\{\psi_\tau(\varepsilon_\tau) = \tau\} + \tau I\{\psi_\tau(\varepsilon_\tau) = \tau - 1\}]p(\mathbf{x})$ holds if and only if $\theta = 0$, where $p(\mathbf{x})$ is the density function of $\mathbf{x}$.

- The family $\{P_\tau(\cdot \mid \theta), \theta \in \Theta\}$ is quadratic mean differentiable (QMD) at $\theta = 0$ with score function $\eta_\tau(Y, \mathbf{x} \mid \theta) = d\log\{p_\tau(Y, \mathbf{x} \mid \theta)\}/d\theta$, see Lehmann and Romano (2005, Definition 12.2.1) for related definitions. That is, $\int_{\mathbb{R}^{1+q}}\{p_\tau^{1/2}(Y, \mathbf{x} \mid \theta) - p_\tau^{1/2}(Y, \mathbf{x} \mid 0) - 2^{-1}\theta\eta_\tau(Y, \mathbf{x} \mid 0)p_\tau^{1/2}(Y, \mathbf{x} \mid 0)\}^2 dY\, d\mathbf{x} = o(\theta^2)$ as $\theta \to 0$.

- The Fisher information exists at $\theta = 0$, that is, $\mathcal{I}_\tau(0) = E\{\eta_\tau^2(Y, \mathbf{x} \mid 0)\} > 0$. Moreover, the score function $\eta_\tau(Y, \mathbf{x} \mid 0)$ is not additively separable such that $\mathrm{cov}[\psi_\tau(\varepsilon_\tau)\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}\eta_\tau(Y, \mathbf{x} \mid 0)] = E[\psi_\tau(\varepsilon_\tau)\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}\eta_\tau(Y, \mathbf{x} \mid 0)] \neq 0$ for some $k \geq 0$.

For a local power analysis, we consider a sequence of local alternatives obtained as $\theta = n^{-1/2}\theta_0$ with some constant $\theta_0 \neq 0$. In this local model, testing the null hypothesis of conditional quantile independence reduces to testing whether θ_0 is zero.

Theorem 3. Let the score function J satisfy the conditions of Proposition 1. Under the local alternative H_{1n} from the model $\{p_\tau(Y, \mathbf{x} \mid \theta_0 n^{-1/2})\}_{\theta_0 \neq 0}$, as $n \to \infty$, $n\,\widehat{V}_\tau$ and $n\,\widehat{U}_\tau$ converge in distribution to $6\sum_{k=1}^{\infty} \lambda_{k,\tau}(N_k + \theta_0\{\tau(1-\tau)\}^{-1/2}\mathrm{cov}[\psi_\tau(\varepsilon_\tau)\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}, \eta_\tau(Y, \mathbf{x} \mid 0)])^2$ and $6\sum_{k=1}^{\infty} \lambda_{k,\tau}\{(N_k + \theta_0\{\tau(1-\tau)\}^{-1/2}\mathrm{cov}[\psi_\tau(\varepsilon_\tau)\phi_k\{\mathbf{G}_\pm(\mathbf{x})\}, \eta_\tau(Y, \mathbf{x} \mid 0)])^2 - 1\}$, respectively. In particular, under H_{1n}, for any $\beta > 0$, there exists a constant C_β depending only on β such that, as long as $\mid \theta_0 \mid > C_\beta$, $\lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{1,\tau,\alpha} = 1) \geq 1 - \beta$ and $\lim_{n\to\infty} \mathrm{pr}(\widehat{T}_{2,\tau,\alpha} = 1) \geq 1 - \beta$.

The results in Theorem 3 show that the rank-based quantile MDD tests $\widehat{T}_{1,\tau,\alpha}$ and $\widehat{T}_{2,\tau,\alpha}$ have non-trivial asymptotic powers against $O(n^{-1/2})$ alternatives, in addition to being consistent against all fixed alternatives and asymptotically distribution-free for large sample sizes.

4. Extension to Allow for Multiple Quantiles

One attractive feature of the proposed methodology is that it enables us to assess the relationship between Y and $\mathbf{x}$ at multiple quantiles. We now discuss testing across multiple quantiles through rank-based quantile MDD.

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Let $\mathfrak{T} = \{\tau_1, \ldots, \tau_L\}$ be a set of quantile levels of interest, where L is finite. Testing that for each $\tau \in \mathfrak{T}$, $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ are independent is equivalent to testing whether each of $\mathrm{MDD}\{\psi_{\tau_l}(\varepsilon_{\tau_l}) \mid \mathbf{G}_\pm(\mathbf{x})\}, 1 \le l \le L$ is zero. To test whether $\mathbf{x}$ has an effect on either of the L-quantiles, we can consider testing $\widetilde{H}_0 : \mathrm{MDD}\{\psi_{\tau_1}(\varepsilon_{\tau_1}) \mid \mathbf{G}_\pm(\mathbf{x})\} = \cdots = \mathrm{MDD}\{\psi_{\tau_L}(\varepsilon_{\tau_L}) \mid \mathbf{G}_\pm(\mathbf{x})\} = 0$ versus $\widetilde{H}_1$: at least one of the $\mathrm{MDD}\{\psi_{\tau_l}(\varepsilon_{\tau_l}) \mid \mathbf{G}_\pm(\mathbf{x})\}$s is non-zero.

To pool information across quantiles, we propose to consider the sum-type test statistics $\widehat{S}_{1,\mathfrak{T}} = \sum_{l=1}^{L} \widehat{V}_{\tau_l}$ and $\widehat{S}_{2,\mathfrak{T}} = \sum_{l=1}^{L} \widehat{U}_{\tau_l}$, where $\widehat{V}_\tau$ and $\widehat{U}_\tau$ are defined in equations (3.2) and (3.3). One may also consider the maximum-type test statistics $\widehat{M}_{1,\mathfrak{T}} = \max_{1\le l\le L} \widehat{V}_{\tau_l}$ and $\widehat{M}_{2,\mathfrak{T}} = \max_{1\le l\le L} \widehat{U}_{\tau_l}$ to combine information across quantiles. We choose the sum-type test statistics as they have good powers against dense alternatives, and $\mathrm{MDD}\{\psi_\tau(\varepsilon_\tau) \mid \mathbf{G}_\pm(\mathbf{x})\}$ is more likely to be non-zero in an interval of τ and thus dense when the alternative is true.

By the joint null convergence result in the next theorem over $\tau \in \mathfrak{T}$, we can extend the distribution-freeness for the test calibration of $\widehat{V}_\tau$ and $\widehat{U}_\tau$ across multiple quantiles.

Theorem 4. Assume that the conditions in Proposition 1 hold. Under the null hypothesis $\widetilde{H}_0$ that for each $\tau \in \mathfrak{T}$, $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ are independent, as $n \to \infty$, the statistics $(n\,\widehat{V}_\tau)_{\tau\in\mathfrak{T}}$ and $(n\,\widehat{U}_\tau)_{\tau\in\mathfrak{T}}$ converge in distribution to

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$\{\tau_l(1-\tau_l)\sum_{k=1}^{\infty}\lambda_k N_{kl}^2\}_{l=1}^L$ and $\{\tau_l(1-\tau_l)\sum_{k=1}^{\infty}\lambda_k(N_{kl}^2-1)\}_{l=1}^L$, respectively, where the λ_k are defined as in Proposition 1, and $(N_{1l})_{l=1}^L, (N_{2l})_{l=1}^L, \ldots$ are mutually independent and identically distributed, each with the multivariate normal distribution with mean $\mathbf{0} \in \mathbb{R}^L$ and covariance matrix $[\{\tau_{l_1}\tau_{l_2}(1-\tau_{l_1})(1-\tau_{l_2})\}^{-1/2}\{\min(\tau_{l_1},\tau_{l_2})-\tau_{l_1}\tau_{l_2}\}]_{l_1,l_2=1}^L \in \mathbb{R}^{L\times L}$.

Theorem 4 gives the joint asymptotic null representation of our rank-based quantile MDD statistics across multiple quantiles. By an application of the continuous-mapping theorem, the random vector $(n\,\widehat{S}_{1,\mathfrak{T}}, n\,\widehat{S}_{2,\mathfrak{T}}, n\,\widehat{M}_{1,\mathfrak{T}}, n\,\widehat{M}_{2,\mathfrak{T}})$ converges in distribution to the vector $\{\sum_{l=1}^L\sum_{k=1}^{\infty}\tau_l(1-\tau_l)\lambda_k N_{kl}^2, \sum_{l=1}^L\sum_{k=1}^{\infty}\tau_l(1-\tau_l)\lambda_k(N_{kl}^2-1), \max_{l=1}^L\sum_{k=1}^{\infty}\tau_l(1-\tau_l)\lambda_k N_{kl}^2, \max_{l=1}^L\sum_{k=1}^{\infty}\tau_l(1-\tau_l)\lambda_k(N_{kl}^2-1)\}$. Moreover, $\lambda_1, \lambda_2, \ldots$ are free of the data generating distributions.

Remark 5. As the joint null convergence result in Theorem 4 over $\tau \in \mathfrak{T}$ is asymptotically distribution-free, this provides a computationally efficient way to compute critical values of the joint tests. Following Remark 4, we generate $(Y_1^{\ddagger}, \mathbf{x}_1^{\ddagger}), \ldots, (Y_n^{\ddagger}, \mathbf{x}_n^{\ddagger})$ independently from uniform distribution, and re-estimate $(\widehat{V}_\tau)_{\tau\in\mathfrak{T}}$ and $(\widehat{U}_\tau)_{\tau\in\mathfrak{T}}$ based on $\{(Y_i^{\ddagger}, \mathbf{x}_i^{\ddagger})\}_{i=1}^n$. We repeat the above procedure for B times and the corresponding simulation-based critical values are given by the upper α quantile of the estimates of $\widehat{S}_{1,\mathfrak{T}}$, $\widehat{S}_{2,\mathfrak{T}}$, $\widehat{M}_{1,\mathfrak{T}}$ and $\widehat{M}_{2,\mathfrak{T}}$ obtained from the randomly generated samples. The

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Dvoretzky-Kiefer-Wolfowitz inequality ensures that this simulation-based procedure approximates the joint null distributions asymptotically, as long as B, say, $B = 1000$ is sufficiently large.

When the dependence between $Q_\tau(Y \mid \mathbf{x})$ and $\mathbf{x}$ holds at a single quantile level $\tau \in \mathfrak{T}$, it follows from Theorem 1 that $n\ \widehat{S}_{1,\mathfrak{T}} = n\ \widehat{V}_\tau + O_p(1)$, $n\ \widehat{S}_{2,\mathfrak{T}} = n\ \widehat{U}_\tau + O_p(1)$, $n\ \widehat{M}_{1,\mathfrak{T}} = \max\{n\ \widehat{V}_\tau, O_p(1)\}$ and $n\ \widehat{M}_{2,\mathfrak{T}} = \max\{n\ \widehat{U}_\tau, O_p(1)\}$. This, together with Theorems 2 and 3, indicates that each of $n\ \widehat{S}_{1,\mathfrak{T}}$, $n\ \widehat{S}_{2,\mathfrak{T}}$, $n\ \widehat{M}_{1,\mathfrak{T}}$ and $n\ \widehat{M}_{2,\mathfrak{T}}$ converges in probability to infinity, under the fixed alternative or the local alternative, which decays at an order slower than $n^{-1/2}$. Therefore, the joint tests across multiple quantiles are consistent and can detect alternatives that tend to the null at the parametric rate $n^{-1/2}$.

5. Numerical Examples

5.1 Monte Carlo simulations

We conduct Monte Carlo simulations to assess the finite sample performance of the proposed rank-based quantile martingale difference divergence tests, which are asymptotically distribution-free and constructed based on $\widehat{V}_\tau$, $\widehat{U}_\tau$, $\widehat{S}_{1,\mathfrak{T}}$, $\widehat{S}_{2,\mathfrak{T}}$, $\widehat{M}_{1,\mathfrak{T}}$ and $\widehat{M}_{2,\mathfrak{T}}$ in Sections 3 and 4. For comparison purposes, we also implement the recent tests proposed by Shao and

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Zhang (2014, SZ_τ) and Zhang et al. (2018, ZYS_τ). Following Shao and Zhang (2014), page 1306, the SZ test statistic can be considered as $\widehat{\mathrm{SZ}}_\tau = -n^{-2}\sum_{i_1,i_2=1}^{n}\widehat{A}_{1,i_1i_2}B_{1,i_1i_2}$, where $\widehat{A}_{1,i_1i_2}$ and B_{1,i_1i_2} are the same as equation (2.4). Let $\mathbf{x}_i=(X_{i1},\ldots,X_{iq})^{\mathrm{T}}$ and define

$$B_{2,i_1i_2}(j) = b_{2,i_1i_2}(j)-(n-2)^{-1}\sum_{i_3=1}^{n}b_{2,i_1i_3}(j)-(n-2)^{-1}\sum_{i_4=1}^{n}b_{2,i_4i_2}(j) +\{(n-1)(n-2)\}^{-1}\sum_{i_3,i_4=1}^{n}b_{2,i_3i_4}(j),$$

with $b_{2,i_1i_2}(j)=\mid X_{i_1j}-X_{i_2j}\mid, j=1,\ldots,q$. Following Zhang et al. (2018), pages 224 and 225, the ZYS test statistic is computed as $\widehat{\mathrm{ZYS}}_\tau=\sum_{j=1}^{q}\{n(n-3)\}^{-1}\sum_{i_1\neq i_2}^{n}\widehat{A}_{2,i_1i_2}B_{2,i_1i_2}(j)$, where $\widehat{A}_{2,i_1i_2}$ is the same as equation (2.5). We use the EDMeasure library of R to compute the SZ and ZYS statistics by the mdd function in that package. The critical value of the SZ or ZYS test is obtained using the wild bootstrap approximation in Section 2.4 of Zhang et al. (2018). The number of bootstrap replications is 500. For the proposed testing procedures at a given quantile level or across multiple quantiles, we suggest the use of the simulation-based methods in Remarks 4 and 5 to provide critical values. We apply each test at a single quantile level $\tau=0.25$, $\tau=0.5$ and $\tau=0.75$, separately. For our joint tests, a set of quantile levels, that is, $\mathfrak{T}=\{0.25,0.5,0.75\}$ is employed. The significance level α is fixed at 0.05, and for each test, the rejection probabilities reported

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in the simulation are computed based on 1000 Monte Carlo replications.

First, generate the data according to

$$\text{A linear model} : Y = \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} + \epsilon, \tag{5.1}$$

$$\text{A transformation model} : Y = \sin(\mathbf{x}^{\mathrm{T}}\boldsymbol{\beta} + \epsilon), \tag{5.2}$$

$$\text{and} \quad \text{A multiple-index model} : Y = \mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_1 + \exp(\mathbf{x}^{\mathrm{T}}\boldsymbol{\beta}_2) + \epsilon. \tag{5.3}$$

In the linear model (5.1) and transformation model (5.2), we consider $q = 2$, $\boldsymbol{\beta} = (0, 1/3)^{\mathrm{T}}$, $X_1 \sim N(0, 1/9)$ or $t(3)$, $X_2 \sim N(0, 1)$ and $\epsilon \sim N(0, 1)$. In the multiple-index model (5.3), we let $q = 4$, $\boldsymbol{\beta}_1 = (0, 1/4, 0, 0)^{\mathrm{T}}$, $\boldsymbol{\beta}_2 = (0, 0, 1/5, 1/5)^{\mathrm{T}}$, $X_1 \sim N(0, 1/9)$ or $t(3)$, $X_2 \sim N(0, 1)$, $X_3 \sim N(0, 1)$, $X_4 \sim N(0, 1)$ and $\epsilon \sim N(0, 1)$. The covariates $X_1, \ldots, X_4$ and ϵ are mutually independent. Under the null, $\boldsymbol{\beta} = (0, 0)^{\mathrm{T}}$ and $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = (0, 0, 0, 0)^{\mathrm{T}}$. For each model, a random sample of size $n = 200$ of $(Y, X_1, \ldots, X_q)$ is generated.

Table 1 displays the critical values of asymptotic null distributions of the center-outward rank-based MDD test statistics for the dimension $q \in \{2, 4\}$ and the score function $J(x) \in \{F^{-1}_{(\chi^2_q)^{1/2}}(x), x, 1\}$ at the significance level $\alpha = 0.05$. Here all the critical values are estimated numerically using the method described in Remarks 4 and 5. We report the empirical sizes of the proposed tests with van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ and the SZ, ZYS tests, presented in Table 2. The simulated size results

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for the proposed tests with Wilcoxon score function $J(x) = x$ and sign score function $J(x) = 1$ are summarized in Table 4. It can be observed that the proposed tests with either rejection threshold as well as their two competitors control the size effectively.

We provide in Table 3 the empirical powers of the proposed tests with van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ as well as those of the SZ and ZYS tests for Models (5.1)-(5.3). For the proposed tests, we further present power results for Wilcoxon score function $J(x) = x$ and sign score function $J(x) = 1$ in Table 5. We can draw several useful conclusions from Tables 3 and 5. The first conclusion is that for heavy-tailed $t(3)$ covariates, the tests via martingale difference divergence with center-outward ranks and signs perform better than the original martingale difference divergence tests. Moreover, our tests have very much comparable powers when all covariates are normal. This is expected since the proposed tests are well-defined without any moment assumptions. The second conclusion is that compared with the other methods at a single quantile level, the proposed tests across quantiles provide relatively high power across all scenarios that are considered. This is not surprising because our joint MDD tests pool information across quantiles. The last but not least conclusion is that the proposed tests based on van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$

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and Wilcoxon score function $J(x) = x$ have more substantial power gain than the MDD-based test with sign score function $J(x) = 1$ does.

We also report the additional simulation results for $X_1 \sim N(0,1)$ in Table 6 when van der Waerden score is adopted, while keeping the rest of the set-up the same as in Models (5.1)-(5.3). These simulations yield the same conclusion as that of Table 3 for $X_1 \sim t(3)$ with $\text{var}(X_1) = 2$. This is anticipated in that X_1 is useless to model the quantiles of the response variable. As the variance of X_1 increases to 1 and 2, both SZ and ZYS tests have substantial power loss. In contrast, owing to the nonparametric property of center-outward ranks and signs, our proposed tests retain the high power under such alternatives.

5.2 Data example

We consider a gene expression dataset, which comes from a study by Scheetz et al. (2006), to illustrate the effectiveness of our proposed tests. This gene expression dataset consists of 120 arrays, each of which contains 31,042 probe sets, which is analyzed on a logarithmic scale. This dataset has been analyzed by a few statisticians, including Fan et al. (2011), Wang et al. (2012) and Shao and Zhang (2014), among others. The empirical observations of previous studies reveal that there exists nonlinear relationship be-

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Table 1: The quantiles table of asymptotic null distributions of the proposed MDD test statistics with different score functions at the nominal level 5%.

	$\tau = 0.25$		$\tau = 0.5$		$\tau = 0.75$		$\mathfrak{T} = \{0.25, 0.5, 0.75\}$			
q	$\widehat{V}_\tau$	$\widehat{U}_\tau$	$\widehat{V}_\tau$	$\widehat{U}_\tau$	$\widehat{V}_\tau$	$\widehat{U}_\tau$	$\widehat{S}_{1,\mathfrak{T}}$	$\widehat{S}_{2,\mathfrak{T}}$	$\widehat{M}_{1,\mathfrak{T}}$	$\widehat{M}_{2,\mathfrak{T}}$
van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$										
2	0.6667	0.3716	0.8489	0.4559	0.6665	0.3722	1.8182	0.8395	0.9416	0.5806
4	0.8585	0.3894	1.1150	0.4937	0.8106	0.3504	2.4455	0.8847	1.1475	0.6010
7	1.0481	0.3992	1.3493	0.4869	1.0669	0.4234	3.0977	0.9207	1.3762	0.5849
Wilcoxon score function $J(x) = x$										
2	0.2818	0.1558	0.3523	0.1851	0.2794	0.1520	0.7717	0.3567	0.3814	0.2347
4	0.2318	0.1001	0.3038	0.1287	0.2191	0.0887	0.6829	0.2489	0.3211	0.1611
7	0.2054	0.0716	0.2686	0.0909	0.2094	0.0759	0.6130	0.1693	0.2759	0.1068
sign score function $J(x) = 1$										
2	0.5837	0.3544	0.7390	0.4323	0.5624	0.3332	1.5631	0.7877	0.8297	0.5309
4	0.4752	0.2282	0.6130	0.2802	0.4487	0.2009	1.3375	0.5101	0.6346	0.3308
7	0.4149	0.1603	0.5368	0.1986	0.4244	0.1679	1.2285	0.3846	0.5567	0.2455

tween gene expression values. The response Y is the gene TRIM32 at probe 1389163_at, which has been found to cause the Bardet-Biedl syndrome (Chiang et al., 2006). For the remaining genes, we rank q probe sets that have the largest variances as the covariates, following Zhou et al. (2024). We choose the dimension of covariates be the same with that used in simulation studies, namely $q = 2, 4, 7$. We apply the proposed tests based on van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ to examine conditional quantile independence between gene expression levels, with comparison to Shao and Zhang (2014) and Zhang et al. (2018). In this analysis, we consider the

Table 2: The empirical sizes of the proposed tests with van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ and the SZ, ZYS tests under different quantiles at the nominal level 5% for Models (5.1)-(5.3).

	normal covariate			$t(3)$ covariate		
Method	Model (5.1)	Model (5.2)	Model (5.3)	Model (5.1)	Model (5.2)	Model (5.3)
			$\tau = 0.25$			
$\widehat{V}_\tau$	0.035	0.036	0.043	0.036	0.034	0.035
$\widehat{U}_\tau$	0.035	0.034	0.044	0.038	0.033	0.036
SZ_τ	0.042	0.040	0.050	0.040	0.041	0.050
ZYS_τ	0.049	0.046	0.052	0.041	0.040	0.047
			$\tau = 0.5$			
$\widehat{V}_\tau$	0.060	0.061	0.048	0.055	0.051	0.052
$\widehat{U}_\tau$	0.061	0.061	0.048	0.055	0.052	0.054
SZ_τ	0.048	0.052	0.045	0.048	0.046	0.051
ZYS_τ	0.052	0.057	0.054	0.038	0.037	0.049
			$\tau = 0.75$			
$\widehat{V}_\tau$	0.043	0.045	0.058	0.045	0.042	0.048
$\widehat{U}_\tau$	0.043	0.044	0.057	0.044	0.042	0.053
SZ_τ	0.058	0.051	0.058	0.046	0.037	0.056
ZYS_τ	0.054	0.051	0.055	0.044	0.042	0.055
			$\mathfrak{T} = \{0.25, 0.5, 0.75\}$			
$\widehat{S}_{1,\mathfrak{T}}$	0.055	0.053	0.046	0.045	0.047	0.047
$\widehat{S}_{2,\mathfrak{T}}$	0.055	0.052	0.046	0.045	0.047	0.048
$\widehat{M}_{1,\mathfrak{T}}$	0.057	0.058	0.053	0.054	0.052	0.055
$\widehat{M}_{2,\mathfrak{T}}$	0.059	0.061	0.051	0.052	0.050	0.049

center-outward van der Waerden tests across three quantiles 0.25, 0.5 and 0.75, and the SZ, ZYS tests at a single quantile level $\tau \in \{0.25, 0.5, 0.75\}$. To evaluate the power performances of the ten tests with nominal significance level 0.05, we randomly select subsets of size 100 from the whole data set, to calculate the test statistics. We repeat this random selection pro-

Table 3: The empirical powers of the proposed tests with van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ and the SZ, ZYS tests under different quantiles at the nominal level 5% for Models (5.1)-(5.3).

	normal covariate			$t(3)$ covariate		
Method	Model (5.1)	Model (5.2)	Model (5.3)	Model (5.1)	Model (5.2)	Model (5.3)
			$\tau = 0.25$			
$\widehat{V}_\tau$	0.779	0.719	0.666	0.748	0.689	0.646
$\widehat{U}_\tau$	0.781	0.720	0.668	0.743	0.691	0.649
SZ_τ	0.823	0.780	0.279	0.141	0.122	0.087
ZYS_τ	0.823	0.775	0.563	0.374	0.310	0.279
			$\tau = 0.5$			
$\widehat{V}_\tau$	0.886	0.861	0.802	0.850	0.839	0.770
$\widehat{U}_\tau$	0.888	0.864	0.802	0.851	0.841	0.773
SZ_τ	0.909	0.887	0.396	0.176	0.164	0.106
ZYS_τ	0.903	0.881	0.700	0.459	0.440	0.389
			$\tau = 0.75$			
$\widehat{V}_\tau$	0.800	0.733	0.761	0.759	0.693	0.716
$\widehat{U}_\tau$	0.803	0.735	0.753	0.763	0.694	0.710
SZ_τ	0.828	0.775	0.341	0.146	0.121	0.090
ZYS_τ	0.820	0.767	0.633	0.380	0.324	0.317
			$\mathfrak{T} = \{0.25, 0.5, 0.75\}$			
$\widehat{S}_{1,\mathfrak{T}}$	0.944	0.918	0.902	0.928	0.892	0.877
$\widehat{S}_{2,\mathfrak{T}}$	0.942	0.919	0.901	0.927	0.892	0.878
$\widehat{M}_{1,\mathfrak{T}}$	0.909	0.883	0.841	0.876	0.848	0.809
$\widehat{M}_{2,\mathfrak{T}}$	0.913	0.889	0.833	0.890	0.866	0.803

cedure 1000 times and report the empirical powers of the tests in Table 7. It suggests that the incorporation of score functions may bring potentially significant efficiency gain for the quantile MDD-based tests.

Table 4: The empirical sizes of the proposed tests with Wilcoxon score function $J(x) = x$ and sign score function $J(x) = 1$ under different quantiles at the nominal level 5% for Models (5.1)-(5.3).

	normal covariate			$t(3)$ covariate		
Method	Model (5.1)	Model (5.2)	Model (5.3)	Model (5.1)	Model (5.2)	Model (5.3)
			Wilcoxon score function $J(x) = x$			
$\widehat{V}_{0.25}$	0.041	0.042	0.048	0.036	0.035	0.041
$\widehat{U}_{0.25}$	0.039	0.041	0.045	0.038	0.034	0.043
$\widehat{V}_{0.5}$	0.053	0.049	0.041	0.049	0.052	0.045
$\widehat{U}_{0.5}$	0.050	0.052	0.043	0.050	0.052	0.046
$\widehat{V}_{0.75}$	0.036	0.036	0.054	0.045	0.044	0.053
$\widehat{U}_{0.75}$	0.038	0.037	0.055	0.047	0.046	0.050
$\widehat{S}_{1,\mathfrak{T}}$	0.043	0.042	0.033	0.043	0.043	0.038
$\widehat{S}_{2,\mathfrak{T}}$	0.040	0.042	0.032	0.041	0.040	0.035
$\widehat{M}_{1,\mathfrak{T}}$	0.045	0.047	0.041	0.050	0.051	0.039
$\widehat{M}_{2,\mathfrak{T}}$	0.042	0.048	0.044	0.048	0.050	0.040
			sign score function $J(x) = 1$			
$\widehat{V}_{0.25}$	0.039	0.041	0.044	0.042	0.040	0.036
$\widehat{U}_{0.25}$	0.037	0.040	0.042	0.043	0.044	0.039
$\widehat{V}_{0.5}$	0.050	0.044	0.045	0.051	0.053	0.053
$\widehat{U}_{0.5}$	0.049	0.046	0.047	0.051	0.055	0.056
$\widehat{V}_{0.75}$	0.043	0.043	0.054	0.050	0.052	0.046
$\widehat{U}_{0.75}$	0.041	0.042	0.053	0.048	0.051	0.044
$\widehat{S}_{1,\mathfrak{T}}$	0.046	0.047	0.045	0.045	0.038	0.051
$\widehat{S}_{2,\mathfrak{T}}$	0.044	0.046	0.043	0.046	0.041	0.048
$\widehat{M}_{1,\mathfrak{T}}$	0.040	0.044	0.051	0.043	0.042	0.052
$\widehat{M}_{2,\mathfrak{T}}$	0.047	0.050	0.058	0.047	0.048	0.054

6. Discussion

This paper provides the martingale-difference-divergence-based framework for specifying conditional quantile dependence measures that leverage the

Table 5: The empirical powers of the proposed tests with Wilcoxon score function $J(x) = x$ and sign score function $J(x) = 1$ under different quantiles at the nominal level 5% for Models (5.1)-(5.3).

	normal covariate			$t(3)$ covariate		
Method	Model (5.1)	Model (5.2)	Model (5.3)	Model (5.1)	Model (5.2)	Model (5.3)
Wilcoxon score function $J(x) = x$						
$\widehat{V}_{0.25}$	0.782	0.730	0.673	0.754	0.690	0.635
$\widehat{U}_{0.25}$	0.781	0.728	0.674	0.758	0.694	0.637
$\widehat{V}_{0.5}$	0.891	0.873	0.809	0.850	0.834	0.779
$\widehat{U}_{0.5}$	0.890	0.872	0.811	0.852	0.835	0.782
$\widehat{V}_{0.75}$	0.797	0.741	0.768	0.759	0.692	0.744
$\widehat{U}_{0.75}$	0.799	0.744	0.763	0.760	0.698	0.735
$\widehat{S}_{1,\mathfrak{T}}$	0.944	0.928	0.890	0.929	0.902	0.866
$\widehat{S}_{2,\mathfrak{T}}$	0.943	0.924	0.885	0.926	0.899	0.861
$\widehat{M}_{1,\mathfrak{T}}$	0.914	0.896	0.821	0.887	0.862	0.789
$\widehat{M}_{2,\mathfrak{T}}$	0.917	0.894	0.830	0.892	0.859	0.796
sign score function $J(x) = 1$						
$\widehat{V}_{0.25}$	0.712	0.637	0.630	0.675	0.617	0.592
$\widehat{U}_{0.25}$	0.710	0.638	0.628	0.674	0.615	0.591
$\widehat{V}_{0.5}$	0.805	0.782	0.781	0.777	0.760	0.744
$\widehat{U}_{0.5}$	0.804	0.780	0.783	0.777	0.759	0.747
$\widehat{V}_{0.75}$	0.743	0.669	0.724	0.693	0.625	0.693
$\widehat{U}_{0.75}$	0.740	0.666	0.727	0.692	0.628	0.690
$\widehat{S}_{1,\mathfrak{T}}$	0.903	0.869	0.876	0.884	0.848	0.846
$\widehat{S}_{2,\mathfrak{T}}$	0.904	0.872	0.880	0.886	0.850	0.847
$\widehat{M}_{1,\mathfrak{T}}$	0.832	0.790	0.819	0.806	0.784	0.780
$\widehat{M}_{2,\mathfrak{T}}$	0.858	0.814	0.831	0.828	0.805	0.792

new concept of center-outward ranks and signs. Asymptotic distributions of the proposed test statistics are studied under the null, local and fixed alternatives by using tools related to U-type and V-type statistics indexed by parameters. The associated independence tests have the strong appeal

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Table 6: The empirical sizes and powers of the proposed tests with van der Waerden score function $J(x) = F^{-1}_{(\chi^2_q)^{1/2}}(x)$ and the SZ, ZYS tests under different quantiles at the nominal level 5% when $X_1 \sim N(0,1)$ under Models (5.1)-(5.3).

	Size			Power		
Method	Model (5.9)	Model (5.10)	Model (5.11)	Model (5.9)	Model (5.10)	Model (5.11)
			$\tau = 0.25$			
$\widehat{V}_\tau$	0.037	0.033	0.046	0.751	0.674	0.657
$\widehat{U}_\tau$	0.040	0.035	0.046	0.752	0.674	0.660
SZ_τ	0.049	0.046	0.040	0.234	0.190	0.111
ZYS_τ	0.039	0.037	0.045	0.491	0.420	0.373
			$\tau = 0.5$			
$\widehat{V}_\tau$	0.056	0.057	0.047	0.853	0.834	0.793
$\widehat{U}_\tau$	0.058	0.061	0.045	0.856	0.836	0.792
SZ_τ	0.052	0.048	0.044	0.308	0.288	0.122
ZYS_τ	0.057	0.055	0.046	0.581	0.548	0.499
			$\tau = 0.75$			
$\widehat{V}_\tau$	0.047	0.045	0.051	0.774	0.708	0.736
$\widehat{U}_\tau$	0.047	0.046	0.048	0.773	0.712	0.736
SZ_τ	0.056	0.060	0.048	0.249	0.201	0.125
ZYS_τ	0.058	0.063	0.049	0.494	0.424	0.447
			$\mathfrak{T} = \{0.25, 0.5, 0.75\}$			
$\widehat{S}_{1,\mathfrak{T}}$	0.049	0.051	0.043	0.922	0.897	0.890
$\widehat{S}_{2,\mathfrak{T}}$	0.050	0.051	0.044	0.922	0.893	0.892
$\widehat{M}_{1,\mathfrak{T}}$	0.049	0.050	0.047	0.871	0.842	0.823
$\widehat{M}_{2,\mathfrak{T}}$	0.046	0.048	0.046	0.881	0.848	0.827

of being asymptotically distribution-free. Via the incorporation of score functions, our framework allows one to construct a variety of consistent center-outward versions of the quantile martingale difference divergence tests. This can lead to significant gains in power, as our numerical experi-

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Table 7: The empirical powers of the center-outward van der Waerden tests across quantiles and the SZ, ZYS tests at a single quantile level in the analysis of the gene expression dataset.

$\widehat{S}_{1,\mathfrak{T}}$	$\widehat{S}_{2,\mathfrak{T}}$	$\widehat{M}_{1,\mathfrak{T}}$	$\widehat{M}_{2,\mathfrak{T}}$	$\mathrm{SZ}_{0.25}$	$\mathrm{SZ}_{0.5}$	$\mathrm{SZ}_{0.75}$	$\mathrm{ZYS}_{0.25}$	$\mathrm{ZYS}_{0.5}$	$\mathrm{ZYS}_{0.75}$
				$q = 2$					
0.688	0.690	0.612	0.650	0.491	0.458	0.307	0.475	0.424	0.256
				$q = 4$					
0.594	0.598	0.562	0.551	0.386	0.422	0.274	0.342	0.419	0.258
				$q = 7$					
0.416	0.428	0.407	0.418	0.319	0.387	0.249	0.462	0.471	0.383

ments demonstrate.

Consider that the convergence rate of empirical center-outward ranks and signs becomes slow as the dimension of covariates increases. When the dimension is moderately large relative to the sample size, the center-outward score tests are expected to suffer from power loss. It can be easily checked that when $q = 1$, $\mathbf{F}_{\pm}$ reduces to $2F - 1$, where F is the usual cumulative distribution function. For $j = 1, \dots, q$, let $\widehat{F}_{X_j}$ be the empirical distribution function of X_j. To enhance the power for large dimension, it would be interesting to follow the idea of Zhang et al. (2018) to construct the QMDD-based score statistic $\sum_{j=1}^{q}\{n(n-3)\}^{-1}\sum_{i_1\neq i_2}^{n}\widehat{A}_{2,i_1i_2}\widehat{B}_{2,i_1i_2}(j)$, where $\widehat{B}_{2,i_1i_2}(j) = \widehat{b}_{2,i_1i_2}(j) - (n-2)^{-1}\sum_{i_3=1}^{n}\widehat{b}_{2,i_1i_3}(j) - (n-2)^{-1}\sum_{i_4=1}^{n}\widehat{b}_{2,i_4i_2}(j) + \{(n-1)(n-2)\}^{-1}\sum_{i_3,i_4=1}^{n}\widehat{b}_{2,i_3i_4}(j)$, with $\widehat{b}_{2,i_1i_2}(j) = \mid \widehat{G}_{X_j}(X_{i_1j}) - \widehat{G}_{X_j}(X_{i_2j}) \mid$

and $\widehat{G}_{X_j}(X_{i_1 j}) = J\{|\ 2\widehat{F}_{X_j}(X_{i_1 j}) - 1\ |\}\text{sign}\{2\widehat{F}_{X_j}(X_{i_1 j}) - 1\}, j = 1, \ldots, q.$ Deriving the asymptotic distribution of the aggregated test statistic in high dimensions is certainly more challenging and is left for future work.

Supplementary Material

The online supplementary material contains all technical proofs, and more numerical results on some aspects of limiting distributions and comparison under moderate dimension.

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References

Baringhaus, L. and C. Franz (2004). On a new multivariate two-sample test. *Journal of Multivariate Analysis 88*, 190–206.

Chernozhukov, V., A. Galichon, M. Hallin, and M. Henry (2017). Monge-kantorovich depth, quantiles, ranks and signs. *The Annals of Statistics 45*, 223–256.

Chiang, A. P., J. S. Beck, H.-J. Yen, M. K. Tayeh, T. E. Scheetz, R. E. Swiderski, D. Y. Nishimura, T. A. Braun, K.-Y. A. Kim, J. Huang, et al. (2006). Homozygosity mapping with snp arrays identifies trim32, an e3 ubiquitin ligase, as a bardet–biedl syndrome gene (bbs11). *Proceedings of the National Academy of Sciences 103*, 6287–6292.

Conde-Amboage, M., C. Sánchez-Sellero, and W. González-Manteiga (2015). A lack-of-fit test for quantile regression models with high-dimensional covariates. *Computational Statistics and Data Analysis 88*, 128–138.

Dong, C., G. Li, and X. Feng (2019). Lack-of-fit tests for quantile regression models. *Journal of the Royal Statistical Society Series B: Statistical Methodology 81*, 629–648.

Escanciano, J. C. and S. C. Goh (2014). Specification analysis of linear quantile models. *Journal of Econometrics 178*, 495–507.

Escanciano, J. C. and C. Velasco (2010). Specification tests of parametric dynamic conditional quantiles. *Journal of Econometrics 159*, 209–221.

Fan, J., Y. Feng, and R. Song (2011). Nonparametric independence screening in sparse ultra-high-dimensional additive models. *Journal of the American Statistical Association 106*, 544–557.

Hallin, M. (2017). On distribution and quantile functions, ranks and signs in r^d: A measure transportation approach. *Available at https://ideas.repec.org/p/eca/wpaper/2013-*

258262.html.

Hallin, M., E. del Barrio, J. Cuesta-Albertos, and C. Matrán (2021). Distribution and quantile functions, ranks and signs in dimension d: A measure transportation approach. *The Annals of Statistics 49*, 1139–1165.

He, X. and L. Zhu (2003). A lack-of-fit test for quantile regression. *Journal of the American Statistical Association 98*, 1013–1022.

Horowitz, J. and V. G. Spokoiny (2002). An adaptive, rate-optimal test of linearity for median regression models. *Journal of the American Statistical Association 97*, 822–835.

Koenker, R. (2005). *Quantile Regression*. Cambridge University Press, New York.

Koenker, R. and G. Bassett (1978). Regression quantiles. *Econometrica 46*, 33–50.

Lee, C. E. and H. Hilafu (2022). Quantile martingale difference divergence for dimension reduction. *Statistica Sinica 32*, 65–87.

Lee, C. E. and X. Shao (2018). Martingale difference divergence matrix and its application to dimension reduction for stationary multivariate time series. *Journal of the American Statistical Association 113*, 216–229.

Lee, C. E., X. Zhang, and X. Shao (2020). Testing the conditional mean independence for functional data. *Biometrika 107*, 331–346.

Lehmann, E. L. and J. P. Romano (2005). *Testing Statistical Hypotheses (third edition)*. Springer, New York.

McKeague, I. and M. Qian (2015). An adaptive resampling test for detecting the presence of significant predictors. *Journal of the American Statistical Association 110*, 1422–1433.

Otsu, T. (2008). Conditional empirical likelihood estimation and inference for quantile regression models. *Journal of Econometrics 142*, 508–538.

Park, T., X. Shao, S. Yao, et al. (2015). Partial martingale difference correlation. *Electronic Journal of Statistics 9*, 1492–1517.

Scheetz, T. E., K.-Y. A. Kim, R. E. Swiderski, A. R. Philp, T. A. Braun, K. L. Knudtson, A. M. Dorrance, G. F. DiBona, J. Huang, T. L. Casavant, et al. (2006). Regulation of gene expression in the mammalian eye and its relevance to eye disease. *Proceedings of the National Academy of Sciences 103*, 14429–14434.

Serfling, R. (1980). *Approximation Theorems of Mathematical Statistics*. New York: Wiley.

Shao, X. and J. Zhang (2014). Martingale difference correlation and its use in high-dimensional variable screening. *Journal of the American Statistical Association 109*, 1302–1318.

Shi, H., M. Drton, and F. Han (2022). Distribution-free consistent independence tests via center-outward ranks and signs. *Journal of the American Statistical Association 117*, 395–410.

Shi, H., M. Hallin, M. Drton, and F. Han (2022). On universally consistent and fully distribution-free rank tests of vector independence. *The Annals of Statistics 50*, 1933–1959.

Stute, W. (1997). Nonparametric model checks for regression. *The Annals of Statistics 25*, 613–641.

Statistica Sinica: Preprint
doi:10.5705/ss.202024.0266

Su, L. and X. Zheng (2017). A martingale-difference-divergence-based test for specification. *Economics Letters 156*, 162–167.

Székely, G., M. Rizzo, and N. Bakirov (2007). Measuring and testing dependence by correlation of distances. *The Annals of Statistics 35*, 2769–2794.

Székely, G. J. and M. L. Rizzo (2009). Brownian distance covariance. *The Annals of Applied Statistics 3*, 1236–1265.

Székely, G. J. and M. L. Rizzo (2014). Partial distance correlation with methods for dissimilarities. *The Annals of Statistics 42*, 2382–2412.

van der Vaart, A. W. (1998). *Asymptotic Statistics*. Cambridge University Press, Cambridge.

Wang, H., I. McKeague, and M. Qian (2018). Testing for marginal linear effects in quantile regression. *Journal of the Royal Statistical Society: Series B 80*, 433–452.

Wang, L., Y. Wu, and R. Li (2012). Quantile regression for analyzing heterogeneity in ultra-high dimension. *Journal of the American Statistical Association 107*, 214–222.

Whang, Y.-J. (2006). Smoothed empirical likelihood methods for quantile regression models. *Econometric Theory 22*, 173–205.

Xu, K. and N. An (2024). A tuning-free efficient test for marginal linear effects in highdimensional quantile regression. *Annals of the Institute of Statistical Mathematics 76*, 93–110.

Xu, K. and F. Chen (2020). Martingale-difference-divergence-based tests for goodness-of-fit in quantile models. *Journal of Statistical Planning and Inference 207*, 138154.

Yao, S., X. Zhang, X. Shao, et al. (2018). Testing mutual independence in high dimension via distance covariance. *Journal of the Royal Statistical Society Series B 80*, 455–480.

Zhang, X., S. Yao, and X. Shao (2018). Conditional mean and quantile dependence testing in high dimension. *The Annals of Statistics 46*, 219–246.

Zheng, J. (1998). A consistent nonparametric test of parametric regression models under conditional quantile restrictions. *Econometric Theory 14*, 123–138.

Zhou, Y., K. Xu, L. Zhu, and R. Li (2024). Rank-based indices for testing independence between two high-dimensional vectors. *The Annals of Statistics 52*, 184–206.

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